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Local assortativity affects the synchronizability of scale-free network

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***Abstract*—Synchronization is critical for system-level behaviour in physical, chemical, biological, and social systems. Empirical evidence has shown that the network topology strongly impacts the synchronizability of the system, and the analysis of their relationship remains an open challenge. We know that the eigenvalue distribution determines a network's synchronizability, but analytical expressions that connect network topology and all relevant eigenvalues (e.g., the extreme values) remain elusive.**

Here, we accurately determine its synchronizability by proposing an analytical method to estimate the extreme eigenvalues using perturbation theory. Our analytical method exposes the role that global and local topology combine to influence synchronizability. We show that the smallest non-zero eigenvalue $\lambda^{(2)}$ which determines synchronizability is estimated by the smallest degree augmented by the inverse degree difference in the least connected nodes. From this, we can conclude that there exists a clear negative relationship between $\lambda^{(2)}$ and the local assortativity of nodes with the smallest degree value. We validate the accuracy of our framework within the setting of a Scale-free (SF) network and can be driven by commonly used ordinary differential equations (ODEs) (e.g., 3-dimensional Rosler dynamics or Hindmarsh-Rose (HR) neuronal circuit). From the results, we demonstrate that the synchronizability of the network can be tuned by rewiring the connections of these particular nodes while maintaining the general degree profile of the network.

***Index Terms*—complex network; synchronizability; network topology; local assortativity; perturbation theory**

I. Introduction

SYNCHRONIZATION, as a collective phenomenon of dynamically coupling units, generally exists in different fields such as power grids [1], wireless communication networks [2], neural networks [3], etc. Realizing that the network topology of the system plays an important role in system's behaviors, the relationship between the network topology and synchronizability has attracted a lot of attention in recent years [4]–[8]. According to intuitive experience, some network topology characteristics are proposed as indicators of synchronizability, such as betweenness centrality [5], the correlation of degrees [9], etc. One problem needs to be pointed out is that when analysing the relationship between synchronizability and topology characteristics, some parameters like number of nodes N of the network, rewiring probability p of SW (small-world) networks need to be adjusted, which would cause other network topology characteristics changing, such as average distance, clustering coefficients, etc. The direct relationship between synchronizability and a given topology characteristic is not clear when other network topology characteristics keep varying. Besides, some topology characteristics provide indicators of synchronization in a network class but fail in other network classes [8]. The previous simulation experiments could reveal the relationship between synchronizability and network topology characteristics in some situation, but are far from clearly explaining it or mathematically abstracting it. It is important for researchers to uncover the behavior of empirical phenomena through experiments and data analysis. More importantly, we need to develop theories that abstract such behavior mathematically and explain the mechanism behind these phenomena [10]. The master stability function relates the global synchronizability to the spectral properties of the Laplacian matrix of the network, which provides the objective criterion for synchronizability [11]. For determined self-dynamics function and coupling dynamics function, the global synchronizability of the network is determined by the spectral properties of the network. Based on this analysis framework, the analysis of synchronizability of complex network could be converted to analyze the bounds of the extreme eigenvalues [12] [13]. There are some main results in previous work for different network models. Here, we mainly discuss the SF network.

In a scale-free (SF) network with large minimum degree or random enough network, the extreme eigenvalues can be bounded by the mean degree, minimum degree, and maximum degree [14]. Perturbation analysis of spectra of SF networks shows that the maximum eigenvalue is approximately equal to the maximum degree [15], but the smallest nonzero eigenvalue $\lambda^{(2)}$ is ensemble averageable by itself. The difficulty in approximating $\lambda^{(2)}$ by perturbation theory is the degeneration of eigenvectors which is caused by the existence of a large number of nodes with the smallest degree. To solve this problem, degenerate perturbation theory is used to estimate the eigenvalues and location of eigenvectors of a random network [16]. Although degenerate perturbation theory can accurately estimate eigenvalues, it requires the global information of the network and the introduction of new eigenvectors several times to solve the degeneration at the first-order perturbation [16]. This will increase the computation complexity in estimation. Besides, it is difficult for us to get any knowledge about the relationship between extreme eigenvalues and network topology characteristics through the complicated process of calculation. Degenerate perturbation theory is an effective method to estimate eigenvalues, but it is difficult to get an analytic equation to help us understand the synchronizability

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TABLE I
METHODS TO ANALYZE THE SYNCHRONIZABILITY AND NETWORK TOPOLOGY

	Method	Advantage	Limitation	Ref.
Discover structure characteristic as an indicator of synchronizability	Mainly use numerical experiments to discover the relationship between synchronizability and network structure characteristic.	Numerical experiment is easy to use and reveal the effect of network structure on synchronizability in some extent.	Numerical method is far from explaining the relationship between network topology and synchronizability; it is not clear when the indicator valid or invalid.	[4]–[9]
Analyze synchronizability by estimating the bounds of extreme eigenvalues	Use graph theory to estimate the bounds of extreme eigenvalues to analyze the synchronizability.	It is a solid mathematic method which can explain why and how network topology affects synchronizability.	In some situation, the difference between upper bound and lower bound estimation is very large. So, it is difficult to determine whether the extreme eigenvalues belong to the stable region.	[12]–[14]
	Perturbation theory to estimate extreme eigenvalues.	It can accurately estimate the largest eigenvalues in some specific network.	It is difficult to estimate the smallest none-zero eigenvalue due to the degeneration of eigenvector caused by the large number of nodes with the same degree.	[15]–[17]
	Specific perturbation theory with link removal method used in this paper.	It can estimate the smallest none-zero eigenvalue by avoiding the degeneration of eigenvector and reveal how local assortativity affects the synchronizability. Also, it provides a strategy to control the synchronizability of the network without changing the degree of nodes.	This method will lose some accuracy in networks with homogeneous degree distribution.	Method in this paper

of the network by this method. Therefore, how to avoid the degeneration of eigenvectors and get a clear analytic equation when estimating the smallest non-zero eigenvalue is necessary. To solve this problem, a link removal method is used to avoid the degeneration of eigenvectors. We prove that the link removal of the node with smallest degree has little effect on the smallest non-zero eigenvalue of the network. The new network after link removal only has one node with smallest degree, which means that the non-degenerate perturbation theory can be used to estimate the smallest non-zero eigenvalue of the new network. The non-degenerate perturbation theory only requires the local information of smallest nodes (node of the minimum degree) and the calculation complexity is much lower compared with the degenerate perturbation theory.

The contribution of this paper is that we propose an analytic framework to get a clear analytic equation when estimating the smallest nonzero eigenvalue of the network (the analytic framework is shown in Fig. 1, and our method is compared with previous methods in Table (I)). There are some advantages compared with previous methods. 1) Our estimation method is more accurate than the graph theory methods, which only provide the bound of extreme eigenvalues. 2) Traditional non-degenerate perturbation theory cannot directly be used to estimate the smallest nonzero eigenvalue due to the degeneration of eigenvectors. We solve this problem by the link removal method and get an analytic estimation expression, which points out how the network topology affects the synchronizability of the network. The analytic expression shows that the smallest non-zero eigenvalue $\lambda^{(2)}$ is mainly determined by the minimum degree of the network and the connection of the smallest nodes in a SF network. According to the analytic expression, there exists a relationship between $\lambda^{(2)}$ and the local assortativity of these nodes. Besides, the analytic expression instructs us how to strengthen or weaken the synchronizability of the network by reconnecting links among nodes with prescribed degree profile. The assumption of this paper is that the network used in this paper is SF network with large minimum degree, and the coupling nodes in the network have identical dynamics.

