IEEE Systems Journal, Volume 15, Number 3, September 2021, pp. 4687-4695
DOI:10.1109/JSYST.2020.3036129

Uncertainty of Resilience in Complex Networks with Nonlinear Dynamics

Giannis Moutsinas, Mengbang Zou*, Weisi Guo, *Senior Member, IEEE*

***Abstract*—Resilience is a system's ability to maintain its function when perturbations and errors occur. Whilst we understand low-dimensional networked systems' behaviour well, our understanding of systems consisting of a large number of components is limited. Recent research in predicting the network level resilience pattern has advanced our understanding of the coupling relationship between global network topology and local nonlinear component dynamics. However, when there is uncertainty in the model parameters, our understanding of how this translates to uncertainty in resilience is unclear for a large-scale networked system. Here we develop a polynomial chaos expansion method to estimate the resilience for a wide range of uncertainty distributions. By applying this method to case studies, we not only reveal the general resilience distribution with respect to the topology and dynamics sub-models, but also identify critical aspects to inform better monitoring to reduce uncertainty.**

***Index Terms*—Uncertainty; Resilience; Dynamic Complex Network**

I. Introduction

ORGANIZED behaviors in economics, infrastructure, ecology and human society often involve large-scale networked systems. These systems couple together relatively simple local component dynamics to achieve sophisticated systematic behaviour. A critical part of the organized behavior is the ability of a system to be resilient - e.g. to recover some desirable performance or state after a perturbation. A system's resilience is a key property and plays a crucial role in reducing risks and mitigating damages [1] [2]. Research on resilience of dynamic networks have arisen in diverse application domains ranging from communication network failures [3], blackout in power systems [4], to loss of biodiversity [5]. Due to the different research contexts, up to now, over 70 detailed definitions of resilience have appeared in scientific research [6]. In this paper, we are interested in the general bi-stable networked system described by ordinary differential equation (ODE) dynamics which are common in social (e.g. population logistic model [7], conflict system [8]), ecological (e.g. soil health [9]), climate (e.g. ocean circulation [10]) systems. In such complex networked systems, resilience is defined as the ability to retain original functionality after a perturbation of failure [11].

G. Moutsinas is with Coventry University, Priory St, Coventry CV1 5FB, U.K. (e-mail: giannismoutsinas@gmail.com).

M. Zou (corresponding author) is with Cranfield University, Cranfield MK43 0AL, U.K. (e-mail: M.Zou@cranfield.ac.uk)

W. Guo is with Cranfield University, Cranfield MK43 0AL, U.K., and also with the Alan Turing Institute, London, NW1 2DB, U.K. (e-mail: weisi.guo@cranfield.ac.uk).

We acknowledge funding from EPSRC grant CoTRE - Complexity Twin for Resilient Ecosystems (EP/R041725/1).

A. Review of Resilience Methods

Existing performance-based methods [12] [13] [14] have been proposed to quantify system's macro resilience with metrics associated with different research domains. These methods proposed before did not use explicit network metrics [15]. Such work promotes our understand of system performance when perturbations happen but do not give us insight into how network topology structure, interaction strength between nodes affect dynamics in complex systems consisting of a large number of components. What they are interested in is the macro performance of the system with optimization metrics but not pay attention to the topology of the networked system. Understanding the relationship between topology and dynamics in complex systems is important for us to augment network topology, design structure or monitor critical nodes to prevent loss of resilience. That is to say, while, current methods make us understand low-dimensional models with a few interacting components well [2], our understanding of complex systems consisting of a large number of components that interact through a complex network is limited.

These limitations are rooted in a theoretical gap that most frameworks are designed to analyze a few interacting components and not suited for complex systems with a large number of components interacting through a complex network. A general network-based theoretical framework is proposed to explore and predict the multiple roots and dimensions of resilience in complex networks [11]. Recent research in predicting the network-level [11] [16] and node-level resilience patterns [17] has advanced our understanding of the coupling relationship between topology and dynamics. In this paper, we propose a network-based method to quantify resilience in complex network systems, which could characterize the relationship between network topology and system resilience in mathematical expressions as well as the effects of uncertainty on system dynamics. The method proposed in this paper could be directly applied in bi-stable systems with ODE dynamics in different domains.

To simulate the dynamics and estimate resilience of complex networks with dynamical effects, we need to define dynamical models with parameter values. However, in practice, uncertainty on the model form and parameters are inherently present. Uncertainty can originate from latent process variables (process noise), e.g., inherent biological variability between cells which are genetically identical [18] or from a parameter estimation procedure based on noisy measurements (measurement or inference noise) as well as from incomplete information of the model. For example, recent research proposed an analytical framework for exactly predicting the critical transition in a complex networked system subjected to noise effects [16]. In this research, the roles of the original large-scale system dynamics, network topology and noise are well separated and the linear noise approximation is used to estimate the effect of noise. Actually, in many cases of networked dynamical systems, uncertainty could exist in system dynamics as well as network topology. In our research, we consider the situation that uncertainty inherently exists in system dynamics and network topology, and Polynomial Chaos Expansion method is used in our research to estimate the effect of these uncertainties.

In recent years, the modelling and numerical simulation of practical problems with uncertainty have received unprecedented attention, which is called Uncertainty Quantification (UQ). UQ methods have been applied in widespread fields like fluid dynamics [19], weather forecasting [20], etc. At present, UQ methods are shown as follows [21]:

B. Review of Uncertainty Quantification

Monte Carlo Methods [22] are based on samples. In these methods, samples are randomly generated according to the probability distribution. For each sample, the problem to be solved becomes a definite problem. By solving these determined problems, representative statistical information about the exact solution can be discovered. These methods are easy to use but need large sample data and computationally expensive [23]. For arbitrarily large dynamical networks, it is difficult to sample appropriately without a foundation UQ theory.

Perturbation Methods [24] expands a function into a Taylor series around its mean value and then make a reasonable truncation. Normally, at most, we can truncate the second-order expansion because, for higher-order cases, the resulting solution system will become very complicated. Besides, it is suitable to be applied in problems with small perturbations since it may magnify the uncertainty.

Moment Equation Methods [25] attempt to directly solve the equations satisfied by the moments of the random solution. These equations about moments need to be derived from the original stochastic problem. For some simple problems, such as linear problems, this method is more effective. But usually, when we derive a certain moment equation, we need to use the information of higher moments. Besides, in most cases, Moment Equation Methods need considerably large computational cost to achieve a good result when it is applied in nonlinear systems, especially in strongly nonlinear systems, such as bi-stable systems [26].

