Letter

Finite-Time Stabilization of Linear Systems With Input Constraints by Event-Triggered Control

Kai Zhang, *Student Member*, Yang Liu, *Member, IEEE*, and Jiubin Tan

Dear editor,

This letter designs the event-triggered control (ETC) to achieve finite-time stabilization (FTS) of linear systems with input constraints. The key idea of the established algorithm is that the designed time-varying high-gain is only scheduled on a specified time determined by an event-triggered mechanism. By exploring the properties of the parametric Lyapunov equation, a positive lower bound of inter-event times associated with the designed ETC can be obtained, such that the Zeno phenomenon is avoided. Moreover, semi-global FTS of linear systems with input constraints is also achieved by the designed ETC. Finally, the application of the spacecraft rendezvous control system verifies the effectiveness of the designed ETC.

Related work: With the advancement of computer and communication technologies, (shared) wired and wireless networked control systems (NCSs) have been an increasing popularity since they have the merits of high flexibility, efficient reconfigurability, and low cost on the installation and maintenance compared to the conventional point-to-point digital control system [1]. However, NCSs may meet some new challenges such as network traffic congestion, which cannot be solved well by using traditional periodic sampling control. Due to the reason, the ETC algorithm that can greatly save communication resources has been paid much attention. Especially the appearance of the literature [2], which greatly promotes the systematic design of ETC methods [3]. Since then, many results for ETC have also been emerged and can be found in [4], [5] and the references therein.

Since the saturation nonlinearity is ubiquitous among every practical system, the ETC of systems subject to input constraints has also received much attention. For example, the local exponential stability of the linear system with input constraint has been achieved by the simultaneous design of a state feedback law and an event-triggering condition in [6]. The global stabilization of general linear systems with input constraints has been well achieved by the nonlinear ETC algorithm in [7]. A controller based on the general Riccati equation was designed in [8], where the event-triggered mechanism is conservative and the solution of the parameters is too complicated. Recently, a series of ETC algorithms based on the parametric Lyapunov equation (PLE) have been designed in [9], [10], where the relationship between the design parameters and the minimal inter-event time (MIET) is clearly and simple.

In addition, the faster convergence (even finite-time convergence) is pursued in reality and some corresponding results have been obtained in [11]–[13]. Since the ETC algorithm with performance of FTS can obtain a rapid convergence rate with less cost on the communication resources, it is very meaningful and has received much attention. For example, an ETC method with an exponential time-varying gain was designed to achieve FTS of nonlinear systems in [14]; global FTS of a class of uncertain nonlinear systems has been achieved by designing an ETC based on the backstepping technique in [15]; finite-time path following of underactuated vessels has been achieved by designing an adaptive neural network ETC in [16]. More results can be found in [17] and the reference therein. Nevertheless, the input constraints are not considered in the above mentioned articles, which is not expectable in practice since any practical actuators can only generate bounded control signals. To the best of our knowledge, how to design an ETC with an easily designable MIET to achieve FTS of linear systems with input constraints has not been well investigated.

Motivated by the above discussion, we aim to establish an ETC algorithm to achieve FTS of linear systems with input constraints. This is done by designing the PLE based time-varying high-gain feedback that is only scheduled at a specified time determined by an event-triggered mechanism. Specially, in the designed ETC, a clear MIET that avoids the complicated relationship with system matrix is given such that the Zeno phenomenon is avoided and a trade-off between communication resources and the stabilized time can be easily found. Finally, the established algorithm is used to the design of the spacecraft rendezvous control system.

Notation: let $\mathbf{N}$ denote the set of nonnegative integers. Let I denote the identity matrix whose dimension is clear from the context, $\mathrm{tr}(A)$ define the trace of A, and $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximal and minimal eigenvalues of P, respectively. $\varphi(A)=\min_{i=1,2,\ldots,n}\{\mathrm{Re}(\lambda_i(A))\}$, where λ_i represent $i$the eigenvalue of the matrix. The saturation function $\sigma_{\varpi}(u_i)=\mathrm{sign}(u_i)\min\{\varpi,|u_i|\}$ for each $i=1,2,\ldots,m$, where $\varpi>0$ is the saturation level.

