Letter

Predefined-Time Sliding Mode Control with Prescribed Convergent Region

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Dear editor,

In recent years, the finite-time and fixed-time control techniques have drawn much attention. This letter will present a new method for designing a predefined-time adaptive sliding mode controller with prescribed convergent region. More specifically, class $\mathcal{K}^1$ function is used to construct the sliding function, and to achieve a real sliding mode, the function is also adopted in designing the adaptive gain without knowing the disturbance's upper bound (DUB). Compared to the existing finite-time and fixed-time controllers, the key superiority of the proposed method is that the system can converge to a prescribed arbitrarily small region in predefined time irrespective of the initial condition. In addition, the control signal is bounded along the settling period, where the settling time instance can be estimated without conservation. The proposed method is applicable to the control of a wide range of uncertain nonlinear systems such as networked control systems with significant network-induced delay.

Related work: Conventional sliding mode control (SMC) requires the disturbance's upper bound (DUB) a priori, which generally results in undesired overly large control gain. The adaptive gain solves this problem but has other shortcomings. For instance, the earliest reported increasing-type adaptive gain [1] ensures convergence to the origin rather than a region, but it overestimates the disturbance since the gain does not decrease when the disturbance becomes small. Since then, new methods have been studied to moderate this issue but cannot yet arbitrarily define an exact convergent time upper bound irrelevant to the disturbance. For example, in the dead-zone algorithm [2], the sliding variable can be bounded within a region defined by the dead-zone size, but its bound size varies with respect to the disturbance's amplitude. Similar issue occurs in the leakage-type (LT) adaptive law [3]. More specifically, the convergent bound size under the LT algorithm is a non-zero constant even when the system is not disturbed. In other words, its convergent bound size cannot be arbitrarily small. As pointed out in [4], the barrier function (BF)-based adaptive gain can be automatically adjusted in accordance with the disturbance variation such that overestimation is removed. In addition, the BF-based adaptive gain ensures that the sliding variable is bounded within a predefined constant size regardless of the disturbance amplitude. However, the settling time is related to the initial condition and thus its upper bound of settling time (UBST) can not be arbitrarily determined as well.

Another shortcoming of conventional SMC is that it not only induces extra reaching time that is often associated with the initial condition but also reduces the control robustness because of the absence of sliding mode invariance in the reaching phase. The conventional integral sliding mode (ISM) removes the reaching phase at the cost of guarantee of asymptotic convergence only. Alternatively, terminal sliding mode (TSM) enhances the convergent speed and control precision without removal of reaching phase though. Further, an integral TSM with recursive structure [5] is proposed to provide the advantages of both ISM and TSM. Unfortunately, this method still requires the DUB and may induce singular control signals. After that, the adaptive control [6] and disturbance observer [7] are developed to solve these issues. However, all the existing methods cannot allow the designer to predefine a desired settling time irrespective of the initial condition. The conventional fixed-time SMC can define a constant UBST, but it is not allowed to be arbitrarily small [8], [9]. The predefined-time SMC provides more flexibility allowing the settling time to be arbitrarily small. Nevertheless, in many existing solutions, the UBST value is overestimated and relates to the initial condition [10], [11]. For the non-overestimation predefined-time SMC, the control signal tends to be unbounded if the state or its derivative is not identically equal to zero when the system approaches to the predefined settling time instant [12]. In addition, in both the conventional finite-time and fixed-time SMCs, they still have the reaching phase and require the DUB a priori [13].

This letter aims to provide an SMC scheme that does not require the DUB a priori and guarantees system's predefined-time stabilization towards a predefined region regardless of the initial condition. It will have the features of removing the reaching phase and allowing arbitrarily small UBST value. To achieve this goal, this letter proposes an adaptive class $\mathcal{K}^1$ function-based predefined-time SMC method. Compared to the state-of-the-art predefined-time controllers, the enhancement of the proposed method lies in four folds. First, the knowledge of the DUB is not required a priori. Second, the system output is bounded within an arbitrarily prescribed region. Third, the UBST can be prespecified as a constant irrespective of the initial condition and without overestimation. Fourth, the control output has no singularity. It is worth noting that the proposed method can achieve the above features simultaneously, which has not yet been available in the literature.

