# Self-testing of symmetric three-qubit states 

Xinhui Li, ${ }^{1}$ Yukun Wang, ${ }^{2, *}$ Yunguang Han, ${ }^{3,4, 母}$ Fei Gao, ${ }^{1,5, ~}{ }^{\text {, }}$ 团 and Qiaoyan Wen ${ }^{1}$<br>${ }^{1}$ State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing, China 100876<br>${ }^{2}$ Department of Electrical \& Computer Engineering,<br>National University of Singapore, Singapore, 117543<br>${ }^{3}$ Department of Physics and Center for Field Theory and Particle Physics, Fudan University, Shanghai, China, 200433<br>${ }^{4}$ State Key Laboratory of Surface Physics, Fudan University, Shanghai, China 200433<br>${ }^{5}$ Center for Quantum Computing, Peng Cheng Laboratory, Shenzhen, China, 518055

(Dated: July 16, 2019)


#### Abstract

Self-testing refers to a device-independent way to uniquely identify the state and the measurement for uncharacterized quantum devices. The only information required comprises the number of measurements, the number of outputs of each measurement, and the statistics of each measurement. Earlier results on self-testing of multipartite state were restricted either to Dicke states or graph states. In this paper, we propose self-testing schemes for a large family of symmetric three-qubit states, namely the superposition of $W$ state and $G H Z$ state. We first propose and analytically prove a self-testing criterion for the special symmetric state with equal coefficients of the canonical basis, by designing subsystem self-testing of partially and maximally entangled state simultaneously. Then we demonstrate for the general case, the states can be self-tested numerically by the swap method combining semi-definite programming (SDP) in high precision.


## I. INTRODUCTION

Entanglement is a critical resource for numerous striking applications of quantum information theory [1]. Furthermore, it is key to comprehend many peculiar properties of quantum many-body systems and has become increasingly important in both theoretical and experimental areas such as teleportation [2] and quantum simulation [3]. Because of the essential role played by symmetry in the field of quantum entanglement, it is of great significance to explore the properties of symmetric states. Also symmetric states are key resources in many experiments such as quantum communication [4], and quantum computing, for instance as an initial state for Grover's algorithm [5]. In addition, restricting analysis to symmetric states can greatly reduce the difficulty of calculations. In this work, we investigate one of verification tasks for symmetric states, namely certification of entanglement state.

A canonical way to approach the problem of certification of quantum states is to exploit tomographic scenario [6]. By repeating the experiment, expectation values of an informationally complete set of measurements allow us to reconstruct the density operator that describes the quantum state. However, such procedure requires a large number fully characterized measurements that scales with the dimension of the quantum state.

An alternative technique which could positively address these problems is self-testing. It is a concept of device independence whose conclusion verdict relies only

[^0]on the observed statistics of measurement outcomes under the sole assumptions of no-signaling and the validity of quantum theory [7]. Consider two players, Alice and Bob, each has a device. Both devices are given classical input ( $x$ and $y$, respectively) which corresponds to the application of measurements inside the devices, and classical output ( $a$ and $b$ ). The devices are physical isolated so that sending signals from one to the other is not possible. The central question is: given observed correlational probabilities $p(a, b \mid x y)$, what can be inferred about the underlying state? Self-testing refers to determining the state completely in such cases.

The idea of self-testing quantum states can be traced back to 1990's, where Popescu and Rohrlich et al. pointed out the maximal violation of the CHSH Bell inequality [8] identifies uniquely the maximally entangled state of two qubits and the corresponding measurements [9]. However, it was not widely known until the works of Mayers and Yao 10], which self-tests the same state with more measurements. Since then self-testing has received substantial attention: self-testing for partially entangled pairs of qubits were presented in [11, 12], while its extension to high dimension partially entangled states was given in [13]. Furthermore, all the criteria for self-testing the maximally entangled pair of qubits were reported in refs [14, 15], where the authors proved a condition for a given binary XOR game to be a robust self-test. The robustness analysis to small deviations from the idea case for self-testing these quantum states and measurements were presented in [16 19], which made self-testing more practical.

Beyond these works focusing on the bipartite scenario, self-testing of multipartite states have recently been studied, such as self-testing of Graph states [20], Dicke states [21], partially entangled GHZ states [21]. Inspired by self-
testing all entangled states in bipartite scenario, one may ask whether all the entanglement states can be self-tested in multipartite scenario? However, the multipartite entanglement is more complicated than bipartite scenario, especially for partially entangled states. The most celebrated example is the case of three-qubit states [22, 23], we consider the particular case of the sates which are equivalent under local unitary transformations to states of the form [24],

$$
|\eta\rangle=a|000\rangle+b|001\rangle+c|010\rangle+d|100\rangle+e|111\rangle
$$

where $a, b, c, d, e$ are normalized coefficients. The two well known inequivalent classes of tripartite genuine entangled states, namely, $W$ states [25] and Greenberger-Horne-Zeilinger $(G H Z)$ states [26] are corresponding to $\frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle)$ and $\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$ respectively. It is obviously that entangled three-qubit states are not only these two kinds of entangled states. One can set any value of the coefficients to define entanglement states. However, what we're interested in is the symmetric entangled states, due to its significant application. We noticed that the symmetric entangled three-qubits states are of the form $\cos \theta|W\rangle+\sin \theta|G H Z\rangle$ under permutations of party labels. The aim of this paper is to investigate the self-testing of these states and where so far only special case has been studied, i.e., self-testing of $W$ state [27] and self-testing of $G H Z$ state [17].

Here we proved analytically the self-testing of a specific symmetric state through projections onto two systems and showed that general cases can be self-tested using fixed Pauli measurements combining the swap method and semidefinite programming (SDP). The paper is structured as following. In Section III we give a review of the two-qubit self-testing, including the whole set of criteria for ideal self-testing of maximally entanglement state and any pure two-qubit state can be self-tested by tilted Bell inequality. In Section III we prove that a symmetric three-qubit state can be self-tested through projections onto two systems, and we show robust selftesting of a more general class of states which is a linear combination of $W$ and $G H Z$ states by the swap method and SDP.

## II. PRELIMINARIES

Let us consider a Bell-type experiment involving two noncommunicating parties. Each has access to a black box with inputs denoted respectively by $x, y \in$ $\{0,1, \ldots, M-1\}$ and outputs $a, b \in\{0,1, \ldots, m-1\}$. Assuming the validity of quantum mechanics, one could model these boxes with an underlying state $|\psi\rangle_{A B}$ and measurement projectors $\left\{M_{x}^{a}\right\}_{x, a}$ and $\left\{M_{y}^{b}\right\}_{y, b}$, which commute for different parties. The state can be taken pure and the measurements can be taken projective without loss of generality, because the dimension of the Hilbert space is not fixed and the possible purification
and auxiliary systems can be given to any of the parties. After sufficiently many repetitions of the experiment one can estimate the joint conditional statistics, also known as the behavior, $p(a, b \mid x, y)=\langle\psi| M_{x}^{a} M_{y}^{b}|\psi\rangle$. Now, we can formally define self-testing in the following way.
Definition 0.1. (Self-testing) We say that the correlations $p(a, b \mid x, y)$ allow for self-testing if for every quantum behavior $\left(|\psi\rangle,\left\{M_{x}^{a}, M_{y}^{b}\right\}\right)$ compatible with $p(a, b \mid x, y)$ there exists a local isometry $\Phi=\Phi_{A} \otimes \Phi_{B}$ such that
$\Phi|\psi\rangle_{A B}|00\rangle_{A^{\prime} B^{\prime}}=|j u n k\rangle_{A B} \otimes|\bar{\psi}\rangle_{A^{\prime} B^{\prime}}$
$\Phi\left(M_{x}^{a} \otimes M_{y}^{b}|\psi\rangle_{A B}|00\rangle_{A^{\prime} B^{\prime}}\right)=|j u n k\rangle_{A B} \otimes \overline{M_{x}^{a}} \otimes \overline{M_{y}^{b}}|\bar{\psi}\rangle_{A^{\prime} B^{\prime}}$,
where $|00\rangle_{A^{\prime} B^{\prime}}$ is the trusted auxiliary qubits attached by Alice and Bob locally into their systems[28]. The isometry must be seen as a virtual protocol: it does not need to be implemented in the laboratory as a part of the procedure of self-testing; all that must be done in laboratory is to query the boxes and derive $p(a, b \mid x, y)$.