Problem statement: We consider the linear system with input constraints

$$\dot{x}=Ax+B\sigma_{\varpi}(u) \tag{1}$$

where $x=x(t)\in\mathbb{R}^n$ is the state, $u=u(t)\in\mathbb{R}^m$ is the control, and $(A,B)\in(\mathbb{R}^{n\times n},\mathbb{R}^{n\times m})$ is controllable. In this letter, we aim to design an ETC algorithm to achieve FTS of input constrained linear system (1). To accomplish this mission, the following important lemma is firstly given.

Lemma [9], [13]: Let (A,B) be controllable and $\gamma>-2\varphi(A)$. Then, the PLE

$$A^TP+PA-PBB^TP+\gamma P=0 \tag{2}$$

has a unique positive definite solution $P=P(\gamma)$ if and only if $P(\gamma)=W^{-1}(\gamma)$, where $W(\gamma)$ is a polynomial function of γ and satisfies the equation $W(A+(\gamma/2)\,I)^T+(A+(\gamma/2)\,I)W=BB^T$. Moreover, we give the following important properties of P in association with PLE (2).

1) Define $\varepsilon_\gamma\triangleq 2\mathrm{tr}(A)+n\gamma$, then

$$\mathrm{tr}(B^TPB)=\varepsilon_\gamma,\ \ PBB^TP\le\varepsilon_\gamma P. \tag{3}$$

2) For any $\gamma\ge\gamma_0>-2\varphi(A)$,

$$0<\frac{P(\gamma)}{\varepsilon_\gamma}\le\mathcal{P}\le\delta_{\mathrm{c}}\frac{P(\gamma)}{\varepsilon_\gamma} \tag{4}$$

where $\mathcal{P}=\mathcal{P}(\gamma)=\mathrm{d}P(\gamma)/\mathrm{d}\gamma$, $\delta_{\mathrm{c}}=\delta_{\mathrm{c}}(\gamma_0)\ge 1$ is a constant of

Corresponding author: Yang Liu.

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K. Zhang and J. B. Tan are with the Key Laboratory of Ultra-Precision Intelligent Instrumentation, Ministry of Industry and Information Technology, and also with the Center of Ultra-Precision Optoelectronic Instrument Engineering, Harbin Institute of Technology, Harbin 150001, China (e-mail: kaizhang0116@163.com; jbtan@hit.edu.cn).

Y. Liu is with the Key Laboratory of Ultra-Precision Intelligent Instrumentation, Ministry of Industry and Information Technology, and the Center of Ultra-Precision Optoelectronic Instrument Engineering, Harbin Institute of Technology, Harbin 150001, and also with the State Key Laboratory of Digital Manufacturing Equipment &Technology, Huazhong University of Science and Technology, Wuhan 430074, China (e-mail: hitlg@hit.edu.cn).

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independent γ.

3) Let $A_{\mathrm{L}} = A - BB^T P/2$, then

$$A_{\mathrm{L}}^T P A_{\mathrm{L}} \leq \left(\frac{3\varepsilon_\gamma^2}{2} - \gamma\varepsilon_\gamma - 2\gamma \mathrm{tr}(A) - 2\mathrm{tr}\left(A^2\right)\right) P \triangleq \pi_\gamma P \quad (5)$$

where π_γ increases as γ increases.

4) Let $\dot{\gamma} \geq 0$ and $P_e = P(\gamma(t)) - P(\gamma(t_i))$. Then, for any $t \geq t_i$,

$$P_e BB^T P_e \leq n(\gamma - \gamma_i) P_e \leq \varepsilon_\gamma P_e \leq \varepsilon_\gamma P. \quad (6)$$

