# Optimal Locally Repairable Linear Codes 

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#### Abstract

Linear erasure codes with local repairability are desirable for distributed data storage systems. An $[n, k, d]$ code having all-symbol $(r, \delta)$-locality, denoted as $(r, \delta)_{a}$, is considered optimal if it also meets the minimum Hamming distance bound. The existing results on the existence and the construction of optimal $(r, \delta)_{a}$ codes are limited to only the special case of $\delta=2$, and to only two small regions within this special case, namely, $m=0$ or $m \geq(v+\delta-1)>(\delta-1)$, where $m=n \bmod (r+\delta-1)$ and $v=k \bmod r$. This paper investigates the existence conditions and presents deterministic constructive algorithms for optimal $(r, \delta)_{a}$ codes with general $r$ and $\delta$. First, a structure theorem is derived for general optimal $(r, \delta)_{a}$ codes which helps illuminate some of their structure properties. Next, the entire problem space with arbitrary $n, k, r$ and $\delta$ is divided into eight different cases (regions) with regard to the specific relations of these parameters. For two cases, it is rigorously proved that no optimal $(r, \delta)_{a}$ could exist. For four other cases the optimal $(r, \delta)_{a}$ codes are shown to exist, deterministic constructions are proposed and the lower bound on the required field size for these algorithms to work is provided. Our new constructive algorithms not only cover more cases, but for the same cases where previous algorithms exist, the new constructions require a considerably smaller field, which translates to potentially lower computational complexity. Our findings substantially enriches the knowledge on $(r, \delta)_{a}$ codes, leaving only two cases in which the existence of optimal codes are yet to be determined.


## I. Introduction

The sheer volume of today's digital data has made distributed storage systems (DSS) not only massive in scale but also critical in importance. Every day, people knowingly or unknowingly connect to various private and public distributed storage systems, include large data centers (such as the Google data centers and Amazon Clouds) and peer-to-peer storage systems (such as OceanStore [1], Total Recall [2], and DHash++ [3]). In a distributed storage system, a data file is stored at a distributed collection of storage devices/nodes in a network. Since any storage device is individually unreliable and subject to failure (i.e. erasure), redundancy must be introduced to provide the much-needed system-level protection against data loss due to device/node failure.

The simplest form of redundancy is replication. By storing $c$ identical copies of a file at $c$ distributed nodes, one copy per node, a $c$-replication system can guarantee the data availability as long as no more than $(c-1)$ nodes fail. Such systems are very easy to implement, but extremely inefficient in storage space utilization, incurring tremendous waste in devices and equipment, building space, and cost for powering and cooling. More sophisticated systems employing erasure coding [4]
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can expect to considerably improve the storage efficiency. Consider a file that is divided into $k$ equal-size fragments. A judiciously-designed $[n, k]$ erasure (systematic) code can be employed to encode the $k$ data fragments (terms systematic symbols in the coding jargon) into $n$ fragments (termed coded symbols) stored in $n$ different nodes. If the $[n, k, d]$ code reaches the Singleton bound such that the minimum Hamming distance satisfies $d=n-k+1$, then the code is maximum distance separable (MDS) and offers redundancy-reliability optimality. With an $[n, k]$ MDS erasure code, the original file can be recovered from any set of $k$ encoded fragments, regardless of whether they are systematic or parity. In other words, the system can tolerate up to $(n-k)$ concurrent device/node failure without jeopardizing the data availability.

Despite the huge potentials of MDS erasure codes, however, practical application of these codes in massive storage networks have been difficult. Not only are simple (i.e. requires very little computational complexity) MDS codes very difficult to construct, but data repair would in general require the access of $k$ other encoded fragments [5], causing considerable input/output (I/O) bandwidth that would pose huge challenges to a typical storage network.


Fig. 1. An example of how a locally repairable linear code is used to construct a distributed storage system: a file $\mathcal{F}$ is first split into five equal packets $\left\{x_{1}, \cdots, x_{5}\right\}$ and then is encoded into 12 packets, using a $(2,3)_{a}$ linear code. These 12 encoded packets are stored at 12 nodes $\left\{\mathrm{v}_{1}, \cdots, \mathrm{v}_{12}\right\}$, which are divided into three groups $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\},\left\{v_{5}, v_{6}, v_{7}, v_{8}\right\}$ and $\left\{v_{9}, v_{10}, v_{11}, v_{12}\right\}$. Each group can perform local repair of up to two nodefailures. For example, if Node $v_{9}$ fails, it can be repaired by any two packets among $\mathrm{v}_{10}, \mathrm{v}_{11}$ and $\mathrm{v}_{12}$. Moreover, the entire file $\mathcal{F}$ can be recovered by five packets from any five nodes $\mathrm{v}_{i_{1}}, \cdots, \mathrm{v}_{i_{5}}$ which intersect each group with at most two packets. For example, $\mathcal{F}$ can be recovered from five packets stored at $\mathrm{v}_{1}, \mathrm{v}_{3}, \mathrm{v}_{7}, \mathrm{v}_{8}$ and $\mathrm{v}_{10}$.

Motivated by the desire to reduce repair cost in the design of erasure codes for distributed storage systems, Gopalan et al. [8] introduced the interesting notion of symbol locality in
linear codes. The $i$ th coded symbol of an $[n, k]$ linear code $\mathcal{C}$ is said to have locality $r(1 \leq r \leq k)$ if it can be recovered by accessing at most $r$ other symbols in $\mathcal{C}$. The concept was further generalized to $(r, \delta)$ locality by Prakash et al. [10], to address the situation of multiple device failures.

According to [10], the $i$ th code symbol $c_{i}, 1 \leq i \leq n$, in an [ $n, k]$ linear code $\mathcal{C}$ is said to have locality $(r, \delta)$ if there exists an index set $S_{i} \subseteq[n]$ containing $i$ such that $\left|S_{i}\right|-\delta+1 \leq r$ and each symbol $c_{j}, j \in S_{i}$, can be reconstructed by any $\left|S_{i}\right|-\delta+1$ symbols in $\left\{c_{\ell} ; \ell \in S_{i}\right.$ and $\left.\ell \neq j\right\}$, where $\delta \geq 2$ is an integer. Thus, when $\delta=2$, the notion of locality in [10] reduces to the notion of locality in [8]. Two cases of $(r, \delta)$ codes are introduced in the literature: An $(r, \delta)_{i}$ code is a systematic linear code whose information symbols all have locality $(r, \delta)$; and an $(r, \delta)_{a}$ code is a linear code all of whose symbols have locality $(r, \delta)$. Hence, an $(r, \delta)_{a}$ code is also referred to as having all-symbol locality $(r, \delta)$, and an $(r, \delta)_{i}$ code is also referred to as having information locality $(r, \delta)$. A symbol with $(r, \delta)$ locality - given that at the most $(\delta-1)$ symbols are erased - can be deduced by reading at most $r$ other unerased symbols.

Clearly, codes with a low symbol locality, such as $r<k$, impose a low I/O bandwidth and repair cost in a distributed storage system. In a DSS system, one can use "group" to describe storage nodes situated in the same physical location which enjoy a higher communication bandwidth and a shorter communication distance than storage nodes belonging to different groups. In the case of node failure, a locally repairable code makes it possible to efficiently recover data stored in the failed node by downloading information from nodes in the same group (or in a minimal number of other groups). Fig. 1 provides a simple example of how an $(r, \delta)_{a}$ code is used to construct a distributed storage system. In this example, $\mathcal{C}$ is a $(2,3)_{a}$ linear code of length 12 and dimension 5 . Note that a failed node can be reconstructed by accessing only two other existing nodes, while it takes five existing nodes to repair a failed node if a $[12,5]$ MDS code is used.

## A. Related Work

Locality was identified as a repair cost metric for distributed storage systems independently by Oggier et al. [7], Gopalan et al. [8] and PaPailiopoulos et al. [9] using different terms. In [8], Gopalan et al. introduced the concept of symbol locality of linear codes and established a tight bound for the redundancy in terms of the message length, the distance, and the locality of information coordinates. A generalized concept, i.e., $(r, \delta)$ locality, was addressed by Prakash et al. [10]. It was proved in [10] that the minimum distance $d$ of an $(r, \delta)_{i}$ linear code $\mathcal{C}$ is upper bounded by

$$
\begin{equation*}
d \leq n-k+1-\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1) \tag{I.1}
\end{equation*}
$$

where $n$ and $k$ are the length and dimension of $\mathcal{C}$ respectively. It was also proved that a class of codes known as pyramid codes [6] achieve this bound. Since an $(r, \delta)_{a}$ code is also an $(r, \delta)_{i}$ code, (I.1) also presents an upper bound for the minimum distance of $(r, \delta)_{a}$ codes.

Locality of general codes (linear or nonlinear) and bounds on the minimum distance for a given locality were presented in parallel and subsequent works [11], [14]. An $(r, \delta)_{a}$ code (systematic or not) is also termed a locally repairable code $(L R C)$, and $(r, \delta)_{a}$ codes that achieve the minimum distance bound are called optimal.

It was proved in [10] that there exists optimal locally repairable linear codes when $(r+\delta-1) \mid n$ and $q>k n^{k}$. Under the condition that $(r+\delta-1) \mid n$, a construction method of optimal locally repairable vector codes was proposed in [14], where maximal rank distance (MRD) codes were used along with MDS array codes. For the special case of $\delta=2$, Tamo et al. [15] proposed an explicit construction of optimal LRCs when

$$
(r+1) \mid n
$$

or

$$
n \bmod (r+1)-1 \geq k \bmod r>0,1
$$

Except for the special case that $n \bmod (r+1)-1 \geq k \bmod r>$ 0 , no results are known about whether there exists optimal $(r, \delta)_{a}$ code when $(r+\delta-1) \nmid n$.

Up to now, designing LRCs with optimal distance remains an intriguing open problem for most coding parameters $n, k, r$ and $\delta$. Since large fields involve rather complicated and expensive computation, a related interesting open problem asks how to limit the design (of optimal LRCs) over relatively smaller fields.

## B. Main Results

In this paper, we investigate the structure properties and the construction of optimal $(r, \delta)_{a}$ linear codes of length $n$ and dimension $k$. A simple property of optimal $(r, \delta)_{a}$ linear codes is proved in Lemma [5, which shows that $\frac{n}{r+k-1} \geq \frac{k}{r}$ for any optimal $(r, \delta)_{a}$ linear code. Hence we impose this condition of $\frac{n}{r+k-1} \geq \frac{k}{r}$ throughout our discussion of optimal $(r, \delta)_{a}$ codes.

The main results of this paper include:
(i) We prove a structure theorem for the optimal $(r, \delta)_{a}$ linear codes for $r \mid k$. This structure theorem indicates that it is possible for optimal $(r, \delta)_{a}$ linear codes, a sub-class of optimal $(r, \delta)_{i}$ linear code, to have a simpler structure than otherwise.
(ii) We prove that there exist no optimal $(r, \delta)_{a}$ linear codes for

$$
\begin{equation*}
(r+\delta-1) \nmid n \text { and } r \mid k \tag{I.2}
\end{equation*}
$$

or

$$
\begin{equation*}
m<v+\delta-1 \text { and } u \geq 2(r-v)+1 \tag{I.3}
\end{equation*}
$$

where $n=w(r+\delta-1)+m$ and $k=u r+v$ such that $0<v<r$ and $0<m<r+\delta-1$ (Theorems 10 and 11).
(iii) We propose a deterministic algorithm for constructing optimal $(r, \delta)_{a}$ linear codes over any field of size $q \geq\binom{ n}{k-1}$ when

$$
\begin{equation*}
(r+\delta-1) \mid n \tag{I.4}
\end{equation*}
$$

[^0]

Fig 2. Summary of existence of optimal $(r, \delta)_{a}$ linear codes.
or

$$
\begin{equation*}
m \geq v+\delta-1 \tag{I.5}
\end{equation*}
$$

where $n=w(r+\delta-1)+m$ and $k=u r+v$ such that $0<v<r$ and $0<m<r+\delta-1$ (Theorem 15 and 16).
(iv) We propose another deterministic algorithm for constructing optimal $(r, \delta)_{a}$ linear codes over any field of size $q \geq\binom{ n}{k-1}$ when

$$
\begin{equation*}
w \geq r+\delta-1-m \text { and } \min \{r-v, w\} \geq u \tag{I.6}
\end{equation*}
$$

or

$$
\begin{equation*}
w+1 \geq 2(r+\delta-1-m) \text { and } \min \{2(r-v), w\} \geq u \tag{I.7}
\end{equation*}
$$

where $n=w(r+\delta-1)+m$ and $k=u r+v$ such that $0<v<r$ and $0<m<r+\delta-1$ (Theorem 26 and 27).

A summary of our results is given in Fig 2. Note that if none of the conditions in (I.2)-I.5) holds, it then follows that

$$
m<v+\delta-1 \text { and } u \leq 2(r-v)
$$

In that case, if condition (I.6) does not hold, we have $w<$ $r+\delta-1-m$ or $r-v<u$; and if condition (I.7) does not hold, we have $w+1<2(r+\delta-1-m)$, i.e., $w<$ $2(r+\delta-1-m)-1$. Hence, if, neither condition (I.6) nor condition (I.7) holds (in addition to (I.2)-(I.5)), then one of the following two conditions must be satisfied:

$$
\begin{equation*}
w<r+\delta-1-m \tag{I.8}
\end{equation*}
$$

or
$r+\delta-1-m \leq w<2(r+\delta-1-m)-1$ and $r-v<u$.

In other words, if none of the conditions (I.2)- (I.7) holds, then either (I.8) or (I.9) will hold. From our existence proof and/or constructive results, the existence of optimal $(r, \delta)_{a}$ linear code is not known only for a limited scope with parameters described by (I.8) and (I.9).

The remainder of the paper is organized as follows. In Section II, we present the notions used in the paper as well as some preliminary results about $(r, \delta)_{a}$ linear codes. In Section III, we investigate the structure of optimal $(r, \delta)_{a}$ linear codes
when $r \mid k$ (should they exist). In Section IV, we consider the non-existence conditions for optimal $(r, \delta)_{a}$ linear codes under conditions (I.2) and (I.3). A construction of optimal $(r, \delta)_{a}$ linear codes for conditions (I.4) and (I.5) is presented in Section V, and a construction of optimal $(r, \delta)_{a}$ linear codes for conditions (I.6) and (I.7) is presented in Section VI. Finally, we conclude the paper in Section VII.

## II. Locality of Linear Codes

For two positive integers $t_{1}$ and $t_{2}\left(t_{1} \leq t_{2}\right)$, we denote $\left[t_{1}, t_{2}\right]=\left\{t_{1}, t_{1}+1, \cdots, t_{2}\right\}$ and $\left[t_{2}\right]=\left\{1,2, \cdots, t_{2}\right\}$. For any set $S$, the size (cardinality) of $S$ is denoted by $|S|$. If $I$ is a subset of $S$ and $|I|=r$, then we say that $I$ is an $r$-subset of $S$. Let $\mathbb{F}_{q}^{k}$ be the $k$-dimensional vector space over the $q$-ary field $\mathbb{F}_{q}$. For any subset $X \subseteq \mathbb{F}_{q}^{k}$, we use $\langle X\rangle$ to denote the subspace of $\mathbb{F}_{q}^{k}$ spanned by $X$.

