is tantamount to $d = 1$, so this statement follows from Proposition 11 of [2].

4) For the study of alignment algorithms for arbitrary digraphs, it is important to observe that the statement of Lemma 2 that "all nontrivial eigenvalues of L have positive real parts" holds true for any digraphs [5, Proposition 9], and not only for strongly connected digraphs or digraphs with spanning converging trees.

In [1, Sec. II-C], a discrete-time counterpart of the consensus algorithm (1) is considered

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j=1}^{n} a_{ij}\big(x_j(k) - x_i(k)\big), \quad i = 1, \ldots, n \tag{2}$$

where $\varepsilon > 0$ is the step size. In the matrix form, (2) is represented as follows:

$$x(k+1) = Px(k) \tag{3}$$

where $P = I - \varepsilon L$ is referred to in [1] as the *Perron matrix with parameter* ε of G.

The matrices $P = I - \varepsilon L$ were studied in [2] and [5]; in particular, (i) of Lemma 3 in [1] actually coincides with Proposition 12 of [2].

Finally, let me mention a few additional results [2], [5] that are applicable to the analysis of consensus algorithms (1) and (3) and flocking algorithms. In the general case where the primitivity of a stochastic matrix P is not guaranteed and the sequence $P, P^2, P^3, \ldots$ may diverge, the *long-run transition matrix* $P^\infty = \lim_{m\to\infty} m^{-1} \sum_{k=1}^{m} P^k$ is considered. P^∞ always exists and, by the *Markov chain tree theorem* [6], [7], it coincides with the *normalized matrix* $\bar{J}$ *of maximal in-forests of* G. $\bar{J}$ is the eigenprojector of L; by Proposition 11 of [2], $\operatorname{rank}(\bar{J}) = d$, where d is the in-forest dimension of G. The columns of $\bar{J}$ span the kernel (null space) of L; as a result, they determine the main properties of the trajectories of (1) and the flocking trajectories [8] in the general case. The elements of $\bar{J}$ were characterized in graph theoretic terms in Theorems $2'$ and 3 of [2]; a finite algebraic method for calculating $\bar{J}$ was proposed in [5] (see also [4]).

Thus, [2], [4], [5] published before the recent avalanche of papers on distributed consensus algorithms ([2] and [5] were sent to J. A. Fax in 2001 and a reference to [4] was sent to R. Olfati-Saber apropos of Lemma 2 in 2003, both on their requests) contained the basic graph theoretic results needed for the analysis of these algorithms. A number of related theorems were proved in [9] and [10]. Some of these results were surveyed in [11]. ■

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Reply to "Comments on "Consensus and Cooperation in Networked Multi-Agent Systems""

R. Olfati-Saber, J. A. Fax, and R. M. Murray

There are essentially four points that Dr. Chebotarev's raises in [1].

Point 1. Chebotarev claims that Lemma 2 is not correct as stated and gives a counterexample consisting of a simple directed tree. This counterexample points out two issues with the lemma as stated.

- The second portion of Lemma 2, referring to the case in which there are c components, only applies to graphs in which there are disjoint components of the graph (no edges between the components). This is clear from the proof of this fact (which simply consists of separating the nodes so that the Laplacian is block diagonal, implying a disjoint set of nodes), but is ambiguous in the statement of the lemma. The conditions that Chebotarev gives are

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- equivalent to what was (implicitly) assumed. This portion of the lemma is not used in the subsequent results in the paper, but is inaccurate as stated.
- In the original paper that contains Lemma 2 (citation [10] in the PROCEEDINGS OF THE IEEE paper), we explicitly assumed that there were no edges between a node and itself (in particular, the (i, i) element of the adjacency matrix was assumed 0). This assumption was not explicitly stated in the PROCEEDINGS OF THE IEEE paper and this could lead to confusion. Chebotarev appears to assume that nodes are connected to themselves in his counter-example of a converging tree (giving $c = n$ components), although this is not essential to his criticism.

Point 2. Chebotarev goes on to state that there are some results for which stronger versions are available in [2]. We agree that these are available and relevant in the case of graphs that are more intricate than those considered in this paper.

Point 3. Chebotarev states that Lemma 3 was already shown in [2]. It appears that Lemma 3 can be derived from his previous results.

Point 4. The final point that Chebotarev makes is that [2], [3], and [4] contained a number of results that were sent to us but that we chose not to cite. We derived the results in our earlier papers using results from standard textbooks (cited in the earlier work, which is subsequently referenced in the PROCEEDINGS OF THE IEEE paper). While it is possible that we could have relied on Chebotarev's results instead, we did not make use of his work and so we cited the sources that we used. We note that these lemmas were simply establishing results that were needed for the main results in the PROCEEDINGS OF THE IEEE paper. ■

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