Letter

Dynamics of the Fractional-Order Lorenz System Based on Adomian Decomposition Method and Its DSP Implementation

Shaobo He, Kehui Sun, and Huihai Wang

Dear Editor,

Dynamics and digital circuit implementation of the fractional-order Lorenz system are investigated by employing Adomian decomposition method (ADM). Dynamics of the fractional-order Lorenz system with derivative order and parameter varying is analyzed by means of Lyapunov exponents (LEs), bifurcation diagram, chaos diagram and phase diagram. Results show that the fractional-order Lorenz system has rich dynamical behaviors and it is a potential model for application. It is also found that the minimum order is affected by numerical algorithm and time step size. Finally, the fractional-order system is implemented on digital signal processor (DSP). The phase diagrams generated by the DSP are consistent with that generated by simulation.

Introduction: The fractional-order Lorenz system with a new set of parameters is firstly analyzed by I. Grigorenko and E. Grigorenko [1], and they reported that the system can generate chaos when the total order is 2.91 by a numerical method they derived. Unfortunately, an error was found among the derived numerical method, thus the result in this letter is not reliable [2]. More recently, Jia *et al*. [3] analyzed dynamics of this system with order q = 0.7, 0.8 and 0.9 and implemented it in analog circuit by employing frequency domain method (FDM) [4]. However, whether this method accurately reflects chaotic characteristics in fractional-order chaotic system was questioned [5]. Another method for solving fractional-order chaotic systems is the Adams-Bashforth-Moulton algorithm (ABM) [6]. It can be used to analyze dynamics with continuous derivative order [7], and some researches of the fractional-order chaos are based on this algorithm [8]. But the calculation speed of this algorithm is very slow, and it consumes too many computer resources [9]. Meanwhile, ADM [10] is employed to obtain numerical solution of the fractional-order chaotic system for its high precision and fast speed of convergence [11]. In addition, based on ADM, LEs of the fractional-order system is calculated [12]. Furthermore, circuit design is essential for application of fractional-order chaotic systems. Although analog circuit implementation is widely reported by researchers, digital circuit realization of the fractional-order chaotic system has better flexibility and repeatability [13]. So, we focus on the dynamics of the fractional-order Lorenz system and its DSP implementation by employing ADM in this letter.

Numerical solution for the fractional order Lorenz system: The fractional-order Lorenz system is presented by [1]

$$\begin{cases} D_{t_0}^{q}x = a(y-x) \\ D_{t_0}^{q}y = cx - xz + dy \\ D_{t_0}^{q}z = xy - bz \end{cases} \tag{1}$$

where a, b, c and d are the system parameters, and q is the derivative order. According to [1] and [3], we investigate dynamics and digital circuit realization of this system by fixing $a = 40$, $b = 3$, $c = 10$, and varying d and q. Because ADM converges very fast [11], we choose the first 6 terms of ADM polynomial for the approximate solution in this letter, and the numerical solution of the fractional-order Lorenz system is denoted by

$$\begin{cases} x_{n+1} = \sum_{j=0}^{6} \kappa_1^j h^{jq}/\Gamma(jq+1) \\ y_{n+1} = \sum_{j=0}^{6} \kappa_2^j h^{jq}/\Gamma(jq+1) \\ z_{n+1} = \sum_{j=0}^{6} \kappa_3^j h^{jq}/\Gamma(jq+1) \end{cases} \tag{2}$$

where h is the integration step-size, $\Gamma(\cdot)$ is the Gamma function, and $k_i^j(\cdot)$ are defined as

$$k_1^0 = x_n,\ \ k_2^0 = y_n,\ \ k_3^0 = z_n \tag{3}$$

$$\begin{cases} \kappa_1^1 = a\left(\kappa_2^0 - \kappa_1^0\right) \\ \kappa_2^1 = c\kappa_1^0 + d\kappa_2^0 - \kappa_1^0\kappa_3^0 \\ \kappa_3^1 = -b\kappa_3^0 + \kappa_1^0\kappa_2^0 \end{cases} \tag{4}$$

