

# On the Achievable Average Power Reduction of MSM Optical Signals

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**Abstract**—In this letter, we consider the achievable average power reduction of multiple subcarrier modulated optical signals by using optimized reserved carriers. Based on Nehari's result we present a lower bound for the maximum average power of the signal after adding the reserved carriers. Simulations show that the mean value of the average required power behaves very close to  $\sqrt{2n \log \log n}$  for binary phase-shift keying (BPSK) constellations where  $n$  is the number of subcarriers. We further remark on evaluating optimum values for reserved carriers using convex optimization and Nehari's result.

**Index Terms**—Average power reduction, intensity modulation (IM), multiple subcarrier modulation, Nehari's theorem, optical communication.

## I. INTRODUCTION

MULTIPLE subcarrier intensity modulation (IM) with direct detection (DD) is an attractive technique for high speed wireless optical communications [1]–[3]. In multiple subcarrier modulation (MSM) with IM/DD, the transmit signal consists of several subcarriers and is modulated onto the optical carrier by intensity modulation. The main disadvantage of MSM with IM/DD is its low optical average power efficiency which is essentially the same problem as the high-peak-to-mean-envelope power ratio (PMEPR) of multicarrier signals in RF communication systems. Since the optical intensity should be always positive, the signal should have a dc component to guarantee that the minimum of the signal is greater than zero. Since this dc component is proportional to the average optical power, as the number of carriers  $n$ , increases and the signal exhibits smaller minimum values, the required average power of the system will be increased.

Previously, the clipping method has been considered to increase the average power efficiency [3]. Recently, You and Kahn [2] proposed block coding and tone reservation to maximize the minimum value of the multiple subcarrier signal. The tone reservation is a method in which the dummy carriers with optimized amplitude are used to increase the average power efficiency of the system. It is worth noting that the tone reservation problem here is a bit different from that of orthogonal frequency division

multiplexing (OFDM) in the sense that we are just interested in the maximization of the minimum value of the signal unlike OFDM in which we are interested in the ratio of the maximum of the signal to the average power [4].

In this letter, we address the achievable average power reduction by using optimum reserved tones appended at the end of the signal. Using Nehari's result on the estimation of a causal function by an anti-causal one, we propose an achievable lower bound on the maximum average power of MSM signals by using reserved tones. We further remark on how to evaluate the optimum values for the reserved carriers using convex optimization and Nehari's result.

## II. MULTIPLE SUBCARRIER MODULATION

MSM signals consist of  $n$  subcarriers where the  $i$ 'th subcarrier is modulated by  $c_i$ . Therefore, the MSM signal can be written as

$$s(\tau) = \text{Re} \left\{ \sum_{i=1}^n c_i e^{j\omega_i \tau} \right\} g(\tau), \quad 0 \leq \tau < T \quad (1)$$

where  $g(t)$  is the transmit pulse shape,  $T$  is the symbol duration, and  $\omega_i$ 's are carrier frequencies. In this letter as in [2], we consider rectangular pulse shape, i.e.,  $g(\tau) = u(\tau) - u(\tau - T)$ , and dense packing for the subcarriers which implies that  $\omega_i = i(2\pi/T)$  for  $i = 1, \dots, n$ . To simplify the mathematical formulation, we normalize the time axis by  $T$  to get

$$s(t) = \text{Re} \left\{ \sum_{i=1}^n c_i e^{j2\pi i t} \right\}, \quad 0 \leq t \leq 1. \quad (2)$$

To make the transmitted signal positive, we have to add a signal  $k(t)$  and guarantee that  $k(t) + s(t)$  is nonnegative. Generally, for each codeword  $(c_1, \dots, c_n)$ ,  $k(t)$  consists of a zero mean part  $k_1(t)$  and a dc part  $k_0$  which is equal to  $-\min\{s(t) + k_1(t)\}$ . Since we require that the average of  $s(t)$  and  $k_1(t)$  be zero, the average optical power will be proportional to  $k_0$ .