This paper is organized as follows. In Sec. II, we introduce the general dynamics model for a networked system and the criteria to characterize synchronizability provided by the master stability function method (MSF). In Sec. III, we propose an analytical method to estimate extreme eigenvalues in SF network by perturbation theory and analyze the relationship between $\lambda^{(2)}$ and local assortativity. In Sec. IV, Rossler system in a SF network and the Hindmarsh-Rose (HR) neuronal in a brain neuron network generated by real data are used to verify our theory. At last, we make a conclusion and discuss the future direction.

II. NETWORK SYNCHRONIZATION

Many infrastructures, social science, or ecology examples amongst same equipment, social atoms or species to have the same dynamic model. For example, in the study of synchronization problem of opinion formation in social networks, the participants (agents) are assumed to have identical dynamics with different initial states [18]. Some types of synchronization of bursting neurons are thought to play a key role in Parkinson's disease, essential tremor, and epilepsies [19]. [20] studies the synchronization of bursting neurons with identical dynamics in a Scale-free neural network. Consider an undirected (or bi-directed) network with N coupled identical nodes (symbols used in this paper are shown in Table (II)).

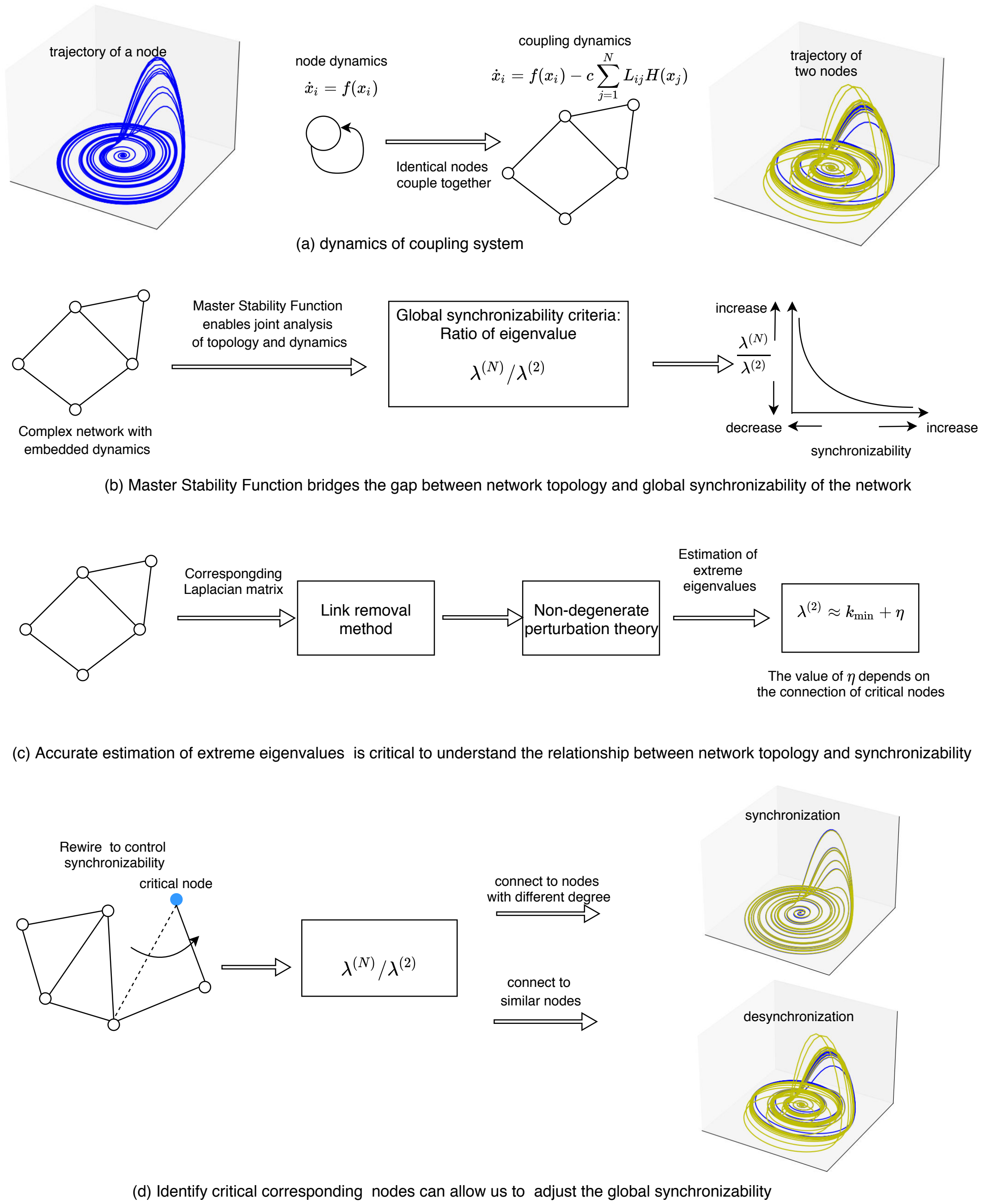


Fig. 1. This figure shows our proposed analysis framework for global synchronizability. (a) describes the dynamics of a node and the dynamics of coupled nodes. (b) shows that Master Stability Function provides us a criterion of global synchronizability, the ratio of extreme eigenvalues. (c) uses non-degenerate perturbation theory to estimate extreme eigenvalues. (d) proposes a strategy to control global synchronizability by rewiring the connections of critical nodes.

The dynamics of each node is described by

$$\dot{x_i} = f(x_i) - c\sum_{j=1}^{N} L_{ij}H(x_j), i = 1, 2, 3, ...N, \tag{1}$$

where $x_i = (x_i^{(1)}, x_i^{(2)}, ..., x_i^{(n)}) \in \mathbb{R}^n$ is the state vector of node i, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ controls self dynamics of node i, $c > 0$ is the coupling strength, $H(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is the inner coupling function. $\mathbf{L}$ is the Laplacian matrix of the network. $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{A}$ is the adjacent matrix and $\mathbf{D}$ is the diagonal matrix of degrees. $D_{ij} = \sum_{j=1}^{N} A_{ij}$ if $i = j$. Otherwise, $D_{ij} = 0$. If an edge exists between node i and node j, $A_{ij} = A_{ji} = 1$. Otherwise, $A_{ij} = 0$. The matrix $\mathbf{L}$ satisfies $L_{ii} = -\sum_{j=1, j\neq i}^{N} L_{ij}, i = 1, 2, ..., N$. If the graph is connected, then $\mathbf{L}$ is irreducible. Zero is an eigenvalue of $\mathbf{L}$ with multiplicity 1 and all other eigenvalues are strictly positive, denoted by

$$0 = \lambda^{(1)} < \lambda^{(2)} \leq \lambda^{(3)} \leq ... \leq \lambda^{(N)}. \tag{2}$$

The nodes are labeled in increasing order of their degrees k_i, such that $k_{\min} = k_1 \leq k_2 \leq \cdots \leq k_N = k_{\max}$. The average degree of the network is $< k >= \sum_{i=1}^{N} k_i/N$.

The system reaches the state of synchronization if $x_1(t) =$

$x_2(t) = ... = x_N(t) = s(t)$ when $t \to \infty$, where $s(t)$ can be an equilibrium point, a periodic orbit, or a chaotic attractor [21]. When the system reaches the synchronized state, a crucial question is whether the synchronization manifold is stable in the presence of small perturbation δx_i [12]. If the system can maintain the synchronized state with the presence of perturbation, then the synchronized state is stable. Otherwise, the state is unstable.