$$\begin{cases} \kappa_1^2 = a\left(\kappa_2^1 - \kappa_1^1\right) \\ \kappa_2^2 = c\kappa_1^1 + d\kappa_2^1 - \kappa_1^0\kappa_3^1 - \kappa_1^1\kappa_3^0 \\ \kappa_3^2 = \kappa_1^1\kappa_2^0 + \kappa_1^0\kappa_2^1 - b\kappa_3^1 \end{cases} \tag{5}$$

$$\begin{cases} \kappa_1^3 = a\left(\kappa_2^0 - \kappa_1^0\right) \\ \kappa_2^3 = c\kappa_1^2 + d\kappa_2^2 - \kappa_1^0\kappa_3^2 - \kappa_1^1\kappa_3^1 \dfrac{\Gamma(2q+1)}{\Gamma^2(q+1)} - \kappa_1^2\kappa_3^0 \\ \kappa_3^3 = \kappa_1^0\kappa_2^2 + \kappa_1^1\kappa_2^1 \dfrac{\Gamma(2q+1)}{\Gamma^2(q+1)} + \kappa_1^2\kappa_2^0 - b\kappa_3^2 \end{cases} \tag{6}$$

$$\begin{cases} \kappa_1^4 = a\left(\kappa_2^3 - \kappa_1^3\right) \\ \kappa_2^4 = c\kappa_1^3 + d\kappa_2^3 - \kappa_1^0\kappa_3^3 - \kappa_1^3\kappa_3^0 \\ \qquad - \left(\kappa_1^2\kappa_3^1 + \kappa_1^1\kappa_3^2\right) \dfrac{\Gamma(3q+1)}{\Gamma(q+1)\Gamma(2q+1)} \\ \kappa_3^4 = \kappa_1^0\kappa_2^3 + \kappa_1^3\kappa_2^0 + b\kappa_3^3 \\ \qquad + \left(\kappa_1^2\kappa_2^1 + \kappa_1^1\kappa_2^2\right) \dfrac{\Gamma(3q+1)}{\Gamma(q+1)\Gamma(2q+1)} \end{cases} \tag{7}$$

$$\begin{cases} \kappa_1^5 = a\left(\kappa_2^4 - \kappa_1^4\right) \\ \kappa_2^5 = c\kappa_1^4 + d\kappa_2^4 - \kappa_1^0\kappa_3^4 \\ \qquad - \left(\kappa_1^3\kappa_3^1 + \kappa_1^1\kappa_3^3\right) \dfrac{\Gamma(4q+1)}{\Gamma(q+1)\Gamma(3q+1)} \\ \qquad - \kappa_1^2\kappa_3^2 \dfrac{\Gamma(4q+1)}{\Gamma^2(2q+1)} - \kappa_1^4\kappa_3^0 \\ \kappa_3^5 = \kappa_1^0\kappa_2^4 + \left(\kappa_1^3\kappa_2^1 + \kappa_1^1\kappa_2^3\right) \dfrac{\Gamma(4q+1)}{\Gamma(q+1)\Gamma(3q+1)} \\ \qquad + \kappa_1^2\kappa_2^2 \dfrac{\Gamma(4q+1)}{\Gamma^2(2q+1)} + \kappa_1^4\kappa_2^0 - b\kappa_3^4 \end{cases} \tag{8}$$

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$$\begin{cases} \kappa_1^6 = a\left(\kappa_2^5 - \kappa_1^5\right) \\ \kappa_2^6 = c\kappa_1^5 + d\kappa_2^5 - \kappa_1^0\kappa_3^5 \\ \qquad - \left(\kappa_1^1\kappa_3^4 + \kappa_1^4\kappa_3^1\right)\dfrac{\Gamma(5q+1)}{\Gamma(q+1)\Gamma(4q+1)} \\ \qquad - \left(\kappa_1^2\kappa_3^3 + \kappa_1^3\kappa_3^2\right)\dfrac{\Gamma(5q+1)}{\Gamma(2q+1)\Gamma(3q+1)} - \kappa_1^5\kappa_3^0 \\ \kappa_3^6 = \kappa_1^0\kappa_2^5 + \kappa_1^5\kappa_2^0 - b\kappa_3^5 \\ \qquad + \left(\kappa_1^1\kappa_2^4 + \kappa_1^4\kappa_2^1\right)\dfrac{\Gamma(5q+1)}{\Gamma(q+1)\Gamma(4q+1)} \\ \qquad + \left(\kappa_1^2\kappa_2^3 + \kappa_1^3\kappa_2^2\right)\dfrac{\Gamma(5q+1)}{\Gamma(2q+1)\Gamma(3q+1)}. \end{cases} \tag{9}$$