Polynomial Chaos Methods [27] are standard methods for UQ in singular dynamical systems. The basic idea is to perform polynomial expansion of the exact solution in random parameter space. This method could solve problems with any type of random parameter inputs. First, we need to perform a finite order expansion of the exact solution in the random parameter space and then take this expansion into the original problem and do Galerkin projection in the expansion polynomial space. After that, we get simultaneous equations about the expansion coefficient. By solving the equations, we can get all the statistical information of the exact solution. If the exact solution has good regularity for random parameters and this method can achieve exponential convergence.

In Table (I), we make a comparison of the above methods according to algorithm computing efficiency and accuracy. Monte Carlo Methods need a large number of samples and are computationally expensive in nonlinear systems. Perturbations Methods are suitable to be applied in small perturbations. Perturbations Methods may enlarge the perturbations in complex systems with nonlinear dynamics and could not achieve high accuracy. Moment Equation Methods are suitable for simple problems, such as linear problems. Considering the nonlinear dynamics of complex systems, Polynomial Chaos Methods could be applied in analysing the uncertainty. Polynomial Chaos Methods could achieve very similar results with computational saving comparing with Monte Carlo Methods [28] and have been successfully used in nonlinear dynamical system [29].

C. Contribution

The contribution of this paper is to propose a method to quantify bi-stable networked system resilience and characterize the explicit mathematical relationship between network topology and resilience. The uncertainty quantification in this space is also lacking. As such, polynomial chaos expansion method is used to quantify uncertain propagation to quantify the uncertainty when estimating the resilience. Then, we analyze how parameters and network topology with uncertainty affect the resilience of dynamic networked systems, which gives us more insight into dynamic networked systems.

II. System Setup

A. Saddle-node bifurcation

The traditional mathematical treatment of resilience used from ecology [30] to engineering [31] approximates the behavior of a complex system with a one-dimensional nonlinear dynamic equation

$$\dot{x} = f(\beta, x). \tag{1}$$

The functional form of $f(\beta, x)$ represents the system's dynamics, and the parameter β captures the changing environment conditions (show in Figure 1 (a)). The system is assumed to be in one of the stable fixed points, x_0 of equation (1), extract from

$$f(\beta, x_0) = 0 \tag{2}$$

$$\left.\frac{df}{dx}\right|_{x=x_0} < 0, \tag{3}$$

where equation (2) provides the system's steady state and equation (3) guarantees its linear stability. The solution of equations (2) and (3) provides the resilience function of $x(\beta)$, which represents the possible states of the system (Figure 1(a)). At some critical point β_c the resilience function may feature a bifurcation (Figure 1(a)), indicating that the system loses its resilience by undergoing a sudden transition to a

TABLE I
UNCERTAINTY QUANTIFICATION METHODS

UQ Methods	Computing efficiency	Accuracy
Monte Carlo Methods	Computationally expensive with large sample data	Accuracy increase with sample data scale
Perturbation Methods	Very complicated for higher-order cases (order $N > 2$)	Accuracy in small perturbation since it may magnify uncertainty
Moment Equation Methods	Effective in simple problems like linear problems but large computational cost in nonlinear such as bi-stable systems	High accuracy in simple problems like linear problems
Polynomial Chaos Methods	Computationally efficient when truncate order is low	High accuracy when probability distributions of uncertainty parameters are defined

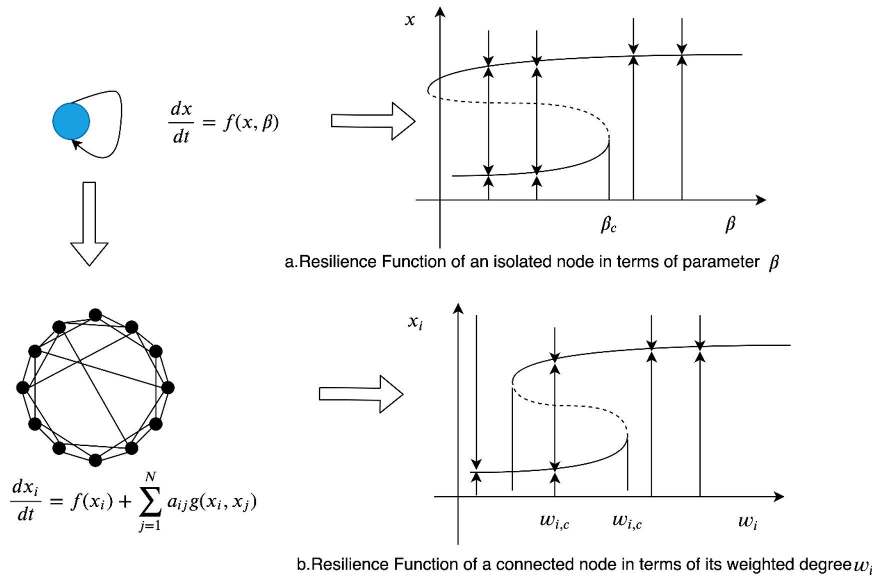


Fig. 1. It shows dynamics of a single node and the coupled dynamics in a complex network. (a) In 1D systems resilience is captured by the resilience function $x(\beta)$, which describes the state(s) of the system as a function of the tunable parameter β. The system exhibits a single stable fixed point for $\beta > \beta_c$ and two (or more) stable fixed points, a desired state and an undesired state for $\beta < \beta_c$. (b)In a coupled dynamic system, the single parameter β is replaced by the complex weighted network w_i, whose characteristics depend on both environmental conditions and the specific pairwise interaction strengths. Consequently, the resilience function, now capturing the behaviour of the vector state $x(w_i)$.

different [32] [33], often undesirable, fixed point of equation (2) [11].

The saddle-node or fold bifurcation is a bifurcation in which two equilibria of a dynamical system collide and annihilate each other. The simplest example of such bifurcation is

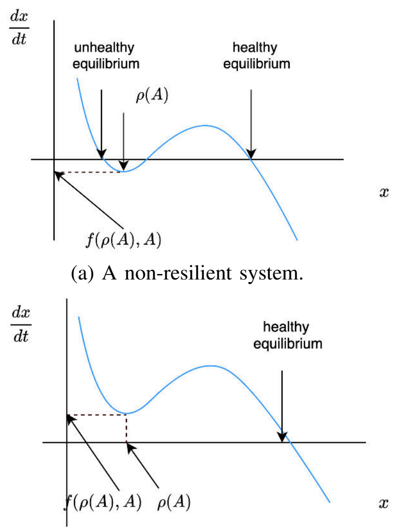
$$\dot{x} = x^2 - c. \tag{4}$$

If $c > 0$, then there are 2 equilibria, stable one at $-\sqrt{c}$ and unstable one at $\sqrt{c}$. If $c < 0$, there are no equilibria for the system since $x^2 - c$ is always positive. For $c = 0$, we have the bifurcation point and only one equilibrium exists, which is not hyperbolic.