In the sequel, whenever we speak of an $(r, \delta)_{a}$ or $(r, \delta)_{i}$ code, we will by default assume it is an $[n, k, d]$ linear code (i.e., its length, dimension and minimum distance are $n, k$ and $d$ respectively).

Suppose $\mathcal{C}$ is an $[n, k, d]$ linear code over $\mathbb{F}_{q}$, and $G=$ $\left(G_{1}, \cdots, G_{n}\right)$ is a generating matrix of $\mathcal{C}$, where $G_{i}, i \in[n]$, is the $i$ th column of $G$. We denote by $\mathcal{G}=\left\{G_{1}, \cdots, G_{n}\right\}$ the collection of columns of $G$. It is well known that the distance property is captured by the following condition (e.g. [18]).

Lemma 1: An $[n, k]$ code $\mathcal{C}$ has a minimum distance $d$, if and only if $|S| \leq n-d$ for every $S \subseteq \mathcal{G}$ having $\operatorname{Rank}(S) \leq$ $k-1$. Equivalently, $\operatorname{Rank}(T)=k$ for every $T \subseteq \mathcal{G}$ of size $n-d+1$.

For any subset $S \subseteq[n]$, let $\left.\mathcal{C}\right|_{S}$ denote the punctured code of $\mathcal{C}$ associated with the coordinate set $S$. That is, $\left.\mathcal{C}\right|_{S}$ is obtained from $\mathcal{C}$ by deleting all symbols $c_{i}, i \in[n] \backslash S$, in each codeword $\left(c_{1}, \cdots, c_{n}\right) \in \mathcal{C}$.

Definition 2 ( $[\boxed{10]})$ : Suppose $1 \leq r \leq k$ and $\delta \geq 2$. The $i$ th code symbol $c_{i}, 1 \leq i \leq n$, in an $[n, k, d]$ linear code $\mathcal{C}$ is said to have locality $(r, \delta)$ if there exists a subset $S_{i} \subseteq[n]$ such that
(1) $\left|S_{i}\right| \leq r+\delta-1$;
(2) The minimum distance of the punctured code $\left.\mathcal{C}\right|_{S_{i}}$ is at least $\delta$.

Remark 3: Let $G=\left(G_{1}, \cdots, G_{n}\right)$ be a generating matrix of $\mathcal{C}$. By Lemma 1 , it is easy to see that the second condition in Definition 2 is equivalent to the following condition
(2') $\operatorname{Rank}\left(\left\{G_{\ell} ; \ell \in I\right\}\right)=\operatorname{Rank}\left(\mathcal{G}_{i}\right)$ for any subset $I \subseteq S_{i}$ of size $|I|=\left|S_{i}\right|-\delta+1$, where $\mathcal{G}_{i}=\left\{G_{\ell} ; \ell \in S_{i}\right\}$;

Moreover, by conditions (1) and (2'), we have

$$
\operatorname{Rank}\left(\mathcal{G}_{i}\right)=\operatorname{Rank}\left(\left\{G_{\ell} ; \ell \in S_{i}\right\}\right) \leq\left|S_{i}\right|-\delta+1 \leq r
$$

That is, $\forall i^{\prime} \in S_{i}$ and $\forall I \subseteq S_{i} \backslash\left\{i^{\prime}\right\}$ of size $|I|=\left|S_{i}\right|-\delta+1$, $G_{i^{\prime}}$ is an $\mathbb{F}_{q^{\prime}}$-linear combination of $\left\{G_{\ell} ; \ell \in I\right\}$. This means that the symbol $c_{i^{\prime}}$ can be reconstructed by the $\left|S_{i}\right|-\delta+1$ symbols in $\left\{c_{\ell} ; \ell \in I\right\}$.

An $(r, \delta)_{a}$ code $\mathcal{C}$ is said to be optimal if the minimum distance $d$ of $\mathcal{C}$ achieves the bound in (I.1).

The following remark follows naturally from Definition 2 and Remark 3 .

Remark 4: If $\mathcal{C}$ is an $(r, \delta)_{a}$ code and $G=\left(G_{1}, \cdots, G_{n}\right)$ is a generating matrix of $\mathcal{C}$, then we can always find a collection $\mathcal{S}=\left\{S_{1}, \cdots, S_{t}\right\}$, where $S_{i} \subseteq[n], i=1, \cdots, t$, such that
(1) $\left|S_{i}\right| \leq r+\delta-1, i=1, \cdots, t$;
(2) $\operatorname{Rank}\left(\left\{G_{\ell} ; \ell \in I\right\}\right)=\operatorname{Rank}\left(\mathcal{G}_{i}\right) \leq r, \forall i \in[t]$ and $I \subseteq S_{i}$ of size $|I|=\left|S_{i}\right|-\delta+1$, where $\mathcal{G}_{i}=\left\{G_{\ell} ; \ell \in S_{i}\right\}$;
(3) $\cup_{i \in[t]} S_{i}=[n]$ and $\cup_{i \in[t] \backslash\{j\}} S_{i} \neq[n], \forall j \in[t]$.

We call the set $\mathcal{S}=\left\{S_{1}, \cdots, S_{t}\right\}$ an $(r, \delta)$-cover set of $\mathcal{C}$.
The following lemma presents a simple property of $(r, \delta)_{a}$ codes.

Lemma 5: An $(r, \delta)_{a}$ code $\mathcal{C}$ satisfies

1) The minimum distance $d \geq \delta$.
2) If $\mathcal{C}$ is an optimal $(r, \delta)_{a}$ code, then $\frac{n}{r+\delta-1} \geq \frac{k}{r}$.

Proof: 1) Let $\mathcal{S}=\left\{S_{1}, \cdots, S_{t}\right\}$ be an $(r, \delta)$-cover set of $\mathcal{C}$. For any $0 \neq\left(c_{1}, \cdots, c_{n}\right) \in \mathcal{C}$, since $\cup_{i \in[t]} S_{i}=[n]$, there is an $i \in[t]$ such that the punctured codeword $\left(c_{j}\right)_{j \in S_{i}}$ is nonzero in $\left.\mathcal{C}\right|_{S_{i}}$. By the second condition of Definition 2, the Hamming weight of $\left(c_{j}\right)_{j \in S_{i}}$ is at least $\delta$. Thus, the Hamming weight of $\left(c_{1}, \cdots, c_{n}\right)$ is at least $\delta$. Since $0 \neq\left(c_{1}, \cdots, c_{n}\right) \in$ $\mathcal{C}$ is arbitrary, the minimum distance $d \geq \delta$.
2) Since $\mathcal{C}$ is an optimal $(r, \delta)_{a}$ code, from the minimum distance bound in (I.1),

$$
n=d+k-1+\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)
$$

From claim 1), $d \geq \delta$; which leads to

$$
n \geq \delta+k-1+\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)
$$

Hence,

$$
\begin{aligned}
n r & \geq r(\delta+k-1)+r\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1) \\
& \geq r(\delta+k-1)+r\left(\frac{k}{r}-1\right)(\delta-1) \\
& =k(r+\delta-1)
\end{aligned}
$$

which implies that $\frac{n}{r+\delta-1} \geq \frac{k}{r}$.

## III. Structure of Optimal $(r, \delta)_{a}$ Code when $r \mid k$

In this section, we prove a structure theorem for optimal $(r, \delta)_{a}$ codes under the condition of $r \mid k$.

Throughout this section, we assume that $\mathcal{C}$ is an $(r, \delta)_{a}$ code over the field $\mathbb{F}_{q}, \mathcal{S}=\left\{S_{1}, \cdots, S_{t}\right\}$ is an $(r, \delta)$-cover set of $\mathcal{C}$, where $S_{i} \subseteq[n], i=1, \cdots t$, and $G=\left(G_{1}, \cdots, G_{n}\right)$ is a generating matrix of $\mathcal{C}$. We denote $\mathcal{G}=\left\{G_{1}, \cdots, G_{n}\right\}$ and $\mathcal{G}_{i}=\left\{G_{\ell} ; \ell \in S_{i}\right\}^{2}$. Then for any $I \subseteq[t]$, we have

$$
\begin{equation*}
\left|\cup_{i \in I} \mathcal{G}_{i}\right|=\left|\left\{G_{i} ; i \in \cup_{\ell \in I} S_{\ell}\right\}\right|=\left|\cup_{i \in I} S_{i}\right| \tag{III.1}
\end{equation*}
$$

and by Remark 4 we get

$$
\begin{equation*}
\cup_{i \in[t]} \mathcal{G}_{i}=\mathcal{G} \text { and } \cup_{i \in[t] \backslash\{j\}} \mathcal{G}_{i} \neq \mathcal{G}, \forall j \in[t] . \tag{III.2}
\end{equation*}
$$

We first give some lemmas to help prove our main results.
Lemma 6: Consider three sets $A, B, X \subseteq \mathbb{F}_{q}^{k}$. If $C$ is a subset of $X$ satisfies: $\operatorname{Rank}(B \cup C)=\operatorname{Rank}(A \cup B \cup C)$, then

$$
\operatorname{Rank}(X \cup A \cup B)-|B| \leq \operatorname{Rank}(X)
$$

Proof: Since $C \subseteq X$ and $\operatorname{Rank}(B \cup C)=\operatorname{Rank}(A \cup B \cup$ $C)$, we have

$$
\begin{aligned}
\operatorname{Rank}(X \cup A \cup B) & =\operatorname{Rank}(X \cup C \cup A \cup B) \\
& =\operatorname{Rank}(X \cup B \cup C) \\
& =\operatorname{Rank}(X \cup B) \\
& \leq \operatorname{Rank}(X)+\operatorname{Rank}(B) \\
& \leq \operatorname{Rank}(X)+|B| .
\end{aligned}
$$

Therefore, $\operatorname{Rank}(X \cup A \cup B)-|B| \leq \operatorname{Rank}(X)$.
Lemma 7: Suppose $\left\{i_{1}, \cdots, i_{\ell}\right\} \subseteq[t]$ such that $\mathcal{G}_{i_{j}} \nsubseteq$ $\left\langle\cup_{\lambda=1}^{j-1} \mathcal{G}_{i_{\lambda}}\right\rangle, j=2, \cdots, \ell$. Then

$$
\left|\cup_{j=1}^{\ell} S_{i_{j}}\right| \geq \operatorname{Rank}\left(\cup_{j=1}^{\ell} \mathcal{G}_{i_{j}}\right)+\ell(\delta-1)
$$

Proof: We prove this lemma by induction.
From Remark 3, $\left|S_{i_{1}}\right| \geq \operatorname{Rank}\left(\mathcal{G}_{i_{1}}\right)+(\delta-1)$. Hence the claim holds for $\ell=1$.

Now consider $\ell \geq 2$. We assume that the claim holds for $\ell-1$, i.e.,

$$
\begin{equation*}
\left|\cup_{j=1}^{\ell-1} S_{i_{j}}\right| \geq \operatorname{Rank}\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right)+(\ell-1)(\delta-1) \tag{III.3}
\end{equation*}
$$

We shall prove that the claim is true for $\ell$.
First, we point out that $\left|\mathcal{G}_{i_{\ell}} \backslash\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right)\right|>\delta-1$. In fact, if $\left|\mathcal{G}_{i_{\ell}} \backslash\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right)\right| \leq \delta-1$, then $\mid \mathcal{G}_{i_{\ell}} \cap\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\left|\geq\left|\mathcal{G}_{i_{\ell}}\right|-(\delta-1)\right.\right.$. From condition (2) of Remark $4 \mathcal{G}_{i_{\ell}} \subseteq\left\langle\mathcal{G}_{i_{\ell}} \cap\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right)\right\rangle \subseteq$ $\left\langle\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right\rangle$, which presents a contradiction to the assumption that $\mathcal{G}_{i_{\ell}} \nsubseteq\left\langle\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right\rangle$. Thus,

$$
\left|\mathcal{G}_{i_{\ell}} \backslash\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right)\right|>\delta-1
$$

[^1]Let $X=\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}$ and $C=\mathcal{G}_{i_{\ell}} \cap\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right)=\mathcal{G}_{i_{\ell}} \cap X$. Let $A$ be a fixed $(\delta-1)$-subset of $\mathcal{G}_{i_{\ell}} \backslash\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right)$ and $B=$ $\left(\mathcal{G}_{i_{\ell}} \backslash \cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right) \backslash A$.

From condition (2) of Remark 4, $\operatorname{Rank}(B \cup C)=\operatorname{Rank}(A \cup$ $B \cup C)$. Then, from Lemma6, we get

$$
\operatorname{Rank}(X \cup A \cup B)-|B| \leq \operatorname{Rank}(X)
$$

i.e.,

$$
\begin{equation*}
\operatorname{Rank}\left(\cup_{j=1}^{\ell} \mathcal{G}_{i_{j}}\right)-|B| \leq \operatorname{Rank}\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right) \tag{III.4}
\end{equation*}
$$

Clearly, $\cup_{j=1}^{\ell} \mathcal{G}_{i_{j}}$ is a disjoint union of $A, B$ and $\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}$. Hence,

$$
\begin{aligned}
\left|\cup_{j=1}^{\ell} \mathcal{G}_{i_{j}}\right| & =\left|\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right|+|A|+|B| \\
& =\left|\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right|+(\delta-1)+|B|
\end{aligned}
$$

and from (III.1), we get

$$
\begin{equation*}
\left|\cup_{j=1}^{\ell} S_{i_{j}}\right|=\left|\cup_{j=1}^{\ell} \mathcal{G}_{i_{j}}\right|=\left|\cup_{j=1}^{\ell-1} S_{i_{j}}\right|+(\delta-1)+|B| . \tag{III.5}
\end{equation*}
$$

Combining (III.3)- (III.5), we have

$$
\begin{aligned}
\left|\cup_{j=1}^{\ell} S_{i_{j}}\right| & =\left|\cup_{j=1}^{\ell-1} S_{i_{j}}\right|+(\delta-1)+|B| \\
& \geq \operatorname{Rank}\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right)+\ell(\delta-1)+|B| \\
& \geq \operatorname{Rank}\left(\cup_{j=1}^{\ell} \mathcal{G}_{i_{j}}\right)-|B|+\ell(\delta-1)+|B| \\
& =\operatorname{Rank}\left(\cup_{j=1}^{\ell} \mathcal{G}_{i_{j}}\right)+\ell(\delta-1)
\end{aligned}
$$

which completes the proof.
Lemma 8: Suppose $\mathcal{C}$ is an optimal $(r, \delta)_{a}$ code. Then

1) $t \geq\left\lceil\frac{n}{r+\delta-1}\right\rceil \geq\left\lceil\frac{k}{r}\right\rceil$.
2) If $J \subseteq[t]$ and $|J| \leq\left\lceil\frac{k}{r}\right\rceil-1$, then $\operatorname{Rank}\left(\cup_{i \in J} \mathcal{G}_{i}\right) \leq k-1$ and $\mathcal{G}_{h} \nsubseteq\left\langle\cup_{i \in J} \mathcal{G}_{i}\right\rangle, \forall h \in[t] \backslash J$.
3) If $J \subseteq[t]$ and $|J|=\left\lceil\frac{k}{r}\right\rceil$, then $\operatorname{Rank}\left(\cup_{i \in J} \mathcal{G}_{i}\right)=k$ and $\left|\cup_{i \in J} S_{i}\right| \geq k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)$
Proof: 1) (Proof by contradiction) Suppose $t \leq$ $\left\lceil\frac{n}{r+\delta-1}\right\rceil-1$. Then from Remark 4

$$
\left|S_{i}\right| \leq r+\delta-1
$$

Hence,

$$
\begin{aligned}
n & =\left|\cup_{i \in[t]} S_{i}\right| \\
& \leq t(r+\delta-1) \\
& \leq\left(\left\lceil\frac{n}{r+\delta-1}\right\rceil-1\right)(r+\delta-1) \\
& <n
\end{aligned}
$$

which presents a contradiction. Hence, it must hold that $t \geq$ $\left\lceil\frac{n}{r+\delta-1}\right\rceil$.