According to (2), the chaotic sequences of the fractional-order Lorenz system are obtained with appropriate initial values. Meanwhile, (2) provides a necessary iterative for DSP implementation of the fractional-order Lorenz system.

Dynamics with parameter varying: In this section, dynamics of the fractional-order Lorenz system with system parameter d and derivate order q varying are investigated. Parameter fixed dynamical analysis method and chaos diagram are used. Here we set $N = 20\,000$ and $h = 0.01$. Three cases are investigated.

1) Fix $d = 25$, and vary derivative order q from 0.75 to 1 with step size of 0.0005. The bifurcation diagram and LEs are shown in Fig. 1. It shows that the system generates chaos for $0.813 \le q < 1$ excepting some periodic windows. Thus the minimum total order for the fractional-order Lorenz system to generate chaos is 2.439 and the corresponding phase diagram is shown in Fig. 2. In addition, the maximum Lyapunov exponent illustrates a decreasing trend as order q increasing.

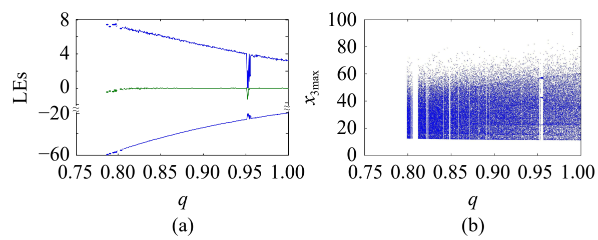

Fig. 1. Dynamics of the fractional-order Lorenz system with $d = 25$ and q varying. (a) LEs; (b) Bifurcation diagram.

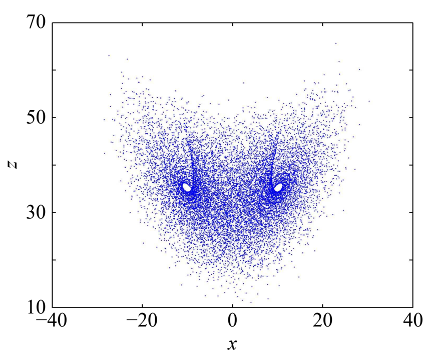

Fig. 2. Phase diagram of the fractional-order Lorenz system with d = 25 and $q = 0.813$.

2) Fix $q = 0.96$ and vary d from 0 to 38 with step size of 0.1. When d decreases from 38, the system presents periodical states until it enters into chaos at $d = 32.1$ by the period-doubling bifurcation as shown in Fig. 3(a). Chaos covers most of the range $d \in [9.8, 32.1]$ with several small periodic widows, like $d \in [14.5, 16.3] \cup [21.1, 21.5]$. Finally, the system becomes convergent at $d = 9.8$ by a tangent bifurcation. To observe dynamics better, phase diagrams are presented in Fig. 4. When d =15, 21.5, and 37, the system is periodic, and the system is chaotic when $d = 20$. It shows that the system presents different states with different parameter d.

3) Vary q from 0.75 to 1 with step size of 0.0025 and vary d from 0 to 38 with step size of 0.38 simultaneously. The maximum Lyapunov exponent based chaos diagram in q-d parameter plane is illustrated in Fig. 5. In this figure, we only draw the case when the maximum Lyapunov exponent is larger than zero. According to Fig. 5, chaos exists in the range of $d \in [10, 32]$. A high complexity region is observed within $d \in [25, 30]$ and $q \in [0.8, 0.97]$, which is favorable for practical application. So, the fractional-order Lorenz system is a good model for real application. The chaos diagram provides a parameter selection basis for the fractional-order Lorenz system in practical application. Compared with bifurcation analysis results based on FDM as shown in [3], results based on ADM are more detailed and accuracy. It also shows that we can analyze dynamics of the system with q varying continuously, but it is difficulty for FDM to do so.