Event-triggered control: In this letter, for a simple clarification, if not specified, we omit the dependence of variables of t in the remaining of this letter, such as, $\gamma = \gamma(t)$, $P = P(\gamma(t))$, $x(t_i) = x_i$, $\gamma_i = \gamma(t_i)$, $P_{\gamma_i} = P(\gamma(t_i))$. We define a measurement error

$$e = x_i - x, \quad \forall t \in [t_i, t_{i+1}), \quad i \in \mathbf{N} \quad (7)$$

and an ellipsoid $\mathcal{E}_{\mathrm{s}}(\gamma) = \left\{x \in \mathbf{R}^n : \varepsilon_\gamma x^T P x \leq 4\varpi^2\right\}$. Moreover, we denote

$$\bar{T} = \frac{1}{2\bar{\alpha}_1} \ln\left(1 + \frac{2\mathrm{tr}(A)}{n\gamma_0}\right), \quad \bar{\alpha}_1 = \frac{(1-\beta)\mathrm{tr}(A)}{n+1}$$

$$\underline{T} = \frac{1}{2\underline{\alpha}_1} \ln\left(1 + \frac{2\mathrm{tr}(A)}{n\gamma_0}\right), \quad \underline{\alpha}_1 = \frac{(1-\beta)\mathrm{tr}(A)}{n+\delta_{\mathrm{c}}}$$

where $\beta \in (0,1)$ is a design parameter. We notice that both $\bar{T}$ and $\underline{T}$ are well defined if $\mathrm{tr}(A) = 0$ [13].

Theorem 1: Let $\gamma_0 > \max\{0, -2\varphi(A)\}$ be a prescribed number. Consider the following bounded control:

$$u(t) = -\frac{B^T P_{\gamma_i} x_i}{2}, \quad t \in [t_i, t_{i+1}) \quad (8)$$

where t_i is determined by the following static event-triggered mechanism (ETM):

$$t_{i+1} = \inf\left\{t > t_i : \beta\gamma x^T P x + f(t) \leq 0\right\} \quad (9)$$

in which $f(t) = x^T PBB^T P_{\gamma_i} e - x^T PBB^T P_e x$, and γ satisfies the scalar differential equation $\dot{\gamma} = (1-\beta)\gamma / \left(\delta_1(t) + \delta_2(t)\right)$, where

$$\delta_1(t) = \frac{n}{\varepsilon_\gamma}, \quad \delta_2(t) = \begin{cases} \dfrac{x^T \mathcal{P} x}{(x^T P x)}, & x \neq 0 \\ 1 + \dfrac{\delta_{\mathrm{c}}}{(2\varepsilon_\gamma)}, & x = 0. \end{cases} \quad (10)$$

Then, for any $x(0) = x_0 \in \mathcal{E}_{\mathrm{s}}(\gamma_0)$, there exists a constant $T_0 = T_0(x_0, \gamma_0)$ satisfying $\bar{T} \leq T_0 \leq \underline{T}$ such that $\lim_{t\to T_0} \|x\| = 0$.

Proof: By using (4), we know from (10) that if $x \neq 0$,

$$\frac{1}{\varepsilon_\gamma} \leq \delta_2 \leq \frac{\delta_{\mathrm{c}}}{\varepsilon_\gamma}$$

if $x = 0$, then $\delta_2 = 1 + \delta_{\mathrm{c}}/2\varepsilon_\gamma$. Thus, we have $1/\varepsilon_\gamma \leq \delta_2 \leq \delta_{\mathrm{c}}/\varepsilon_\gamma$ for any x. It then follows from the definition of γ that:

$$(1-\beta)\frac{\varepsilon_\gamma \gamma}{(n+\delta_{\mathrm{c}})} \leq \dot{\gamma} \leq (1-\beta)\frac{\varepsilon_\gamma \gamma}{(n+1)}. \quad (11)$$

Then, we define

$$\bar{\gamma} = \frac{\exp\left(2\bar{\alpha}_1 \bar{T}\right) - 1}{\exp\left(2\bar{\alpha}_1 \left(\bar{T} - t\right)\right) - 1}\gamma_0, \quad \forall t \in \left(0, \bar{T}\right),$$

$$\underline{\gamma} = \frac{\exp\left(2\underline{\alpha}_1 \underline{T}\right) - 1}{\exp\left(2\underline{\alpha}_1 \left(\underline{T} - t\right)\right) - 1}\gamma_0, \quad \forall t \in (0, \underline{T}). \quad (12)$$