Moreover, from Claim 2) of Lemma 5] $\frac{n}{r+\delta-1} \geq \frac{k}{r}$. Thus,

$$
t \geq\left\lceil\frac{n}{r+\delta-1}\right\rceil \geq\left\lceil\frac{k}{r}\right\rceil
$$

2) From Remark 3, $\operatorname{Rank}\left(\mathcal{G}_{i}\right) \leq r, \forall i \in[t]$. Hence, if $|J| \leq$ $\left\lceil\frac{k}{r}\right\rceil-1$, then

$$
\operatorname{Rank}\left(\cup_{i \in J} \mathcal{G}_{i}\right) \leq r|J| \leq r\left(\left\lceil\frac{k}{r}\right\rceil-1\right)<r \frac{k}{r}=k
$$

i.e., $\operatorname{Rank}\left(\cup_{i \in J} \mathcal{G}_{i}\right) \leq k-1$.

Now, suppose $\mathcal{G}_{h} \subseteq\left\langle\cup_{i \in J} \mathcal{G}_{i}\right\rangle$, and we will see a contradiction results. First, we can find a subset $J_{0}=\left\{i_{1}, \cdots, i_{s}\right\} \subseteq J$ such that $\mathcal{G}_{h} \subseteq\left\langle\cup_{\lambda=1}^{s} \mathcal{G}_{i_{s}}\right\rangle$ and $\mathcal{G}_{h} \nsubseteq\left\langle\cup_{i \in J^{\prime}} \mathcal{G}_{i}\right\rangle$ for any proper subset $J^{\prime}$ of $J_{0}$. In particular, we have

$$
\mathcal{G}_{i_{j}} \nsubseteq\left\langle\cup_{\lambda=1}^{j-1} \mathcal{G}_{i_{\lambda}}\right\rangle, j=2, \cdots, s
$$

Note that $\left|J_{0}\right| \leq|J| \leq\left\lceil\frac{k}{r}\right\rceil-1$. By the proved result, we have

$$
\operatorname{Rank}\left(\cup_{i \in J_{0}} \mathcal{G}_{i}\right) \leq k-1
$$

Next, we can find a sequence $\mathcal{G}_{i_{1}}, \cdots, \mathcal{G}_{i_{s}}, \mathcal{G}_{i_{s+1}}, \cdots, \mathcal{G}_{i_{\ell}}$ such that $\ell \geq\left\lceil\frac{k}{r}\right\rceil, \operatorname{Rank}\left(\cup_{j=1}^{\ell} \mathcal{G}_{i_{j}}\right)=k$ and $\mathcal{G}_{i_{j}} \nsubseteq\left\langle\cup_{\lambda=1}^{j-1} \mathcal{G}_{i_{\lambda}}\right\rangle, j=$ $2, \cdots, \ell$. In particular, $\operatorname{Rank}\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right) \leq k-1$. Therefore, there exists a $\mathcal{G}_{i_{\ell}}^{\prime} \subseteq \mathcal{G}_{i_{\ell}}$ such that $\operatorname{Rank}\left(\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right) \cup \mathcal{G}_{i_{\ell}}^{\prime}\right)=$ $k-1$. Denote $\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right) \cup \mathcal{G}_{i_{\ell}}^{\prime}=S$. Then $\operatorname{Rank}(S)=k-1$ and

$$
\begin{align*}
\left|\mathcal{G}_{i_{\ell}}^{\prime} \backslash \cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right| & \geq \operatorname{Rank}(S)-\operatorname{Rank}\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right) \\
& =(k-1)-\operatorname{Rank}\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right) \tag{III.6}
\end{align*}
$$

From Lemma 7

$$
\begin{equation*}
\left|\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right| \geq \operatorname{Rank}\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right)+(\ell-1)(\delta-1) \tag{III.7}
\end{equation*}
$$

Then by equations (III.6) and (III.7),

$$
\begin{align*}
|S| & =\left|\mathcal{G}_{i_{\ell}}^{\prime} \backslash \cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right|+\left|\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right| \\
& \geq(k-1)+(\ell-1)(\delta-1) \\
& \geq k-1+\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1) \tag{III.8}
\end{align*}
$$

Since $h \in[t] \backslash J, \mathcal{G}_{h} \neq \mathcal{G}_{i_{j}}, j=1, \cdots, s$. Moreover, since $\mathcal{G}_{h} \subseteq\left\langle\cup_{\lambda=1}^{s} \mathcal{G}_{i_{s}}\right\rangle$ and $\mathcal{G}_{i_{j}} \nsubseteq\left\langle\cup_{\lambda=1}^{j-1} \mathcal{G}_{i_{\lambda}}\right\rangle, j=2, \cdots, \ell$, so $\mathcal{G}_{h} \neq$ $\mathcal{G}_{i_{j}}, j=s+1, \cdots, \ell$. From equation (III.2), we have $\mathcal{G}_{h} \nsubseteq$ $\cup_{j=1}^{\ell} \mathcal{G}_{i_{j}}$. Then, from equation (III.8), we get

$$
\left|\mathcal{G}_{h} \cup S\right|>|S| \geq k-1+\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)
$$

Since we assumed $\mathcal{G}_{h} \subseteq\left\langle\cup_{\lambda=1}^{s} \mathcal{G}_{i_{s}}\right\rangle \subseteq\langle S\rangle$, then $\operatorname{Rank}\left(\mathcal{G}_{h} \cup\right.$ $S)=\operatorname{Rank}(S)=k-1$. By Lemma 1, we have

$$
d \leq n-\left|\mathcal{G}_{h} \cup S\right|<n-k+1-\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)
$$

which contradicts the assumption that $\mathcal{C}$ is an optimal $(r, \delta)_{a}$ code. Hence, it must be that $\mathcal{G}_{h} \nsubseteq\left\langle\cup_{i \in J} \mathcal{G}_{i}\right\rangle 3^{3}$
3) Suppose $J=\left\{i_{1}, \cdots, i_{s}\right\}$, where $s=\left\lceil\frac{k}{r}\right\rceil$. By claim 2),

$$
\mathcal{G}_{i_{j}} \nsubseteq\left\langle\cup_{\lambda=1}^{j-1} \mathcal{G}_{i_{\lambda}}\right\rangle, j=2, \cdots, s
$$

First, we have $\operatorname{Rank}\left(\cup_{i \in J} \mathcal{G}_{i}\right)=k$. Otherwise, as in the proof of claim 2), we can find a sequence $\mathcal{G}_{i_{1}}, \cdots, \mathcal{G}_{i_{s}}$, $\mathcal{G}_{i_{s+1}}, \cdots, \mathcal{G}_{i_{\ell}}\left(\ell>s=\left\lceil\frac{k}{r}\right\rceil\right)$ and a set $S=\left(\cup_{j=1}^{\ell-1} \mathcal{G}_{i_{j}}\right) \cup$ $\mathcal{G}_{i_{\ell}}^{\prime}\left(\mathcal{G}_{i_{\ell}}^{\prime} \subseteq \mathcal{G}_{i_{\ell}}\right)$ such that
$|S| \geq k-1+(\ell-1)(\delta-1)>k-1+\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)$.
By Lemma 1

$$
d \leq n-|S|<n-k+1-\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)
$$

[^2]which contradicts the assumption that $\mathcal{C}$ is an optimal $(r, \delta)_{a}$ code. Therefore, we have $\operatorname{Rank}\left(\cup_{i \in J} \mathcal{G}_{i}\right)=k$.

Now, by Lemma 7 ,

$$
\begin{aligned}
\left|\cup_{i \in J} S_{i}\right| & \geq \operatorname{Rank}\left(\cup_{i \in J} \mathcal{G}_{i}\right)+\left\lceil\frac{k}{r}\right\rceil(\delta-1) \\
& =k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)
\end{aligned}
$$

This completes the proof.
We now present our main theorem of this section.
Theorem 9: Suppose $\mathcal{C}$ is an optimal $(r, \delta)_{a}$ linear code. If $r \mid k$ and $r<k$, then the following conditions hold:

1) $S_{1}, \cdots, S_{t}$ are mutually disjoint;
2) $\left|S_{i}\right|=r+\delta-1, \forall i \in[t]$, and the punctured code $\left.\mathcal{C}\right|_{S_{i}}$ is an $[r+\delta-1, r, \delta]$ MDS code.
In particular, we have $(r+\delta-1) \mid n$.
Proof: Since $r \mid k$ and $r<k$, then $k=\ell r$ for some $\ell \geq 2$. By 1) of Lemma $8, t \geq\left\lceil\frac{k}{r}\right\rceil=\ell$. Let $\left\{i_{1}, i_{2}\right\} \subseteq[t]$ be arbitrarily chosen. Let $J$ be an $\ell$-subset of $[t]$ such that $\left\{i_{1}, i_{2}\right\} \subseteq J$. Then by 3 ) of Lemma 8,

$$
\begin{equation*}
\operatorname{Rank}\left(\cup_{i \in J} \mathcal{G}_{i}\right)=k=\ell r \tag{III.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\cup_{i \in J} \mathcal{S}_{i}\right| \geq k+\ell(\delta-1)=\ell(r+\delta-1) \tag{III.10}
\end{equation*}
$$

Since $\left|S_{i}\right| \leq r+\delta-1$ and by Remark 4 $\operatorname{Rank}\left(\mathcal{G}_{i}\right) \leq r$, then equations (III.9) and (III.10) imply that $\operatorname{Rank}\left(\mathcal{G}_{i}\right)=r$, $\left|S_{i}\right|=r+\delta-1$, and $\left\{S_{i}\right\}_{i \in J}$ are mutually disjoint.

In particular, $\operatorname{Rank}\left(\mathcal{G}_{i_{1}}\right)=\operatorname{Rank}\left(\mathcal{G}_{i_{2}}\right)=r, \mathcal{G}_{i_{1}} \cap \mathcal{G}_{i_{2}}=\emptyset$ and $\left|S_{i_{1}}\right|=\left|S_{i_{2}}\right|=r+\delta-1$. Since $i_{1}$ and $i_{2}$ are arbitrarily chosen, we have proved that $\operatorname{Rank}\left(\mathcal{G}_{i}\right)=r,\left|S_{i}\right|=r+\delta-1$, and $\left\{S_{i}\right\}_{i \in J}$ are mutually disjoint. Hence, $(r+\delta-1) \mid n$. Moreover, by Lemma 1 and Remark $3,\left.\mathcal{C}\right|_{S_{i}}$ is an $[r+\delta-1, r, \delta]$ MDS code.

In [10], it was proved that if $\mathcal{C}$ is an optimal $(r, \delta)_{i}$ code, then there exists a collection $\left\{S_{1}, \cdots, S_{a}\right\} \subseteq\left\{S_{1}, \cdots, S_{t}\right\}$ which has the same properties in Theorem 9 where $a$ is a properly-defined value. Thus, Theorem 9 shows that as a subclass of optimal $(r, \delta)_{i}$ codes, optimal $(r, \delta)_{a}$ codes tend to have a simpler structure than otherwise.

## IV. Non-Existence Conditions of Optimal $(r, \delta)_{a}$ Linear Codes

In this section, we derive two sets of conditions under which there exists no optimal $(r, \delta)_{a}$ linear codes. From the minimum distance bound in (I.1), we know that when $r=k$, optimal $(r, \delta)_{a}$ linear codes are exactly MDS codes. Hence, in this section, we focus on the case of $r<k$.

The first result is obtained directly from Theorem 9
Theorem 10: If $(r+\delta-1) \nmid n$ and $r \mid k$, then there exist no optimal $(r, \delta)_{a}$ linear codes.

Proof: If $\mathcal{C}$ is an optimal $(r, \delta)_{a}$ linear code and $r \mid k$, then by Theorem $9(r+\delta-1) \mid n$, which contradicts the condition
that $(r+\delta-1) \nmid n$. Hence, there exist no optimal $(r, \delta)_{a}$ linear codes when $(r+\delta-1) \nmid n$ and $r \mid k$.

When $(r+\delta-1) \nmid n$ and $r \nmid k$, we provide in the below a set of conditions under which no optimal $(r, \delta)_{a}$ code exists.

Theorem 11: Suppose $n=w(r+\delta-1)+m$ and $k=u r+v$, where $0<m<r+\delta-1$ and $0<v<r$. If $m<v+\delta-1$ and $u \geq 2(r-v)+1$, then there exist no optimal $(r, \delta)_{a}$ codes.

Proof: We prove this theorem by contradiction.
Suppose $\mathcal{C}$ is an optimal $(r, \delta)_{a}$ code over the field $\mathbb{F}_{q}$ and $\mathcal{S}=\left\{S_{1}, \cdots, S_{t}\right\}$ is an $(r, \delta)$-cover set of $\mathcal{C}$. Then by claim 1) of Lemma 8 , we have

$$
\begin{equation*}
t \geq\left\lceil\frac{n}{r+\delta-1}\right\rceil=w+1 \tag{IV.1}
\end{equation*}
$$

Moreover, by 3) of Lemma 8, for any $\left\lceil\frac{k}{r}\right\rceil$-subset $J$ of $[t]$,

$$
\left|\cup_{i \in J} S_{i}\right| \geq k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)
$$

For each $i \in[t]$, if $\left|S_{i}\right|<r+\delta-1$, let $T_{i} \subseteq[n]$ be such that $S_{i} \subseteq T_{i}$ and $\left|T_{i}\right|=r+\delta-1$; If $\left|S_{i}\right|=r+\delta-1$, let $T_{i}=S_{i}$. Then clearly,

$$
\cup_{i \in[t]} T_{i}=\cup_{i \in[t]} S_{i}=[n]
$$

and for any $\left\lceil\frac{k}{r}\right\rceil$-subset $J$ of $[t]$,

$$
\begin{equation*}
\left|\cup_{i \in J} T_{i}\right| \geq k+\left\lceil\frac{k}{r}\right\rceil(\delta-1) \tag{IV.2}
\end{equation*}
$$

Let $M=\left(m_{i, j}\right)_{t \times n}$ be a $t \times n$ matrix such that $m_{i, j}=1$ if $j \in T_{i}$, and $m_{i, j}=0$ otherwise. For each $j \in[n]$, let

$$
A_{j}=\left\{i \in[t] ; m_{i, j}=1\right\} .
$$

Then $\left|A_{j}\right|$ is the number of $T_{i}(i \in[t])$ satisfying $j \in T_{i}$, and this number equals the number of 1 s in the $j$ th column of $M$. Since $\cup_{i \in[t]} T_{i}=[n]$, then $\left|A_{j}\right|>0, \forall j \in[n]$. On the other hand, by the construction of $M$, for each $i \in[t]$, $T_{i}=\left\{j \in[n] ; m_{i, j}=1\right\}$. Thus, the number of the 1 s in each row of $M$ is $r+\delta-1$. It then follows that the total number of the 1 s in $M$ is

$$
\begin{equation*}
\sum_{j=1}^{n}\left|A_{j}\right|=\sum_{i=1}^{t}\left|T_{i}\right|=t(r+\delta-1) \tag{IV.3}
\end{equation*}
$$

Combining (IV.1) and (IV.3), we have

$$
\begin{align*}
\sum_{j=1}^{n}\left|A_{j}\right| & \geq(w+1)(r+\delta-1) \\
& =n+(r+\delta-1-m) \tag{IV.4}
\end{align*}
$$

Since $m<v+\delta-1$, then

$$
r+\delta-1-m>r-v
$$

Hence from IV.4, we have

$$
\begin{equation*}
\sum_{j=1}^{n}\left|A_{j}\right| \geq n+(r-v+1) \tag{IV.5}
\end{equation*}
$$

Let $P=\left\{j \in[n] ;\left|A_{j}\right|>1\right\}$. From (IV.5), $P \neq \emptyset$ and

$$
\sum_{j \in P}\left|A_{j}\right| \geq|P|+(r-v+1) .
$$

Without loss of generality, assume $P=\{1, \cdots, \ell\}$. Since $\left|A_{j}\right|>1, \forall j \in P$, we can find a number $\lambda \in\{1, \cdots, \ell\}$ such that $\sum_{j=1}^{\lambda-1}\left|A_{j}\right|<\lambda+(r-v)$ and $\sum_{j=1}^{\lambda}\left|A_{j}\right| \geq \lambda+(r-v+1)$. This means that we can find a subset $B_{\lambda} \subseteq A_{\lambda}$ such that $\left|B_{\lambda}\right|>1$ and

$$
\begin{equation*}
\sum_{j=1}^{\lambda-1}\left|A_{j}\right|+\left|B_{\lambda}\right|=\lambda+r-v+1 \tag{IV.6}
\end{equation*}
$$

Also note that

$$
\begin{equation*}
\lambda \leq r-v+1 \tag{IV.7}
\end{equation*}
$$

because otherwise, $\sum_{j=1}^{\lambda-1}\left|A_{j}\right|+\left|B_{\lambda}\right| \geq 2 \lambda>\lambda+r-v+1$, which contradicts (IV.6).