Preliminaries: Consider the following autonomous system:

$$\dot{\boldsymbol{x}} = \boldsymbol{h}(\boldsymbol{x};\boldsymbol{\rho}),\ \ \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{1}$$

where $\boldsymbol{x}:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ is the system state and the vector $\boldsymbol{\rho}\in\mathbb{R}^m$ stands for the tunable parameters of (1). $\boldsymbol{h}:\mathbb{R}^n\to\mathbb{R}^n$ is a function such that the solution of (1) exists and is unique in the sense of Filippov.

Definition 1 (Fixed-time stable [14]): The system (1) is finite-time stable if the settling-time function of (1), $T(\boldsymbol{x}_0)$, is bounded on $\mathbb{R}^n$, i.e., there exists $T_{\max}$ such that $\sup_{\boldsymbol{x}_0\in\mathbb{R}^n} T(\boldsymbol{x}_0)\leq T_{\max}<\infty$.

Definition 2 (Predefined-time stable [14]): The system (1) is predefined-time stable if it is fixed-time stable and for any $T_c\in\mathbb{R}_+$, there exists some $\boldsymbol{\rho}\in\mathbb{R}^m$ such that the settling-time function of (1) satisfies

$$\sup_{\boldsymbol{x}_0\in\mathbb{R}^n} T(\boldsymbol{x}_0)\leq T_c. \tag{2}$$

Definition 3 (PTUBPB [15]): A solution $\boldsymbol{\Phi}(t,\boldsymbol{x}_0)$ of (1) is predefined-time ultimately bounded with predefined bound (PTUBPB), if for any $T_c,\eta\in\mathbb{R}_+$, there exist some $\boldsymbol{\rho}\in\mathbb{R}^m$ such that for any $\boldsymbol{x}_0\in\mathbb{R}^n$, $\|\boldsymbol{\Phi}(t,\boldsymbol{x}_0)\|\leq\eta$, $\forall t\geq T_c$.

Definition 4 (Class $\mathcal{K}^1$ function [11]): A continuous scalar function $\kappa(y):\mathbb{R}_{\geq 0}\to[0,1)$ belongs to class $\mathcal{K}^1$ as denoted by $\kappa(y)\in\mathcal{K}^1$, if it is strictly increasing, and $\kappa(0)=0$ and $\lim_{y\to\infty}\kappa(y)\to 1$.

Property 1: The inverse of $\kappa(y)$ as denoted by $\kappa^{-1}(y):[0,1)\to\mathbb{R}_{\geq 0}$, is strictly increasing. Furthermore, $\kappa^{-1}(0)=0$ and $\lim_{y\to 1^-}\kappa(y)\to\infty$.

Main results: Consider the following second-order uncertain

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system:

$$\ddot{x} = f(x,\dot{x}) + g(x,\dot{x})u + \xi \tag{3}$$

where u is control input, x is system state, $f(x,\dot{x})$ and $g(x,\dot{x})$ are known smooth positive functions with $g(x,\dot{x}) \neq 0$; and ξ is an unknown function that comprises the parameter uncertainty, modelling error and unknown disturbance and satisfies that $|\xi| \le \bar{\xi}$ with $\bar{\xi}$ being an unknown positive constant.

First, consider the system (4) and let the sliding functions be

$$\sigma(t) = \dot{x} + \kappa_{\rm m}^{-1}(t)x \tag{4}$$

$$s(t) = \sigma(t) + z(t) \tag{5}$$

where $\kappa_{\rm m}^{-1}(t)$ is given by

$$\kappa_{\rm m}^{-1}(t) = \min\{\kappa^{-1}(\alpha t), \lambda\} \tag{6}$$

with $\alpha > 0$, $\lambda = \kappa^{-1}(\alpha t_x)$ and $0 < t_x < \alpha^{-1}$. As shown in Fig. 1, the function $\kappa_{\rm m}^{-1}(t)$ starts from zero at $t = 0$ and monotonically increases to λ as $t \to t_x$, and remains at that level for any $t \ge t_x$. $z(t)$ is an auxiliary function satisfying

1) $z(0) = -\sigma(0) = -\dot{x}(0)$;
2) $z(t) \to 0$ as $t \to t_f$ and $z(t) = 0$ for $t \ge t_f$;
3) $\dot{z}(t)$ exists and is bounded, and $\dot{z}(t_f) = 0$;

with $0 < t_f < t_x$.