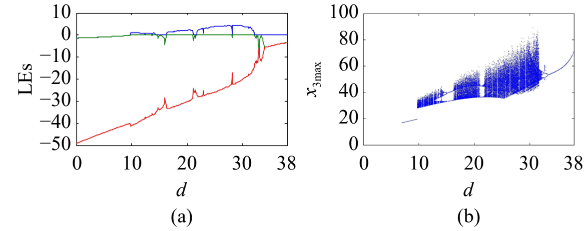

Fig. 3. Dynamics of the fractional-order Lorenz system with q = 0.96 and d varying. (a) LEs; (b) Bifurcation diagram.

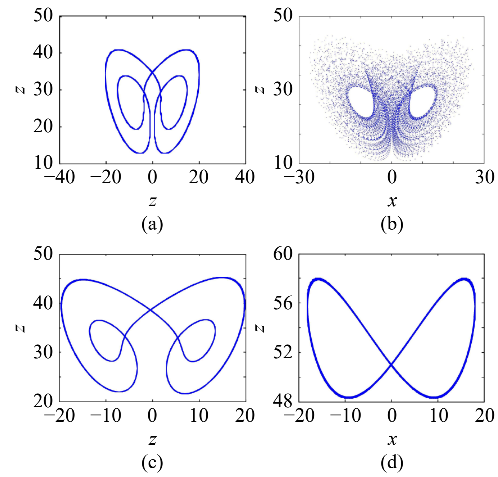

Fig. 4. Phase diagrams of the fractional-order Lorenz system with q = 0.96 and d varying. (a) d = 15; (b) d = 20; (c) d = 21.5; (d) d = 37.

Discussion about the minimum order: Obviously, the minimum order for chaos is different for different system parameter. But it is also different when the numerical solution algorithm or time step size h is different [14]. Thus these two aspects are discussed as follows.

1) Compared with other approaches, chaotic system has a much lower order if it is solved by ADM algorithm. The equilibrium point of this system is (0, 0, 0) and $\left(\pm\sqrt{b(c+d)}, \pm\sqrt{b(c+d)}, c+d\right)$. When $d = 25$, the eigenvalues at (0, 0, 0) are $\lambda_1 = -45.6608$, $\lambda_2 = 30.6608$, $\lambda_3 = -3.0000$, and the eigenvalues at $\left(\pm\sqrt{105}, \pm\sqrt{105}, 35\right)$ are $\lambda_1 = -25.2415$, $\lambda_2 = 3.6207 + 17.8795i$ and $\lambda_3 = 3.6207 - 17.8795i$. According to the stability theory, the lowest order q to generate chaos is

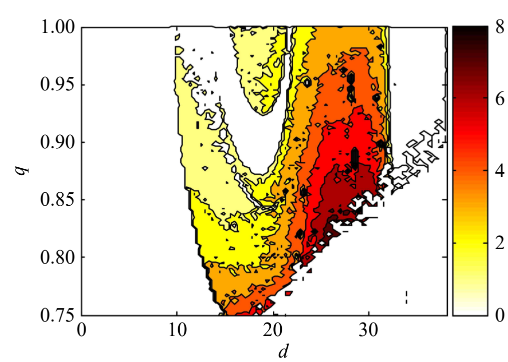

Fig. 5. Maximum Lyapunov exponent based chaos diagram.

q = 0.8726. It is not difficult to find out that ABM satisfies this result. However, FDM and ADM do not. According to [3], when q = 0.7, the system has rich dynamics and chaos still exists by applying FDM. According to Fig. 1, the minimum order of the system is q = 0.813 by applying ADM. Actually, the stability theory [15] is proposed to analyze fractional-order linear systems. For fractional-order nonlinear systems, Li *et al*. [16] proved that the stability theory does not always work when the specified matrix $J(X)$ is time-varying. We think that it is more complex to analyze stability of fractional-order nonlinear system. Besides, although FDM and ADM do not satisfy the stability theory, they are widely used and accepted by researchers. In addition, it shows in [17] that different results of a fractional-order system may be achieved when simulations are performed based on different numerical methods. Since FDM and ADM can obtain chaos at a much lower order, they extend the parameter space of fractional-order chaotic systems.