Let us review some previous results on the self-testing of two-qubit state which are used as building blocks of our work.

## A. All the self-testings of the singlet for two binary measurements

In the ref. 15], Wang et al. proposed the whole set of criteria for the ideal self-testing of singlet. Consider four unknown operators $A_{x}$ and $B_{y}$ for $i, j \in\{1,2\}$ with binary outcomes labelled $\pm 1$ and satisfy $\left[A_{x}, B_{y}\right]=0$. Denote

$$
\begin{equation*}
E_{x y}=\langle\psi| A_{x} B_{y}|\psi\rangle=\cos \alpha_{x y} \tag{2}
\end{equation*}
$$

where $\alpha_{x y}$ is the angle between the two vectors $A_{x}|\psi\rangle$ and $B_{y}|\psi\rangle$. The observed correlations $E_{x y}$ self-test the singlet if and only if they satisfy one of the conditions

$$
\begin{equation*}
\sum_{(x, y) \neq(i, j)} \arcsin \left(E_{x y}\right)-\arcsin \left(E_{i, j}\right)=\xi \pi \tag{3}
\end{equation*}
$$

with $\arcsin \left(E_{x y}\right)_{\{x, y\} \in\{1,2\}} \in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for $i, j \in\{1,2\}, \xi \in$ $\{+1,-1\}$. The eight equations in (3) are equivalent in the sense that each one can be transformed into the other by relabelling the measurements and outcomes. Without loss of generality, consider the case of $i=1, j=2$ and $\xi=+1$, that is $\alpha_{11}+\alpha_{21}=\alpha_{12}-\alpha_{22}$. It means $A_{1}|\psi\rangle$, $A_{2}|\psi\rangle, B_{1}|\psi\rangle, B_{2}|\psi\rangle$ are in the same plane.

This all self-testing criteria for singlet state is proved to be equivalent to a binary nonlocal XOR game defined by the figure of merit $\sum_{(x, y) \in\{1,2\}^{2}} f_{x y} E_{x y}$ if $E_{x y}=\cos \alpha_{x y}$ satisfy $\alpha_{11}+\alpha_{21}=\alpha_{12}-\alpha_{22}$ and the coefficients $f_{x y}$ are constructed by

$$
\left(\begin{array}{l}
f_{11}  \tag{4}\\
f_{12} \\
f_{21} \\
f_{22}
\end{array}\right)=\left(\begin{array}{c}
\frac{1}{\sin \alpha_{11}} \\
-\frac{1}{\sin \left(\alpha_{11}+\alpha_{21}+\alpha_{22}\right)} \\
\frac{1}{\sin \alpha_{21}} \\
\frac{1}{\sin \alpha_{22}}
\end{array}\right)
$$

## B. Self-testing of pure partially entangled two-qubit state

It has been shown that any pure two-qubit state in their Schmidt form

$$
\begin{equation*}
|\psi\rangle=\cos \theta|00\rangle+\sin \theta|11\rangle \tag{5}
\end{equation*}
$$

can be self-tested by observing the maximum violation of the tilted CHSH inequality [11, 12, 18]

$$
\begin{align*}
I^{\alpha}\left(\alpha, A_{1}, A_{2}, B_{1}, B_{2}\right)= & \alpha A_{1}+A_{1} B_{1}+A_{1} B_{2} \\
& +A_{2} B_{1}-A_{2} B_{2} \leq 2+\alpha \tag{6}
\end{align*}
$$

where $\alpha$ is defined through $\sin 2 \theta=\sqrt{\frac{4+\alpha^{2}}{4-\alpha^{2}}} . A_{1}, A_{2}$, $B_{1}$ and $B_{2}$ are the unknown measurements by Alice and Bob, respectively. The maximal quantum violation of this inequality is given by $I_{Q}^{\alpha}=\sqrt{8+2 \alpha^{2}}$, achievable with the measurement settings

$$
\begin{align*}
& A_{1}=\sigma_{z}, \quad A_{2}=\sigma_{x} \\
& B_{1}=\cos \mu \sigma_{z}+\sin \mu \sigma_{x} \\
& B_{2}=\cos \mu \sigma_{z}-\sin \mu \sigma_{x} \tag{7}
\end{align*}
$$

where $\tan \mu=\sin 2 \theta$.

## III. SELF-TESTING OF SYMMETRIC THREE-QUBIT STATES

The work in [22, 23] gave a generalization of the Schmidt decomposition for three-qubit pure states and proved that for any pure three-qubit state the existence of local bases which allow one to build a set of five orthogonal product states in terms of which can be written in a unique form. The local bases product states can be given as three inequivalent sets

$$
\begin{align*}
& \{|000\rangle,|001\rangle,|010\rangle,|100\rangle,|111\rangle\}, \\
& \{|000\rangle,|010\rangle,|110\rangle,|100\rangle,|111\rangle\}, \\
& \{|000\rangle,|100\rangle,|110\rangle,|101\rangle,|111\rangle\}, \tag{8}
\end{align*}
$$

whereas the first set is symmetric under permutation of parties, the other two are not.

In this paper, we consider the self-testing of symmetric three-qubit states based on the fist set of (8) with different kinds of coefficients.

## A. Self-testing of a symmetric three-qubit state

The specific case we consider is the state with equal coefficient of the basis, reads as:

$$
\begin{equation*}
|\psi\rangle=\frac{1}{\sqrt{5}}(|001\rangle+|010\rangle+|100\rangle+|000\rangle+|111\rangle) \tag{9}
\end{equation*}
$$

The basic idea of self testing this state is to project the state onto two kinds of subsystem entangled states by one


FIG. 1. The swap circuit. The local isometry used to self-test the $|\psi\rangle$ state. $H$ is the standard Hadamard gate, $Z$ and $X$ are controlled by the auxiliary qubit. The trusted ancillary qubits are prepared in the state $|0\rangle$.
party's measurement $Z$. More precisely, after measuring one party in the $Z$ basis, the remaining two parties can achieve the maximal violation of tailored Bell inequalities simultaneously using the same measurement settings conditioned on the outcome being either " 0 " or " 1 ".