Theorem 1: For the sliding functions (5) and (6), if s is bounded within the region of $(-\varepsilon, \varepsilon)$, $\forall t \ge 0$, then there exists a sufficiently large α such that the system state x will be terminally bounded within the region of $(-\varepsilon/\lambda, \varepsilon/\lambda)$, $\forall t \ge T_c$ with T_c an arbitrarily small constant satisfying $T_c > t_x$.

Proof: It can be seen that due to the operation of $z(t)$ in (6), we have $\sigma = s \in (-\varepsilon, \varepsilon)$, $\forall t \ge t_f$. Then, (5) and (6) can be rewritten as

$$\dot{x} + \kappa_{\rm m}^{-1}(t)x = s. \tag{7}$$

Define a positive constant $\zeta = \int_{t_f}^{t_x} \kappa^{-1}(\alpha t)dt$. Then, $\forall t \ge t_f$, we have

$$\begin{aligned}
|x(t)| &= e^{-\int_{t_f}^{t}\kappa_{\rm m}^{-1}(t)dt}\left|x(t_f) + \int_{t_f}^{t} s(t)e^{\int_{t_f}^{t}\kappa_{\rm m}^{-1}(t)dt}dt\right| \\
&= e^{-\zeta-\lambda(t-t_x)}\left|x(t_f) + \int_{t_f}^{t} s(t)e^{\zeta+\lambda(t-t_x)}dt\right| \\
&< \left|x(t_f)\right|e^{-\lambda(t-t_x)} + \varepsilon\int_{t_f}^{t} e^{-\lambda(t-\tau)}d\tau \\
&= |x(t_f)|e^{-\lambda(t-t_x)} + \frac{\varepsilon}{\lambda}\left(1 - e^{-\lambda(t-t_f)}\right).
\end{aligned} \tag{8}$$

The above inequality implies that $|x(t)|$ is exponentially-decreasing. It can be seen that for any t_x, there exists a sufficiently large $\alpha < t_x^{-1}$ such that $\lambda = \kappa^{-1}(\alpha t_x)$ is sufficiently large. Therefore, for an arbitrarily small $T_c > t_x$, the initial state $x(t_f)$ is not associated with (8), i.e., $|x(t)| < \varepsilon/\lambda$, $\forall t \ge T_c$. ■

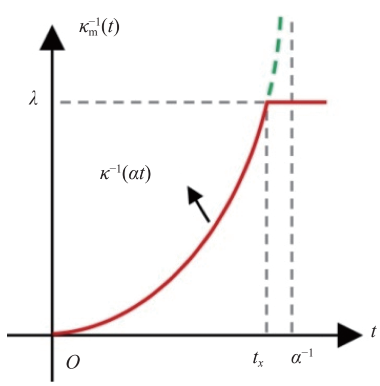

Fig. 1. Plot of the function (6).

Next, we propose the following adaptive control input:

$$u = -\frac{1}{g}\left(\dot{\kappa}_{\rm m}^{-1}(t)x + \kappa_{\rm m}^{-1}(t)\dot{x} + \dot{z}(t) + f + \kappa^{-1}(\varepsilon^{-1}|s|)\mathrm{sgn}(s)\right). \tag{9}$$

Theorem 2: Consider the system (4) under the proposed control (9) with its sliding functions as designed in (5) and (6). Then, the closed-loop system will enter a real sliding mode [2] from the initial time, i.e., the sliding variable satisfying $|s| < \varepsilon$, $\forall t \ge 0$.