We are in dynamics system $\dot{x} = f(x, \mathbf{A})$, with f smooth. We will assume that this system always has a stable equilibrium $x_d > 0$ that is not close to the origin and the saddle-node bifurcation can happen close to the origin, see Figure 2. Note that here $\mathbf{A}$ denotes a vector of parameters and not just one.

The stable equilibrium away from the origin is a desirable state of the system and will it be called *healthy*. The possible stable equilibrium close to the origin is an undesirable state of the system and it will be called *unhealthy*. If in the system the unhealthy equilibrium is absent, then we say that the system is resilient. We illustrate this concept by exploring the abundance of species in an ecological network [34]. When there only exists a health equilibrium away from the origin, in which the average abundance is high, the system maintains its resilience. However, when there exists an unhealthy equilibrium close to the origin and a healthy equilibrium, a bifurcation will happen, resulting in a desirable high-abundance state and an undesirable low-abundance state. Under these conditions the system loses its resilience, potentially transitioning to the undesirable low-abundance state.

As it can be seen from the Figure 2, in order to detect whether the system is resilient or not, we can look at the value of the local minimum and check its sign. If it is negative, then we are in the case shown in Figure 2(a). If it is positive, then we are in the case shown in Figure 2(b). We do this by simply finding the smallest positive root of the equation $f^{'}(x, \mathbf{A}) = 0$, we will denote this by $\rho(\mathbf{A})$.



(a) A non-resilient system.

(b) A resilient system

Fig. 2. In Figure 2(a) we can see a system before the saddle-node bifurcation, where both the unhealthy and the healthy equilibria are present. In Figure 2(b), we see a system after the saddle-node bifurcation, where the unhealthy equilibrium has been annihilated

B. Dynamics on graph

Real systems are usually composed of numerous components linked via a complex set of weighted, often directed, interactions(show in Figure 1(b)). Let G be a weighted directed graph of n vertices and m edges and let M be its weighted adjacency matrix. Using G we couple n one-dimensional dynamical systems. The dynamics of each one-dimensional system is described by the differential equation $\dot{x} = f(x, \mathbf{A})$, where f is a smooth function and $\mathbf{A}$ is a vector of parameters. The coupling term is described by a smooth function $g(x, y, \mathbf{B})$, where $\mathbf{B}$ is a vector of parameters. $\mathbf{A} = \{a_1, ..., a_i\}$, $\mathbf{B} = \{b_{11}, ..., b_{ij}\}$. The dynamics of the system is described by

$$\dot{x_i} = f(x_i, a_i) + \sum_{j=1}^{n} M_{ji} g(x_i, x_j, b_{ij}). \tag{5}$$

We assume that each parameter of the equation (5) is a i.i.d random variable that gets a different realization on each node. This assumption is suitable for homogeneous models but not for heterogeneous models.

We denote that $\mathbf{X} = \{x_1, ..., x_n\} \in R^N$ and we define $F : R^N \to R^N$ by

$$(F(\mathbf{X}, \mathbf{A}, \mathbf{B}))_i = f(x_i, a_i) + \sum_{j=1}^{n} M_{ji} g(x_i, x_j, b_{ij}). \tag{6}$$

Then the system of equations (6) can be written as

$$\dot{\mathbf{X}} = F(\mathbf{X}, \mathbf{A}, \mathbf{B}). \tag{7}$$

The equilibrium of the system satisfies $F(\mathbf{X}_e, \mathbf{A}, \mathbf{B}) = 0$.

Generally, we do not know very well when $\dot{\mathbf{X}} = F(\mathbf{X}, \mathbf{A}, \mathbf{B})$ will be resilient in a large-scale network. It is more difficult to know the resilience of $\dot{\mathbf{X}}$ when considering uncertainty on parameters of vectors $\mathbf{A}, \mathbf{B}$ and uncertainty on topology (e.g. properties of M_{ij}) in dynamic network.

III. Approach and Methodology

A. Dynamic network with uncertainty

Uncertainty in dynamic network may exist in self-dynamics of each component in $f(x_i, a_i)$ and each component in coupling term $g(x_i, x_j, b_{ij})$ as well as the network topology. We assume that each parameter is a random variable that gets a different realization on each node and moreover the value of any parameters has to be within a range of its true value. So we have $a_i = a_i(1+e_1 u_i)$, $b_{ij} = b_{ij}(1+e_2 v_{ij}), M = M(1+e_3 r)$,where u_i, v_{ij}, r are random variables uniform in $[a, b]$ and e_1, e_2, e_3 are constants. $\mathbf{U} = \{u_1, ..., u_i\}$, $\mathbf{V} = \{v_{11}, ..., v_{ij}\}$. The mathematics model of dynamic network with uncertainty is showed as:

$$\begin{aligned} \dot{x_i} &= f(x_i, a_i(1+e_1 u_i)) + \\ &\sum_{j}^{n} M_{ji}(1+e_3 r) g(x_i, x_j, b_{ij}(1+e_2 v_{ij})). \end{aligned} \tag{8}$$

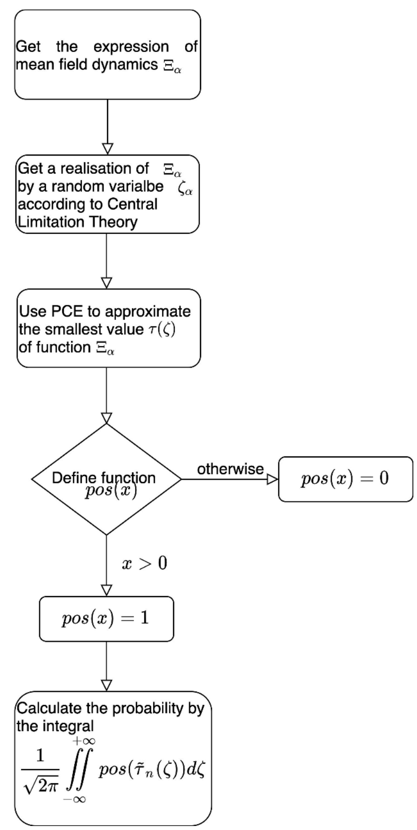


Fig. 3. Steps to estimate resilience with uncertainty

B. Two-step method to estimate resilience with uncertainty

The proposed method to estimate resilience with uncertainty is shown in a flowchart (Figure (3)). The first step is to use mean-field dynamics and central limit theorem to get the expression which describes the probability of resilience of dynamic networked systems. The second step is to use Polynomial Chaos Expression (PCE) to calculate the probability.