If partition the three parties into $A \mid B C$, we have

$$
\begin{align*}
|\psi\rangle= & \frac{1}{\sqrt{5}}\left[\sqrt{3}|0\rangle_{A} \otimes \frac{1}{\sqrt{3}}(|00\rangle+|01\rangle+|10\rangle)_{B C}\right. \\
& \left.+\sqrt{2}|1\rangle_{A} \otimes \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)_{B C}\right] \tag{10}
\end{align*}
$$

We denote

$$
\left|\varphi_{0}\right\rangle:=\frac{1}{\sqrt{3}}(|00\rangle+|01\rangle+|10\rangle)
$$

Using Schmidt decomposition, the state $\left|\varphi_{0}\right\rangle$ can be written as

$$
\begin{equation*}
\left|\varphi_{0}\right\rangle=\cos \beta\left|0^{\prime} 0^{\prime}\right\rangle+\sin \beta\left|1^{\prime} 1^{\prime}\right\rangle \tag{11}
\end{equation*}
$$

here $\cos \beta=\sqrt{\frac{3+\sqrt{5}}{6}}, \sin \beta=\sqrt{\frac{3-\sqrt{5}}{6}}$ and $\left\{\left|0^{\prime}\right\rangle,\left|1^{\prime}\right\rangle\right\}$ are the new basis (see the detail in Appendix). Following the results given in sec IIB this state can be self-tested by violating the tilted CHSH inequality maximally

$$
\begin{equation*}
I_{Q}^{\alpha}=\sqrt{8+2 \alpha^{2}}=\frac{12}{\sqrt{13}} \tag{12}
\end{equation*}
$$

with $\alpha=2 \sqrt{\frac{1-\sin ^{2} 2 \beta}{1+\sin ^{2} 2 \beta}}$. The optimal measurements are set according to (7), with $\tan \mu=\frac{1}{3}$.

At the same time, singlet is invariant under basis transformation

$$
\begin{equation*}
\left|\varphi_{1}\right\rangle=\frac{\left|0^{\prime} 0^{\prime}\right\rangle+\left|1^{\prime} 1^{\prime}\right\rangle}{\sqrt{2}} \tag{13}
\end{equation*}
$$

using the same bases and optimal measurements settings with $\left|\varphi_{0}\right\rangle$ would satisfy $\alpha_{11}+\alpha_{21}=\alpha_{12}-\alpha_{22}$ where
$\alpha_{11}=\mu, \alpha_{12}=-\mu, \alpha_{21}=\frac{\pi}{2}-\mu$ and $\alpha_{22}=-\frac{\pi}{2}-\mu$. So, $\left|\varphi_{1}\right\rangle$ can be self-tested by winning the binary nonlocal XOR game [15]:

$$
\begin{equation*}
\sum_{(x, y) \in\{1,2\}^{2}} f_{x y} \cos \alpha_{x y}=\frac{4}{\sin 2 \mu} \tag{14}
\end{equation*}
$$

and the coefficients $f_{x y}$ are constructed as (4).
Hence, the states $\left|\varphi_{0}\right\rangle$ and $\left|\varphi_{1}\right\rangle$ conditioned on the outcome " 0 " and " 1 " after the measurement in the $Z$ basis of $A$ violate the tilted CHSH inequality and XOR game maximally using the same measurements, respectively. This also holds when the first measured party is $C$.

The following result sums it up.
Result 1. Alice, Bob, and Charlie, spatially separated, each perform three measurements denoted as $\left\{A_{i}, B_{j}, C_{k}\right\}$ ( $i, j, k \in\{0,1,2\}$ ) with binary outcomes on an unknown shared quantum state $|\psi\rangle$. The target state is self-tested if the following statistics are observed:

$$
\begin{align*}
&\left\langle P_{A}^{0} P_{B}^{0} P_{C}^{0}\right\rangle=\left\langle P_{A}^{0} P_{B}^{0} P_{C}^{1}\right\rangle=\left\langle P_{A}^{0} P_{B}^{1} P_{C}^{0}\right\rangle \\
&=\left\langle P_{A}^{1} P_{B}^{0} P_{C}^{0}\right\rangle=\left\langle P_{A}^{1} P_{B}^{1} P_{C}^{1}\right\rangle=\frac{1}{5}  \tag{15}\\
&\left\langle P_{C}^{i} A_{1}\left(B_{2}-B_{1}\right)\right\rangle=0  \tag{16a}\\
& \cos \omega\left\langle P_{C}^{i} A_{1}\right\rangle+\sin \omega\left\langle P_{C}^{i} A_{2}\right\rangle=\left\langle P_{C}^{i} A_{0}\right\rangle  \tag{16b}\\
& \cos \omega\left\langle P_{A}^{i} C_{1}\right\rangle+\sin \omega\left\langle P_{A}^{i} C_{2}\right\rangle=\left\langle P_{A}^{i} C_{0}\right\rangle \tag{17a}
\end{align*}
$$

$$
\begin{equation*}
\left\langle P_{C}^{i} A_{1}\left(B_{1}-B_{2}\right)\right\rangle=0 \tag{18a}
\end{equation*}
$$

$$
\begin{equation*}
\frac{\sin (\omega+\mu)\left\langle P_{C}^{i} B_{2}\right\rangle}{\sin 2 \mu}-\frac{\sin (\omega-\mu)\left\langle P_{C}^{i} B_{1}\right\rangle}{\sin 2 \mu}=\left\langle P_{C}^{i} B_{0}\right\rangle \tag{18b}
\end{equation*}
$$

for $i \in\{0,1\}$, and

$$
\begin{equation*}
\sin \omega\left\langle P_{x}^{0} P_{y}^{0} T_{2}\right\rangle-\cos \omega\left\langle P_{x}^{0} P_{y}^{0} T_{1}\right\rangle=\frac{2}{5} \tag{19a}
\end{equation*}
$$

$$
\begin{equation*}
\frac{\cos (\omega-\mu)\left\langle P_{A}^{0} P_{C}^{0} B_{1}\right\rangle}{\sin 2 \mu}-\frac{\cos (\omega+\mu)\left\langle P_{A}^{0} P_{C}^{0} B_{2}\right\rangle}{\sin 2 \mu}=\frac{2}{5} \tag{19b}
\end{equation*}
$$

for $(x, y, T)=\{(A, B, C),(B, C, A)\}$, and

$$
\begin{align*}
\left\langle P_{C}^{0} I^{\alpha}\left(\alpha, A_{1}, A_{2}, B_{1}, B_{2}\right)\right\rangle & =\frac{3}{5} \sqrt{8+2 \alpha^{2}}  \tag{20a}\\
\left\langle P_{A}^{0} I^{\alpha}\left(\alpha, C_{1}, C_{2}, B_{1}, B_{2}\right)\right\rangle & =\frac{3}{5} \sqrt{8+2 \alpha^{2}} \tag{20b}
\end{align*}
$$

$$
\begin{equation*}
\sum_{i, j=\{1,2\}} f_{i, j}\left\langle P_{A}^{1} B_{i} C_{j}\right\rangle=\sum_{i, j=\{1,2\}} f_{i j}\left\langle P_{C}^{1} A_{i} B_{j}\right\rangle=\frac{8}{5 \sin 2 \mu} \tag{21}
\end{equation*}
$$

where $P^{0}=\frac{1+Z}{2}$ and $P^{1}=\frac{1-Z}{2}$ are projectors for the $Z$ measurement, $\alpha=2 \sqrt{\frac{5}{13}}, \tan \mu=\frac{2}{3}, \cos \omega=\frac{1}{\sqrt{5}}$ and $\sin \omega=\frac{2}{\sqrt{5}}$.