Proof: Defining a Lyapunov function $V = 0.5s^2$ and combining (4) with (1)–(6) and (9), we have

$$\begin{aligned}
\dot{V} &= s\left(\ddot{x} + \dot{\kappa}_{\rm m}^{-1}(t)x + \kappa_{\rm m}^{-1}(t)\dot{x} + \dot{z}(t)\right) \\
&= s\left(-\kappa^{-1}(\varepsilon^{-1}|s|)\mathrm{sgn}(s) + \xi\right) \\
&\le -\left(\kappa^{-1}(\varepsilon^{-1}|s|) - \bar{\xi}\right)|s|.
\end{aligned} \tag{10}$$

It is clear that $s(0) = 0$, and in the domain of $[0, \varepsilon)$, there must exist a positive number $\bar{s} = \varepsilon\kappa(\bar{\xi}) < \varepsilon$ such that $\kappa^{-1}(\varepsilon^{-1}\bar{s}) = \bar{\xi}$. Then, we have $\kappa^{-1}(\varepsilon^{-1}|s|) > \bar{\xi}$ for any $|s| > \bar{s}$, i.e., $\dot{V} < 0$. In other words, the sliding variable s is bounded by $|s| \le \bar{s} < \varepsilon$ from the initial time and remains in that region thereafter. ■

Theorem 3: Consider the system (4) under the proposed control (9) with its sliding functions as designed in (5) and (6). Then, the system state x is PTUBPB. More specific, for any ε/λ and an arbitrarily small $T_c > t_x$, there exists a sufficiently large $\alpha < t_x^{-1}$ such that for any initial condition $x(0)$, x will be terminally bounded within a region of $(-\varepsilon/\lambda, \varepsilon/\lambda)$, $\forall t \ge T_c$.

Theorem 3 can be straightly obtained based on Theorems 1 and 2, and Definition 3; and the proof is thus omitted.

Remark 1: Although the actual settling time T_c cannot be explicitly specified, one can adjust it to be arbitrarily close to t_x. From the proof, it can be seen that an arbitrarily small T_c satisfying $T_c > t_x$ can be obtained by choosing a sufficiently large α. This means that we can tune the settling time estimation error $T_c - t_x$ to be arbitrarily small up to $T_c \approx t_x$.

Remark 2: Selecting $\kappa_{\rm m}^{-1}(t) = \kappa^{-1}(\alpha t)$ in (7) reduces (5) and (6) as

$$\dot{x} + \kappa^{-1}(\alpha t)x = s. \tag{11}$$

Denote $\chi(t) = \int_{t_f}^{t} \kappa^{-1}(\alpha t)dt$, then we have

$$\begin{aligned}
|x(t)| &= e^{-\chi(t)}\left|x(t_f) + \int_{t_f}^{t} s(t)e^{\chi(t)}dt\right| \\
&< \left|x(t_f)\right|e^{-\chi(t)} + \varepsilon\int_{t_f}^{t} e^{-\chi(t)+\chi(\tau)}d\tau \\
&< \left|x(t_f)\right|e^{-\chi(t)} + \varepsilon/\kappa^{-1}(\alpha t)
\end{aligned} \tag{12}$$

for any $t_f \le t < \alpha^{-1}$. Since $\lim_{t\to(\alpha^{-1})^-}\kappa^{-1}(\alpha t) \to \infty$ and $\lim_{t\to(\alpha^{-1})^-}\chi(t) \to \infty$, it implies that $\lim_{t\to(\alpha^{-1})^-}|x(t)| \to 0$. In other words, the system state x will terminally approach the origin as $t \to (\alpha^{-1})^-$ without overestimation of UBST regardless of ε. However, unbounded control signal may be induced in the control (9) if x or $\dot{x}$ is not identically equal to zero.

Remark 3: To remove the requirement for the DUB, no extra modification is imposed on the adaptive gain $\kappa^{-1}(\varepsilon^{-1}|s|)$ in (9). From (10), it can be seen that there always exists a sufficiently large $\kappa^{-1}(\varepsilon^{-1}|s|) > \bar{\xi}$ in the domain of $(-\varepsilon, \varepsilon)$ to maintain the stability regardless of the DUB. In fact, one can also constrain the output of $\kappa^{-1}(\varepsilon^{-1}|s|)$ as that in [4], but it would conversely require the DUB.