1) Mean field dynamics: In order to find the mean field approximation of the equilibrium of the system, we define $\mathbf{1} := 1, ...1 \in R^N$

$$\Xi(x) := Mean[F(x\mathbf{1}, \mathbf{A}, \mathbf{B})] = \\ \frac{1}{n}\sum_{i=1}^{n}(f(x, a_i)) + \frac{1}{n}\sum_{i,j=1}^{n} M_{ji} g(x, x, b_{ij}). \tag{9}$$

Note that $\Xi(x)$ depends on parameters in $\mathbf{A}$ and $\mathbf{B}$. Since parameters in $\mathbf{A}$ and $\mathbf{B}$ are random variables, for any x, $\Xi(x)$ is a function depending on these random variables. Then we search for r such that $\Xi(x) = 0$.

Because, the parameters a_i are assumed to be iid random variables, for fixed x, $f(x, a_i)$ are also iid random variables. We define

$$\mu_{f(x)} := \mathbf{E}[f(x, a_i)] \tag{10}$$

$$\delta_{f(x)} := \sqrt{\mathbf{Var}[f(x, a_i)]}. \tag{11}$$

This means that by Central Limit Theorem, for big enough n, $\frac{1}{n}\sum_{i=1}^{n} f(x, a_i)$ can be approximated by a normally distributed random variable with mean $\mu_{f(x)}$ and standard deviation $\frac{1}{n}\delta_{f(x)}$, i.e

$$\frac{1}{n}\sum_{i=1}^{n} f(x, a_i) \sim \mathbf{N}(\mu_{f(x)}, \frac{1}{n}\delta^2_{f(x)}). \tag{12}$$

Similarly, the random variables $g(x, x, b_{ij})$ are i.i.d, we define

$$\mu_{g(x)} := \mathbf{E}[g(x, x, b_{ij})] \tag{13}$$

$$\delta_{g(x)} := \sqrt{\mathbf{Var}[g(x, x, b_{ij})]}. \tag{14}$$

Then we have

$$\frac{1}{n}\sum_{i,j=1}^{n} M_{ji} g(x, x, b_{ij}) \sim \mathbf{N}(\frac{m}{n}\mu_{g(x)}, \frac{m}{n^2}\delta^2_{f(x)}). \tag{15}$$

For dynamic network with uncertainty, we define the auxiliary functions:

$$\phi(x, \mathbf{U}) = f(x, \mathbf{E}[\mathbf{A}](1 + e_1\mathbf{U})) \tag{16}$$

$$\varphi(x, r, \mathbf{V}) = \mathbf{E}[M](1 + e_3 r) g(x, x, \mathbf{E}[\mathbf{B}](1 + e_2\mathbf{V})). \tag{17}$$

Let k be the dimension of $\mathbf{A}$ and l be the dimension of $\mathbf{B}$, then for the function f we define

$$\mu_{f(x)} = \int_{[a,b]^k} \frac{1}{(b-a)^k}\phi(x, \mathbf{U}) d\mathbf{U} \tag{18}$$

and

$$\delta^2_{f(x)} = \int_{[a,b]^k} \frac{1}{(b-a)^k}(\phi(x, \mathbf{U})^2 - \mu^2_{f(x)}) d\mathbf{U}. \tag{19}$$

Similarly, for g we define

$$\mu_{g(x)} = \int_{[a,b]^{l+1}} \frac{1}{(b-a)^{l+1}}\varphi(x, r, \mathbf{V}) dr d\mathbf{V} \tag{20}$$

and

$$\delta^2_{g(x)} = \int_{[a,b]^{l+1}} \frac{1}{(b-a)^{l+1}}(\varphi(x, r, \mathbf{V})^2 - \mu^2_{g(x)}) dr d\mathbf{V}. \tag{21}$$

Since $\Xi(x)$ is the sum of 2 normally distributed random variables, when we combine the above we get

$$\Xi(x) \sim \mathbf{N}(\mu_{f(x)} + \frac{m}{n}\mu_{g(x)}, \frac{1}{n}\delta^2_{f(x)} + \frac{m}{n^2}\delta^2_{g(x)}). \tag{22}$$

We can get a realisation of $\Xi_\alpha(x)$ by drawing ζ_α from $\mathbf{N}(0, 1)$. and setting

$$\Xi_\alpha(x) = \mu_{f(x)} + \frac{m}{n}\mu_{g(x)} + \sqrt{\frac{1}{n}\delta^2_{f(x)} + \frac{m}{n^2}\delta^2_{g(x)}}\zeta_\alpha. \tag{23}$$

We assume that every realisation of $\Xi(x)$ has the shape described in Figure 2, i.e. it is close to a saddle-node bifurcation. We find that the smallest positive root ρ of $\Xi^{'}(x)$. Finally we set $\tau = \Xi(\rho)$.

Since $\Xi(x)$ is a random variable, both ρ and τ are functions based on this random variable. Moreover, τ is an indicator for the saddle-node bifurcation. For a given realization of ζ_α, if $\tau_\alpha > 0$, then there is only one equilibrium and the dynamics is resilient and if $\tau_\alpha < 0$, then there are three equilibria and the dynamics is non-resilient. Thus the probability of the system being resilient is $\mathbf{P}(\tau > 0)$. We can use a Polynomial chaos expansion (PCE) truncated to degree n to approximate $\tau(\zeta)$, we will denote this PCE by $\widetilde{\tau}_n(\zeta)$. We define the function

$$pos(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{24}$$

Then, the probability that the system is resilient is given by the integral

$$\frac{1}{\sqrt{2\pi}} \iint_{-\infty}^{+\infty} pos(\widetilde{\tau}_n(\zeta))\, d\zeta. \tag{25}$$

2) Polynomial chaos expansion: Let Ξ be random variable with known probability distribution function (PDF) w. Moreover let $X = \phi(\zeta)$, with ϕ a function that is square integrable on $\mathbf{R}$ with w as weight function, let us call this space L^2_w. Our goal is to approximate X by a polynomial series of ζ.

For this we need a family of polynomials P_n such that P_0 is not 0, for all n the polynomial P_n has degree n and are orthogonal with respect to w, i.e. the inner product

$$< P_n, P_m >_w = \int_{-\infty}^{+\infty} P_m(x) P_n(x) w(x)\, dx \tag{26}$$

is 0 when $m \neq n$. Moreover we assume that P_0 is normalized so that $< P_0, P_0 >_w = 1$. The polynomials P_n can be used as a basis for L^2_w. So we can write

$$\phi(\zeta) = \sum_{n \geq 0} c_n P_n(\zeta). \tag{27}$$

In order to get the expression of $\phi(\zeta)$, we need to define the orthogonal basis P_n and the coefficients c_n. What kind of orthogonal basis should be chosen depends on the distribution of random variable ζ. If random variable ζ obeys a Gaussian distribution, we can choose the Hermite polynomial as the orthogonal basis. If random variable ζ obeys uniform distribution, we can choose Legendre polynomial as the basis [35].