Let $B=\left(\cup_{j=1}^{\lambda-1} A_{j}\right) \cup B_{\lambda}$. Then from (IV.6),

$$
|B|=\left|\left(\cup_{j=1}^{\lambda-1} A_{j}\right) \cup B_{\lambda}\right| \leq \sum_{j=1}^{\lambda-1}\left|A_{i}\right|+\left|B_{\lambda}\right| \leq 2(r-v+1)
$$

Since $u \geq 2(r-v)+1$, then $2(r-v+1) \leq u+1$, we get

$$
|B| \leq u+1=\left\lceil\frac{k}{r}\right\rceil .
$$

Let $J$ be a $\left\lceil\frac{k}{r}\right\rceil$-subset of $[t]$ such that $B \subseteq J$. By the construction of $M$ and $B$, for each $j \in\{1, \cdots, \lambda-1\}$, there are at least $\left|A_{j}\right|$ subsets in $\left\{T_{i} ; i \in B\right\}$ containing $j$, and there are at least $\left|B_{\lambda}\right|$ subsets in $\left\{T_{i} ; i \in B\right\}$ containing $\lambda$. Hence,

$$
\begin{equation*}
\left|\cup_{i \in J} T_{i}\right| \leq|J|(r+\delta-1)-\left(\sum_{j=1}^{\lambda-1}\left|A_{j}\right|+\left|B_{\lambda}\right|-\lambda\right) \tag{IV.8}
\end{equation*}
$$

Combining (IV.6) and IV.8, we have

$$
\begin{aligned}
\left|\cup_{i \in J} T_{i}\right| & \leq\left\lceil\frac{k}{r}\right\rceil(r+\delta-1)-(r-v+1) \\
& =u r+v-1+\left\lceil\frac{k}{r}\right\rceil(\delta-1) \\
& =k-1+\left\lceil\frac{k}{r}\right\rceil(\delta-1) .
\end{aligned}
$$

which contradicts (IV.2).
Thus, we can conclude that there exist no optimal $(r, \delta)_{a}$ linear codes when $m<v+\delta-1$ and $u \geq 2(r-v)+1$.

Example: We now provide an example to help illustrate the method used in the proof of Theorem 11. Let $n=13, r=$ $\delta=2$ and $k=7$. Suppose $T_{1}=\{1,2,3\}, T_{2}=\{4,5,6\}$, $T_{3}=\{7,8,9\}, T_{4}=\{10,11,12\}, T_{5}=\{1,5,13\}$ and $T_{6}=$ $\{5,8,13\}$. Following the notations in the proof of Theorem 11) we have

$$
M=\left(\begin{array}{lllllllllllll}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Therefore, $A_{1}=\{1,5\}, A_{5}=\{2,5,6\}, A_{8}=\{3,6\}, A_{13}=$ $\{5,6\}$, and $P=\{1,5,8,13\}$. Note that $\left|A_{1}\right|+\left|A_{5}\right|=5>$ $2+(r-v+1)$. Let $B_{2}=\{2,5\} \subseteq A_{5}$ and $B=A_{1} \cup B_{2}=$ $\{1,2,5\}$; then $|B|<4=\left\lceil\frac{k}{r}\right\rceil$. Let $J=\{1,2,3,5\} \supseteq B$, then $\cup_{i \in J} T_{i}=\{1,2,3,4,5,6,7,8,9,13\}$. Hence, $\left|\cup_{i \in J} T_{i}\right|=$ $10<11=k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)$. (See the illustration of $M$ below.)

$$
M=\left(\begin{array}{lllllllllllll}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1
\end{array}\right)
$$

More generally, in this example, for any $t \geq 5$ and $\left\{T_{1}, \cdots, T_{t}\right\}$ such that $\left|T_{i}\right|=r+\delta-1=3$ and $\cup_{i=1}^{t} T_{i}=$ $[n]=\{1, \cdots, 13\}$, we can always find a $J \subseteq[t]$ such that $\left|\cup_{i \in J} T_{i}\right|<11=k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)$.
In general, since $0<v<r$, then $r-v \leq r-1$. If $k>$ $2 r^{2}+r$, then we have $u \geq 2(r-1)+1 \geq 2(r-v)+1$. Hence, when $0<n \bmod (r+\delta-1)<(k \bmod r)+\delta-1$ and $k>2 r^{2}+r$, then by Theorem 11 there exist no optimal $(r, \delta)_{a}$ codes.

## V. Construction of Optimal $(r, \delta)_{a}$ Codes: ALGORITHM 1

In this section, we propose a deterministic algorithm for constructing optimal $(r, \delta)_{a}$ linear codes over the field of size $q \geq\binom{ n}{k-1}$, when $(r+\delta-1) \mid n$ or $m \geq v+\delta-1$, where $n=w(r+\delta-1)+m$ and $k=u r+v$ satisfying $0<v<r$ and $0<m<r+\delta-1$. Recall that when $(r+\delta-1) \mid n$, it was proved in [10] that optimal $(r, \delta)_{a}$ linear codes exist over the field of size $q>k n^{k}$. Note that our method requires a much smaller field than what's shown in [10], and hence it also has a lower complexity for implementation.

To present our method, we will use the following definitions and notations, most of which follow from [8].

Definition 12: Let $\mathcal{S}=\left\{S_{1}, \cdots, S_{t}\right\}$ be a partition of $[n]$ and $\delta \leq\left|S_{i}\right| \leq r+\delta-1, \forall i \in[t]$. A subset $S \subseteq[n]$ is called an $(\mathcal{S}, r)$-core if $\left|S \cap S_{i}\right| \leq\left|S_{i}\right|-\delta+1, \forall i \in[t]$. If $S$ is an $(\mathcal{S}, r)$-core and $|S|=k$, then $S$ is called an $(\mathcal{S}, r, k)$-core.

Clearly, if $S \subseteq[n]$ is an $(\mathcal{S}, r)$-core and $S^{\prime} \subseteq S$, then $S^{\prime}$ is also an $(\mathcal{S}, r)$-core. In particular, if $S \subseteq[n]$ is an $(\mathcal{S}, r)$-core and $S^{\prime}$ is a $k$-subset of $S$, then $S^{\prime}$ is an $(\mathcal{S}, r, k)$-core.

Before presenting our construction method, we first give a lemma, which will take an important role in our discussion.

Lemma 13: Let $X_{1}, \cdots, X_{\ell}$ and $X$ be $\ell+1$ subspaces of $\mathbb{F}_{q}^{k}$ and $X \nsubseteq X_{i}, \forall i \in[\ell]$. If $q \geq \ell$, then $X \nsubseteq \cup_{i=1}^{\ell} X_{i}$.

Proof: We prove this lemma by induction.
Clearly, the claim is true when $\ell=1$.
Now, we suppose that the claim is true for $\ell-1$, i.e.,

$$
X \nsubseteq \cup_{i=1}^{\ell-1} X_{i}
$$

Then there exists an $x \in X$ such that $x \notin \cup_{i=1}^{\ell-1} X_{i}$. If $x \notin X_{\ell}$, then $x \notin \cup_{i=1}^{\ell} X_{i}$ and $X \nsubseteq \cup_{i=1}^{\ell} X_{i}$. So we assume $x \in X_{\ell}$.

Since $X \nsubseteq X_{\ell}$, there exists a $y \in X$ such that $y \notin X_{\ell}$. Then for any $\left\{a, a^{\prime}\right\} \subseteq \mathbb{F}_{q}$ and $i \in\{1, \cdots, \ell-1\}$,

$$
\left\{a x+y, a^{\prime} x+y\right\} \nsubseteq X_{i}
$$

(Otherwise, $\left(a-a^{\prime}\right) x=(a x+y)-\left(a^{\prime} x+y\right) \in X_{i}$, which contradicts to the assumption that $x \notin \cup_{i=1}^{\ell-1} X_{i}$.)

Since $q \geq \ell$, we can pick a subset $\left\{a_{1}, \cdots, a_{\ell}\right\} \subseteq \mathbb{F}_{q}$. Then $\left\{a_{1} x+y, \cdots, a_{\ell} x+y\right\} \nsubseteq \cup_{i=1}^{\ell-1} X_{i}$. (Otherwise, by the Pigeonhole principle, there is a subset $\left\{a_{i_{1}}, a_{i_{2}}\right\} \subseteq\left\{a_{1}, \cdots, a_{\ell}\right\}$ and a $j \in\{1, \cdots, \ell-1\}$ such that $\left\{a_{i_{1}} x+y, a_{i_{2}} x+y\right\} \subseteq X_{j}$, which contradicts to the proven result that for any $\left\{a, a^{\prime}\right\} \subseteq \mathbb{F}_{q}$ and $i \in\{1, \cdots, \ell-1\},\left\{a x+y, a^{\prime} x+y\right\} \nsubseteq X_{i}$.) Without loss of generality, assume $a_{1} x+y \notin \cup_{i=1}^{\ell-1} X_{i}$. Note that $x \in X_{\ell}$ and $y \notin X_{\ell}$, then $a_{1} x+y \notin X_{\ell}$. Hence, $a_{1} x+y \notin \cup_{i=1}^{\ell} X_{i}$. On the other hand, since $x, y \in X$, then $a_{1} x+y \in X$. So $X \nsubseteq \cup_{i=1}^{\ell} X_{i}$, which completes the proof.

We present our construction method in the following theorem.

Theorem 14: Let $\mathcal{S}=\left\{S_{1}, \cdots, S_{t}\right\}$ be a partition of $[n]$ and $\delta \leq\left|S_{i}\right| \leq r+\delta-1, \forall i \in[t]$. Suppose $t \geq\left\lceil\frac{k}{r}\right\rceil$ and for any $\left\lceil\frac{k}{r}\right\rceil$-subset $J$ of $[t],\left|\cup_{i \in J} S_{i}\right| \geq k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)$. If $q \geq\binom{ n}{k-1}$, then there exists an optimal $(r, \delta)_{a}$ linear code over $\mathbb{F}_{q}$.

Proof: For each $i \in[t]$, let $U_{i}$ be an $\left(\left|S_{i}\right|-\delta+1\right)$-subset of $S_{i}$. Let $\Omega_{0}=\cup_{i \in[t]} U_{i}$ and $L=\left|\Omega_{0}\right|$. Let $J$ be a $\left\lceil\frac{k}{r}\right\rceil$ subset of $[t]$. Since $\cup_{i \in J} U_{i} \subseteq \Omega_{0}$, from the assumptions of this theorem,

$$
L=\left|\Omega_{0}\right| \geq\left|\cup_{i \in J} U_{i}\right|=\left|\cup_{i \in J} S_{i}\right|-\left\lceil\frac{k}{r}\right\rceil(\delta-1) \geq k
$$

The construction of an optimal $(r, \delta)_{a}$ code consists of the following two steps:

Step 1: Construct an $[L, k]$ MDS code $\mathcal{C}_{0}$ over $\mathbb{F}_{q}$. Since $q \geq\binom{ n}{k-1} \geq n>L$, such an MDS code exists over $\mathbb{F}_{q}$. Let $G^{\prime}$ be a generating matrix of $\mathcal{C}_{0}$. We index the columns of $G^{\prime}$ by $\Omega_{0}$, i.e., $G^{\prime}=\left(G_{\ell}\right)_{\ell \in \Omega_{0}}$, where $G_{\ell}$ is a column of $G^{\prime}$ for each $\ell \in \Omega_{0}$.

Step 2: Extend $\mathcal{C}_{0}$ to an optimal $(r, \delta)_{a}$ code $\mathcal{C}$ over $\mathbb{F}_{q}$. This can be achieved by the following algorithm.

## Algorithm 1:

1. Let $\Omega=\Omega_{0}$.
$i$ runs from 1 to $t$.
2. While $S_{i} \backslash \Omega \neq \emptyset$ :
3. Pick a $\lambda \in S_{i} \backslash \Omega$ and let $G_{\lambda} \in\left\langle\left\{G_{\ell} ; \ell \in S_{i} \cap \Omega\right\}\right\rangle$ be such that for any $(\mathcal{S}, r, k)$-core $S \subseteq \Omega \cup\{\lambda\}$, $\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent.
4. $\Omega=\Omega \cup\{\lambda\}$.
5. Let $\mathcal{C}$ be the linear code generated by the matrix $G=$ $\left(G_{1}, \cdots, G_{n}\right)$.

To complete the proof of Theorem 14 we need to prove three claims: In Claim 1 and Claim 2 below we show that the code $\mathcal{C}$ output by Algorithm 1 is indeed an optimal $(r, \delta)_{a}$ linear code over $\mathbb{F}_{q}$; In Claim 3, we prove that the vector $G_{\lambda}$ described in Line 4 of Algorithm 1 can always be found,
hence the algorithm does terminate successfully.
Claim 1: The code $\mathcal{C}$ output by Algorithm 1 is an $(r, \delta)_{a}$ linear code over $\mathbb{F}_{q}$.