The stability of the synchronized manifold $x_1 = x_2 = ... = x_N$ can be determined by the master stability equation [11]

$$\dot{\zeta} = [Df(s) + \gamma DH(s)]\zeta, \tag{3}$$

where ζ is the collection of variations and $\zeta = (\zeta_1, \zeta_2, ..., \zeta_N)$. ζ_i is the variation on the i-th node. $Df(s)$ and $DH(s)$ are the Jacobian matrix of functions f and H at s. Then we need to calculate the largest Lyapunov exponent $\Omega_{\max}$. $\Omega_{\max}$ is a function of γ, where $\gamma = -c\lambda^{(i)}$. The evolution of small ζ is described on average as $\|\zeta(t)\| \sim e^{\Omega_{\max}(\gamma)t}$, and the state is stable with $\|\zeta(t)\| \to 0$, if $\Omega_{\max}(\gamma) < 0$. $\Omega_{\max}(\gamma)$ can be calculated by $\Omega_{\max} = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\zeta_t\|}{\|\zeta_0\|}$ [12].

The region S of γ which makes $\Omega_{\max}$ negative is called the synchronized region. If the eigenvalue $\lambda^{(i)}$ of matrix $\mathbf{L}$ satisfies

$$-c\lambda^{(i)} \in S, i = 2, 3, \cdots N, \tag{4}$$

then (1) is asymptotically stable. The stable synchronized region S could be an unbounded region $(-\infty, \gamma_1)$, a bounded region (γ_1, γ_2), an empty set or a union of several subregions.

When the synchronized region $S = (\gamma_1, \gamma_2)$, where γ_1, γ_2 are both negative real numbers, the eigenvalues of matrix $\mathbf{L}$ need to satisfy

$$-c\lambda^{(N)} > \gamma_1, -c\lambda^{(2)} < \gamma_2 \tag{5}$$

TABLE II
LIST OF SYMBOLS USED IN THIS PAPER

Symbol	Describtion
x_i	state of node i
$f(\cdot)$	self-dynamics function
$H(\cdot)$	inner coupling function
c	coupling strength
$\mathbf{L}$	Laplacian matrix of the network
$\mathbf{D}$	diagonal matrix of degrees
$\mathbf{A}$	adjacent matrix
$< k >$	the average degree of the network
ζ_i	the variation on node i
γ	$\gamma = c * \lambda^{(i)}$
S	the synchronized region
$\Omega_{\max}$	the largest Lyapunov exponent
L_{ij}	element of matrix $\mathbf{L}$
A_{ij}	element of matrix $\mathbf{A}$
$\lambda^{(i)}$	eigenvalue of matrix $\mathbf{L}$
k_i	degree of node i
$Df(s)$	the Jacobian matrix of function f at s
$DH(s)$	the Jacobian matrix of function H at s
ϵ	the expansion parameter which tends to be small
$\vec{\xi}^{(i)}$	eigenvector corresponding to $\lambda^{(i)}$
$\xi_\alpha^{(i)}$	the α-th element of eigenvector $\vec{\xi}^{(i)}$
$\vec{\xi}^{(i)tr}$	the transpose of $\vec{\xi}^{(i)}$
δ_{ij}	Kronecker delta function
ρ_i	local assortativity of node i
r	assortativity of the network
θ_i	degree difference between node i and its neighbours

to make the synchronized region asymptotically stable. (5) can be written as

$$\frac{\lambda^{(N)}}{\lambda^{(2)}} < \frac{\gamma_1}{\gamma_2}. \tag{6}$$

The ratio of $\frac{\lambda^{(N)}}{\lambda^{(2)}}$ characterizes the synchronizability in this case. When the synchronized region $S = (-\infty, \gamma_1)$, the eigenvalues must satisfy $-c\lambda^{(2)} < \gamma_1$.

III. ANALYSE SYNCHRONIZABILITY BY PERTURBATION THEORY

Since the synchronizability of networks depends on the extreme eigenvalues $\lambda^{(2)}$ and $\lambda^{(N)}$, we use the non-degenerate perturbation theory to estimate $\lambda^{(N)}$. Similar perturbation methods are used in [15] [16] to estimate eigenvalues and eigenvectors[1].

First, we introduce the expansion parameter $\epsilon =< k >^{-1}$. The Laplacian matrix $\mathbf{L}$ could be rewritten as $\mathbf{L} = \mathbf{L}_0 + \epsilon\mathbf{L}_1$, where $\mathbf{L}_0 = \mathbf{D}$ and $\mathbf{L}_1 = - < k > \mathbf{A}$. $\vec{\xi}^{(i)} = \{\xi_1^{(i)}, \xi_2^{(i)} \cdots \xi_N^{(i)}\}^T$ represents the Laplacian eigenvector of the i-th mode and $\lambda^{(j)}$ is the corresponding eigenvalue. Since $\mathbf{L}$ is a real symmetric matrix, the eigenvectors can be orthonormalized to $\sum_{\alpha=1}^N \xi_\alpha^{(i)} \xi_\alpha^{(j)} = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta function. $\mathbf{L}_0$ is considered as an unperturbed matrix, and $\mathbf{L}_1$ is considered as a perturbation. We can get

$$\sum_{\beta=1}^N L_{\alpha\beta}\xi_\beta^{(i)} = \lambda^{(i)}\xi_\alpha^{(i)}. \tag{7}$$

We expand the eigenvector and eigenvalues with ϵ as

$$\begin{aligned} \vec{\xi}^{(i)} &= \vec{\xi}_0^{(i)} + \epsilon\vec{\xi}_1^{(i)} + \epsilon^2\vec{\xi}_2^{(i)} + \cdots, \\ \lambda^{(i)} &= \lambda_0^{(i)} + \epsilon\lambda_1^{(i)} + \epsilon^2\lambda_2^{(i)} + \cdots. \end{aligned} \tag{8}$$

We assume that the unperturbed eigenvectors are orthonormalized and the higher-order perturbation vectors are orthogonal to the unperturbed eigenvectors. According to (7) and (8), we can obtain the following equations up to $O(\epsilon^2)$[2]:

$$\begin{aligned} &(\mathbf{L}_0 - \lambda_0^{(i)})\vec{\xi}_0^{(i)} = 0, \\ &(\mathbf{L}_0 - \lambda_0^{(i)})\vec{\xi}_1^{(i)} = (\lambda_1^{(i)} - \mathbf{L}_1)\vec{\xi}_0^{(i)}, \\ &(\mathbf{L}_0 - \lambda_0^{(i)})\vec{\xi}_2^{(i)} = (\lambda_1^{(i)} - \mathbf{L}_1)\vec{\xi}_1^{(i)} + \lambda_2^{(i)}\vec{\xi}_0^{(i)}, \\ &(\mathbf{L}_0 - \lambda_0^{(i)})\vec{\xi}_3^{(i)} = (\lambda_1^{(i)} - \mathbf{L}_1)\vec{\xi}_2^{(i)} + \lambda_2^{(i)}\vec{\xi}_1^{(i)} + \lambda_3^{(i)}\vec{\xi}_0^{(i)}. \end{aligned} \tag{9}$$

It is easy to obtain

$$\begin{aligned} \lambda_0^{(i)} &= k_i, \\ \lambda_1^{(i)} &= - < k > A_{ii} = 0. \end{aligned} \tag{10}$$

Furthermore, it can be deduced

$$\begin{aligned} \lambda_2^{(i)} &= \vec{\xi}_0^{(i)tr}\mathbf{L}_1\vec{\xi}_1^{(i)}, \\ \lambda_3^{(i)} &= \vec{\xi}_0^{(i)tr}\mathbf{L}_1\vec{\xi}_2^{(i)}, \end{aligned} \tag{11}$$

[1] except the eigenvalue $\lambda^{(1)} = 0$ and its corresponding eigenvector, which have exceptional characteristics and is excluded from the analysis [16]