Because P_n is an orthogonal basis, we can get the coefficients by projecting on each basis vector

$$c_n = \frac{< \phi, P_n >_w}{< P_n, P_n >_w}. \tag{28}$$

In order to do any computation with a PCE series, we need to truncate it. First, we notice that if the series converges, then the size of each coefficient goes to 0 if we take the limit of any index to infinity. This means that for such convergent series we can ignore terms with order higher than some N. However, for a given problem it is not trivial to find which exactly this N is. Usually, this is done by trial and error, where we can calculate more terms until the size of the new terms is smaller than the precision we need.

For the computation of the coefficient, we will use a non-intrusive method. We start by truncating the series to an arbitrary order N, $\phi_n(\zeta) = \sum_{n=0}^{N} c_n P_n(\zeta)$ and assume that this is enough for the wanted precision. Then we observe that this is a linear relation with respect to c_n. So we generate $M > N$ instances of the random variable ζ, $\{\zeta_1, \zeta_2, ..., \zeta_M\}$. Then for every ζ_i we have the equation

$$\phi(\zeta_i) = \sum_{n=0}^{N} c_n P_n(\zeta_i). \tag{29}$$

Notice that $\phi(\zeta_i)$ and $P_n(\zeta_i)$ are just numbers and now we can compute the coefficients c_n by solving a linear regression. After that we compute $\sup_\zeta |c_n P_n(\zeta)|$ and if it is smaller than the precision we stop, otherwise we increase N and repeat the process.

IV. Results

The method proposed in this paper could be directly applied in bi-stable systems with ODE dynamics in different domains. Application examples include population logistic models, soil health ecology, etc. For other dynamical systems, the definition of a healthy equilibrium and unhealthy equilibrium may need to be adjusted. The limitation of proposed method is that it is not suitable for complex system with Partial Differential Equation (PDE) at this point in development.

A. Case study: mutualistic dynamics

We will apply the above method in the case of mutualistic dynamics among species in the plant-pollinator network. Here equation (30) tracks the abundance $x_i(t)$ of species i, following [11]. We set

$$f(x, B, C, K) = B + x(\frac{x}{C} - 1)(1 - \frac{x}{K}) \tag{30}$$

$$g(x, y, D, E, H) = \frac{xy}{D + Ex + Hy}, \tag{31}$$

where B, C, K, D, E and H are positive parameters. The first term on the right hand of equation(30) account for the incoming migration of species at a rate B from neighbouring ecosystems. The second term describes logistic growth with the Allee effect C, according to which for low abundance ($x < C$), the system features negative growth [36]. The third term describes the system carrying capability K, according to which for high abundance ($x > K$), the system features negative growth [37]. Equation (31) describes mutualistic interactions, captured by a response function that saturates for large x, y, indicating that y's positive contribution to x is bounded [11]. We assume that some of them are random variables that get different realization on each node. We set $\mathbf{E}[B] = 0.1$, $\mathbf{E}[C] = 1$, $\mathbf{E}[D] = 5$, $\mathbf{E}[K] = 5$, $E = 0.9$, $H = 0.1$. We moreover assume that the value of any parameter has to be within 10% its mean, so we have $B = \mathbf{E}[B](1 + 0.1U)$, $C = \mathbf{E}[C](1 + 0.1U)$ and so on, where U a random variable uniform in $[-1, 1]$.

We define auxiliary functions

$$\begin{aligned} \phi(x, U_1, U_2, U_3) &= f(x, \mathbf{E}[B](1 + 0.1U_1), \\ &\mathbf{E}[C](1 + 0.1U_2), \mathbf{E}[K](1 + 0.1U_3)) \end{aligned} \tag{32}$$

and

$$\varphi(x, U_4, U_5) = \frac{\mathbf{E}[M](1 + 0.1U_5)x^2}{\mathbf{E}[D](1 + 0.1U_4) + Ex + Hx}. \tag{33}$$

Then for the function f we define

$$\mu_{f(x)} := \iiint\limits_{[-1,1]^3} \frac{1}{8} \phi(x, U_1, U_2, U_3)\, dU_1\, dU_2\, dU_3 \tag{34}$$

and

$$\delta^2_{f(x)} := \iiint\limits_{[-1,1]^3} \frac{1}{8} (\phi(x, U_1, U_2, U_3)^2 - \mu^2_{f(x)})\, dU_1\, dU_2\, dU_3. \tag{35}$$

Similarly for g we define

$$\mu_{g(x)} := \iint\limits_{[-1,1]^2} \frac{1}{4} \varphi(x, U_4, U_5)\, dU_4\, dU_5 \tag{36}$$

and

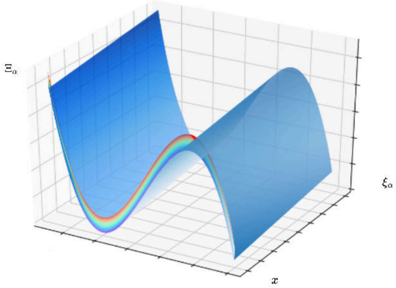
$$\delta^2_{g(x)} := \iint\limits_{[-1,1]^2} \frac{1}{16} (\varphi(x, U_4, U_5)^2 - \mu^2_{g(x)})\, dU_4\, dU_5. \tag{37}$$

According to the above method, we can get a realisation of $\Xi_\alpha(x) = \mu_{f(x)} + \frac{m}{n}\mu_{g(x)} + \sqrt{\frac{1}{n}\delta^2_{f(x)} + \frac{m}{n^2}\delta^2_{g(x)}}\zeta_\alpha$. The figure of the function $\Xi_\alpha(x)$ is shown in Figure 4 when ζ_α has different values.

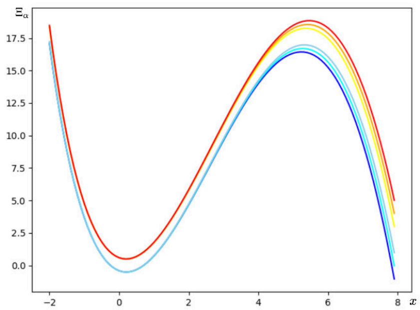
So we can see that every realisation of $\Xi(x)$ has the shape described in Figure 2. We can then find the smallest positive root ρ of $\Xi^{'}(x)$, then use PCE to approximate $\tau(\zeta)$.