By the comparison lemma [18], it follows from (11) and (12) that:

$$\gamma \leq \bar{\gamma}, \ \forall t \in (0, \bar{T}); \ \gamma \geq \underline{\gamma}, \ \forall t \in (0, \underline{T})$$

from which it follows that there exists a $T_0 = T_0(x_0, \gamma_0) \in [\bar{T}, \underline{T}]$ such that $\lim_{t\to T_0} \gamma = \infty$. The closed-loop system consisting of (1) and (8) is

$$\dot{x} = Ax + B\sigma_\varpi\left(-\frac{B^T P_{\gamma_i} x_i}{2}\right), \quad \forall t \in [t_i, t_{i+1}). \quad (13)$$

We define $B = [b_1 \ b_2 \ \cdots \ b_m]$, then according to the proof of Theorem 1 in [10], for any $k \in \mathbf{I}[1,m]$, we have $\varepsilon_\gamma x^T P x \leq \varpi^2 \Rightarrow \sigma_\varpi\left(-(1/2)b_k^T P_{\gamma_i} x_i\right) = -(1/2)b_k^T P_{\gamma_i} x_i$. This implies that (13) can be continued as

$$\dot{x} = Ax - \frac{1}{2}BB^T P_{\gamma_i} x_i = Ax - \frac{1}{2}B\left(B^T P x - B^T P_e x + B^T P_{\gamma_i} e\right) \quad (14)$$

where we have noted the definition of P_e and e. Then, we choose the Lyapunov-like function $V(x,t) = \varepsilon_\gamma x^T P x$. It follows from the definition of γ and (11) that the time-derivative of $V(x(t),t)$ along system (14) can be given as:

$$\begin{aligned} \dot{V}(x,t) &= \dot{\gamma}(nx^T P x + \varepsilon_\gamma x^T \mathcal{P} x) - \gamma V(x,t) - \varepsilon_\gamma f(t) \\ &\leq -\varepsilon_\gamma(\beta\gamma x^T P x + f(t)), \ \forall t \in [t_i, t_{i+1}). \end{aligned} \quad (15)$$

By noting (9), we have $\beta\gamma x^T P x + f(t) \geq 0, \ \forall t \in [t_i, t_{i+1})$. This together with (15) indicates that

$$V(x,t) \leq \varpi^2 \Rightarrow \dot{V}(x,t) \leq 0, \ \forall t \in [0, \mathrm{T}_0). \quad (16)$$

Therefore, the closed-loop system consisting of (1) and (8) can be written as (14), and (16) is satisfied for all $t \in [0, T_0)$. Moreover, we have $V(x(t),t) = \varepsilon_\gamma x^T P x \geq \varepsilon_\gamma \lambda_{\min}(P(\gamma_0))\|x\|^2$ and $V(x(0),0) \leq \varepsilon_{\gamma_0}\|P(\gamma_0)\|\|x(0)\|^2$, where we have used $\mathcal{P} > 0$ and $\gamma \geq \gamma_0$. Theses implies $\|x\| \leq \sqrt{\varepsilon_{\gamma_0}\|P(\gamma_0)\| / \left(\varepsilon_\gamma \lambda_{\min}(P(\gamma_0))\right)}\|x(0)\|$, from which it follows that $\lim_{t\to T_0}\|x\| = 0$ since $\lim_{t\to T_0}\gamma = \infty$. This implies that the closed-loop system is locally finite-time stabilized with $\mathcal{E}_{\mathrm{s}}(\gamma_0)$ contained in the domain of attraction.