[2] the value of ϵ tends to be small and second order approximation is sufficient

where $\vec{\xi}_0^{(i)tr}$ is the transpose matrix of $\vec{\xi}_0^{(i)}$. The first-order and second-order corrections $\vec{\xi}_1^{(i)}, \vec{\xi}_2^{(i)}$ of the i-th vector can be obtained by

$$\begin{aligned} \vec{\xi}_1^{(i)} &= \sum_{j\neq i} \frac{\vec{\xi}_0^{(j)tr}\mathbf{L}_1\vec{\xi}_0^{(i)}}{\lambda_0^{(i)} - \lambda_0^{(j)}}\vec{\xi}_0^{(j)}, \\ \vec{\xi}_2^{(i)} &= \sum_{h\neq i} \frac{\vec{\xi}_0^{(h)tr}\mathbf{L}_1\vec{\xi}_1^{(i)}}{\lambda_0^{(i)} - \lambda_0^{(h)}}\vec{\xi}_0^{(h)}. \end{aligned} \tag{12}$$

(12) reveals that $\vec{\xi}_1^{(i)}$ is determined by the nodes connected to node i which are called the first-order neighbour nodes of node i. $\vec{\xi}_2^{(i)}$ is determined by the nodes connected to neighbour nodes of node i. These nodes are called the second-order neighbour nodes of node i. Furthermore, we can get

$$\lambda_2^{(i)} = \sum_{i\neq j} \frac{A_{ij}^2}{k_i - k_j}. \tag{13}$$

$\lambda_2^{(i)}$ is determined by the first-order neighbour nodes of node i. Also, (11) reveals that $\lambda_3^{(i)}$ is determined by the second-order neighbour nodes of node i. $\lambda_3^{(i)} = 0$, since the second-order neighbour nodes of node i are not directly connected with node i. In a similar way, we can deduce that $\lambda_3^{(i)}, \lambda_4^{(i)} ... = 0$. Therefore, according to our perturbation analysis, $\lambda^{(i)}$ is mainly determined by its first-order neighbour nodes and is almost not affected by other nodes.

So $\lambda^{(N)}$ could be estimated by perturbation expansion to the second order as

$$\lambda^{(N)} \simeq k_N + \sum_{j\neq N} \frac{A_{Nj}^2}{k_N - k_j}. \tag{14}$$

The second-order term can be expanded as $\sum_j (A_{Nj})^2(\frac{1}{k_N} + \frac{k_j}{k_N^2} + \cdots) = 1 + k_N^{\text{ave}}/k_N + \cdots$, where k_N^{ave} is the average degree of the nearest neighbours of node N. For large N, $k_N^{\text{ave}}/k_N \ll 1$ [15]. Therefore, $\lambda^{(N)} \simeq k_N + 1$, which means that $\lambda^{(N)}$ is mainly decided by $k_{\max}$. On the other hand, $\lambda^{(2)}$ should be estimated by degenerate theory [22], since several nodes may have the same node degree with $\lambda^{(2)}$ in the SF network. However, the difficulty is that some eigenvectors remain degenerate after several estimation steps. So, the eigenvectors and eigenvalues cannot be determined from (9). To solve this problem, we remove one link from the node with the lowest degree to get a new network in which $k_{\min} - 1 = k_1 < k_2 \leq \cdots \leq k_N = k_{\max}$. So the non-degenerate perturbation theory could be used to estimate $\lambda^{(2)}$ of the new network. If the effect of link removal on eigenvalues is very small, then we think that $\lambda^{(2)}$ of the new network extremely approaches to the original one. Therefore, we can obtain $\lambda^{(2)}$ of the original network by estimating $\lambda^{(2)}$ of the new network.

The link removal method is shown in Fig. 2. Perturbation theory can consider how matrix functions, such as eigenvalues or singular changes when the matrix is subject to perturbations [23]. Link removal is a kind of perturbation on the graph, which will slightly modify the elements of the Laplacian matrix as well as the eigenvalues. Therefore, it is reasonable for us to use perturbation theory to analyze the effect of link removal on Laplacian eigenvalues. The change of eigenvalues can be estimated by

$$(\mathbf{L} + \Delta\mathbf{L})(\vec{\xi}^{(i)} + \Delta\vec{\xi}^{(i)}) = (\lambda^{(i)} + \Delta\lambda^{(i)})(\vec{\xi}^{(i)} + \Delta\vec{\xi}^{(i)}), \tag{15}$$

where $\Delta\mathbf{L}, \Delta\vec{\xi}^{(i)}, \Delta\lambda^{(i)}$ represent the changes in $L, \vec{\xi}^{(i)}, \lambda^{(i)}$. Multiplying (15) by the transpose of $\vec{\xi}^{(i)}$, $\vec{\xi}^{(i)tr}$, we can get

$$\Delta\lambda^{(i)} = \frac{\vec{\xi}^{(i)tr}\Delta\mathbf{L}\vec{\xi}^{(i)} + \vec{\xi}^{(i)tr}\Delta\mathbf{L}\Delta\vec{\xi}^{(i)}}{\vec{\xi}^{(i)tr}\vec{\xi}^{(i)} + \vec{\xi}^{(i)tr}\Delta\vec{\xi}^{(i)}}. \tag{16}$$

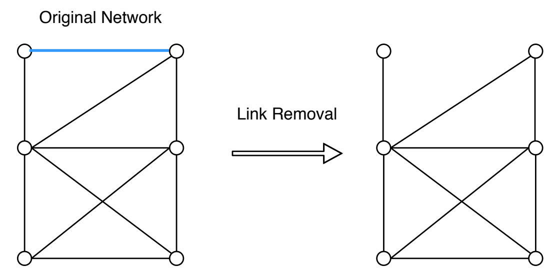


Fig. 2. The link removal method is removing the edge between the node with the smallest degree and its neighbour. We can get a network by applying the link removal method to the original network. The smallest non-zero eigenvalues of the original network and the new network are almost the same.

For a large complex network, it is reasonable to assume that the removal of only a link has small effects on the network as well as the eigenvector $\vec{\xi}^{(i)tr}$, which means that $\Delta\vec{\xi}^{(i)tr} \simeq 0$ [24] [25]. If the link between node k and node m is removed, then

$$\Delta\lambda^{(i)} \simeq \frac{2\xi_k^{(i)}\xi_m^{(i)} - \xi_k^{(i)}\xi_k^{(i)} - \xi_m^{(i)}\xi_m^{(i)}}{\vec{\xi}^{(i)tr}\vec{\xi}^{(i)}}. \tag{17}$$

Since $\vec{\xi}^{(i)tr}\vec{\xi}^{(i)} = 1$, $\Delta\lambda^{(i)} \simeq -(\xi_k^{(i)} - \xi_m^{(i)})^2$. Therefore, the perturbation of $\Delta\lambda^{(2)}$ mainly depends on the Fielder vector [26] (eigenvector corresponding to the smallest nonzero eigenvalue $\lambda^{(2)}$). The Fiedler vector could be obtained by minimizing the degree-adjusted Rayleigh quotient [26] [27]. It is not difficult to find that $(\xi_k^{(i)} - \xi_m^{(i)})^2 < 2$. Actually, $(\xi_k^{(i)} - \xi_m^{(i)})^2 \ll 1$, which means that $\Delta\lambda^{(2)} \ll \lambda^{(2)}$ (shown in Fig. 3). Therefore, we can ignore $\Delta\lambda^{(2)}$ and obtain $\lambda^{(2)}$ of the original network by estimating $\lambda^{(2)}$ of the new network according to

$$\lambda^{(2)} \simeq \min\left((k_i - 1) + \sum_{j\neq i} \frac{A_{ij}^2}{(k_i - 1) - k_j}\bigg|_{k_i = k_{\min}} \right), \tag{18}$$

where $k_{\min}$ is the minimum degree of the original network, A_{ij} represents the connections of node i.