By Definition 2 and Remark 3, we aim to show that for every $i \in[t]$ and for every subset $I \subset S_{i}$ with $|I|=\left|S_{i}\right|-\delta+1$, it holds that

$$
\begin{equation*}
\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in I}\right)=\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in S_{i}}\right) \tag{V.1}
\end{equation*}
$$

Since in Line 4 of Algorithm 1, we choose $G_{\lambda} \in\left\langle\left\{G_{\ell} ; \ell \in\right.\right.$ $\left.\left.S_{i} \cap \Omega\right\}\right\rangle$, we have

$$
\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in\left(S_{i} \cap \Omega\right) \cup\{\lambda\}}\right)=\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in S_{i} \cap \Omega}\right)
$$

By induction,

$$
\begin{align*}
\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in S_{i}}\right) & =\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in S_{i} \cap \Omega_{0}}\right) \\
& =\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in U_{i}}\right)  \tag{V.2}\\
& =\left|S_{i}\right|-\delta+1
\end{align*}
$$

Suppose $i \in[t]$ and $I \subseteq S_{i}$ such that $|I|=\left|S_{i}\right|-\delta+1$. Then $|I|=\left|S_{i}\right|-\delta+1 \leq r \leq k$. Since $t \geq\left\lceil\frac{k}{r}\right\rceil$, we can find a $\left\lceil\frac{k}{r}\right\rceil$-subset $J^{\prime}$ of $[t]$ such that $i \in J^{\prime}$. For each $j \in J^{\prime}$, let $W_{j}$ be an $\left(\left|S_{j}\right|-\delta+1\right)$-subset of $S_{j}$ such that $W_{i}=I$. Clearly, $\cup_{j \in J^{\prime}} W_{j}$ is an $(\mathcal{S}, r)$-core. From the assumption of this lemma,

$$
\left|\cup_{j \in J^{\prime}} S_{j}\right| \geq k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)
$$

Hence

$$
\left|\cup_{j \in J^{\prime}} W_{j}\right|=\left|\cup_{j \in J^{\prime}} S_{j}\right|-\left\lceil\frac{k}{r}\right\rceil(\delta-1) \geq k
$$

Let $S$ be a $k$-subset of $\cup_{j \in J^{\prime}} W_{j}$ such that $I \subseteq S$, then $S$ is an $(\mathcal{S}, r, k)$-core. Therefore, $\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent, which in turn implies that $\left\{G_{\ell} ; \ell \in I\right\}$ is also linearly independent. Therefore,

$$
\begin{equation*}
\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in I}\right)=|I|=\left|S_{i}\right|-\delta+1 \tag{V.3}
\end{equation*}
$$

Combining (V.2) and V.3) we obtain V.1.
Claim 2: The code $\mathcal{C}$ output by Algorithm 1 has minimum distance achieving the upper bound (I.1), and hence is an optimal $(r, \delta)_{a}$ linear code.

According to Lemma 11 and (I.1), it suffices to prove that for any subset $T \subseteq[n]$ of size $|T|=k+\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)$,

$$
\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in T}\right)=k
$$

Let

$$
J=\left\{j \in[t] ;\left|T \cap S_{j}\right| \geq\left|S_{j}\right|-\delta+1\right\}
$$

For each $j \in J$, let $W_{j}$ be an $\left(\left|S_{j}\right|-\delta+1\right)$-subset of $T \cap S_{j}$; For each $j \in[t] \backslash J$, let $W_{j}=T \cap S_{j}$. Then $\cup_{j \in[t]} W_{j}$ is an ( $\mathcal{S}, r$ )-core. We consider the following two cases:

Case 1: $|J| \geq\left\lceil\frac{k}{r}\right\rceil$. Without loss of generality, assume that $|J|=\left\lceil\frac{k}{r} \sqrt[4]{4}\right.$. Since $\left|\cup_{j \in J} S_{j}\right| \geq k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)$, then

$$
\left|\cup_{j \in[t]} W_{j}\right| \geq\left|\cup_{j \in J} W_{j}\right| \geq k
$$

${ }^{4}$ If $|J|>\left\lceil\frac{k}{r}\right\rceil$, then pick a $\left\lceil\frac{k}{r}\right\rceil$-subset $J_{0}$ of $J$, and replace $J$ by $J_{0}$ in our discussion.

Case 2: $|J| \leq\left\lceil\frac{k}{r}\right\rceil-1$. In that case,
$\left|\cup_{j \in[t]} W_{j}\right| \geq|T|-|J|(\delta-1) \geq|T|-\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1) \geq k$.
In both cases, $\left|\cup_{j \in[t]} W_{j}\right| \geq k$. Let $S$ be a $k$-subset of $\cup_{j \in J} W_{j}$, then $S$ is an $(\mathcal{S}, r, k)$-core. Therefore, $\left\{G_{\ell} ; \ell \in S\right\}$ are linearly independent and

$$
\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in T}\right)=\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in S}\right)=k
$$

From equation (I.1) and Lemma 1, we get

$$
d=n-k+1-\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)
$$

where $d$ is the minimum distance of $\mathcal{C}$. Thus, $\mathcal{C}$ is an optimal $(r, \delta)_{a}$ code.

Claim 3: The vector $G_{\lambda}$ in Line 4 of Algorithm 1 can always be found.

The proof of this claim is based on a classical technique in network coding (e.g., [16], [17]). Since $G^{\prime}=\left(G_{\ell}\right)_{\ell \in \Omega_{0}}$ is a generating matrix of the MDS code $\mathcal{C}_{0}$, then for any ( $\mathcal{S}, r, k$ )-core $S \subseteq \Omega_{0},\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent. By induction, we can assume that for any $(\mathcal{S}, r, k)$-core $S \subseteq \Omega$, $\left\{G_{\ell} ; \ell \in S\right\}$ are linearly independent.

Let $\Lambda$ be the set of all $S_{0} \subseteq \Omega$ such that $S_{0} \cup\{\lambda\}$ is an ( $\mathcal{S}, r, k$ )-core. By Definition 12, for any $S_{0} \in \Lambda$,

$$
\begin{gathered}
\left|S_{0}\right|=k-1 \\
\left|S_{0} \cap S_{j}\right| \leq\left|S_{j}\right|-\delta+1, \forall j \in[t] \backslash\{i\}
\end{gathered}
$$

and

$$
\left|S_{0} \cap S_{i}\right| \leq\left|S_{i}\right|-\delta
$$

Note that

$$
U_{i} \subseteq S_{i} \cap \Omega_{0} \subseteq S_{i} \cap \Omega
$$

Hence

$$
\left|S_{i} \cap \Omega\right| \geq\left|U_{i}\right|=\left|S_{i}\right|-\delta+1
$$

Thus, there is an $\eta \in\left(S_{i} \cap \Omega\right) \backslash S_{0}$. Since $S_{1}, \cdots, S_{t}$ are mutually disjoint, $\eta \notin S_{j}, \forall j \in[t] \backslash\{i\}$. Therefore,

$$
\left|\left(S_{0} \cup\{\eta\}\right) \cap S_{j}\right| \leq\left|S_{j}\right|-\delta+1, j=1, \cdots, t
$$

Then $S_{0} \cup\{\eta\} \subseteq \Omega$ is an $(\mathcal{S}, r, k)$-core. By assumption, $\left\{G_{\ell}\right\}_{\ell \in S_{0} \cup\{\eta\}}$ is linearly independent. Hence

$$
G_{\eta} \notin\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{0}}\right\rangle
$$

and

$$
\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{i} \cap \Omega}\right\rangle \nsubseteq\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{0}}\right\rangle
$$

Since $q \geq\binom{ n}{k-1} \geq|\Lambda|$, by Lemma 13 ,

$$
\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{i} \cap \Omega}\right\rangle \nsubseteq\left(\cup_{S_{0} \in \Lambda}\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{0}}\right\rangle\right)
$$

Let $G_{\lambda}$ be a vector in $\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{i} \cap \Omega}\right\rangle \backslash\left(\cup_{S_{0} \in \Lambda}\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{0}}\right\rangle\right)$. Then for any $S_{0} \in \Lambda,\left\{G_{\ell}\right\}_{\ell \in S_{0} \cup\{\lambda\}}$ are linearly independent.

Suppose $S \subseteq \Omega \cup\{\lambda\}$ is an $(\mathcal{S}, r, k)$-core. If $\lambda \notin S$, then $S \subseteq \Omega$ and by assumption, $\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent. If $\lambda \in S$, then $S_{0}=S \backslash\{\lambda\} \in \Lambda$ and by the selection of $G_{\lambda},\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent. Hence
we always have that $\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent. Thus, the vector $G_{\lambda}$ satisfies the requirement of Algorithm 1.

From the proof of Theorem 14, we can see that $\mathcal{S}=$ $\left\{S_{1}, \cdots, S_{t}\right\}$ is in fact an $(r, \delta)$-cover set of the code $\mathcal{C}$, where $\mathcal{C}$ is the output of Algorithm 1. The following example demonstrates how does Algorithm 1 work.

Example: We now construct an optimal $(r, \delta)_{a}$ linear code with $r=\delta=2, k=3$ and $n=6$. Let $S_{1}=\{1,2,3\}, S_{2}=$ $\{4,5,6\}$ and $\mathcal{S}=\left\{S_{1}, S_{2}\right\}$. Let $U_{1}=\{1,2\}, U_{2}=\{4,5\}$ and $\Omega_{0}=U_{1} \cup U_{2}=\{1,2,4,5\}$. Our construct involves the following two steps.

Step 1: Construct a $[4,3]$ MDS code, where $4=\left|\Omega_{0}\right|$. Let $G^{\prime}=\left(G_{1}, G_{2}, G_{4}, G_{5}\right)$ be a generating matrix of such code.

Step 2: Extend $G^{\prime}=\left(G_{1}, G_{2}, G_{4}, G_{5}\right)$ to a matrix $G=$ $\left(G_{1}, G_{2}, G_{3}, G_{4}, G_{5}, G_{6}\right)$ such that $G$ is a generating matrix of an optimal $(2,2)_{a}$ linear code.

It remains to determine $G_{3}$ and $G_{6}$ via two iterations.

1) $i=1: \Omega=\{1,2,4,5\}$ and $S_{1} \backslash \Omega=\{3\}$. We can verify that $\{1,4,3\},\{1,5,3\},\{2,4,3\},\{2,5,3\}$ and $\{4,5,3\}$ are all subsets of $\{1,2,3,4,5\}$ which is an $(\mathcal{S}, r, k)$ core and contains the index 3 . Let $\Lambda=\{\{1,4\},\{1,5\}$, $\{2,4\},\{2,5\},\{4,5\}\}$. Since $G^{\prime}=\left(G_{1}, G_{2}, G_{4}, G_{5}\right)$ generates an MDS code, then $G_{1}, G_{2}$ and $G_{4}$ are linearly independent. So $\left\langle G_{1}, G_{2}\right\rangle \nsubseteq\left\langle G_{1}, G_{4}\right\rangle$. Similarly, $\left\langle G_{1}, G_{2}\right\rangle \nsubseteq\left\langle G_{i}, G_{j}\right\rangle, \forall\{i, j\} \in \Lambda$. By Lemma 13, if $q \geq|\Lambda|=5$, then $\left\langle G_{1}, G_{2}\right\rangle \nsubseteq \cup_{\{i, j\} \in \Lambda}\left\langle G_{i}, G_{j}\right\rangle$. Note that $S_{1} \cap \Omega=\{1,2\}$. Therefore, let

$$
G_{3} \in\left\langle G_{1}, G_{2}\right\rangle \backslash\left(\cup_{\{i, j\} \in \Lambda}\left\langle G_{i}, G_{j}\right\rangle\right)
$$

Then for any $(\mathcal{S}, r, k)$-core $S \subseteq\{1,2,3,4,5\},\left\{G_{\ell} ; \ell \in\right.$ $S\}$ is linearly independent.
2) $i=2: \Omega=\{1,2,3,4,5\}$ and $S_{2} \backslash \Omega=\{6\}$. Similarly, we can verify that $\{1,2,6\},\{1,3,6\},\{2,3,6\}$, $\{1,4,6\},\{1,5,6\},\{2,4,6\}$ and $\{2,5,6\}$ are all subsets which is an $(\mathcal{S}, r, k)$-core and contains the index 6 . Let $\Lambda=\{\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{1,5\},\{2,4\}$, $\{2,5\}\}$. Clearly, $\left\langle G_{4}, G_{5}\right\rangle \nsubseteq\left\langle G_{i}, G_{j}\right\rangle, \forall\{i, j\} \in \Lambda$. By Lemma [13, if $q \geq|\Lambda|=7$, then $\left\langle G_{4}, G_{5}\right\rangle \nsubseteq$ $\cup_{\{i, j\} \in \Lambda}\left\langle G_{i}, G_{j}\right\rangle$. As $S_{2} \cap \Omega=\{4,5\}$, let

$$
G_{6} \in\left\langle G_{4}, G_{5}\right\rangle \backslash\left(\cup_{\{i, j\} \in \Lambda}\left\langle G_{i}, G_{j}\right\rangle\right)
$$

Then for any $(\mathcal{S}, r, k)$-core $S$, $\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent. Thus, we can obtain a matrix $G=$ $\left(G_{1}, G_{2}, G_{3}, G_{4}, G_{5}, G_{6}\right)$ such that for any $(\mathcal{S}, r, k)$ core $S$, $\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent. Let $\mathcal{C}$ be the linear code generated by $G$. Then $\mathcal{C}$ is an optimal $(2,2)_{a}$ linear code.
We can in fact employ a smaller field than $\mathbb{F}_{7}$. The following is a generating matrix of an optimal $(2,2)_{a}$ linear code:

$$
G=\left(\begin{array}{llllll}
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & \alpha & \alpha \\
0 & 0 & 0 & 1 & 1 & \alpha
\end{array}\right)
$$

over the field $\mathbb{F}_{4}=\{0,1, \alpha, 1+\alpha\}$, where $\alpha^{2}=1+\alpha$.
In the rest of this section, we shall use Theorem 14 to prove that optimal $(r, \delta)_{a}$ linear codes exist over a field of size $q \geq$
$\binom{n}{k-1}$ when $(r+\delta-1) \mid n$ or $m \geq v+\delta-1$, where $n=$ $w(r+\delta-1)+m$ and $k=u r+v$ satisfying $0<v<r$ and $0<m<r+\delta-1$. By Claim 2) of Lemma 5 $\frac{n}{r+\delta-1} \geq \frac{k}{r}$ is a necessary condition for the existence of optimal $(r, \delta)_{a}$ linear codes. For this reason, we assume $\frac{n}{r+\delta-1} \geq \frac{k}{r}$ holds in both cases.

Theorem 15: Suppose $(r+\delta-1) \mid n$. If $q \geq\binom{ n}{k-1}$, then there exists an optimal $(r, \delta)_{a}$ linear code over $\mathbb{F}_{q}$.

Proof: Let $n=t(r+\delta-1)$. Note that we have assumed that $\frac{n}{r+\delta-1} \geq \frac{k}{r}$. Then

$$
t=\left\lceil\frac{n}{r+\delta-1}\right\rceil \geq\left\lceil\frac{k}{r}\right\rceil
$$

Let $\left\{S_{1}, \cdots, S_{t}\right\}$ be a partition of $\{1, \cdots, n\}$ such that $\left|S_{i}\right|=$ $r+\delta-1, i=1, \cdots, t$.

For any $J \subseteq[t]$ of size $|J|=\left\lceil\frac{k}{r}\right\rceil$,

$$
\left|\cup_{i \in J} S_{i}\right|=\left\lceil\frac{k}{r}\right\rceil(r+\delta-1) \geq k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)
$$

By Theorem 14, if $q \geq\binom{ n}{k-1}$, then there exists an optimal $(r, \delta)_{a}$ code over $\mathbb{F}_{q}$.