Proof. We start from observation (15) which implies that

$$
\begin{align*}
\left\langle P_{A}^{0} P_{B}^{0} P_{C}^{0}\right\rangle & +\left\langle P_{A}^{0} P_{B}^{0} P_{C}^{1}\right\rangle+\left\langle P_{A}^{0} P_{B}^{1} P_{C}^{0}\right\rangle \\
& +\left\langle P_{A}^{1} P_{B}^{0} P_{C}^{0}\right\rangle+\left\langle P_{A}^{1} P_{B}^{1} P_{C}^{1}\right\rangle=1 \tag{22}
\end{align*}
$$

Therefore $P_{A}^{a} P_{B}^{b} P_{C}^{c}|\psi\rangle=0$ for other three projectors.
For convenience, define the operators for each party as

$$
\begin{align*}
& Z_{A}=A_{0}, \quad Z_{A}^{\prime}=A_{1}, \quad X_{A}^{\prime}=A_{2} \\
& Z_{B}=B_{0}, \quad Z_{B}^{\prime}=\frac{B_{1}+B_{2}}{2 \cos \mu}, \quad X_{B}^{\prime}=\frac{B_{1}-B_{2}}{2 \sin \mu} \\
& Z_{C}=C_{0}, \quad Z_{C}^{\prime}=C_{1}, \quad X_{C}^{\prime}=C_{2} \tag{23}
\end{align*}
$$

Following the self-testing of nonmaximally entangled qubits from the ref. [12], maximal violation of the tilted Bell inequality (20a) implies

$$
\begin{align*}
P_{C}^{0} Z_{A}^{\prime}|\psi\rangle & =P_{C}^{0} Z_{B}^{\prime}|\psi\rangle  \tag{24a}\\
P_{C}^{0} Z_{A}^{\prime} X_{A}^{\prime}|\psi\rangle & =-P_{C}^{0} X_{A}^{\prime} Z_{A}^{\prime}|\psi\rangle  \tag{24b}\\
p_{C}^{0} X_{A}^{\prime}\left(I+Z_{B}^{\prime}\right)|\psi\rangle & =\frac{1}{\tan \theta} P_{C}^{0} X_{B}^{\prime}\left(1-Z_{A}^{\prime}\right)|\psi\rangle \tag{24c}
\end{align*}
$$

From (24a) and (24c), we get

$$
\begin{align*}
& P_{C}^{0} X_{B}^{\prime}\left(1+Z_{A}^{\prime}\right)|\psi\rangle=\frac{1}{\tan \theta} P_{C}^{0} X_{A}^{\prime}\left(1-Z_{B}^{\prime}\right)|\psi\rangle  \tag{25a}\\
& P_{C}^{0} X_{B}^{\prime}\left(1-Z_{B}^{\prime}\right)|\psi\rangle=\tan \theta P_{C}^{0} X_{A}^{\prime}\left(1+Z_{A}^{\prime}\right)|\psi\rangle \tag{25b}
\end{align*}
$$

On the other hand, the following equation holds with (24a),

$$
\begin{align*}
P_{C}^{0} X_{B}^{\prime}|\psi\rangle & =P_{C}^{0} X_{B}^{\prime} \frac{\left(1+Z_{A}^{\prime}\right)}{2}|\psi\rangle+P_{C}^{0} X_{B}^{\prime} \frac{\left(1-Z_{A}^{\prime}\right)}{2}|\psi\rangle \\
& =\frac{1}{\tan \theta} P_{C}^{0} X_{A}^{\prime} \frac{\left(1-Z_{B}^{\prime}\right)}{2}|\psi\rangle+\tan \theta P_{C}^{0} X_{A}^{\prime} \frac{\left(1+Z_{B}^{\prime}\right)}{2} \tag{26}
\end{align*}
$$

Multiply by operator $\left(1+Z_{B}^{\prime}\right)$ on both sides of (26) such that

$$
\begin{align*}
P_{C}^{0}\left(1+Z_{B}^{\prime}\right) X_{B}^{\prime}|\psi\rangle= & \frac{1}{\tan \theta} P_{C}^{0} X_{A}^{\prime}\left(1+Z_{B}^{\prime}\right) \frac{\left(1-Z_{B}^{\prime}\right)}{2}|\psi\rangle \\
& +\tan \theta P_{C}^{0} X_{A}^{\prime}\left(1+Z_{B}^{\prime}\right)|\psi\rangle \\
= & \tan \theta P_{C}^{0} X_{A}^{\prime}\left(1+Z_{A}^{\prime}\right)|\psi\rangle \\
= & P_{C}^{0} X_{B}^{\prime}\left(1-Z_{B}^{\prime}\right)|\psi\rangle \tag{27}
\end{align*}
$$

holds with (24a), i.e.

$$
\begin{equation*}
P_{C}^{0} Z_{B}^{\prime} X_{B}^{\prime}|\psi\rangle=-P_{C}^{0} X_{B}^{\prime} Z_{B}^{\prime}|\psi\rangle \tag{28}
\end{equation*}
$$

Observation (16a) implies

$$
\begin{equation*}
\left\langle P_{C}^{0} A_{2}\left(B_{2}-B_{1}\right)\right\rangle=0 \Rightarrow P_{C}^{0} X_{A}^{\prime}|\psi\rangle \perp P_{C}^{0} Z_{B}^{\prime}|\psi\rangle \tag{29}
\end{equation*}
$$

and combine the first relation (24a) from tilted Bell inequality, we have

$$
\begin{equation*}
P_{C}^{0} X_{A}^{\prime}|\psi\rangle \perp P_{C}^{0} Z_{A}^{\prime}|\psi\rangle \tag{30}
\end{equation*}
$$

Then $Z_{A}|\psi\rangle$ in the subsapce of projector $P_{C}^{0}$ can be written as

$$
\begin{equation*}
P_{C}^{0} Z_{A}|\psi\rangle=\cos \omega P_{C}^{0} Z_{A}^{\prime}|\psi\rangle+\sin \omega P_{C}^{0} X_{A}^{\prime}|\psi\rangle \tag{31}
\end{equation*}
$$

by equations (16b) and (30). So, one can define the vector $X_{A}|\psi\rangle$ orthogonal to $Z_{A}|\psi\rangle$ as

$$
\begin{equation*}
P_{C}^{0} X_{A}|\psi\rangle=\sin \omega P_{C}^{0} Z_{A}^{\prime}|\psi\rangle-\cos \omega P_{C}^{0} X_{A}^{\prime}|\psi\rangle \tag{32}
\end{equation*}
$$

Since operators $Z_{A}^{\prime}$ and $X_{A}^{\prime}$ are hermitian, unitary and anti-commutation in the subspace of $P_{C}^{0}$ by (24a) and (24b), we get the anti-commutation relations

$$
\begin{equation*}
P_{C}^{0} Z_{A} X_{A}|\psi\rangle=-P_{C}^{0} X_{A} Z_{A}|\psi\rangle \tag{33}
\end{equation*}
$$

Following the self-testing of maximally entangled qubits from observation [15], maximal violation of the XOR game (21) implies

$$
\begin{align*}
P_{C}^{1} Z_{A}^{\prime}|\psi\rangle & =P_{C}^{1} Z_{B}^{\prime}|\psi\rangle \\
P_{C}^{1} X_{A}^{\prime}|\psi\rangle & =P_{C}^{1} X_{B}^{\prime}|\psi\rangle \tag{34}
\end{align*}
$$

and the anti-commutation relations

$$
\begin{align*}
P_{C}^{1} Z_{A}^{\prime} X_{A}^{\prime}|\psi\rangle & =-P_{C}^{1} X_{A}^{\prime} Z_{A}^{\prime}|\psi\rangle \\
P_{C}^{1} Z_{B}^{\prime} X_{B}^{\prime}|\psi\rangle & =-P_{C}^{1} X_{B}^{\prime} Z_{A}^{\prime}|\psi\rangle \tag{35}
\end{align*}
$$

Observation (16a) implies

$$
\begin{equation*}
\left\langle P_{C}^{1} A_{2}\left(B_{2}-B_{1}\right)\right\rangle=0 \Rightarrow P_{C}^{1} X_{A}^{\prime}|\psi\rangle \perp P_{C}^{0} Z_{B}^{\prime}|\psi\rangle \tag{36}
\end{equation*}
$$

and combine the first relation in (34), we have

$$
\begin{equation*}
P_{C}^{1} X_{A}^{\prime}|\psi\rangle \perp P_{C}^{1} Z_{A}^{\prime}|\psi\rangle \tag{37}
\end{equation*}
$$