In a given network, degree of nodes is determined, so $\lambda^{(2)}$ is mainly affected by $\sum_{i\neq j} \frac{A_{ij}^2}{(k_i-1)-k_j}$. We assume that there are z nodes with minimum degree $k_{\min}$ in the original network. Since these z nodes have their own different connections, $\sum_{i\neq j} \frac{A_{ij}^2}{(k_i-1)-k_j}$ varies according to different nodes. So $\lambda^{(2)}$ is determined by the node with the smallest $\sum_{i\neq j} \frac{A_{ij}^2}{(k_i-1)-k_j}$ of these z nodes. (18) shows that connections of nodes with smallest degree mainly affect $\lambda^{(2)}$, otherwise, nodes with large degree do not have much effects on $\lambda^{(2)}$. That is, the node with

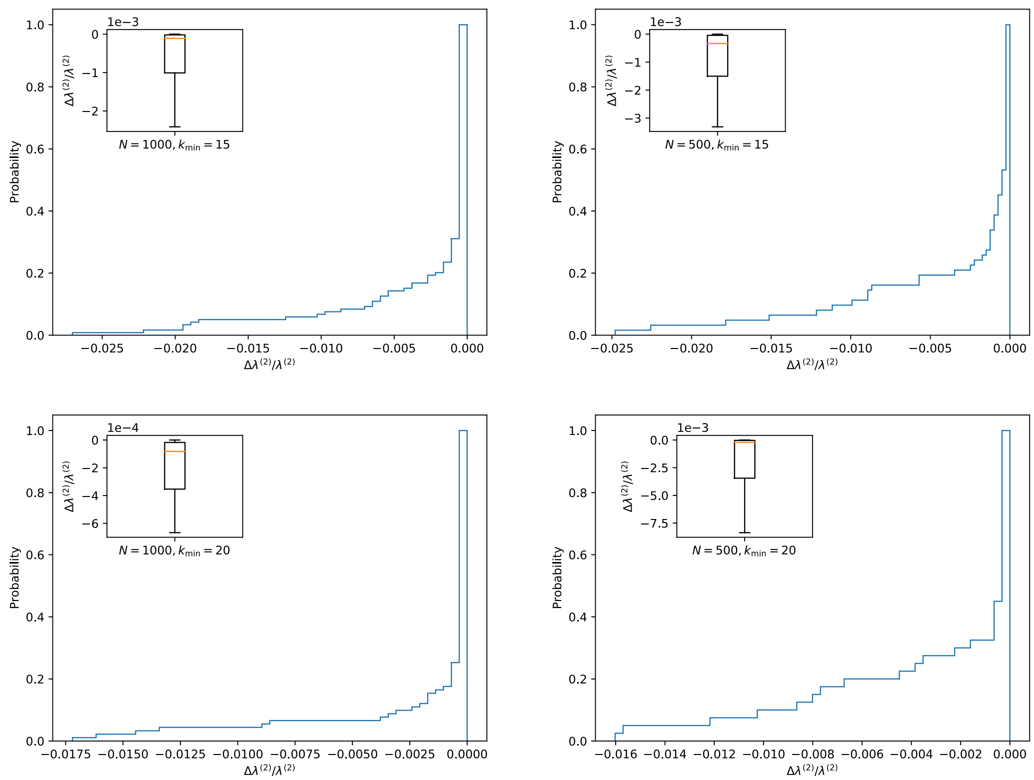


Fig. 3. These four figures demonstrate the cumulative distribution of $\Delta\lambda^{(2)}/\lambda^{(2)}$ of different networks. The box-plots show the extreme values and median of $\Delta\lambda^{(2)}/\lambda^{(2)}$ in different networks. From the distribution and extreme value of $\Delta\lambda^{(2)}/\lambda^{(2)}$, we can know that the link's removal of nodes with smallest degree almost does not affect $\lambda^{(2)}$ of the network, $\Delta\lambda^{(2)} \ll \lambda^{(2)}$.

the smallest degree connecting to nodes with similar degree will decrease $\lambda^{(2)}$. This implies that there exists a relationship between local assortativity and $\lambda^{(2)}$. Local assortativity is a property of a single node and indicates how similar a node is to its neighbours [28]. A lot of methods are proposed to calculate local assortativity. A simple method proposed in [29] to calculate local assortativity ρ_i is used in this paper:

$$\rho_i = \frac{r+1}{N} - \bar{\theta}_i, \tag{19}$$

where $\bar{\theta}_i = \theta_i / \sum_i^N \theta_i$, r is the assortativity of the network and θ_i is calculated by

$$\theta_i = \frac{1}{k_i} \sum_{i=1}^{N} A_{ij} \left| k_i - k_j \right| . \tag{20}$$

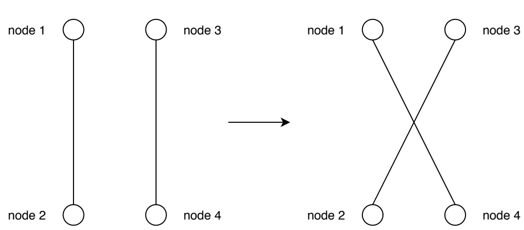
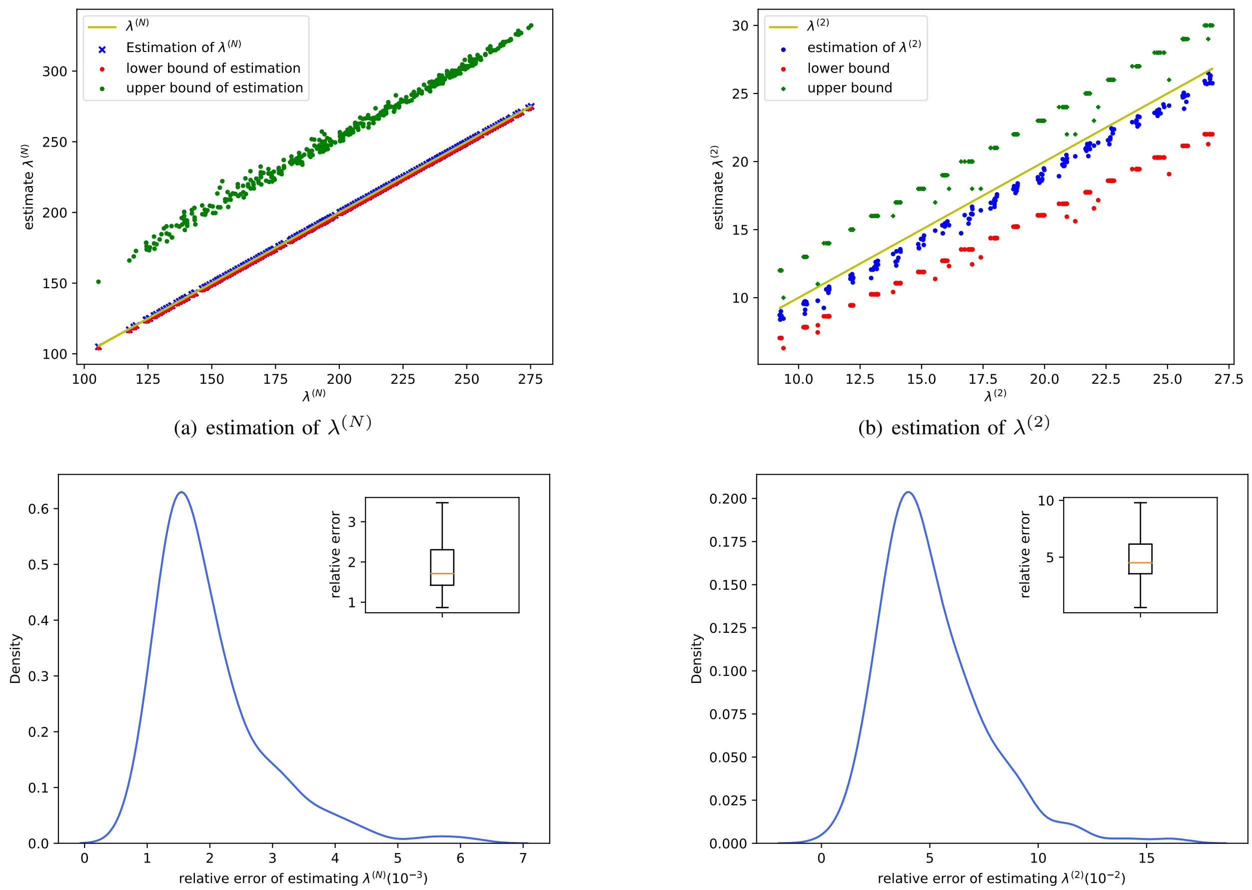