Remark 1: Although FTS can be achieved by Theorem 1, it may possess some numerical problems in computing u when $t \to T_0$. To solve these problems, we will replace γ in Theorem 1 with

$$\dot{\gamma}(t) = \begin{cases} \dfrac{(1-\beta)\gamma}{(\delta_1(t) + \delta_2(t))}, & \gamma \leq \gamma_* \\ 0, & \gamma > \gamma_* \end{cases}$$

where γ_* is a prescribed sufficiently large number. By this we can see that $\gamma = \gamma_*$ when $t \geq t^*$ where t^* is the minimal time such that $\gamma(t^*) = \gamma_*$. Moreover, since $\dot{\gamma} = 0$ when $t > t^*$, $P_e = 0$ and the static ETM (9) equals to

$$t_{i+1} = \inf\left\{t > t_i : \beta\gamma_* x^T P_{\gamma_*} x + x^T P_{\gamma_*} BB^T P_{\gamma_*} e \leq 0\right\}.$$

Since $\dot{\gamma} = 0$, we have from (15) that $\dot{V}(x,t) \leq (1-\beta)\gamma_* V(x,t)$, $t > t^*$, which implies $\mathcal{E}_{\mathrm{s}}(\gamma_0)$ is still an invariant set and $\|x\| \leq \sqrt{\lambda_{\max}(P(\gamma_*))/\lambda_{\min}(P(\gamma_*))}\exp^{[-((1-\beta)\gamma_*/2)(t-t_*)]} \ \|x(t^*)\|$, $t > t^*$. Thus the state converges to zero exponentially with a fast convergence rate after t_* (which can be called as practical FTS) [12]. ■

Theorem 2: The IETs associated with the designed ETC are bounded below by

$$\tau_e = \begin{cases} \tau_1 = \displaystyle\int_0^{\frac{\beta^2\gamma_0^2}{2\varepsilon_{\gamma_*}^2}} \frac{1}{(\pi_{\gamma_*} + \varepsilon_{\gamma_*} + \gamma_* + (1 + \gamma_* + 2\varepsilon_{\gamma_*})\tau + \frac{3\varepsilon_{\gamma_*}\tau^2}{4})d\tau}, \\ t \leq t^* \\ \tau_2 = \displaystyle\int_0^{\frac{\beta^2\gamma_*^2}{\varepsilon_{\gamma_*}^2}} \frac{1}{(\pi_{\gamma_*} + (1 + \gamma_* + \frac{4\varepsilon_{\gamma_*}}{3})\tau + \frac{3\varepsilon_{\gamma_*}\tau^2}{4})d\tau}, \\ t > t^*. \end{cases}$$

Proof: For $t \leq t^*$, by using (3) and (6), we have

$$\beta\gamma x^T Px + f(t) \geq \frac{\beta\gamma_0}{2} x^T Px - \frac{\varepsilon_{\gamma_*}^2}{\beta\gamma_0}\left(e^T P_{\gamma_i} e + x^T P_e x\right)$$

where we have noted $\gamma_0 \leq \gamma \leq \gamma_*$. This indicates that the inter-event times (IETs) associated with the static ETM (9) are bounded below by the time needed by the function $E_1 = E_1(t) = (e^T P_{\gamma_i} e + x^T P_e x)/x^T Px$ going from 0 to $\beta^2\gamma_0^2/(2\varepsilon_{\gamma_*}^2)$. By noting the definition of e and P_e, we have from (14) that

$$\dot{e} = -\dot{x} = -A_{\mathrm{L}_i} x + BB^T P_{\gamma_i} \frac{e}{2}$$

where $A_{\mathrm{L}_i} = A - BB^T P_{\gamma_i}/2$. With this we obtain

$$2e^T P_{\gamma_i} \dot{e} \leq \pi_{\gamma_i} x^T Px + (1+\varepsilon_{\gamma_i}) e^T P_{\gamma_i} e \tag{17}$$

where we have used (3) and (5). Again by the triangle inequality, (2) and (3), we can obtain from (14) that