Then $Z_{A}|\psi\rangle$ and its orthogonal vector in the subsapce of projector $P_{C}^{1}$ can be written as

$$
\begin{align*}
P_{C}^{1} Z_{A}|\psi\rangle & =\cos \omega P_{C}^{1} Z_{A}^{\prime}|\psi\rangle+\sin \omega P_{C}^{1} X_{A}^{\prime}|\psi\rangle \\
P_{C}^{1} X_{A}|\psi\rangle & =\sin \omega P_{C}^{1} Z_{A}^{\prime}|\psi\rangle-\cos \omega P_{C}^{1} X_{A}^{\prime}|\psi\rangle \tag{38}
\end{align*}
$$

by equations (16b) and (37). Moreover, we can obtain the equivalence relations

$$
\begin{align*}
P_{C}^{1} Z_{A}|\psi\rangle & =P_{C}^{1} Z_{B}|\psi\rangle \\
P_{C}^{1} X_{A}|\psi\rangle & =P_{C}^{1} X_{B}|\psi\rangle \tag{39}
\end{align*}
$$

and anti-commutation relations

$$
\begin{align*}
P_{C}^{1} Z_{A} X_{A}|\psi\rangle & =-P_{C}^{1} X_{A} Z_{A}|\psi\rangle \\
P_{C}^{1} Z_{B} X_{B}|\psi\rangle & =-P_{C}^{1} X_{B} Z_{B}|\psi\rangle \tag{40}
\end{align*}
$$

for party A in the subspace of $P_{C}^{1}$.
After some manipulations similar to party A, we can obtain the relations between $Z_{B(C)}|\psi\rangle, X_{B(C)}|\psi\rangle$ and $Z_{B(C)}^{\prime}|\psi\rangle, X_{B(C)}^{\prime}|\psi\rangle$ for party B (and C) in the subspace of the projectors $P_{C}^{0}\left(P_{A}^{0}\right)$ and $P_{C}^{1}\left(P_{A}^{1}\right)$

$$
\begin{align*}
Z_{B(C)}|\psi\rangle & =\cos \omega Z_{B(C)}^{\prime}|\psi\rangle+\sin \omega X_{B(C)}^{\prime}|\psi\rangle \\
X_{B}|\psi\rangle & =\cos \omega X_{B}^{\prime}|\psi\rangle-\sin \omega Z_{B}^{\prime}|\psi\rangle \\
X_{C}|\psi\rangle & =\sin \omega Z_{C}^{\prime}|\psi\rangle-\cos \omega X_{C}^{\prime}|\psi\rangle \tag{41}
\end{align*}
$$

Then the anti-commutation relations

$$
\begin{align*}
P_{C}^{0} Z_{B} X_{B}|\psi\rangle & =-P_{C}^{0} X_{B} Z_{B}|\psi\rangle \\
P_{C}^{1} Z_{B} X_{B}|\psi\rangle & =-P_{C}^{1} X_{B} Z_{B}|\psi\rangle \\
P_{A}^{0} Z_{C} X_{C}|\psi\rangle & =-P_{C}^{0} X_{C} Z_{C}|\psi\rangle \\
P_{A}^{1} Z_{C} X_{C}|\psi\rangle & =-P_{A}^{1} X_{C} Z_{C}|\psi\rangle \tag{42}
\end{align*}
$$

in subspace hold.
Observations (19a) and (19b) imply that

$$
\begin{align*}
P_{A}^{0} P_{B}^{0}|\psi\rangle & =P_{A}^{0} P_{B}^{0} X_{C}|\psi\rangle  \tag{43a}\\
P_{B}^{0} P_{C}^{0}|\psi\rangle & =P_{B}^{0} P_{C}^{0} X_{A}|\psi\rangle  \tag{43b}\\
P_{A}^{0} P_{C}^{0}|\psi\rangle & =P_{A}^{0} P_{C}^{0} X_{B}|\psi\rangle \tag{43c}
\end{align*}
$$

Now the isometry can be constructed by $Z, X$ and $H$ per party as ref. 27] as shown in Fig.1. The formula of $Z$ and $X$ are based on the measurement operators for each party,

$$
\begin{align*}
& Z_{A}=A_{0}, \quad X_{A}=\sin \omega A_{1}-\cos \omega A_{2} \\
& Z_{B}=B_{0}, \quad X_{B}=\frac{\cos (\omega-\mu) B_{1}}{\sin 2 \mu}-\frac{\cos (\omega+\mu) B_{2}}{\sin 2 \mu} \\
& Z_{C}=C_{0}, \quad X_{C}=\sin \omega C_{1}-\cos \omega C_{2} \tag{44}
\end{align*}
$$

The output after the isometry can be written as

$$
\begin{align*}
\left|\Psi^{\prime}\right\rangle= & \Phi|\psi\rangle_{A B C}|000\rangle_{A^{\prime} B^{\prime} C^{\prime}} \\
= & \frac{1}{8}\left[\left(I+Z_{A}\right)\left(I+Z_{B}\right)\left(I+Z_{C}\right)|\psi\rangle|000\rangle\right. \\
& +\left(I+Z_{A}\right)\left(I+Z_{B}\right) X_{C}\left(I-Z_{C}\right)|\psi\rangle|001\rangle \\
& +\left(I+Z_{A}\right) X_{B}\left(I-Z_{B}\right)\left(I+Z_{C}\right)|\psi\rangle|010\rangle \\
& +X_{A}\left(I-Z_{A}\right)\left(I+Z_{B}\right)\left(I+Z_{C}\right)|\psi\rangle|100\rangle \\
& +X_{A}\left(I-Z_{A}\right) X_{B}\left(I-Z_{B}\right) X_{C}\left(I-Z_{C}\right)|\psi\rangle|111\rangle \\
& +\left(I+Z_{A}\right) X_{B}\left(I-Z_{B}\right) X_{C}\left(I-Z_{C}\right)|\psi\rangle|011\rangle \\
& +X_{A}\left(I-Z_{A}\right)\left(I+Z_{B}\right) X_{C}\left(I-Z_{C}\right)|\psi\rangle|101\rangle \\
& \left.+X_{A}\left(I-Z_{A}\right) X_{B}\left(I-Z_{B}\right)\left(I+Z_{C}\right)|\psi\rangle|110\rangle\right] . \tag{45}
\end{align*}
$$

For the second term in (45), we could prove it is equal to $P_{A}^{0} P_{B}^{0} P_{C}^{0} X_{C}|\psi\rangle$ using (42). Then, it can be replaced with $P_{A}^{0} P_{B}^{0} P_{C}^{0}|\psi\rangle$ because of (43a). The third and forth terms are similar.

For the fifth term in (45), one moves $X_{A}$ and $X_{B}$ to the right using (40), which is in turn equal to
$X_{C} P_{C}^{1} P_{B}^{0} X_{B} P_{A}^{0} X_{A}|\psi\rangle$. Then replace $X_{B}$ with $X_{A}$ by (41) and this line becomes $P_{A}^{0} P_{B}^{0} X_{C} P_{C}^{1}|\psi\rangle$. After moving $X_{C}$ to the right using (42), this term is equal to $P_{A}^{0} P_{B}^{0} P_{C}^{0}|\psi\rangle$ from 43a).

Remind that the last three terms in (45) equal to zeros.
Therefore, all these properties of the operators deduced from the measurement requirements will help to reduce the general output (45) to
$\left|\Psi^{\prime}\right\rangle=P_{A}^{0} P_{B}^{0} P_{C}^{0}|\psi\rangle(|000\rangle+|001\rangle+|010\rangle+|100\rangle+|111\rangle)$.
This state can be normalized into the form of $|j u n k\rangle_{A B C} \otimes|\psi\rangle_{A^{\prime} B^{\prime} C^{\prime}}$, here $|j u n k\rangle=\sqrt{5} P_{A}^{0} P_{B}^{0} P_{C}^{0}|\psi\rangle$.