Fig. 4. Rewire connections between nodes. This method includes two steps: delete connections and add new connections. By this method, the neighbour of these four nodes have been changed, but their degree remains the same. So, we can use this method to adjust the network structure while maintaining the degree of each node.

In (19), we can see that ρ_i is determined by $\frac{r+1}{N}$ and $\bar{\theta}_i$. Generally speaking, similar connection (connection between similar degree nodes) will increase r. The effect of similar connection on $\bar{\theta}_i$ needs to be analyzed. In a large-scale SF network with N nodes, where N is a large number, we maintain the degree sequence of the network and only rewire the connections of nodes shown in Fig. 4. It is assumed that node 1 and node 4 have small degree, while node 2 and node 3 have large degree. $k_1 \simeq k_4 \ll k_2 \simeq k_3$. Therefore, the connection between node 1 and node 4 as well as connection between node 2 and node 3 could be seen as a similar connection. Since we analyse the local assortativity of the node with smallest degree, we assume $k_1 = k_{\min}$. Let $\theta_{\text{sum}} = \sum_i^N \theta_i$. The change of θ_{sum} from left connections to right connections in Fig. 4 is $\Delta\theta_{\text{sum}}$ and the change of θ_1 is $\Delta\theta_1$. In the left connections, $\bar{\theta_1} = \theta_1/\theta_{\text{sum}}$; in the right connections, $\bar{\theta_1'} = (\theta_1 + \Delta\theta_1)/(\theta_{\text{sum}} + \Delta\theta_{\text{sum}})$. Then

$$\begin{aligned} \Delta\bar{\theta_1} &= -\bar{\theta_1} + \bar{\theta_1'} = \frac{-\theta_1}{\theta_{\text{sum}}} + \frac{\theta_1 + \Delta\theta_1}{(\theta_{\text{sum}} + \Delta\theta_{\text{sum}})} \\ &= \frac{\theta_{\text{sum}}\Delta\theta_1 - \theta_1\Delta\theta_{\text{sum}}}{(\theta_{\text{sum}} + \Delta\theta_{\text{sum}})\theta_{\text{sum}}}; \end{aligned} \tag{21}$$

$$\Delta\theta_1 = \left| \frac{k_4 - k_1}{k_1} \right| - \left| \frac{k_2 - k_1}{k_1} \right| \simeq - \left| \frac{k_2 - k_1}{k_1} \right| < 0; \tag{22}$$



(a) estimation of $\lambda^{(N)}$

(b) estimation of $\lambda^{(2)}$

(c) estimation accuracy of $\lambda^{(N)}$

(d) estimation accuracy of $\lambda^{(2)}$

Fig. 5. (a) shows that the estimation of $\lambda^{(N)}$, k_N+1, is very close to the real value of $\lambda^{(N)}$. Therefore, k_N+1 is a good estimation of $\lambda^{(N)}$ in SF network. (b) shows the estimation of $\lambda^{(2)}$ in different SF netwoks by the method proposed in this paper. The red and blue scatter graphs represent the lower bound and upper bound estimation of $\lambda^{(2)}$ and $\lambda^{(N)}$ by previous method $k_{\max} \leq \lambda_N \leq k_{\max}(1+2/\sqrt{<k>})$ and $k_{\min}(1-2/\sqrt{<k>}) \leq \lambda_2 \leq k_{\min}$. The yellow line represents the real values of $\lambda^{(2)}$. We can see that the estimation of extreme eigenvalues by our method is very close to the real value. (c) (d) is separately the probability density function plot of relative error of estimating $\lambda^{(N)}$ and $\lambda^{(2)}$. The subplots in (c) and (d) are the boxplots of relative error. In most situations, the relative error of estimating $\lambda^{(N)}$ is less than 0.2% and the relative error of $\lambda^{(2)}$ is less than 5%

$$\Delta\theta_{\text{sum}} = \left(\left|\frac{k_4-k_1}{k_1}\right| + \left|\frac{k_4-k_1}{k_4}\right| + \left|\frac{k_3-k_2}{k_3}\right| + \left|\frac{k_3-k_2}{k_2}\right|\right) - \left(\left|\frac{k_2-k_1}{k_1}\right| + \left|\frac{k_2-k_1}{k_2}\right| + \left|\frac{k_3-k_4}{k_3}\right| + \left|\frac{k_3-k_4}{k_4}\right|\right) \simeq 0. \tag{23}$$

Since $\Delta\theta_{\text{sum}} > 0$ and $\Delta\theta_1 < 0$, we get $\Delta\bar{\theta_1} < 0$. The similar connection causes the increase of ρ_i and the decrease of $\lambda^{(2)}$, which means a negative relationship exists between local assortativity and $\lambda^{(2)}$.

Here, we add some discussion on how to extend our method to other types of networks such as star-coupled networks and small-world networks, and also discuss the limitation of our method in different types of networks. In a network, if the degree of the largest node is much larger than the average degree of the neighbours of this node $\sum_j (A_{Nj})^2(\frac{1}{k_N} + \frac{k_j}{k_N^2} + \cdots) = 1 + k_N^{\text{ave}}/k_N + \cdots \simeq 1$. So, in a Star-coupled network or SF network, $\lambda^{(N)} \simeq k_N + 1$ is a good estimation. While, in a small-world network, the degree of the largest node is not always much larger than the average degree of this node, $\lambda^{(N)} \simeq k_N+1$ is not accurate. But we can still use this method to determine the bound of $\lambda^{(N)}$ in small-world networks. Since $0 \leq \sum_{j\neq N} \frac{A_{Nj}^2}{k_N-k_j} \leq k_N, k_N \leq \lambda^{(N)} \leq 2k_N$. Small-world networks can be obtained by randomly rewiring links to a regular network where each node is connected to its $2M$ nearest neighbours. Here, we set the rewiring probability as p. If $2Mp \ll 1$, the small-world network is similar to the original regular network, which means that almost all nodes have the same degree. In this situation, the second-order estimation equation $\lambda_2^{(i)} = \sum_{i\neq j}(A_{ij}^2)/(k_i - k_j)$ is not accurate enough. This is because when k_j is close to k_i, the small perturbation on k_j will greatly affect the value of $A_{ij}^2/(k_i-k_j)$ and cause some inaccuracy. $\sum_{i\neq j}(A_{ij}^2)/(k_i-k_j)$ will enlarge the inaccuracy if a node connects to many similar nodes. However, we can still use our method to determine the bounds $0 < \lambda^{(N)} \lesssim k_{\min}$. If $2Mp$ is large enough, the small-world network has many shortcuts, which means that the degree of nodes is not as homogeneous as the regular network. In this situation, $\lambda^{(N)} \simeq k_N + 1$ and $\lambda^{(2)} \simeq \min\left((k_i - 1) + \sum_{j \neq i} \frac{A_{ij}^2}{(k_i-1)-k_j}\Big|_{k_i=k_{\min}}\right)$ are good estimation of extreme eigenvalues.