If $(r+\delta-1) \mid n$ and $\delta \leq d$, then following a similar line of proof in 〔10], we can show that $t=\left\lceil\frac{n}{r+\delta-1}\right\rceil \geq\left\lceil\frac{k}{r}\right\rceil$. Under these two conditions, it was proved in [10] that there exists an optimal $(r, \delta)_{a}$ code over the field $\mathbb{F}_{q}$ of size $q>k n^{k}$. Our method requires a field of size only $\binom{n}{k-1}$, which is at the largest a fraction $\frac{1}{k!}$ of $k n^{k}$.

Theorem 16: Suppose $n=w(r+\delta-1)+m$ and $k=u r+v$, where $0<m<r+\delta-1$ and $0<v<r$. Suppose $m \geq v+\delta-1$ and $d \geq \delta$. If $q \geq\binom{ n}{k-1}$, then there exists an optimal $(r, \delta)_{a}$ linear code over $\mathbb{F}_{q}$.

Proof: Let $t=w+1$. Since we have assumed that $\frac{n}{r+\delta-1} \geq \frac{k}{r}$, we get

$$
t=w+1=\left\lceil\frac{n}{r+\delta-1}\right\rceil \geq\left\lceil\frac{k}{r}\right\rceil=u+1
$$

Note that $n-m=w(r+\delta-1)$. Let $\left\{S_{1}, \cdots, S_{w}\right\}$ be a partition of $\{1, \cdots, n-m\}$ and $S_{t}=[n-m+1, n]$.

For any $J \subseteq[t]$ of size $|J|=\left\lceil\frac{k}{r}\right\rceil$, we have the following two cases:

Case 1: $t \notin J$. Then

$$
\left|\cup_{i \in J} S_{i}\right|=\left\lceil\frac{k}{r}\right\rceil(r+\delta-1) \geq k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)
$$

Case 2: $t \in J$. Since $m \geq v+\delta-1$, then

$$
\begin{aligned}
\left|\cup_{i \in J} S_{i}\right| & =\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(r+\delta-1)+m \\
& \geq\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(r+\delta-1)+v+\delta-1 \\
& =k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)
\end{aligned}
$$

Hence, for any $\left\lceil\frac{k}{r}\right\rceil$-subset $J$ of $[t],\left|\cup_{i \in J} S_{i}\right| \geq k+\left\lceil\frac{k}{r}\right\rceil(\delta-$ 1). By Theorem 14 if $q \geq\binom{ n}{k-1}$, there exists an optimal $(r, \delta)_{a}$ code over $\mathbb{F}_{q}$.

When $\delta=2$, the conditions of Theorem 15 and Theorem 16 become $(r+1) \mid n$ and $n \bmod (r+1)-1 \geq k \bmod r>0$ respectively. For this special case, Tamo et al. [15] introduced a different construction method which is very easy to implement. However, the method in [15] requires the field size $q=O\left(n^{k}\right)$, which is larger than the field size $q=\binom{n}{k-1}$ of our method.

## VI. Construction of Optimal $(r, \delta)_{a}$ Codes: <br> Algorithm 2

In this section, we present yet another method for constructing optimal $(r, \delta)_{a}$ codes. This constructive method also points out two other sets of coding parameters where optimal $(r, \delta)_{a}$ codes exist. As the method in Section V, this method construct an optimal $(r, \delta)_{a}$ code which has a given set $\mathcal{S}$ as its $(r, \delta)$ cover set. The difference is that the set $\mathcal{S}$ used by this method has a more complicated structure. We again borrow the notion of core from [8].

Definition 17: Let $\mathcal{S}=\left\{S_{1}, \cdots, S_{t}\right\}$ be a collection of $(r+$ $\delta-1)$-subsets of $[n], \mathcal{A}=\left\{A_{1}, \cdots, A_{\alpha}, B\right\}$ be a partition of $[t]$ and $\Psi=\left\{\xi_{1}, \cdots, \xi_{\alpha}\right\} \subseteq[n]$. We say that $\mathcal{S}$ is an $(\mathcal{A}, \Psi)$ frame over the set $[n]$, if the following two conditions are satisfied:
(1) For each $j \in[\alpha],\left\{\xi_{j}\right\}=\cap_{\ell \in A_{j}} S_{\ell}$ and $\left\{S_{i} \backslash\left\{\xi_{j}\right\} ; i \in A_{j}\right\}$ are mutually disjoint;
(2) $\left\{\cup_{\ell \in A_{j}} S_{\ell} ; j \in[\alpha]\right\} \cup\left\{S_{j} ; j \in B\right\}$ is a partition of $[n]$.

Example 18: Let $\mathcal{S}=\left\{S_{1}, \cdots, S_{8}\right\}$ be what's shown in Fig 3) Clearly $\mathcal{S}$ is an $(\mathcal{A}, \Psi)$-frame over $[n]$, where the subsets $S_{1}, S_{2}, S_{3}$ have a common element $\xi_{1}=1$, and the subsets $S_{4}, S_{5}$ have a common element $\xi_{2}=14$.

$$
\begin{aligned}
& A_{1}\left\{\begin{array}{l|lllll}
S_{1} & 1, & 2, & 3, & 4, & 5 \\
S_{2} & 1, & 6, & 7, & 8, & 9 \\
S_{3} & 1, & 10, & 11, & 12, & 13 \\
\cline { 3 - 5 } & 14, & 15, & 16, & 17, & 18
\end{array}\right. \\
& A_{2}\left\{\begin{array}{ll|}
S_{4} & 14,15,16,17,18 \\
S_{5} & 14,19,20,21,22 \\
S_{5} & 12,24,10,12,13
\end{array}\right. \\
& B\left\{\begin{array}{l|l|}
S_{6} & 23,24,25,26,27 \\
S_{7} & 28,29,30,31,32 \\
S_{8} & 33,34,35,36,37
\end{array}\right.
\end{aligned}
$$

Fig 3. An $(\mathcal{A}, \Psi)$-frame, where $n=37, r=\delta=3, t=8, A_{1}=\{1,2,3\}$, $A_{2}=\{4,5\}, B=\{6,7,8\}, \mathcal{A}=\left\{A_{1}, A_{2}, B\right\}$ and $\Psi=\{1,14\}$.

Definition 19: A subset $S \subseteq[n]$ is said to be an $(\mathcal{S}, r)$-core if the following three conditions hold:
(1) If $j \in[\alpha]$ and $\xi_{j} \in S$, then $\left|S \cap S_{i}\right| \leq r, \forall i \in A_{j}$;
(2) If $j \in[\alpha]$ and $\xi_{j} \notin S$, then there is an $i_{j} \in A_{j}$ such that $\left|S \cap S_{i_{j}}\right| \leq r$ and $\left|S \cap S_{i}\right| \leq r-1, \forall i \in A_{j} \backslash\left\{i_{j}\right\} ;$
(3) If $i \in B$, then $\left|S \cap S_{i}\right| \leq r$.

Additionally, if $S \subseteq[n]$ is an $(\mathcal{S}, r)$-core and $|S|=k$, then $S$ is called an $(\mathcal{S}, r, k)$-core.

Clearly, if $S \subseteq[n]$ is an $(\mathcal{S}, r)$-core and $S^{\prime} \subseteq S$, then $S^{\prime}$ is also an $(\mathcal{S}, r)$-core. In particular, if $S \subseteq[n]$ is an $(\mathcal{S}, r)$-core and $S^{\prime}$ is a $k$-subset of $S$, then $S^{\prime}$ is an $(\mathcal{S}, r, k)$-core.

Example 18 continued: In Example 18, let $k=7$. Then $\{1,2,3,6,7,10,11\}$ and $\{2,3,4,6,7,28,33\}$ are both $(\mathcal{S}, r, k)$-core. However, $S=\{2,3,4,6,7,8,28\}$ and $S^{\prime}=$ $\{2,6,15,23,24,25,26\}$ are not $(\mathcal{S}, r)$-core, because $S$ does not satisfy Condition (2) and $S^{\prime}$ does not satisfy Condition (3) of Definition 19

Lemma 20: Let $\mathcal{S}$ be an $(\mathcal{A}, \Psi)$-frame as in Definition 17 Suppose $t \geq\left\lceil\frac{k}{r}\right\rceil$ and for any $\left\lceil\frac{k}{r}\right\rceil$-subset $J$ of $[t],\left|\cup_{i \in J} S_{i}\right| \geq$ $k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)$. Then the following hold:

1) If $T \subseteq[n]$ has size $|T| \geq k+\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)$, then there is an $S \subseteq T$ such that $S$ is an $(\mathcal{S}, r, k)$-core.
2) For any $i \in[t]$ and $I \subseteq S_{i}$ of size $|I|=r$, there is an $(\mathcal{S}, r, k)$-core $S$ such that $I \subseteq S$.

## Proof: 1) Let

$$
J=\left\{\ell \in[t] ;\left|T \cap S_{\ell}\right| \geq r\right\}
$$

For each $j \in[\alpha]$ and $\ell \in A_{j}$, we pick a subset $W_{\ell} \subseteq T$ as follows:
i) If $J \cap A_{j}=\emptyset$, then let $W_{\ell}=T \cap S_{\ell}$ for each $\ell \in A_{j}$.
ii) If $J \cap A_{j} \neq \emptyset$ and $\xi_{j} \in T$, then for each $\ell \in J \cap A_{j}$, let $W_{\ell}$ be an $r$-subset of $T \cap S_{\ell}$ satisfying $\xi_{j} \in W_{\ell}$, and for each $\ell \in A_{j} \backslash J$, let $W_{\ell}=T \cap S_{\ell}$.
iii) If $J \cap A_{j} \neq \emptyset$ and $\xi_{j} \notin T$, then fix an $\ell_{j} \in J \cap A_{j}$, and let $W_{\ell_{j}}$ be an $r$-subset of $T \cap S_{\ell_{j}}$, let $W_{\ell}$ be an $(r-1)$-subset of $T \cap S_{\ell}$ for each $\ell \in J \cap A_{j} \backslash\left\{\ell_{j}\right\}$, and let $W_{\ell}=T \cap S_{\ell}$ for each $\ell \in A_{j} \backslash J$.

Moreover, for each $\ell \in J \cap B$, let $W_{\ell}$ be an $r$-subset of $T \cap S_{\ell}$, and for each $\ell \in B \backslash J$, let $W_{\ell}=T \cap S_{\ell}$.

Let $W=\cup_{\ell \in[t]} W_{\ell}$, then by Definition $19 W$ is an $(\mathcal{S}, r)$ core. We now prove that $|W| \geq k$. Let

$$
\Theta(J)=\left\{j \in[\alpha] ; J \cap A_{j} \neq \emptyset\right\}
$$

We need to consider the following two cases:
Case 1: $|J| \geq\left\lceil\frac{k}{r}\right\rceil$. Without loss of generality, assume $|J|=$ $\left\lceil\frac{k}{r}\right\rceil^{5}$. Then from the assumption of this lemma,

$$
\begin{equation*}
\left|\cup_{\ell \in J} S_{\ell}\right| \geq k+|J|(\delta-1) \tag{VI.1}
\end{equation*}
$$

By Definition 17

$$
\begin{align*}
\left|\cup_{\ell \in J} S_{\ell}\right|= & \sum_{j \in \Theta(J)}\left|J \cap A_{j}\right|(r+\delta-2) \\
& +|\Theta(J)|+|J \cap B|(r+\delta-1) \tag{VI.2}
\end{align*}
$$

Since $\mathcal{A}=\left\{A_{1}, \cdots, A_{\alpha}, B\right\}$ is a partition of $[t],\left\{J \cap A_{j} ; j \in\right.$ $\Theta(J)\} \cup\{J \cap B\}$ is a partition of $J$ and

$$
\begin{equation*}
|J|=\sum_{j \in \Theta(J)}\left|J \cap A_{j}\right|+|J \cap B| . \tag{VI.3}
\end{equation*}
$$

Combining VI.1)-VI.3), we have

$$
\begin{equation*}
\sum_{j \in \Theta(J)}\left|J \cap A_{j}\right|(r-1)+|\Theta(J)|+|J \cap B| r \geq k \tag{VI.4}
\end{equation*}
$$

[^3]By the construction of $W$, we have

$$
\begin{equation*}
\left|\cup_{\ell \in J} W_{\ell}\right|=\sum_{j \in \Theta(J)}\left|J \cap A_{j}\right|(r-1)+|\Theta(J)|+|J \cap B| r \tag{VI.5}
\end{equation*}
$$

Equations VI.4) and VI.5) imply that

$$
|W| \geq\left|\cup_{\ell \in J} W_{\ell}\right| \geq k
$$

Case 2: $|J|<\left\lceil\frac{k}{r}\right\rceil$. By the construction of $W$, for each $j \in[\alpha]$ and $\ell \in J \cap A_{\ell}, W_{\ell}$ is obtained by deleting at most ( $\delta-1$ ) elements from $T \cap S_{\ell}$. We thus have

$$
\left|\cup_{\ell \in A_{j}} W_{\ell}\right| \geq\left|T \cap\left(\cup_{\ell \in A_{j}} S_{\ell}\right)\right|-\left|J \cap A_{j}\right|(\delta-1)
$$

Moreover,

$$
\left|\cup_{\ell \in B} W_{\ell}\right| \geq\left|\cup_{\ell \in B}\left(T \cap S_{\ell}\right)\right|-|J \cap B|(\delta-1)
$$

Then

$$
|W|=\left|\cup_{\ell \in[t]} W_{\ell}\right| \geq|T|-|J|(\delta-1)
$$

Note that $|T| \geq k+\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)$ and $|J|<\left\lceil\frac{k}{r}\right\rceil$. Therefore

$$
|W| \geq|T|-\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)=k
$$

Gathering both cases, we always have $|W| \geq k$. Let $S$ be a $k$-subset of $W$. Note that $W$ is an $(\mathcal{S}, r)$-core. So $S \subseteq W \subseteq T$ is an $(\mathcal{S}, r, k)$-core.
2) To prove the second claim of Lemma 20, note that $t \geq$ $\left\lceil\frac{k}{r}\right\rceil$, and hence we can always find a $\left\lceil\frac{k}{r}\right\rceil$-subset $J$ of $[t]$ such that $i \in J$. Similar to the proof of 1 ), for each $\ell \in J$, we can pick a $W_{\ell}$ such that $W_{i}=I, \cup_{\ell \in J} W_{\ell}$ is an $(\mathcal{S}, r)$-core and $\left|\cup_{\ell \in J} W_{\ell}\right| \geq k$. Let $S$ be a $k$-subset of $\cup_{\ell \in J} W_{\ell}$ such that $I \subseteq S$. Then $S$ is an $(\mathcal{S}, r, k)$-core and $I \subseteq S$.

Example 18 further continued: Consider the $(\mathcal{A}, \Psi)$-frame $\mathcal{S}$ in Example 18, Let $k=7$. Then $\mathcal{S}$ satisfies the conditions of Lemma 20. We consider the following two instances:

Instance 1: $T=\{2,3,4,6,7,8,14,15,16,17,19,23,24,28\}$. As in the proof of Lemma 20, $J=\left\{\ell ;\left|T \cap S_{\ell}\right| \geq r\right\}=$ $\{1,2,4\}$ and $|J|=3=\left\lceil\frac{k}{r}\right\rceil$. Let $W_{1}=\{2,3,4\}$, $W_{2}=\{6,7\}, W_{4}=\{14,15,16\}, W_{5}=\{19\}$, $W_{6}=\{23,24\}, W_{7}=\{28\}$ and $W_{\ell}=\emptyset$ for $\ell \in\{3,8\}$. Then $|W|=\left|\cup_{\ell=1}^{8} W_{\ell}\right| \geq\left|\cup_{\ell \in J} W_{\ell}\right| \geq k=7$.