Thus, we have proven that, with these requirements (15) -(21) on the measurement results indeed self-test the unknown state as target state (9).

Further more, we also consider the robustness for each observation in (15)-(21) has a deviation at most equal to $\epsilon$ around the perfect value. The robustness bound is given in the next section together (see Fig. 3).

## B. Robust self-testing of more general pure three-qubit states

The previous work on self-testing of $W$ and $G H Z$ states proved that both representatives of the two inequivalent local operations and classical communication classes of three-qubit [25] can be self-tested. The question then remains whether one can self-test every pure three-qubit state. Here we explicitly shows how one can self-test a large family of three-qubit states using swap method and Navascués-Pironio-Acín (NPA) hierarchy [29]. The target state we consider is given as

$$
\begin{equation*}
|\psi(\theta)\rangle=\cos \theta|W\rangle+\sin \theta|G H Z\rangle \tag{46}
\end{equation*}
$$

here $|W\rangle=\frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle)$ and $|G H Z\rangle=$ $\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle), \theta \in\left[0, \frac{\pi}{2}\right]$ is a parameter .

Result 2. Alice and Bob each party performs two dichotomic measurements, Charlie performs three measurements with binary outcomes on an unknown shared quantum state. The state $\left|\psi_{\theta}\right\rangle$ can be self-tested using the full statistics for $\theta \in\left[0, \frac{\pi}{2}\right]$. Moreover, the self-testing is robust.

We consider the $[2,2,3]$ scenario that Alice and Bob each performs two dichotomic measurements $Z$ and $X$ with binary outcomes as $\pm 1$, while Charlie performs three dichotomic measurements denoted as $Z, X$, and $D$ with outcomes. Suppose that the observed behavior exhibits the following two groups of full-body statistics

$$
\begin{align*}
& \left\langle Z_{i} Z_{j} Z_{k}\right\rangle=-\cos \theta^{2}, \quad\left\langle Z_{i} X_{j} X_{k}\right\rangle=\frac{2 \cos ^{2} \theta}{3}-\frac{\sin 2 \theta}{\sqrt{6}} \\
& \left\langle Z_{i} Z_{j} X_{k}\right\rangle=\frac{\sin 2 \theta}{\sqrt{6}}, \quad\left\langle X_{i} X_{j} X_{k}\right\rangle=\sin \theta^{2} \tag{47}
\end{align*}
$$

up to permutations of Alice, Bob and Charlie, (i.e. $i, j, z \in\{A, B, C\}, i \neq j \neq z)$ and

$$
\begin{align*}
\left\langle Z_{i} Z_{j} D_{C}\right\rangle & =-\frac{\cos ^{2} \theta}{\sqrt{2}}+\frac{\sin 2 \theta}{2 \sqrt{3}},\left\langle Z_{i} X_{j} D_{C}\right\rangle=\frac{\sqrt{2} \cos ^{2} \theta}{3} \\
\left\langle X_{i} X_{j} D_{C}\right\rangle & =\frac{5 \sqrt{2}-\sqrt{2} \cos 2 \theta-2 \sqrt{3} \sin 2 \theta}{12} \tag{48}
\end{align*}
$$

up to permutations of Alice and Bob (i.e. $i, j \in\{A, B\}$, $i \neq j$ ).

These are the statistics that one would obtain for the $|\psi\rangle$ for $\theta \in\left[0, \frac{\pi}{2}\right]$ if $Z=\sigma_{z}, X=\sigma_{x}$ and $D=\frac{\sigma_{x}+\sigma_{z}}{\sqrt{2}}$.

We consider the same isometry as sec III A as shown in Fig. 1. The isometry can be re-written as a swap operator $U=S_{A A^{\prime}} \otimes S_{B B^{\prime}} \otimes S_{C C^{\prime}}$ with $S_{A A^{\prime}}=$ $W_{A A^{\prime}} V_{A A^{\prime}} W_{A A^{\prime}}$ and

$$
\begin{align*}
W_{A A^{\prime}} & =I_{A} \otimes|0\rangle\langle 0|+X_{A} \otimes|1\rangle\langle 1| \\
V_{A A^{\prime}} & =\frac{I_{A}+Z_{A}}{2} \otimes I+\frac{I_{A}-Z_{A}}{2} \otimes \sigma_{x} \tag{49}
\end{align*}
$$

and the same for $S_{B B^{\prime}}$ and $S_{C C^{\prime}}$. After this isometry, the trusted auxiliary systems will be left in the state

$$
\begin{align*}
\rho_{\text {swap }} & =\operatorname{tr}_{A B C}\left[U \rho_{A B C} \otimes|000\rangle\left\langle\left. 000\right|_{A^{\prime} B^{\prime} C^{\prime}} U^{\dagger}\right]\right. \\
& =\sum C_{i j k l s t}|i\rangle\langle j| \otimes|k\rangle\langle l| \otimes|s\rangle\langle t| \tag{50}
\end{align*}
$$

where

$$
\begin{equation*}
C_{i j k l s t}=\frac{1}{64} \operatorname{tr}_{A B C}\left[M_{j, i}^{A} \otimes M_{l, k}^{B} \otimes M_{t, s}^{C} \rho_{A B C}\right] \tag{51}
\end{equation*}
$$

and $M_{j, i}^{A}=\left(I+Z_{A}\right)^{j+1}\left(X_{A}-X_{A} Z_{A}\right)^{j}\left(I+Z_{A}\right)^{i+1}\left(X_{A}-\right.$ $\left.X_{A} Z_{A}\right)^{i}$ for $i, j \in\{0,1\}, M_{l, k}^{B}$ and $M_{t, s}^{C}$ are analogous.

Finally, the closeness of the unknown resource to the target state can be then captured by the fidelity

$$
\begin{equation*}
f=\langle\psi| \rho_{\text {swap }}|\psi\rangle \tag{52}
\end{equation*}
$$

as a linear function of $\theta$, observed behavior and some non-observable correlations which involve different measurements on the same party, such as $\langle\psi| M_{x}^{a} M_{x^{\prime}}^{a^{\prime}}|\psi\rangle$ with $x \neq x^{\prime}$ which are left as variables. The terms in fidelity that are not determined should be compatible with a quantum realisation. As well known, this requirement can't be formulated as an efficient constraint, but it can be relaxed to a family of semi-definite constraints [29]. Since the objective function $f$ is linear, the optimisation can then be cast as a semi-definite programming (SDP) [18, 19]:

$$
\begin{array}{cl}
\min & f=\langle\psi| \rho_{\text {swap }}|\psi\rangle \\
\text { s.t. } & \Gamma \geq 0,  \tag{53}\\
& \text { equations (47) and (48), }
\end{array}
$$

where $\Gamma$ is a matrix with NPA hierarchy characterization of the quantum behaviors. This moment matrix corresponding to $q$-local level 1 (which includes any products with at most one operator per party) has size $74 \times 74$
and is augmented by necessary terms(like $\left\langle Z_{A} X_{A} Z_{A}\right\rangle$, $\left\langle Z_{B} X_{B} Z_{C} X_{C}\right\rangle,\left\langle Z_{C} X_{C} Z_{A} X_{A} Z_{A}\right\rangle$ and so on) to express all the average values $\langle\cdot\rangle$ that appear in the expression of fidelity. For all $\theta \in\left[0, \frac{\pi}{2}\right]$, the SDP returns $f>99.96 \%$ (Fig.2). We believe that the deviation from 1 is due to the limitation of the SDP relaxation.