2) The effect of time step h should be further investigated. As for ADM, when h = 0.01, the lowest order to generate chaos is q = 0.813. We also find that the lowest order decrease with the decrease of the time step size h. As shown in Fig. 6, when h = 0.001, the lowest order is q = 0.505, and the lowest order is q = 0.402 for h = 0.0001. The system generates chaos with lower order when time step h is smaller, but more memory and computing resources are needed. It is not good for the real application. We think h = 0.01 is a suitable choice for general cases. However, the reason why the lowest order decreases with the decrease of time h needs further study.

According to the discussion above, when a minimum order for chaos generation of a fractional-order chaotic system is presented, the certain set of parameters, numerical algorithm and time step size should also be specified.

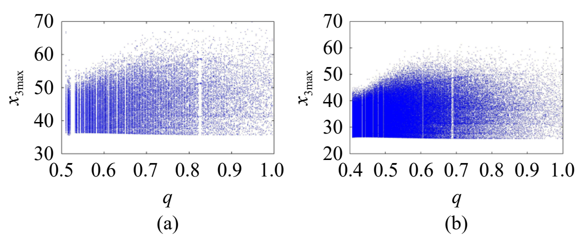

Fig. 6. Bifurcation diagrams of the fractional-order Lorenz system under different h. (a) h = 0.001; (b) h = 0.0001.

Digital circuit implementation: The DSP implementation method for the fractional-order chaotic systems is described in [13]. In this section, the fractional-order Lorenz system is realized according to (2)−(9). Here, the initial value is x_0 = [1 2 3]. Setting q = 0.8130, d = 25, the phase diagram is shown in Fig. 7(a). The corresponding MATLAB simulation result is illustrated in Fig. 2. Setting q = 0.96, d = 15, the phase diagram is shown in Fig. 7(b), and its MATLAB simulation counterpart is presented in Fig. 4(a). Setting q = 0.96 and varying d (d = 20 and d = 37), the phase diagrams are shown in Figs. 7(c) and 7(d). It can be seen that they consist with phase diagrams as shown in Figs. 4(b) and 4(d). It shows that the fractional-order Lorenz system is implemented on the DSP platform successfully. It lays a hardware foundation for the applications of the fractional-order Lorenz chaotic system.

Conclusion: In this letter, based on ADM algorithm, we investigated the dynamics of the fractional-order Lorenz system. It shows that the fractional-order Lorenz system has rich dynamical characteristics. The system is more complex with smaller derivate order q as the maximum Lyapunov exponent decreases with the increase of q. The lowest order for chaos generation is different according to different numerical algorithms. The fractional-order Lorenz system has a much lower order for chaos if it is solved by ADM algorithm. Meanwhile, the lowest order for chaos is smaller when the time step size h is smaller. Finally, the system is implemented by employing DSP, and phase diagrams generated by the DSP device are consistent with the simulation results. Our further work will focus on real applications of the fractional-order Lorenz system.

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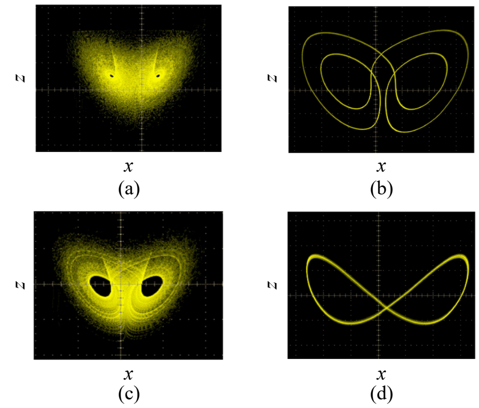

Fig. 7. Phase diagrams of the fractional-order Lorenz system recorded by the oscilloscope (a) q = 0.8130 and d = 25; (b) q = 0.96 and d = 15; (c) q = 0.96 and d = 20; (d) q = 0.96 and d = 37.

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