$$\begin{aligned} 2x^T P_e \dot{x} = & -\gamma x^T Px + x^T PBB^T Px + \gamma_i x^T P_{\gamma_i} x \\ & - x^T P_{\gamma_i} BB^T P_{\gamma_i} x - x^T P_e BB^T P_{\gamma_i} x_i \\ \leq & \; x^T PBB^T Px - x^T P_{\gamma_i} BB^T P_{\gamma_i} x \\ & + \frac{1}{2}\left(2x^T P_e BB^T P_e x + \frac{x_i^T P_{\gamma_i} BB^T P_{\gamma_i} x_i}{2}\right) \\ \leq & \; x^T PBB^T Px - x^T P_{\gamma_i} BB^T P_{\gamma_i} x + x^T P_e BB^T P_e x \\ & + \frac{\left(4x^T P_{\gamma_i} BB^T P_{\gamma_i} x + \frac{4e^T P_{\gamma_i} BB^T P_{\gamma_i} e}{3}\right)}{4} \\ \leq & \; \varepsilon_\gamma x^T Px + \varepsilon_\gamma x^T P_e x + \frac{\varepsilon_{\gamma_i}}{3} e^T P_{\gamma_i} e \end{aligned} \tag{18}$$

where we have used (6) and noted $x_i = e + x$. Hence, it follows from (17) and (18) that:

$$\begin{aligned} 2e^T P_{\gamma_i} \dot{e} + 2x^T P_e \dot{x} \leq & (\varepsilon_{\gamma_*} + \pi_{\gamma_*}) x^T Px \\ & + \left(1 + \frac{4\varepsilon_{\gamma_*}}{3}\right)(x^T P_e x + e^T P_{\gamma_i} e) \end{aligned} \tag{19}$$

in which we have noted $\gamma \leq \gamma_*$. Similar, by using (2), (3), (6) and the triangle inequality again, we also have

$$-2x^T P\dot{x} \leq \left(\gamma_* + \frac{2}{3}\varepsilon_{\gamma_*}\right) x^T Px + \frac{3\varepsilon_{\gamma_*}}{4}\left(e^T P_{\gamma_i} e + x^T P_e x\right) \tag{20}$$

where we have noted $\gamma \leq \gamma_*$. In addition, for any $x \neq 0$, we have from the definition of $\dot{\gamma}$ and $\dot{P}_e = \dot{P}$ that

$$\frac{x^T \dot{P}_e x}{x^T Px} = \dot{\gamma}\frac{x^T \mathcal{P} x}{x^T Px} \leq \frac{(1-\beta)\gamma}{\delta_2(t)} \frac{x^T \mathcal{P} x}{x^T Px} = (1-\beta)\gamma \leq \gamma_*.$$

This together with (19) and (20) indicates that

$$\dot{E}_1 \leq \pi_{\gamma_*} + \varepsilon_{\gamma_*} + \gamma_* + \left(1 + \gamma_* + 2\varepsilon_{\gamma_*}\right) E_1 + \frac{3\varepsilon_{\gamma_*}}{4} E_1^2$$

where we have noted $\dot{P} \geq 0$. It then follows from the comparison lemma [18] that $E_1 \leq h_1$, where $h_1 = h_1(t)$ is the solution of:

$$\dot{h}_1 = \pi_{\gamma_*} + \varepsilon_{\gamma_*} + \gamma_* + \left(1 + \gamma_* + 2\varepsilon_{\gamma_*}\right) h_1 + \frac{3\varepsilon_{\gamma_*} h_1^2}{4} \tag{21}$$

with $h_1(t_i) = E_1(t_i) = 0$. Let $t_i + \tau_1$ be the solution to the equation $h_1(t_i + \tau_1) = \beta^2\gamma_0^2/\left(2\varepsilon_{\gamma_*}^2\right)$. Since $h_1(t)$ is an increasing function that, we have $E_1(t) \leq h_1(t) < \beta^2\gamma_0^2/\left(2\varepsilon_{\gamma_*}^2\right)$, $t \in [t_i, t_i + \tau_1)$. This together with (20) indicates that the IETs are bounded below by τ_1. Hence, the Zeno behavior is avoided.