FIG. 2. Red line represents the lower bound on the fidelity obtained with varying parameter $\theta$ from 0 to $\frac{\pi}{2}$.

Interesting as the above result is in itself, it relies on observing the measurement statistics in (47) and (48) exactly, which is not possible due to inevitable experimental uncertainties. To investigate the robustness of self-testing induced by these statistics, we shall consider mixing them with white noise, that is by multiplying each term by $(1-\epsilon)$ and $\epsilon$ represents the deviation of the observed behavior from the ideal values. As examples, we plot four special values for $\theta=\left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \arccos \sqrt{\frac{3}{5}}\right\}$ which correspond to $|W\rangle, \frac{|W\rangle+|G H Z\rangle}{\sqrt{2}},|G H Z\rangle$ and superposition state investigated in sec III A

$$
\frac{1}{\sqrt{5}}(|000\rangle+|001\rangle+|010\rangle+|100\rangle+|111\rangle)
$$

respectively (Fig.3).
Note that even though calculation of deducing the expression of fidelity does not contain any moment involving the measurement $D$, its appearance in the matrix $\Gamma$ makes it necessary and useful to bound the fidelity.

In particular, the robustness for self-testing the specific state analysed in sec III A can be also obtained by SDP. Let us first look at the ideal quantum realization to design our swap circuit. To construct the swap circuit, set $Z_{A}=A_{0}, Z_{B}=B_{0}$ and $Z_{C}=C_{0}$, and we'd rather need $\sigma_{x}$ per party, which in the ideal case should have the forms as (44). However, written with the unknown measurement operators, this expression may not define a unitary operator. A method to circumvent this obstacle has been presented in previous works [15, 18]: one defines


FIG. 3. (Color Online) The plot of lower bounds on the fidelity between the measured state and the target state. $\epsilon$ represents the deviation of the observed behavior from the ideal values.
a fourth dichotomic operator $\widetilde{X}_{i}$ for $i \in\{A, B, C\}$ such that

$$
\begin{align*}
\widetilde{X}_{A}\left(\sin \omega A_{1}-\cos \omega A_{2}\right) & \geq 0 \\
\widetilde{X}_{B} \frac{\cos (\omega-\mu) B_{1}-\cos (\omega+\mu) B_{2}}{\sin (2 \mu)} & \geq 0 \\
\widetilde{X}_{C}\left(\sin \omega C_{1}-\cos \omega C_{2}\right) & \geq 0 \tag{54}
\end{align*}
$$

Since these equations are not SDP constraints, one relaxes each equation to the positivity of a "localizing matrix".

In order to make the lower bound tight, we add a localizing matrix for $\sigma_{z}$ each party. Then run the SDP using NPA matrix size $163 \times 163$ and augmented by six localizing matrices (two per party), to minimize the fidelity with the target state for each observation in (15)- (21) with a deviation $\epsilon$ from the perfect value. The result is summarized in Fig.3.

## IV. CONCLUSION

We proposed self-testing schemes for a large family of symmetric three-qubit states. The target states we mainly focused on is the superposition of $W$ state and $G H Z$ state due to the simple form and their wide applications in quantum information tasks. We provided two different approaches applying to these states.

For the special case where the state has equal coefficients of the canonical basis, our approach is constructed by combining bipartite self-testing schemes. Through projecting the state onto two kinds of subsystem entangled states by one party's measurement $Z$, the remaining parties can reach the self-testing criteria for these
bipartite entangled states simultaneously using the same measurements settings. The bound is robust against the inevitable experimental errors.

For the general case, we demonstrated that these states can be self-tested using fixed measurements numerically. Here in our work, only the simplest Pauli measurements in $[2,2,3]$ scenario are used. This is quite helpful in the experiments. It would be of interest to study whether this result could be generalized to generic multipartite states. Previous work on multipartite states are usually realized by constructing tailored Bell inequalities for the target states. The complexity of self-testing multipartite states would decrease significantly if our approach holds. A comprehensive study of these questions remains open for other states and scenarios.

## V. APPENDIX

This appendix provides the details of the transformational relations between the Pauli matrices and measurements operators by Schmidt decomposition.

Rewrite the state

$$
\left|\varphi_{0}\right\rangle:=\frac{1}{\sqrt{3}}(|00\rangle+|01\rangle+|10\rangle) .
$$

So we can denote the coefficient matrix as

$$
C=\frac{1}{\sqrt{3}}\left[\begin{array}{ll}
1 & 1  \tag{55}\\
1 & 0
\end{array}\right]
$$

which has Schmidt decomposition $C=U S V$, where

$$
S=\left[\begin{array}{cc}
\sqrt{\frac{1}{6}(3+\sqrt{5})} & 0  \tag{56}\\
0 & \sqrt{\frac{1}{6}(3-\sqrt{5})}
\end{array}\right]
$$

is diagonal, $U$ and $V$ are unitary matrices:

$$
U=\left[\begin{array}{ll}
\frac{3+\sqrt{5}}{2 \sqrt{5+2 \sqrt{5}}} & \frac{3-\sqrt{5}}{2 \sqrt{5-2 \sqrt{5}}}  \tag{57}\\
\frac{1+\sqrt{5}}{2 \sqrt{5+2 \sqrt{5}}} & \frac{1-\sqrt{5}}{2 \sqrt{5-2 \sqrt{5}}}
\end{array}\right], \quad V=\left[\begin{array}{cc}
\frac{1+\sqrt{5}}{2 \sqrt{5+\sqrt{5}}} & \frac{1-\sqrt{5}}{\sqrt{10-2 \sqrt{5}}} \\
\sqrt{\frac{2}{5+\sqrt{5}}} & \frac{5+\sqrt{5}}{10}
\end{array}\right] .
$$

Define $\left|j_{A}^{\prime}\right\rangle \equiv \sum_{j} U_{j i}|j\rangle,\left|k_{B}^{\prime}\right\rangle \equiv \sum_{k} V_{i k}|k\rangle$ and $\lambda_{i}=S_{i i}$, for $i, j, k \in\{0,1\}$. Obviously, $\left\{\left|j^{\prime}\right\rangle\right\}_{A}$ and $\left\{\left|k^{\prime}\right\rangle\right\}_{B}$ are two groups new standard orthogonal basis. Then the
state $\left|\varphi_{0}\right\rangle$ can be written as

$$
\begin{align*}
\left|\varphi_{0}\right\rangle & =\sum_{i j k} U_{j i} S_{i i} V_{i k}|j\rangle_{A}|k\rangle_{B} \\
& =\cos \beta\left|0_{A}^{\prime} 0_{B}^{\prime}\right\rangle+\sin \beta\left|1_{A}^{\prime} 1_{B}^{\prime}\right\rangle, \tag{58}
\end{align*}
$$

here $\cos \beta=\sqrt{\frac{3+\sqrt{5}}{6}}$ and $\sin \beta=\sqrt{\frac{3-\sqrt{5}}{6}}$.
Now, consider the relation between operator $Z^{\prime}$ and $X^{\prime}$ in the new basis and $\sigma_{z}$ and $\sigma_{x}$ for part A,

$$
\begin{align*}
Z_{A}^{\prime} & =\left|0^{\prime}\right\rangle\left\langle\left. 0^{\prime}\right|_{A}-\mid 1^{\prime}\right\rangle\left\langle\left. 1^{\prime}\right|_{A}\right. \\
& =\sum_{j_{1}, j_{2} \in\{0,1\}^{2}} U_{j_{1} 0} U_{j_{2} 0}\left|j_{1}\right\rangle\left\langle j_{2}\right|-\sum_{j_{1}, j_{2} \in\{0,1\}^{2}} U_{j_{1} 1} U_{j_{2} 1}\left|j_{1}\right\rangle\left\langle j_{2}\right| \\
& =\frac{1}{\sqrt{5}}(|0\rangle\langle 0|-|1\rangle\langle 1|)+\frac{2}{\sqrt{5}}(|0\rangle\langle 1|+|1\rangle\langle 0|) \\
& =\cos \omega \sigma_{z}+\sin \omega \sigma_{x}, \tag{59}
\end{align*}
$$