Numerical example: Consider an inverted pendulum system with network-induced time delay as follows:

$$\ddot{\theta} = \frac{g\sin\theta - mla\dot{\theta}^2\cos\theta\sin\theta}{l\left(\frac{4}{3} - ma\cos^2\theta\right)} + \frac{a\cos\theta}{l\left(\frac{4}{3} - ma\cos^2\theta\right)}u(t-\tau) + d \tag{13}$$

where $a = 1/(m + m_c)$ with $m = 0.1$ kg and $m_c = 1$ kg the pendulum and cart masses, respectively. θ represents the pendulum angle, $g = 9.81$ m/s² is the gravity acceleration, u is the control force imposed on the cart, τ is the time delay; and d is the disturbance.

Select the class $\mathcal{K}^1$ function as $\kappa(y) = (2/\pi)\arctan(y)$ and the auxiliary function as

$$z(t) = \begin{cases} -\dfrac{\sigma(0)}{t_f^2}(t-t_f)^2, & \text{if } 0 \le t < t_f \\ 0, & \text{if } t \ge t_f. \end{cases} \tag{14}$$

The control parameters are chosen as $\varepsilon = 0.05$, $t_f = 1$ s, $\alpha = 0.2$, and $t_x = 4.9$ s. In the simulation, the initial condition is set as $\theta(0) = 1$ rad, $\dot{\theta}(0) = 0.3$ rad/s; the disturbance is $d = 5\sin(2\pi t)$ N/kg; the time delay is $\tau = 0.002$ s; and the sampling period is 0.0002 s. It can be seen from Fig. 2 (a) that θ is bounded within $|\theta| < \varepsilon/\lambda = 0.0016$ rad from 4.9 s and thereafter. The control input in Fig. 2 (b) varies in accordance with disturbance frequency for disturbance suppression. In addition, Fig. 2(c) shows that the sliding variable s starts from zero and is consistently bounded within $|s| < \varepsilon = 0.05$ as desired. Similarly, Fig. 2 (d) indicates that the adaptive parameter $\kappa^{-1}(\varepsilon^{-1}|s|)$ is updated according to the variation of $|s|$.

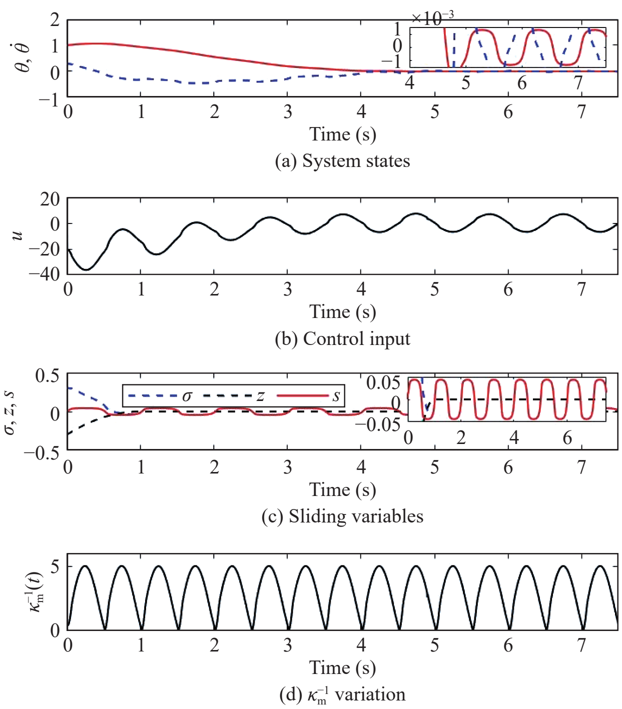

Fig. 2. Response of the system (13).

Conclusions: In this letter, based on the definition of class $\mathcal{K}^1$ function, a new class of sliding function is first constructed to guarantee the system's predefined-time convergence. Next, a class $\mathcal{K}^1$ function-based adaptive law is proposed to realize real sliding mode for the closed-loop system without knowing the DUB. The proposed adaptive controller ensures the system to be bounded within an arbitrarily prescribed region and in a predefined convergent time regardless of the initial condition.

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