For $t > t^*$, as mentioned in Remark 1, we have $\gamma(t) = \gamma_*$ is a bounded constant and the ETM (9)

$$\beta\gamma x^T P_{\gamma_*} x + x^T P_{\gamma_*} BB^T P_{\gamma_*} e \geq \frac{\beta\gamma_*}{2} x^T P_{\gamma_*} x - \frac{\varepsilon_{\gamma_*}^2}{2\beta\gamma_*} e^T P_{\gamma_*} e.$$

This implies that the IETs associated with the designed ETC for $t > t^*$ are bounded below by the time needed by the function $E_2 = E_2(t) = e^T P_{\gamma_*} e/(x^T P_{\gamma_*} x)$ going from 0 to $\beta^2\gamma_*^2/\varepsilon_{\gamma_*}^2$. Since $P_{\gamma_i} = P_{\gamma_*}$ and $P_e = 0$, we have $2e^T P_{\gamma_*} \dot{e} \leq \pi_{\gamma_*} x^T P_{\gamma_*} x + (1+\varepsilon_{\gamma_*}) e^T P_{\gamma_*} e$, and

$$-2x^T P_{\gamma_*} \dot{x} \leq \gamma_* x^T Px + \frac{\varepsilon_{\gamma_*} x^T P_{\gamma_*} x}{3} + \frac{3\varepsilon_{\gamma_*} e^T P_{\gamma_*} e}{4}$$

where (17) and (19) have been noted. With this we can obtain

$$\begin{aligned} \dot{E}_2 = & \frac{2e^T P_{\gamma_*} \dot{e}}{x^T P_{\gamma_*} x} - E_2 \frac{2x^T P_{\gamma_*} \dot{x}}{x^T P_{\gamma_*} x} \leq \pi_{\gamma_*} \\ & + \left(1 + \gamma_* + \frac{4\varepsilon_{\gamma_*}}{3}\right) E_2 + \frac{3\varepsilon_{\gamma_*}}{4} E_2^2. \end{aligned}$$

Similar to $t \leq t^*$, we can also obtain that the IETs associated with the designed ETC for $t > t^*$ are bounded below by τ_2. We can easily from the definition of τ that τ_1/τ_2 increases as β increases and $\tau_1 \leq \tau_2$. ■

In addition, the designed ETC can also achieve the semi-global FTS of the input constrained linear systems in the following corollary.

Corollary (Semi-global FTS by ETC): Assume that $(A, B) \in \left(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times m}\right)$ is controllable and all the eigenvalues of A are on the imaginary axis. For any arbitrarily large but bounded set $\Omega \subset \mathbb{R}^n$, let $\gamma_0 > 0$ be such that $\Omega \subseteq \mathcal{E}_{\mathrm{s}}(\gamma_0)$ and γ satisfy the definition in Theorem 1. Then, for any $x(0) \in \Omega$, there exists a constant $T_0 = T_0(x_0, \gamma_0)$ satisfying $(n+1)/((1-\beta)n\gamma_0) \leq T_0 \leq (n+\delta_{\mathrm{c}})/((1-\beta)n\gamma_0)$, such that the closed-loop system (13) with the ETM (9) is finite-time stabilized. Moreover, the IETs are bounded below by

$$\tau_{\mathrm{ce}} = \int_0^{\frac{\beta^2}{2n^2}} \frac{1}{\pi_{\gamma_*} + (n+1)\gamma_* + (1 + (2n+1)\gamma_*)\tau + 3n\gamma_*\tau^2/4} d\tau.$$

Proof: The existence of γ_0 follows from $\lim_{\gamma\to 0^+} P(\gamma) = 0$ since $Re(\lambda_i(A)) = 0$, $i = 1, 2, \ldots, n$. Then, by combining the corresponding semi-global stabilization results in [10] and Theorems 1 and 2, the proof of Corollary 1 is straightforward and thus is omitted here. ■

Numerical simulation: In this part, the spacecraft rendezvous control system studied in [13], [19] is considered again. The relative motion between the target and chaser can be described by the following Newton's equations [20]:

$$\begin{cases} \ddot{x}_1 = 2\omega\dot{y} + \omega^2(R + x_1) - \theta\eta(R + x_1) + \sigma_\varpi(u_1) \\ \ddot{x}_2 = -2\omega\dot{x}_1 + \omega^2 x_2 - \theta\eta x_2 + \sigma_\varpi(u_2) \\ \ddot{x}_3 = -\theta\eta x_3 + \sigma_\varpi(u_3) \end{cases} \tag{22}$$

where $\omega = \eta^{1/2}R^{3/2}$ is the orbit rate of target orbit, η represents the gravitational parameter [19].