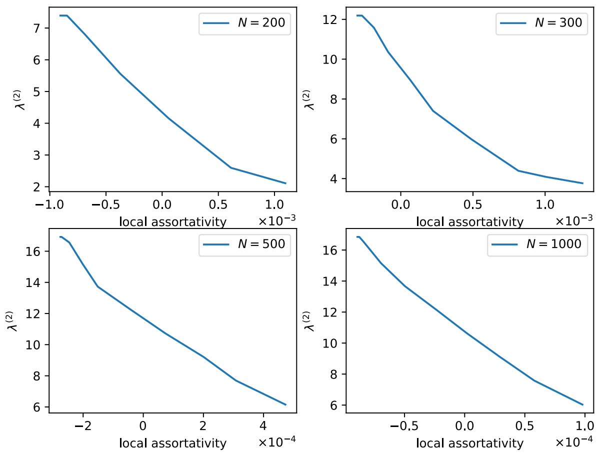


Fig. 6. Relationship between local assortativity and $\lambda^{(2)}$ in BA networks with different size. All of these figures show that $\lambda^{(2)}$ decreases as the increase of the local assortativity.

IV. Results and analysis

Here, we consider the Rossler system [30] for our nodes. The dynamic function of the i-th oscillator is

$$\begin{aligned} \dot{x_i}^{(1)} &= -(x_i^{(2)} + x_i^{(3)}) \\ \dot{x_i}^{(2)} &= x_i^{(1)} + a x_i^{(2)} \\ \dot{x_i}^{(3)} &= b + x_i^{(3)}(x_i^{(1)} - d). \end{aligned} \tag{24}$$

We assume that $a = 0.2, b = 0.2, d = 6.0$, the coupling strength $c = 0.03$. Also, nodes are coupled by $x^{(1)}$. The inner coupling function H is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{25}$$

According to the master stability function, the stability region is: $S = (-5.44, -0.199)$.

We consider different sizes of SF networks generated by Barabási–Albert (BA) model [31]. The estimation method proposed above will be used to estimate the extreme eigenvalues and the accuracy of the method will be verified by simulations. In Fig. 5, it shows that $k_N + 1$ is very close to the real value $\lambda^{(N)}$ in different SF network. Therefore, $k_N + 1$ is a good estimation of $\lambda^{(N)}$. Estimation of $\lambda^{(2)}$ by the proposed method is shown in Fig. 5. Compared with the previous estimation $k_{\min}(1 - 2/\sqrt{<k>}) \leq \lambda_2 \leq k_{\min}$ or $\lambda^{(2)} \simeq k_{\min}$ our estimation is closer to the real value of $\lambda^{(2)}$. To quantify the estimation accuracy, we calculate the relative error of estimation in different networks. The relative error is $\left|\lambda^{(i)} - \lambda^{(i)'}\right| / \lambda^{(i)}$, where $\lambda^{(i)'}$ is the estimation of the real value $\lambda^{(i)}$. The results are shown in Fig. 5.

To verify the relationship between local assortativity, the rewire connection method in Fig. 4 is applied in BA networks. Fig. 6 shows the relationship between local assortativity and $\lambda^{(2)}$ in different BA networks with $N = 200, 300, 500, 1000$. $\lambda^{(2)}$ decreases with the increase of local assortativity. Since the rewire connection method does not change the degree sequence of the network, $\lambda^{(N)} \simeq k_N + 1$ will not change according to the above analysis. Therefore, the ratio of $\frac{\lambda^{(N)}}{\lambda^{(2)}}$ will increase with the local assortativity, which indicates the decrease of synchronizability of the network.

Therefore, (18) provides us with a good strategy to control synchronizability while maintaining the degree sequence of the network. If the node with smallest degree connects to nodes with similar degree, $\lambda^{(2)}$ will decrease and this action will weaken the synchronizability. On the other hand, if the node with smallest degree connect to nodes with large degree, $\lambda^{(2)}$ will increase and the synchronizability will be strengthened. Here, a BA network with Rossler dynamics described in (24) will be used to verify the effectiveness of the strategy. The network of 100 nodes is grown by attaching new nodes with 10 edges that are preferent attached to existing nodes with high degree. In this network $G1$, $\lambda^{(N)} = 45.6$ and $\lambda^{(2)} = 7.6$. $-c\lambda^{(2)} = -0.228 \in S, -c\lambda^{(N)} = -1.368 \in S$. Then, for every eigenvalue of this network, $-c\lambda^{(i)} \in S$. According to the above analysis before, every node of this network will achieve synchronization (The three pictures at the bottom of Fig. 7 show the synchronization process of nodes). Then, according to the above controlling strategy, we rewire connections of the node with similar nodes and get a new network $G2$. The extreme eigenvalues of $G2$, $\lambda^{(2)} = 2.6$ and $\lambda^{(N)} = 45.6$. Then, $-c\lambda^{(N)} \in S$ but $-c\lambda^{(2)} \notin S$. Nodes in $G2$ cannot achieve global synchronization (the top three pictures in Fig. 7 show the synchronization process of some nodes in $G2$). Therefore, rewiring connections of nodes with the smallest degree is a valid strategy to enhance or weaken the synchronization of the whole network.

In the second case, we extend our work to analyze a real-world networked system which is not a classic SF network. We investigate the synchronizability of coupled neuronal circuits. The presence of synchronized rhythms has been experimentally observed in electroencephalograph recordings of electrical activity in the brain. Moreover, some types of synchronization of bursting neurons are thought to be related to some diseases like Parkinson's disease, essential tremor, and epilepsies [19]. The Hindmarsh-Rose (HR) neuronal circuit is considered as the node dynamics, which is described by equations

$$\begin{aligned} \dot{x_i}^{(1)} &= x_i^{(2)} - a x_i^{(1)3} + b x_i^{(1)2} - x_i^{(3)} + I_e \\ \dot{x_i}^{(2)} &= d - e x_i^{(1)2} - x_i^{(2)} \\ \dot{x_i}^{(3)} &= r[s(x_i^{(1)} + 1.6) - x_i^{(3)}]. \end{aligned} \tag{26}$$

This model is widely used in modeling the firing activities of neurons [32] [33] [34]. a, b, d, e, r, s are the system parameters which are set as $a = 1, b = 3, d = 1, e = 5, r = 6 * 10^{-2}, s = 4$. $I_e = 320\mu A$ is the external forcing current. The inner coupling function H is the same as (25) and the coupling strength c is set as 0.1. The network of this system is from a real dataset, the cat brain network data [35] [36] (the data