here $\tan \omega=2$. It is easy to get $X_{A}^{\prime}=\sin \omega \sigma_{z}-\cos \omega \sigma_{x}$. This is also hold for part C. Similarly, we have

$$
\begin{align*}
Z_{B}^{\prime} & =\left|0^{\prime}\right\rangle\left\langle\left. 0^{\prime}\right|_{B}-\mid 1^{\prime}\right\rangle\left\langle\left. 1^{\prime}\right|_{B}\right. \\
& =\sum_{k_{1}, k_{2} \in\{0,1\}^{2}} V_{0 k_{1}} V_{0 k_{2}}\left|k_{1}\right\rangle\left\langle k_{2}\right|-\sum_{k_{1}, k_{2} \in\{0,1\}^{2}} V_{1 k_{1}} V_{1 k_{2}}\left|k_{1}\right\rangle\left\langle k_{2}\right| \\
& =\frac{1}{\sqrt{5}}(|0\rangle\langle 0|-|1\rangle\langle 1|)-\frac{2}{\sqrt{5}}(|0\rangle\langle 1|+|1\rangle\langle 0|) \\
& =\cos \omega \sigma_{z}-\sin \omega \sigma_{x} \tag{60}
\end{align*}
$$

and $X_{B}^{\prime}=\sin \omega \sigma_{z}+\cos \omega \sigma_{x}$.
Hence, if Alice and Bob each performs optimal operators as (7) using the new basis, then the measurements can be transformed into Pauli matrices

$$
\left\{\begin{array} { l } 
{ \sigma _ { z } = \operatorname { c o s } \omega A _ { 1 } + \operatorname { s i n } \omega A _ { 2 } , }  \tag{61}\\
{ \sigma _ { x } = \operatorname { s i n } \omega A _ { 1 } - \operatorname { c o s } \omega A _ { 2 } ; }
\end{array} \quad \left\{\begin{array}{l}
\sigma_{z}=\frac{\sin (\omega+\mu) B_{2}-\sin (\omega-\mu) B_{1}}{\sin 2 \mu} \\
\sigma_{x}=\frac{\cos (\omega-\mu) B_{1}-\cos (\omega+\mu) B_{2}}{\sin 2 \mu}
\end{array}\right.\right.
$$

Charlie is analogue to Alice.

## ACKNOWLEDGMENT

We would like to thank Valerio Scarani, Huangjun Zhu and Yu Cai for useful discussions. This work is funded by National Nature Science Foundation of China (Grant Nos. 61671082, 61572081, 61672110, 11875110).
[1] Horodecki R, Horodecki P, Horodecki M, et al. Quantum entanglement. Reviews of modern physics, 2009, 81(2): 865.
[2] Bennett C H, Brassard G, Crépeau C, et al. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Physical review letters, 1993, 70(13): 1895.
[3] Reichardt B W, Unger F, Vazirani U. Classical command of quantum systems. Nature, 2013, 496(7446): 456.
[4] Brandao F G S L, Oppenheim J. Public quantum communication and superactivation. IEEE Transactions on Information Theory, 2012, 59(4): 2517-2526.
[5] Ivanov S S, Ivanov P A, Linington I E, et al. Scalable quantum search using trapped ions. Physical Review A,

2010, 81(4): 042328.
[6] Kosaka H, Inagaki T, Rikitake Y, et al. Spin state tomography of optically injected electrons in a semiconductor. Nature, 2009, 457(7230): 702.
[7] Scarani V. The device-independent outlook on quantum physics. Acta Physica Slovaca, 2012, 62(4): 347-409.
[8] Clauser J F, Horne M A, Shimony A, et al. Proposed experiment to test local hidden-variable theories. Physical review letters, 1969, 23(15): 880.
[9] Tsirelson B S. Some results and problems on quantum Bell-type inequalities. Hadronic Journal Supplement, 1993, 8(4): 329-345.
[10] Mayers D, Yao A. Self testing quantum apparatus. arXiv preprint quant-ph/0307205, 2003.
[11] Yang T H, Navascués M. Robust self-testing of unknown quantum systems into any entangled two-qubit states. Physical Review A, 2013, 87(5): 050102.
[12] Bamps C, Pironio S. Sum-of-squares decompositions for a family of Clauser-Horne-Shimony-Holt-like inequalities and their application to self-testing. Physical Review A, 2015, 91(5): 052111.
[13] Coladangelo A, Goh K T, Scarani V. All pure bipartite entangled states can be self-tested. Nature communications, 2017, 8: 15485.
[14] Miller C A, Shi Y. Optimal robust quantum selftesting by binary nonlocal XOR games. arXiv preprint arXiv:1207.1819, 2012.
[15] Wang Y, Wu X, Scarani V. All the self-testings of the singlet for two binary measurements. New Journal of Physics, 2016, 18(2): 025021.
[16] McKague M, Yang T H, Scarani V. Robust self-testing of the singlet. Journal of Physics A: Mathematical and Theoretical, 2012, 45(45): 455304.
[17] Kaniewski J. Analytic and nearly optimal self-testing bounds for the Clauser-Horne-Shimony-Holt and Mermin inequalities. Physical review letters, 2016, 117(7): 070402.
[18] Bancal J D, Navascué M, Scarani V, et al. Physical characterization of quantum devices from nonlocal correlations. Physical Review A, 2015, 91(2): 022115.
[19] Yang T H, Vértesi T, Bancal J D, et al. Robust and versatile black-box certification of quantum devices. Physical review letters, 2014, 113(4): 040401.
[20] McKague M. Self-testing graph states. Conference on Quantum Computation, Communication, and Cryptography. Springer, Berlin, Heidelberg, 2011: 104-120.
[21] Supić I, Coladangelo A, Augusiak R, et al. Self-testing multipartite entangled states through projections onto two systems. New Journal of Physics, 2018, 20(8): 083041.
[22] Acín A, Andrianov A, Costa L, et al. Generalized Schmidt decomposition and classification of three-quantum-bit states. Physical Review Letters, 2000, 85(7): 1560.
[23] Acín A, Andrianov A, Jané E, et al. Three-qubit purestate canonical forms. Journal of Physics A: Mathematical and General, 2001, 34(35): 6725.
[24] Linden N, Popescu S, Wootters W K. Almost every pure state of three qubits is completely determined by its twoparticle reduced density matrices. Physical review letters, 2002, 89(20): 207901.
[25] Dür W, Vidal G, Cirac J I. Three qubits can be entangled in two inequivalent ways. Physical Review A, 2000, 62(6): 062314.
[26] Brunner N, Sharam J, Vertesi T. Testing the structure of multipartite entanglement with Bell inequalities. Physical review letters, 2012, 108(11): 110501.
[27] Wu X, Cai Y, Yang T H, et al. Robust self-testing of the three-qubit W state. Physical Review A, 2014, 90(4): 042339.
[28] Šupić I, Bowles J. Self-testing of quantum systems: a review. arXiv preprint arXiv:1904.10042, 2019.
[29] Navascués M, Pironio S, Acín A. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. New Journal of Physics, 2008, 10(7): 073013.


[^0]:    * wykun06@gmail.com
    $\dagger$ hyg@fudan.edu.cn
    $\ddagger$ Gaof@bupt.edu.cn