Instance 2: $T=\{2,3,4,6,7,8,10,11,14,15,19,23,24,28\}$. Then $J=\left\{\ell ;\left|T \cap S_{\ell}\right| \geq r\right\}=\{1,2\}$ and $|J|<\left\lceil\frac{k}{r}\right\rceil$. Let $W_{1}=\{2,3,4\}, W_{2}=\{6,7\}, W_{3}=\{10,11\}, W_{4}=$ $\{14,15\}, W_{5}=\{19\}, W_{6}=\{23,24\}, W_{7}=\{28\}$ and $W_{8}=\emptyset$. Then $|W|=\left|\cup_{\ell=1}^{8} W_{\ell}\right| \geq|T|-|J|(\delta-1) \geq k=7$.

Remark 21: Let $\mathcal{S}$ be an $(\mathcal{A}, \Psi)$-frame as in Definition 17 , For each $j \in[\alpha]$ and $i \in A_{j}$, let $U_{i}$ be an $r$-subset of $S_{i}$ such that $\xi_{j} \in U_{i}$. For each $i \in B$, let $U_{i}$ be an $r$-subset of $S_{i}$. Let

$$
\Omega_{0}=\cup_{i \in[t]} U_{i}
$$

Then by Definition 19, $\Omega_{0}$ is an $(\mathcal{S}, r)$-core. Clearly,

$$
\left|\Omega_{0}\right|=n-t(\delta-1)=\left|\cup_{j=1}^{\alpha} A_{j}\right|(r-1)+\alpha+|B| r
$$

Example 22: In Example 18, let $k=7$, then $\Omega_{0}=\{1$, $2,3,6,7,10,11,14,15,16,19,20,23,24,25,28,29,30,33,34$, $35\}$ is an $(\mathcal{S}, r)$-core obtained by the process of Remark 21 .

Lemma 23: Let $\mathcal{S}$ be an $(\mathcal{A}, \Psi)$-frame as defined in Definition 17 and $\Omega_{0}$ be what's described in Remark 21 Suppose $\Omega_{0} \subseteq \Omega \subseteq[n], S_{0} \subseteq \Omega$ and $i \in[t]$. If $\lambda \in S_{i} \backslash \Omega$ and $S_{0} \cup\{\lambda\}$ is an $(\mathcal{S}, r, k)$-core, then there exists an $\eta \in S_{i} \cap \Omega$ such that $S_{0} \cup\{\eta\}$ is an $(\mathcal{S}, r, k)$-core.

Proof: By the construction of $\Omega_{0},\left|S_{i} \cap \Omega_{0}\right|=r$. Since $\Omega_{0} \subseteq \Omega$, so

$$
\left|S_{i} \cap \Omega\right| \geq r
$$

Since $S_{0} \cup\{\lambda\}$ is an $(\mathcal{S}, r, k)$-core, by Definition 19

$$
\left|S_{0}\right|=k-1
$$

and

$$
\left|S_{0} \cap S_{i}\right| \leq r-1
$$

Thus, we can find an $\eta \in\left(S_{i} \cap \Omega\right) \backslash S_{0}$.
If $i \in B$, then by Definition $17 \geqslant \nexists S_{i^{\prime}}, \forall i^{\prime} \in[t] \backslash\{i\}$. Then $S_{0} \cup\{\eta\}$ is an $(\mathcal{S}, r, k)$-core.

Now, suppose $i \in A_{j}$ for some $j \in[\alpha]$. We need to consider the following two cases.

Case 1: $\xi_{j} \in S_{0}$. Since $\eta \in\left(S_{i} \cap \Omega\right) \backslash S_{0}$, then $\eta \neq \xi_{j}$ and $\eta \notin S_{i^{\prime}}, \forall i^{\prime} \in[t] \backslash\{i\}$. Then $S_{0} \cup\{\eta\}$ is an $(\mathcal{S}, r, k)$-core.

Case 2: $\xi_{j} \notin S_{0}$. Since $S_{0} \cup\{\lambda\}$ is an $(\mathcal{S}, r, k)$-core, from Definition 19, we differentiate the following two sub-cases:

Subcase 2.1: $\left|S_{0} \cap S_{i^{\prime}}\right| \leq r-1, \forall i^{\prime} \in A_{j}$. In that case, it is clear that $S_{0} \cup\{\eta\}$ is an $(\mathcal{S}, r, k)$-core.

Subcase 2.2: There is an $i_{j} \in A_{j} \backslash\{i\}$ such that $\left|S_{0} \cap S_{i_{j}}\right|=$ $r,\left|S_{0} \cap S_{i}\right| \leq r-2$ and $\left|S_{0} \cap S_{i^{\prime}}\right| \leq r-1, \forall i^{\prime} \in A_{j} \backslash\left\{i_{j}, i\right\}$. In that case, we have

$$
\left|\left(S_{i} \cap \Omega\right) \backslash S_{0}\right| \geq 2
$$

Let $\eta \in\left(S_{i} \cap \Omega\right) \backslash\left(S_{0} \cup\left\{\xi_{j}\right\}\right)$, then $\eta \neq \xi_{j}$ and $\eta \notin S_{i^{\prime}}, \forall i^{\prime} \in$ $[t] \backslash\{i\}$. It then follows that $S_{0} \cup\{\eta\}$ is an $(\mathcal{S}, r, k)$-core.

Example 18 and 22 continued: Consider again Example 18 . Let $k=7, \Omega=\Omega_{0} \cup\{4,5,8\}$ and $\lambda=9 \in S_{2}$, where $\Omega_{0}$ is as in Example 22 We can easily verify the following:
Let $S_{0}=\{1,2,3,6,10,14\}$; Then $S_{0} \cup\{9\}$ is an $(\mathcal{S}, r, k)$ core. If we further let $\eta=7 \in S_{2}$, then $S_{0} \cup\{\eta\}$ is also an $(\mathcal{S}, r, k)$-core.
Let $S_{0}^{\prime}=\{2,3,6,7,14,15\}$; Then $S_{0}^{\prime} \cup\{9\}$ is an $(\mathcal{S}, r, k)$ core. If we further let $\eta^{\prime}=8 \in S_{2}$, then $S_{0}^{\prime} \cup\{\eta\}$ is also an $(\mathcal{S}, r, k)$-core.

Let $S_{0}^{\prime \prime}=\{2,3,4,10,11,15,23\}$; Then $S_{0}^{\prime \prime} \cup\{9\}$ is an $(\mathcal{S}, r, k)$-core. If we further let $\eta^{\prime \prime}=6 \in S_{2}$, then $S_{0}^{\prime \prime} \cup\left\{\eta^{\prime \prime}\right\}$ is also an $(\mathcal{S}, r, k)$-core.

Lemma 24: Let $\mathcal{S}$ be an $(\mathcal{A}, \Psi)$-frame defined in Definition 17 and $\Omega_{0}$ be what's defined in Remark 21 Let $\Omega_{0} \subseteq \Omega \subseteq[n]$ and $\mathcal{G}=\left\{G_{\ell} \in \mathbb{F}_{q}^{k} ; \ell \in \Omega\right\}$ such that for any $(\mathcal{S}, r, k)$-core $S \subseteq \Omega$, the vectors in $\left\{G_{\ell} ; \ell \in S\right\}$ are linearly independent. Suppose $i \in[t]$ and $S_{i} \backslash \Omega \neq \emptyset$. If $q \geq\binom{ n}{k-1}$, then for any $\lambda \in S_{i} \backslash \Omega$, there is a $G_{\lambda} \in\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{i} \cap \Omega}\right\rangle$ such that for any $(\mathcal{S}, r, k)$-core $S \subseteq \Omega \cup\{\lambda\}$, the vectors in $\left\{G_{\ell} ; \ell \in S\right\}$ are linearly independent.

Proof: Let $\Lambda$ be the set of all $S_{0} \subseteq \Omega$ such that $S_{0} \cup\{\lambda\}$ is an $(\mathcal{S}, r, k)$-core. For any $S_{0} \in \Lambda$, by Lemma 23, there is an $\eta \in S_{i} \cap \Omega$ such that $S_{0} \cup\{\eta\}$ is an $(\mathcal{S}, r, k)$-core. From the assumptions, $\left\{G_{\ell}\right\}_{\ell \in S_{0} \cup\{\eta\}}$ is linearly independent. Hence

$$
G_{\eta} \notin\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{0}}\right\rangle .
$$

Thus,

$$
\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{i} \cap \Omega}\right\rangle \nsubseteq\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{0}}\right\rangle
$$

Since $q \geq\binom{ n}{k-1} \geq|\Lambda|$, by Lemma 13 ,

$$
\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{i} \cap \Omega}\right\rangle \nsubseteq\left(\cup_{S_{0} \in \Lambda}\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{0}}\right\rangle\right) .
$$

Let $G_{\lambda} \in\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{i} \cap \Omega}\right\rangle \backslash\left(\cup_{S_{0} \in \Lambda}\left\langle\left\{G_{\ell}\right\}_{\ell \in S_{0}}\right\rangle\right)$. Then for any $(\mathcal{S}, r, k)$-core $S \subseteq \Omega \cup\{\lambda\}$, the vectors in $\left\{G_{\ell} ; \ell \in S\right\}$ are linearly independent.

The second construction method for optimal $(r, \delta)_{a}$ codes is illustrated in the proof of the following theorem.

Theorem 25: Let $\mathcal{S}$ be an $(\mathcal{A}, \Psi)$-frame in Definition 17 Suppose $t \geq\left\lceil\frac{k}{r}\right\rceil$ and for any $\left\lceil\frac{k}{r}\right\rceil$-subset $J$ of $[t],\left|\cup_{i \in J} S_{i}\right| \geq$ $k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)$. If $q \geq\binom{ n}{k-1}$, then there exists an optimal $(r, \delta)_{a}$ linear code over $\mathbb{F}_{q}$.

Proof: Let $\Omega_{0}$ be what's described in Remark 21 and $L=\left|\Omega_{0}\right|$. Clearly,

$$
L=n-t(\delta-1) .
$$

Since $t \geq\left\lceil\frac{k}{r}\right\rceil$, let $J$ be a $\left\lceil\frac{k}{r}\right\rceil$-subset of $[t\rceil$; then from the assumptions,

$$
\left|\cup_{i \in J} S_{i}\right| \geq k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)=k+|J|(\delta-1)
$$

By Remark 21, $\cup_{i \in J} U_{i} \subseteq \Omega_{0}$. Hence

$$
L=\left|\Omega_{0}\right| \geq\left|\cup_{i \in J} U_{i}\right|=\cup_{i \in J} S_{i}|-|J|(\delta-1) \geq k
$$

The construction of an optimal $(r, \delta)_{a}$ code consists of the following two steps.

Step 1: Construct an $[L, k]$ MDS code $\mathcal{C}_{0}$ over $\mathbb{F}_{q}$. Such an MDS code exists when $q \geq\binom{ n}{k-1} \geq n>L$. Let $G^{\prime}$ be a generating matrix of $\mathcal{C}_{0}$. We index the columns of $G^{\prime}$ by $\Omega_{0}$, i.e., $G^{\prime}=\left(G_{\ell}\right)_{\ell \in \Omega_{0}}$, where $G_{\ell}$ is a column of $G^{\prime}, \forall \ell \in \Omega_{0}$.

Step 2: Extend the code $\mathcal{C}_{0}$ to an optimal $(r, \delta)_{a}$ code $\mathcal{C}$. This can be achieved by the following algorithm, which appears similar to Algorithm 1 (on the surface) but is actually different (in details).

## Algorithm 2:

1. Let $\Omega=\Omega_{0}$.
$i$ runs from 1 to $t$.
While $S_{i} \backslash \Omega \neq \emptyset$ :
2. Pick a $\lambda \in S_{i} \backslash \Omega$ and let $G_{\lambda} \in\left\langle\left\{G_{\ell} ; \ell \in S_{i} \cap \Omega\right\}\right\rangle$ be such that for any $(\mathcal{S}, r, k)$-core $S \subseteq \Omega \cup\{\lambda\}$, $\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent.
$\Omega=\Omega \cup\{\lambda\}$.
3. Let $\mathcal{C}$ be the linear code generated by the matrix $G=$ $\left(G_{1}, \cdots, G_{n}\right)$.

Since $G^{\prime}=\left(G_{\ell}\right)_{\ell \in \Omega_{0}}$ is a generating matrix of the MDS code $\mathcal{C}_{0}$, so for any $(\mathcal{S}, r, k)$-core $S \subseteq \Omega_{0}$, $\left\{G_{\ell} ; \ell \in S\right\}$ is
linearly independent. Then in Algorithm 2, by induction, we can assume that for any $(\mathcal{S}, r, k)$-core $S \subseteq \Omega$, $\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent. By Lemma 24 in line 4 of Algorithm 2, we can always find a $G_{\lambda}$ satisfying the requirement. Hence, by induction, the collection $\left\{G_{\ell} ; \ell \in[n]\right\}$ satisfies the condition that for any $(\mathcal{S}, r, k)$-core $S \subseteq[n],\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent. Moreover, since in line 4 of Algorithm 2, we can choose a $G_{\lambda} \in\left\langle\left\{G_{\ell} ; \ell \in S_{i} \cap \Omega\right\}\right\rangle$, which satisfies

$$
\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in\left(S_{i} \cap \Omega\right) \cup\{\lambda\}}\right)=\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in S_{i} \cap \Omega}\right)
$$

By induction,

$$
\begin{aligned}
\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in S_{i}}\right) & =\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in S_{i} \cup \Omega_{0}}\right) \\
& =\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in U_{i}}\right) \\
& =r .
\end{aligned}
$$

For any $i \in[t]$ and $I \subseteq S_{i}$ of size $|I|=r$, by Claim 2) of Lemma 20, there is an $(\mathcal{S}, r, k)$-core $S$ such that $I \subseteq S$. Hence $\left\{G_{\ell} ; \ell \in S\right\}$ is linearly independent. Thus,

$$
\operatorname{Rank}\left(\left\{G_{\ell}\right\}_{\ell \in I}\right)=r
$$

Therefore, by Definition 2 and Remark 3 $\mathcal{C}$ is an $(r, \delta)_{a}$ code.
Finally, we prove that the minimum distance of $\mathcal{C}$ is $d=$ $n-k+1-\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)$.

Suppose $T \subseteq[n]$ and $|T|=k+\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)$. By 1) of Lemma 20, there is an $S \subseteq T$ which is an $(\mathcal{S}, r, k)$-core. Therefore,

$$
\operatorname{Rank}\left(\left\{G_{\ell} ; \ell \in T\right\}\right)=\operatorname{Rank}\left(\left\{G_{\ell} ; \ell \in S\right\}\right)=k
$$

By the minimum distance bound in (I.1) and Lemma 1 the minimum distance of $\mathcal{C}$ is

$$
d=n-k+1-\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)
$$

Hence $\mathcal{C}$ is an optimal $(r, \delta)_{a}$ code.
Example 18 continued: Consider the $(\mathcal{A}, \Psi)$-frame $\mathcal{S}$ in Example 18 Let $k=7$. Then it is obvious $\mathcal{S}$ satisfies the conditions of Theorem 25. Thus, we can use Algorithm 2 to construct an optimal $(r, \delta)_{a}$ linear code over the field of size $q \geq\binom{ n}{k-1}=\binom{37}{6}$. Note that $r=\delta=3$. Hence, $(r+\delta-1) \nmid n$ and this is a new optimal $(r, \delta)_{a}$ code.