B. Convergence test of PCE

Since ζ obeys Gaussian distribution, we choose Hermite polynomial as the orthogonal basis. We truncate the series to arbitrary orders N from 2 to 5 shown in Figure 5. Increasing the order (N) of the polynomial improves the convergence of the function. However, increasing the order of the polynomial means that a substantially higher number of simulations is required. Therefore, a compromise between accuracy and the required computational time is necessary.



(a) graph of function Ξ_α



(b) ζ_α has different values

Fig. 4. (a) graph of function $\Xi_\alpha(x)$ (b) Graph of function $\Xi_\alpha(x)$ projects to XZ plane. When ζ_α has different values, graphs of function $\Xi_\alpha(x)$ are different and the smallest positive root ρ are different. Whether the system is resilient could be estimated through the figure.

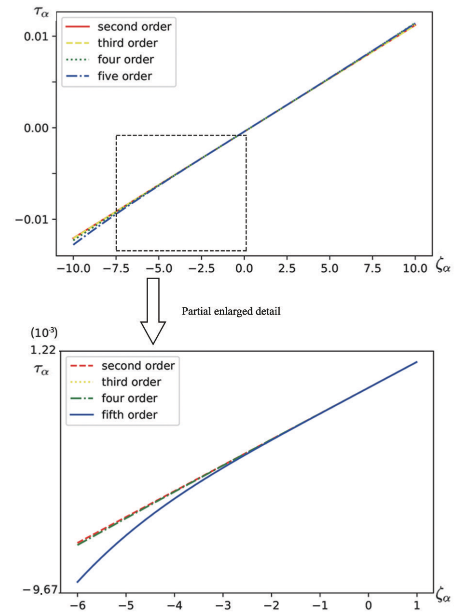


Fig. 5. Approximate $\tau(\zeta)$ by Hermite Polynomials. We truncate the series of polynomial to arbitrary orders N from 2 to 5 and estimate the smallest value of $\Xi_\alpha(x)$ when ζ_α has different values. In order to show the difference between estimation when the order N has different value, we enlarge the partial details of the above figure.

Reference to the graph in Figure 5, it is impossible to infer which order of N yields sufficient convergence of the PCE process. According to PCE in Figure 5, we can get the PDF with different truncation order in Figure 6. We can easily find the difference among different order especially $N = 2$. In order to estimate the probability of resilience, we obtain a graph of Cumulative Distribution Function (CDF) with different truncation in Figure 7. It can be seen that the results for $N = 3, N = 4, N = 5$ almost overlap while there is significant difference for $N = 2$ in comparison to $N = 3$. When $N = 2$, the result is 0.3576. The results are respectively 0.357182, 0.357134 and 0.35707 for $N = 3, N = 4, N = 5$.

Therefore, $N = 3$ can be considered as the appropriate choice for the polynomial order since choosing higher order polynomials substantially increases the required simulation time with only minor effects on improving the accuracy of the results.

C. Analysis

In order to know how topology of network influence resilience of the system, we need to do parameter sensitivity analysis of the system, such as weight of edges. In Figure 8(a), we can see that probability of resilience is correlated to the weight of the system. Strong connectivity promotes resilience since the effect of perturbation are eliminated through inputs from the broader system. In the mutualistic system, the first term on the right-hand side of equation (30) accounts for the incoming migration at a rate B from neighbour ecosystems. In Figure 8(b), we can see that the probability of resilience is positively correlated to the parameter B, which means that

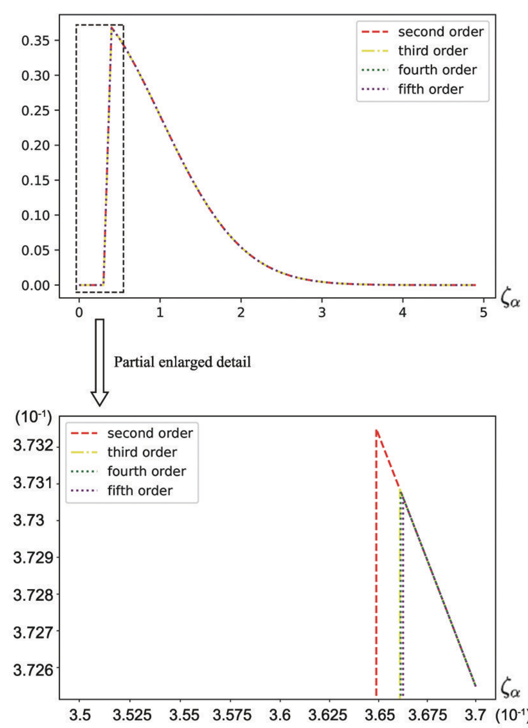


Fig. 6. According to the PCE of $\tau(\zeta)$, we can get the PDF of resilience of the system. We truncate the series of polynomial to arbitrary orders N from 2 to 5 and get the PDF of resilience of the system. In order to show the difference between estimation when the order N has different value, we enlarge the partial details of the above figure.

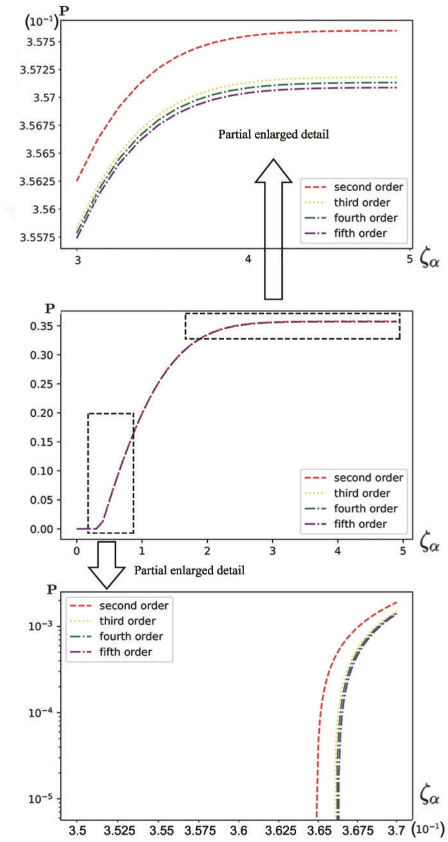
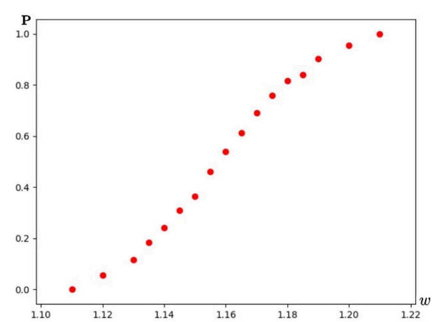
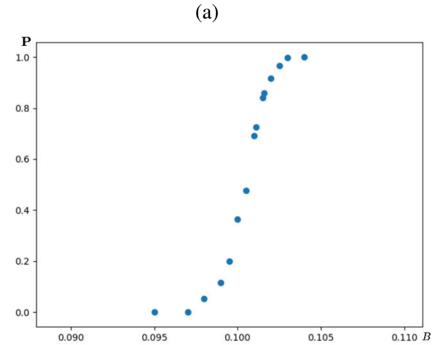


Fig. 7. Get CDF of resilience of system by PDF. When the order N has different value, the probability of resilience estimated by PCE is different. And it is clearly show that the results for $N = 3, N = 4, N = 5$ almost overlap while it is significant different for $N = 2$ in comparison to $N = 3, 4, 5$.