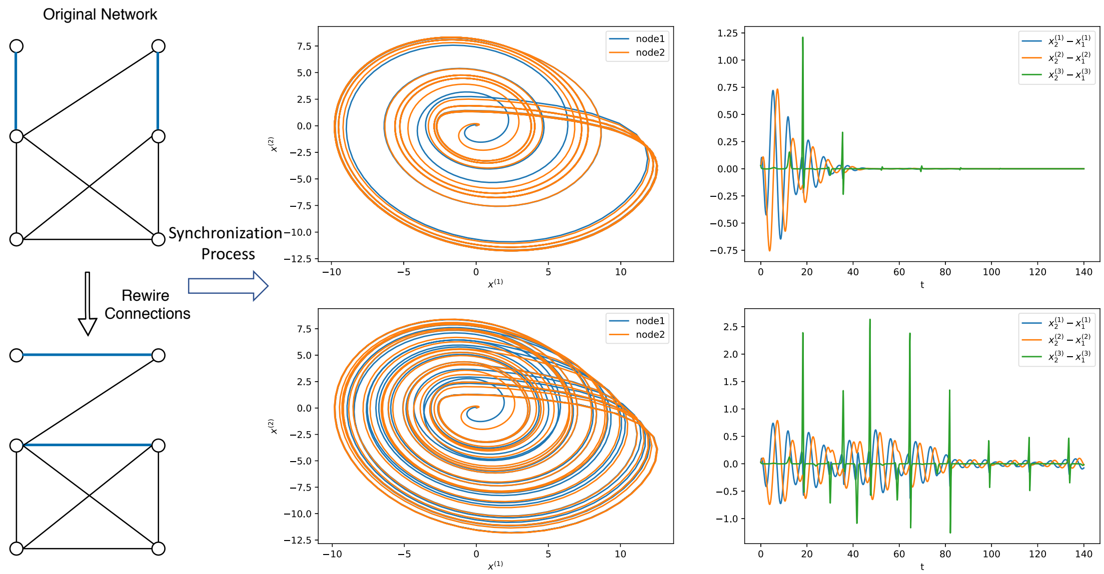


Fig. 7. Control the synchronizability of SF network by rewiring connections of nodes with the smallest degree. From the top three subfigures, we can see that the original network has good synchronizability, and the nodes' trajectories eventually synchronize. In the three bottom subfigures, it shows that the network structure has been adjusted by rewiring connections. The node with the smallest degree connects to similar nodes. The trajectories of these nodes do not achieve synchronization finally.

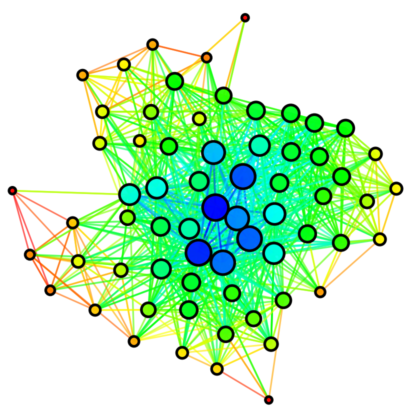


Fig. 8. Cat brain neuron network generated from real network data. The size of nodes is weighted by the degree of nodes. $k_{\min} = 3, k_{\max} = 45$ and $< k >= 22.46$

is available in networkrepository). We convert the data into an undirected network shown in Fig. 8. According to the master stability function, the stability region is $(-\infty, -0.23)$. Since there only exists one smallest node, the degeneration of the eigenvector will not happen. So the smallest nonzero eigenvalue can be directedly estimated by the equation $\lambda^{(2)} \simeq k_i + \sum_{j\neq i} \frac{A_{ij}^2}{k_i - k_j}\Big|_{k_i = k_{\min}}$. The estimate of $\lambda^{(2)}$ by our proposed method is 2.8726. The real value of $\lambda^{(2)}$ is 2.8842. The relative error of our estimation is 0.402%. Since $-c\lambda^{(2)} \in S$, the system synchronizes in a stable state, and the process is shown in Fig. 9. After we rewire the connections of the smallest degree nodes, $\lambda^{(2)}$ becomes 2.286 and $-c\lambda^{(2)} \notin S$, which indicates the desynchronization of the system shown in the top three subfigures of Fig. 9. Therefore, the rewiring method we propose is an effective way to adjust the synchronizability. Also, we calculate the local assortativity of the original network $\rho_i = -0.018$ and the network after reconnection $\rho_i = -0.00327$. This proves the negative relationship between local assortativity and synchronizability as we analyzed before.

V. Conclusions

In this paper, we have proposed an analytical method to estimate the extreme eigenvalues of scale-free (SF) networks and add some discussion on how to extend the framework to other types of networks. To avoid the degeneration of eigenvectors when estimating the smallest nonzero eigenvalue of the SF network, a link removal method has been used. Then the non-degenerate perturbation theory can be used to do the estimation and only requires the local information of smallest nodes in the network. The non-degenerate perturbation theory can give us an analytic equation of estimation, which indicates that there exists a negative relationship between the smallest nonzero eigenvalue $\lambda^{(2)}$ and the local assortativity of the smallest node. Furthermore, the equation informs us how to control the synchronizability of the network by rewiring the connections of the smallest nodes. By simulation of Rossler system and Hindmarsh-Rose (HR) neuronal circuit in different networks, the method has been verified. Therefore, this paper helps us understand the relationship between the connections of smallest nodes and the global synchronizability of the network. And also it can inform us how to control the

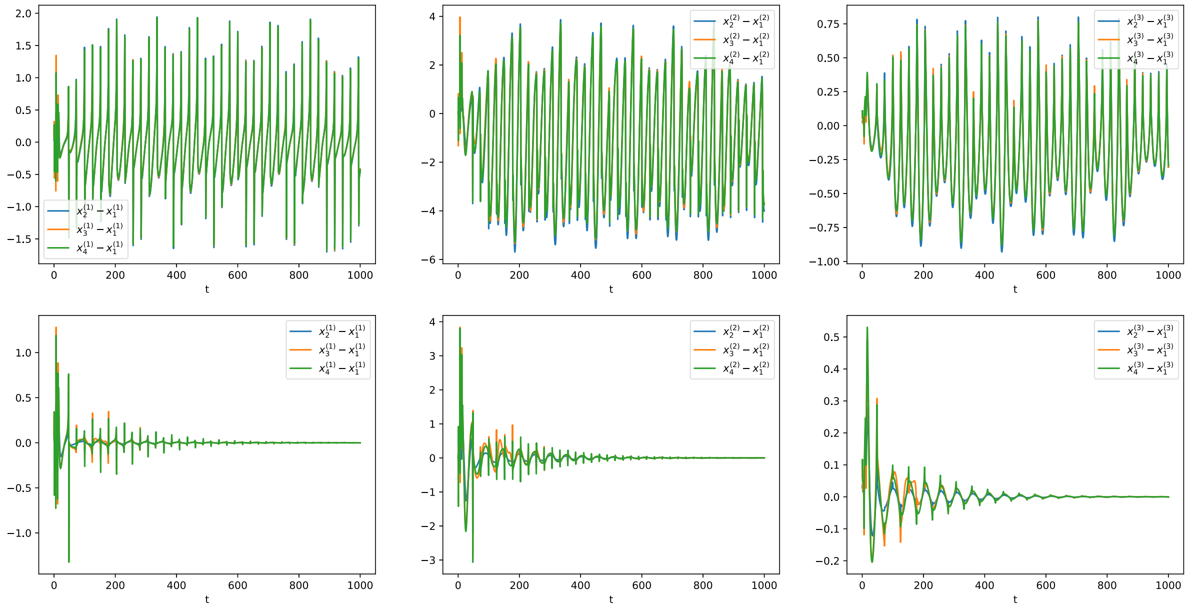


Fig. 9. It shows the synchronization and desynchronization process of Hindmarsh-Rose (HR) neuronal circuit. When the dynamics system embedded in the original network with $\lambda^{(2)} = 2.8842$, the system reaches synchronization. The synchronization process is shown in the three bottom pictures. When we rewire connections of critical nodes, the system loses synchronization as shown in the top three figures.

synchronizability of the network by rewiring connections of nodes while maintaining the degree sequence of the network. In the future, we expect to explore what causes the difference between the estimation value and the true value of the extreme eigenvalues. The difference may reveal potential network topology that affects global synchronizability.

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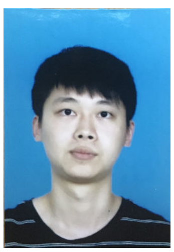

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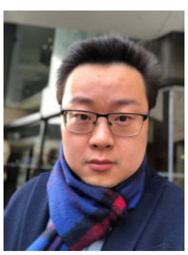

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Local assortativity affects the synchronizability of scale-free network

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