As applications of Theorem 25, in the following, we show that optimal $(r, \delta)_{a}$ codes exist for two other sets of coding parameters. From Claim 2) of Lemma 5, we know that $\frac{n}{r+\delta-1} \geq \frac{k}{r}$ is a necessary condition for the existence of optimal $(r, \delta)_{a}$ linear codes. Thus we will assume $\frac{n}{r+\delta-1} \geq \frac{k}{r}$ in the following discussion.

Theorem 26: Suppose $n=w(r+\delta-1)+m$ and $k=$ $u r+v$, where $0<m<r+\delta-1$ and $0<v<r$. Suppose $w \geq r+\delta-1-m$ and $r-v \geq u$. If $q \geq\binom{ n}{k-1}$, then there exists an optimal $(r, \delta)_{a}$ linear code over $\mathbb{F}_{q}$.

Proof: Let $t=w+1$. Note that we have assumed that $\frac{n}{r+\delta-1} \geq \frac{k}{r}$. Then

$$
t=w+1=\left\lceil\frac{n}{r+\delta-1}\right\rceil \geq\left\lceil\frac{k}{r}\right\rceil=u+1
$$

Let

$$
\ell=r+\delta-1-m
$$

and

$$
\begin{equation*}
L=(\ell+1)(r+\delta-2)+1 \tag{VI.6}
\end{equation*}
$$

Then from the assumptions, $w \geq(r+\delta-1)-m=\ell$. Therefore

$$
t=w+1 \geq \ell+1
$$

and

$$
\begin{align*}
n-L & =(w-\ell)(r+\delta-1) \\
& =(t-\ell-1)(r+\delta-1) \tag{VI.7}
\end{align*}
$$

From equation VI.6, $L-1=(\ell+1)(r+\delta-2)$. The set $[2, L]$ can be partitioned into $\ell+1$ mutually disjoint subsets, say, $T_{1}, \cdots, T_{\ell+1}$, each of size $r+\delta-2$. Let

$$
S_{i}=\{1\} \cup T_{i}, i=1, \cdots, \ell+1
$$

Moreover, from equation VI.7), the set $[L+1, n]$ can be partitioned into $t-(\ell+1)$ mutually disjoint subsets, say, $S_{\ell+2}, \cdots, S_{t}$, each of size $r+\delta-1$.

Let $\alpha=1$ and $A_{1}=\{1, \cdots, \ell+1\}, B=\{\ell+1, \cdots, t\}$, $\mathcal{A}=\left\{A_{1}, B\right\}$, and $\Psi=\{1\}$. Then $\mathcal{S}=\left\{S_{1}, \cdots, S_{t}\right\}$ is an $(\mathcal{A}, \Psi)$-frame. For any $\left\lceil\frac{k}{r}\right\rceil$-subset $J$ of $[t]$, since $r-v \geq u$, then

$$
|J|=\left\lceil\frac{k}{r}\right\rceil=u+1 \leq r-v+1
$$

Let $J_{1}=J \cap\{1, \cdots, \ell+1\}$, and $J_{2}=J \backslash\{1, \cdots, \ell+1\}$. By the construction of $\mathcal{S}$, we have

$$
\begin{aligned}
\left|\cup_{i \in J} S_{i}\right| & =\left|J_{1}\right|(r+\delta-2)+1+\left|J_{2}\right|(r+\delta-1) \\
& =|J|(r+\delta-1)-\left|J_{1}\right|+1 \\
& \geq|J|(r+\delta-1)-|J|+1 \\
& \geq|J|(r+\delta-1)-(r-v+1)+1 \\
& =(|J|-1) r+v+|J|(\delta-1) \\
& =u r+v+\left\lceil\frac{k}{r}\right\rceil(\delta-1) \\
& =k+\left\lceil\frac{k}{r}\right\rceil(\delta-1) .
\end{aligned}
$$

By Theorem 25, if $q \geq\binom{ n}{k-1}$, then there exists an optimal $(r, \delta)_{a}$ code over $\mathbb{F}_{q}$.

Theorem 27: Suppose $n=w(r+\delta-1)+m$ and $k=$ $u r+v$, where $0<m<r+\delta-1$ and $0<v<r$. Suppose $w+1 \geq 2(r+\delta-1-m)$ and $2(r-v) \geq u$. If $q \geq\binom{ n}{k-1}$, then there exists an optimal $(r, \delta)_{a}$ linear code over $\mathbb{F}_{q}$.

Proof: Let $t=w+1$. Note that we have assumed that $\frac{n}{r+\delta-1} \geq \frac{k}{r}$. Then

$$
t=w+1=\left\lceil\frac{n}{r+\delta-1}\right\rceil \geq\left\lceil\frac{k}{r}\right\rceil=u+1
$$

Let

$$
\ell=(r+\delta-1)-m
$$

and

$$
\begin{equation*}
L=\ell(2(r+\delta-1)-1) \tag{VI.8}
\end{equation*}
$$

Then by assumption, $t=w+1 \geq 2(r+\delta-1-m)=2 \ell$. It then follows that

$$
\begin{equation*}
n-L=(t-2 \ell)(r+\delta-1) \geq 0 \tag{VI.9}
\end{equation*}
$$

From equation VI.8, the set $[L]$ can be partitioned into $\ell$ mutually disjoint subsets, say, $T_{1}, \cdots, T_{\ell}$, each of size $2(r+$ $\delta-1)-1$. For each $i \in\{1, \cdots, \ell\}$, we can find two subsets $S_{2 i-1}, S_{2 i}$ of $T_{i}$ such that

$$
\left|S_{2 i-1}\right|=\left|S_{2 i}\right|=r+\delta-1
$$

and

$$
S_{2 i-1} \cup S_{2 i}=T_{i}
$$

Then

$$
\left|S_{2 i-1} \cap S_{2 i}\right|=1
$$

Let $S_{2 i-1} \cap S_{2 i}=\left\{\xi_{i}\right\}$ and $\Psi=\left\{\xi_{1}, \cdots, \xi_{\ell}\right\}$.
Moreover, from Equation VI.9, the set $[L+1, n]$ can be partitioned into $t-2 \ell$ mutually disjoint subsets, say $S_{2 \ell+1}, \cdots, S_{t}$, each of size $r+\delta-1$.

Let $A_{i}=\{2 i-1,2 i\}, i=1, \cdots, \ell, B=[2 \ell+1, t]$ and $\mathcal{A}=\left\{A_{1}, \cdots, A_{\ell}, B\right\}$. Then $\mathcal{S}=\left\{S_{1}, \cdots, S_{t}\right\}$ is an $(\mathcal{A}, \Psi)$ frame. For any $\left\lceil\frac{k}{r}\right\rceil$-subset $J$ of $[t]$. Since $2(r-v) \geq u$, then

$$
\begin{equation*}
|J|=\left\lceil\frac{k}{r}\right\rceil=u+1 \leq 2(r-v)+1 \tag{VI.10}
\end{equation*}
$$

Let $\Gamma(J)=\left\{j \in[\ell] ; A_{j} \subseteq J\right\}$. Then

$$
\begin{equation*}
|J| \geq\left|\cup_{j \in \Gamma(J)} A_{j}\right|=2|\Gamma(J)| \tag{VI.11}
\end{equation*}
$$

Combining VI.10) an VI.11, we have

$$
|\Gamma(J)| \leq \frac{|J|}{2} \leq \frac{2(r-v)+1}{2}=r-v+\frac{1}{2}
$$

Since $|\Gamma(J)|$ is an integer, then

$$
|\Gamma(J)| \leq r-v
$$

By the construction of $\mathcal{S}$, we have

$$
\begin{aligned}
\left|\cup_{i \in J} S_{i}\right| & =|J|(r+\delta-1)-|\Gamma(J)| \\
& \geq|J|(r+\delta-1)-(r-v) \\
& =(|J|-1) r+v+|J|(\delta-1) \\
& =k+\left\lceil\frac{k}{r}\right\rceil(\delta-1)
\end{aligned}
$$

By Theorem 25, if $q \geq\binom{ n}{k-1}$, then there exists an optimal $(r, \delta)_{a}$ code over $\mathbb{F}_{q}$.

We now provide some discussions of Theorem 27 Since $0<m<r+\delta-1$, then $2(r+\delta-1-m)<2(r+\delta-1)$. Given $k, r$ and $\delta$, let $\alpha=\max \left\{2(r+\delta-1),\left\lceil\frac{k}{r}\right\rceil\right\}$. Then the conditions $w+1 \geq 2(r+\delta-1-m)$ and $w \geq u$ can always be satisfied when $n \geq \alpha(r+\delta-1)$. On the other hand, when $\frac{k}{3}<r<k$ and $r \neq \frac{\bar{k}}{2}$, then $u=1$ or 2 and $r-v \geq 1$, which leads to $2(r-v) \geq u$. By Theorem 27 there exist optimal $(r, \delta)_{a}$ codes when $n \geq \alpha(r+\delta-1), \frac{k}{3}<r<k$ and $r \neq \frac{k}{2}$.

| $r \backslash k$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 2 | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ |
| 3 | $\mathrm{~N}_{11}$ | $\mathrm{~N}_{10}$ | $\mathrm{E}_{27}$ | $\mathrm{E}_{27}$ | $\mathrm{~N}_{10}$ | $\mathrm{~N}_{11}$ | $\mathrm{~N}_{11}$ | $\mathrm{~N}_{10}$ | $\mathrm{~N}_{11}$ | $\mathrm{~N}_{11}$ |
| 4 | $\mathrm{E}_{27}$ | $\mathrm{~N}_{10}$ | $\mathrm{E}_{27}$ | $\mathrm{E}_{27}$ | $\mathrm{~N}_{11}$ | $\mathrm{~N}_{10}$ | $\mathrm{E}_{27}$ | $\mathrm{E}_{27}$ | $\mathrm{~N}_{11}$ | $\mathrm{~N}_{10}$ |
| 5 | $\mathrm{E}_{16}$ | $\mathrm{E}_{27}$ | $\mathrm{E}_{27}$ | $\mathrm{E}_{27}$ | $\mathrm{~N}_{10}$ | $\mathrm{E}_{27}$ | $\mathrm{E}_{27}$ | $\mathrm{E}_{27}$ | $\mathrm{~N}_{12}$ | $\mathrm{~N}_{10}$ |
| 6 | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ |
| 7 | $\mathrm{E}_{26}$ | $\mathrm{E}_{26}$ | $\mathrm{E}_{26}$ | $\mathrm{~N}_{10}$ | $\mathrm{E}_{26}$ | $\mathrm{E}_{26}$ | $\mathrm{E}_{26}$ | $\mathrm{E}_{26}$ | $\mathrm{E}_{26}$ | $\sim$ |
| 8 | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ |
| 9 | $\mathrm{E}_{16}$ | $\mathrm{E}_{16}$ | $\mathrm{E}_{16}$ | $\mathrm{E}_{26}$ | $\mathrm{E}_{26}$ | $\mathrm{E}_{26}$ | $\mathrm{E}_{26}$ | $\mathrm{~N}_{10}$ | $\mathrm{E}_{16}$ | $\mathrm{E}_{16}$ |
| 10 | $\sim$ | $\sim$ | $\sim$ | $\sim$ | $\sim$ | $\sim$ | $\sim$ | $\sim$ | $\sim$ | $\mathrm{~N}_{10}$ |
| 11 | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ | $\mathrm{E}_{M}$ |

Table 1. Existence of optimal $(r, \delta)_{a}$ codes for parameters $n=60, \delta=5,2 \leq r \leq 11$ and $11 \leq k \leq 20$.

## VII. Conclusions

We have investigated the structure properties and construction methods of optimal $(r, \delta)_{a}$ linear codes, whose length and dimension are $n$ and $k$ respectively. A structure theorem for optimal $(r, \delta)_{a}$ code with $r \mid k$ is first obtained. We next derived two sets of parameters where no optimal $(r, \delta)_{a}$ linear codes could exist (over any field), as well as identified four sets of parameters where optimal $(r, \delta)_{a}$ linear codes exist over any field of size $q \geq\binom{ n}{k-1}$. Some of these existence conditions were reported in the literature before, but the minimum field size we derived is (considerably) smaller than those derived in the previous works. Our results have considerably substantiated the results in terms of constructing optimal $(r, \delta)_{a}$ codes, and there are now only two small holes (two subcases with specific parameters) where the existence results are unknown. Except for these two small subcases, for all the other cases, given each tuple of $(n, k, r, \delta)$, either an optimal $(r, \delta)_{a}$ linear code does not exist or an optimal $(r, \delta)_{a}$ linear code can be constructed using a deterministic algorithm.

As an illustrative summary of our results, we also provide in Table 1 an example of the existence of optimal $(r, \delta)_{a}$ linear codes for the parameters of $n=60, \delta=5,2 \leq r \leq 11$ and $11 \leq k \leq 20$. In this table, $\mathrm{E}_{M}$ means that optimal $(r, \delta)_{a}$ linear codes can be constructed by the method in [10] or by our Theorem 15 and Algorithm 1 (which requires a substantially smaller field); $\mathrm{E}_{16}$ (resp. $\mathrm{E}_{26}, \mathrm{E}_{27}$ ) means optimal $(r, \delta)_{a}$ linear codes can be constructed by Theorem 16 (resp. Theorem 26, Theorem 27); $\mathrm{N}_{10}$ (resp. $\mathrm{N}_{11}$ ) means optimal $(r, \delta)_{a}$ linear codes do not exist according to Theorem 10 (resp. Theorem11); and $\sim$ means we do not yet know whether an optimal $(r, \delta)_{a}$ linear code exists or not.

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[^0]:    ${ }^{1}$ Note that this condition is equivalent to the condition that $m \geq v+1$, where $n=w(r+1)+m$ and $k=u(r+1)+v$ satisfying $0<m<r+1$ and $0<v<r$.

[^1]:    ${ }^{2}$ When $G_{i}$ and $G_{j}$ are viewed as vectors of $\mathbb{F}_{q}^{k}$, it is possible for $G_{i}=G_{j}$ where $i \neq j$. However, when treating them as two different columns of $G$, we shall view $G_{i}$ and $G_{j}$ as two separate elements in $\mathcal{G}$ (even though they may be identical).

[^2]:    ${ }^{3}$ In this proof, for any $(r, \delta)_{a}$ code $\mathcal{C}$, we obtain a subset $S \subseteq \mathcal{G}$ such that $|S| \geq k-1+\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)$ and $\operatorname{Rank}(S)=k-1$. Then by Lemma (1) the minimum distance of $\mathcal{C}$ is $d \leq n-k+1-\left(\left\lceil\frac{k}{r}\right\rceil-1\right)(\delta-1)$, which also provides a proof of the minimum distance bound ${ }^{r}$ in (I.1).

[^3]:    ${ }^{5}$ If $|J|>\left\lceil\frac{k}{r}\right\rceil$, then pick a $\left\lceil\frac{k}{r}\right\rceil$-subset $J_{0}$ of $J$, and replace $J$ by $J_{0}$ in our discussion.