(a)



(b)

Fig. 8. (a) shows that probability of resilience is positive correlated to the weight of network. (b) shows that probability of resilience is positive correlated to the parameter B.

the increase of incoming migration from neighbour ecosystems could make the system more possible to be resilient. This is because incoming migration from neighbour ecosystems could help the abundance of species recover from perturbation. To show the advantage of PCE, Monte Carlo Method is used to estimate the uncertainty and we compare the results of Monte Carlo Method and PCE. When the sample size is larger than 1000, Monte Carlo Method can achieve an accuracy result [38]. In Figure (9), it shows the results when we use different sample size from 1000 to 10000. We know that for Monte Carlo Method, the accuracy of the result increases with the sample size. The probability is 0.3570 when the sample size is 10000. The computational cost increases rapidly with the increasing of sample size. We can see that PCE could achieve a very approaching result for $N = 3$. Therefore, comparing with Monte Carlo Method, PCE could achieve an approaching result with much more computationally efficient.

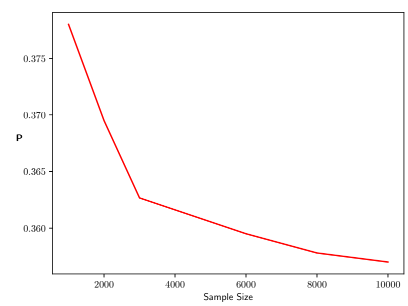


Fig. 9. Probability of resilience estimated by Monte Carlo Method. The sample size is from 1000 to 10000. It is clearly show that convergence of Monte Carlo Method is slow.

V. Conclusion and Future Work

Currently, we do not understand how to estimate the resilience of dynamic networked systems with multiple model parameter uncertainty. In this paper, we built a mean-field informed Polynomial Chaos Expansion (PCE) model to quantify the uncertainty for a wide range of uncertainty distributions. This approach can effectively estimate the resilience behaviour of an arbitrarily large networked system and analyze the effect of both topological and dynamical parameters on the system. The current research has developed the framework to analysis the relationship between macroscopic dynamics, like network-level resilience and network topology. However, we still do not understand the effect of mesoscopic topology, like the community structure of some components, on the local and global dynamics, even though these components with different mesoscopic topology may share the same network-level dynamics. Therefore, in the future, we will develop multi-resolution algorithms to achieve local to global resilience prediction.

.

References

[1] R. Cohen, K. Erez, D. Ben-Avraham, and S. Havlin, "Resilience of the internet to random breakdowns," *Physical review letters*, vol. 85, no. 21, p. 4626, 2000.

[2] R. V. Sole and M. Montoya, "Complexity and fragility in ecological networks," *Proceedings of the Royal Society of London. Series B: Biological Sciences*, vol. 268, no. 1480, pp. 2039–2045, 2001.

[3] G. Moutsinas and W. Guo, "Probabilistic stability of traffic load balancing on wireless complex networks," *IEEE Systems Journal*, 2019.

[4] R. Arghandeh, A. Von Meier, L. Mehrmanesh, and L. Mili, "On the definition of cyber-physical resilience in power systems," *Renewable and Sustainable Energy Reviews*, vol. 58, pp. 1060–1069, 2016.

[5] C. N. Kaiser-Bunbury, J. Mougal, A. E. Whittington, T. Valentin, R. Gabriel, J. M. Olesen, and N. Blüthgen, "Ecosystem restoration strengthens pollination network resilience and function," *Nature*, vol. 542, no. 7640, pp. 223–227, 2017.

[6] L. Fisher, "More than 70 ways to show resilience," *Nature*, vol. 518, no. 7537, pp. 35–35, 2015.

[7] M. Lundqvist, A. Compte, and A. Lansner, "Bistable, irregular firing and population oscillations in a modular attractor memory network," *PLoS Comput Biol*, vol. 6, no. 6, p. e1000803, 2010.

[8] G. Aquino, W. Guo, and A. Wilson, "Nonlinear dynamic models of conflict via multiplexed interaction networks," *arXiv preprint arXiv:1909.12457*, 2019.

[9] L. Todman, F. Fraser, R. Corstanje, J. Harris, M. Pawlett, K. Ritz, and A. Whitmore, "Evidence for functional state transitions in intensively-managed soil ecosystems," *Scientific reports*, vol. 8, no. 1, pp. 1–10, 2018.

[10] R. Marsh, A. Yool, T. Lenton, M. Gulamali, N. Edwards, J. Shepherd, M. Krznaric, S. Newhouse, and S. Cox, "Bistability of the thermohaline circulation identified through comprehensive 2-parameter sweeps of an efficient climate model," *Climate Dynamics*, vol. 23, no. 7-8, pp. 761–777, 2004.

[11] J. Gao, B. Barzel, and A.-L. Barabási, "Universal resilience patterns in complex networks," *Nature*, vol. 530, no. 7590, pp. 307–312, 2016.

[12] E. D. Vugrin, D. E. Warren, M. A. Ehlen, and R. C. Camphouse, "A framework for assessing the resilience of infrastructure and economic systems," in *Sustainable and resilient critical infrastructure systems*. Springer, 2010, pp. 77–116.

[13] D. Henry and J. E. Ramirez-Marquez, "Generic metrics and quantitative approaches for system resilience as a function of time," *Reliability Engineering & System Safety*, vol. 99, pp. 114–122, 2012.

[14] R. Francis and B. Bekera, "A metric and frameworks for resilience analysis of engineered and infrastructure systems," *Reliability Engineering & System Safety*, vol. 121, pp. 90–103, 2014.

[15] S. N. Naghshbandi, L. Varga, A. Purvis, R. Mcwilliam, E. Minisci, M. Vasile, M. Troffaes, T. Sedighi, W. Guo, E. Manley, and D. H. Jones, "A review of methods to study resilience of complex engineering and engineered systems," *IEEE Access*, vol. 8, pp. 87 775–87 799, 2020.