We choose $x = [x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3]^T$, then the linearized system of (21) can be written as (1), in which $u = [u_1 \;\; u_2 \;\; u_3]^T$ and

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & -2\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_3 \end{bmatrix}.$$

Then, we carry out simulations under different value of to verify the corresponding results. All the controller algorithms will act

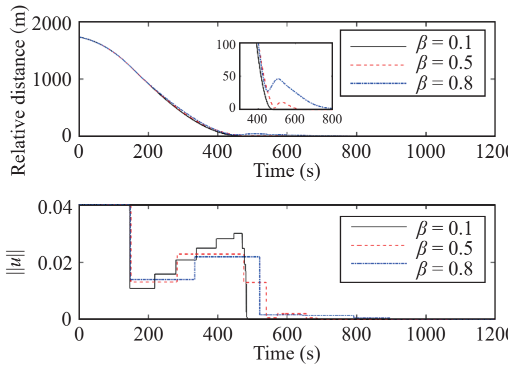

Fig. 1. The relative distance and control signals under different values of β.

directly on the nonlinear system (31), although the controller design is based on the linear system (1). The corresponding parameters are the same as the literature [13], [19], that is, $R = 4.2241 \times 10^7$ m, $\omega = 7.2722{\times}10^{-5}$ rad/s, $\varpi = 0.1$ m/s^2, and $\delta_c = 20.5$. Let the initial value be $x(0) = x_0 = [-1000\ 1000\ 1000\ 2\ -2\ 2\,]^T$, and the sampling interval be 0.01 s. $\gamma_0 = 0.006\,9099$, can be solved by the equation $\varepsilon^T_{\gamma_0} x^T_0 P_{\gamma_0} x_0 = 4$. The relative distance $\|x_1, x_2, x_3\|$ and 2-norm of the control signals $\|u_1, u_2, u_3\|$ are showed in Fig. 1. Moreover, for easier comparisons, some key indices associated with IETs are showed in Table 1, where TNs (times), MIET (s) and AIET (s) represent the trigger numbers, the minimal and average inter-event time, respectively. Table 1 shows the MIET and AIET (TNs) increases (decreases) as β increases. Hence, it follows from Table 1 and Fig. 1 that a trade-off between the stabilized time and IETs can be easily found by appropriately selecting the value of β.

Table 1. The MIET/AIET/TNS Associated With Theorem 1

β	0.1	0.5	0.8
Theorem 1	0.56/3.1/386	0.74/5.6/214	69.7/127/9

Next, we carry out simulation to compare three different finite-time stabilized methods. Method 1: An ETC in Theorem 1 with and $\gamma^* = 1$. Method 2: The finite-time control algorithm associated with Theorem 2 in [19] with $T^* = T - 0.1$. Method 3: The finite-time control algorithm associated Theorem 2 in [13] with $\gamma^* = 1000$. These parameters are chosen to guarantee the best control performance in terms of the rendezvous time. It can be observed from Fig. 3 that our solution not only saves communication resources but also has the minimal rendezvous time among others.

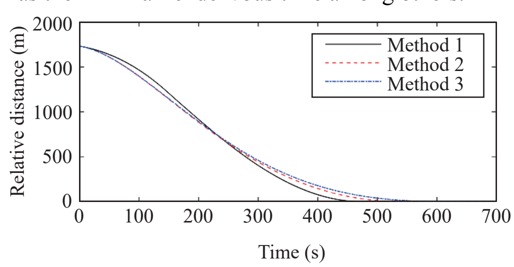

Fig. 2. The relative distance for different methods.

Conclusion: This letter designed an ETC to achieve finite-time stabilization of linear systems with input constraints. The key idea is that an event-triggered mechanism is designed to update the control law, where the control gain based on the solution of the parametric Lyapunov equation will approach to infinity at finite time. The Zeno phenomenon was avoided for the ETC algorithm. In the future, we will design the dynamic event-triggered control to increase the inter-event times and self-triggered control algorithm to avoid continuously monitor of system states.

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