[16] J. Liang, Y. Hu, G. Chen, and T. Zhou, "A universal indicator of critical state transitions in noisy complex networked systems," *Scientific reports*, vol. 7, p. 42857, 2017.

[17] G. Moutsinas and W. Guo, "Node-level resilience loss in dynamic complex networks," *Scientific Reports*, vol. 10, no. 1, pp. 1–12, 2020.

[18] M. Kaern, T. C. Elston, W. J. Blake, and J. J. Collins, "Stochasticity in gene expression: from theories to phenotypes," *Nature Reviews Genetics*, vol. 6, no. 6, pp. 451–464, 2005.

[19] H. N. Najm, "Uncertainty quantification and polynomial chaos techniques in computational fluid dynamics," *Annual review of fluid mechanics*, vol. 41, pp. 35–52, 2009.

[20] S. Fernández-González, M. Martín, A. Merino, J. Sánchez, and F. Valero, "Uncertainty quantification and predictability of wind speed over the iberian peninsula," *Journal of Geophysical Research: Atmospheres*, vol. 122, no. 7, pp. 3877–3890, 2017.

[21] T. Tang and T. Zhou, "Recent developments in high order numerical methods for uncertainty quantification," *Scientia Sinica Mathematica*, vol. 45, no. 7, pp. 891–928, 2015.

[22] G. Fishman, *Monte Carlo: concepts, algorithms, and applications*. Springer Science & Business Media, 2013.

[23] J. A. Paulson, E. A. Buehler, and A. Mesbah, "Arbitrary polynomial chaos for uncertainty propagation of correlated random variables in dynamic systems," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 3548–3553, 2017.

[24] C. Zhao, S. Xie, X. Chen, M. P. Jensen, and M. Dunn, "Quantifying uncertainties of cloud microphysical property retrievals with a perturbation method," *Journal of Geophysical Research: Atmospheres*, vol. 119, no. 9, pp. 5375–5385, 2014.

[25] D. Zhang, *Stochastic methods for flow in porous media: coping with uncertainties*. Elsevier, 2001.

[26] P. Green, K. Worden, K. Atallah, and N. Sims, "The benefits of duffing-type nonlinearities and electrical optimisation of a mono-stable energy harvester under white gaussian excitations," *Journal of Sound and Vibration*, vol. 331, no. 20, pp. 4504–4517, 2012.

[27] N. Wiener, "The homogeneous chaos," *American Journal of Mathematics*, vol. 60, no. 4, pp. 897–936, 1938.

[28] S. Hosder, R. Walters, and R. Perez, "A non-intrusive polynomial chaos method for uncertainty propagation in cfd simulations," in *44th AIAA aerospace sciences meeting and exhibit*, 2006, p. 891.

[29] M. Abbaszadeh, G. Moutsinas, P. J. Thomas, and W. Guo, "Uncertainty quantification in molecular signals using polynomial chaos expansion," *IEEE Transactions on Molecular, Biological and Multi-Scale Communications*, vol. 4, no. 4, pp. 248–256, 2018.

[30] R. M. May, "Thresholds and breakpoints in ecosystems with a multiplicity of stable states," *Nature*, vol. 269, no. 5628, pp. 471–477, 1977.

[31] A. M. Lyapunov, "The general problem of the stability of motion," *International journal of control*, vol. 55, no. 3, pp. 531–534, 1992.

[32] R. M. May, S. A. Levin, and G. Sugihara, "Ecology for bankers," *Nature*, vol. 451, no. 7181, pp. 893–894, 2008.

[33] M. Scheffer, S. R. Carpenter, T. M. Lenton, J. Bascompte, W. Brock, V. Dakos, J. Van de Koppel, I. A. Van de Leemput, S. A. Levin, E. H. Van Nes *et al.*, "Anticipating critical transitions," *science*, vol. 338, no. 6105, pp. 344–348, 2012.

[34] J. N. Holland, D. L. DeAngelis, and J. L. Bronstein, "Population dynamics and mutualism: functional responses of benefits and costs," *The American Naturalist*, vol. 159, no. 3, pp. 231–244, 2002.

[35] D. Xiu and G. E. Karniadakis, "The wiener–askey polynomial chaos for stochastic differential equations," *SIAM journal on scientific computing*, vol. 24, no. 2, pp. 619–644, 2002.

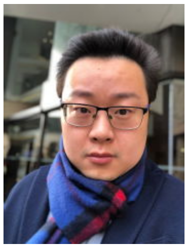
[36] W. C. Allee, O. Park, A. E. Emerson, T. Park, K. P. Schmidt *et al.*, "Principles of animal ecology," Saunders Company Philadelphia, PA, Tech. Rep., 1949.

[37] C. Hui, "Carrying capacity, population equilibrium, and environment's maximal load," *Ecological Modelling*, vol. 192, no. 1-2, pp. 317–320, 2006.

[38] A. Matala, "Sample size requirement for monte carlo simulations using latin hypercube sampling," *Helsinki University of Technology, Department of Engineering Physics and Mathematics, Systems Analysis Laboratory*, 2008.

Giannis Moutsinas received the degree in physics (with a major in mathematical physics) from the Aristotle University of Thessaloniki in 2008, the M.Sc. degree in mathematics from Utrecht University in 2011, and the Ph.D. degree in mathematics from the University of Warwick in 2016, studying the asymptotic behavior of dynamical systems. Currently, he is a Lecturer with the School of Computing, Electronics and Mathematics, Coventry University. His research interests include dynamical systems, complexity theory, and uncertainty quantification.

Mengbang Zou received his M.Sc degree from Huazhong University of Science and Technology in Mechanical Science and Engineering in 2019. He is now a Ph.D. student in Aerospace School of Cranfield University. He is interested in complex system with nonlinear dynamics and complex network theory. His current research is supported by China Scholarship Council (CSC).



Weisi Guo (S07, M11, SM17) received his MEng, MA, and Ph.D. degrees from the University of Cambridge. He is Chair Professor of Human Machine Intelligence at Cranfield University. He has published over 170 papers and is PI on over 12 research projects from EPSRC, Royal Society, EC H2020, and InnovateUK. His research has won him several international awards (IET Innovation 15, Bell Labs Prize Finalist 14 and Semi-Finalist 16 and 19). He is a Turing Fellow at the Alan Turing Institute and Fellow of Royal Statistical Society.

2020-11-24

Uncertainty of resilience in complex networks with nonlinear dynamics

Moutsinas, Giannis

IEEE