In this letter instead of maximizing the minimum of  $s(t) + k_1(t)$  to minimize the average optical power, we consider a stronger condition and we minimize the maximum value of  $|\sum_{i=1}^n c_i e^{j2\pi i t} + b(t)|$  where  $k_1(t) = \text{Re}\{b(t)\}$  and  $b(t)$  has zero average. This condition limits the variations of the signal both in the negative and positive sides. From a practical point of view, this restriction on the envelope rather than the real part of the signal, will both reduce the average power requirement and eliminate high instantaneous intensity peaks from the transmitted signal.

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### III. NEHARI'S PROBLEM AND ITS IMPLICATIONS ON THE RESERVED CARRIER METHOD

A well-known method in OFDM systems is to use optimized reserved carriers placed at frequencies larger than  $n$  to minimize the ratio of the peak to average power of the multicarrier signal [2], [4]. However, here for MSM optical signals, we just need to maximize the minimum of the signal over the amplitude of the reserved carriers.

We can formulate the problem as the following: Let  $c_i$ 's be given and define  $T(z) = \sum_{i=1}^n c_i z^{-i}$  and  $B(z) = \sum_{i=n+1}^M b_i z^{-i}$  where  $z = e^{-j2\pi t}$ . We would like to find the optimum values of  $b_i$ 's such that  $\|T(z) + B(z)\|_\infty$  is minimized. This also raises the question of how much reduction can we get on  $\|T(z) + B(z)\|_\infty$  by using as many reserved carrier as we want, in other words, given the  $c_i$ 's what is the best  $\gamma$  such that, we have  $\|\sum_{i=1}^n c_i z^{-i} + \sum_{i=n+1}^\infty b_i z^{-i}\|_\infty \leq \gamma$ .

A slightly different problem in functional analysis, known as Nehari's problem, states the following result.

*Theorem 1:* (Nehari's Theorem [5]) Let  $T_1(z) = \sum_{i=1}^n c_{n-i+1} z^i$  be an anti-causal function. Then

$$\inf_{\text{causal } K_1(z)} \|T_1(z) - K_1(z)\|_\infty = \sigma(\mathcal{H}_T) \quad (3)$$

where  $K_1(z) = \sum_{i=0}^\infty d_i z^{-i}$  and  $\mathcal{H}_T$  is the Hankel operator

$$\mathcal{H}_T = \begin{bmatrix} c_n & c_{n-1} & \dots & c_1 \\ c_{n-1} & \dots & c_1 & 0 \\ \vdots & & & \\ c_1 & 0 & \dots & \end{bmatrix} \quad (4)$$

and  $\sigma(\cdot)$  is the maximum singular value of its argument.

It is worth mentioning that Nehari also showed that the optimum sequence  $d_i$  exponentially decreases in  $i$ . Thus, in practice, we can truncate the infinite series  $K_1(z)$  to its first  $M - n$  coefficients. To find the solution to Nehari's problem, we can use Theorem 12.8.2 of [6] to find  $K_1(z)$  numerically. The computation is straightforward and the details can be found in [7].

Using Theorem 1, we can state a lower bound for the maximum average power reduction by using reserved carriers at the end of the signal. Substituting  $T_1(z) = z^{n+1}T(z)$  in (3), for all  $z$  over the unit circle and any codeword  $(c_1, \dots, c_n)$ , we get

$$\begin{aligned} \sigma(\mathcal{H}_T) &= \inf_{\text{causal } K_1(z)} \|T_1(z) - K_1(z)\|_\infty \\ &= \inf_{d_0, d_1, \dots} \left\| \sum_{i=1}^n c_i z^{-i} - \sum_{i=n+1}^\infty d_{i-n-1} z^{-i} \right\|_\infty \\ &\leq \left\| \sum_{i=1}^n c_i z^{-i} + \sum_{i=n+1}^M b_i z^{-i} \right\|_\infty \end{aligned} \quad (5)$$

for any sequence  $b_i$ 's and any  $M$ . Therefore given the  $c_i$ 's, the maximum average power reduction by placing reserved carriers at the end of the signal is bounded below by  $\sigma(\mathcal{H}_T)$  where  $\mathcal{H}_T$  is as defined in (4).

For simplicity let us assume  $c_i$ 's are chosen from a MPSK constellation, i.e.,  $c_i \in \{e^{j2\pi k/M} : k = 0, \dots, M-1\}$ . We can easily bound the maximum singular value of  $\mathcal{H}_T$  over all the possible codewords by using the fact that the sum square of

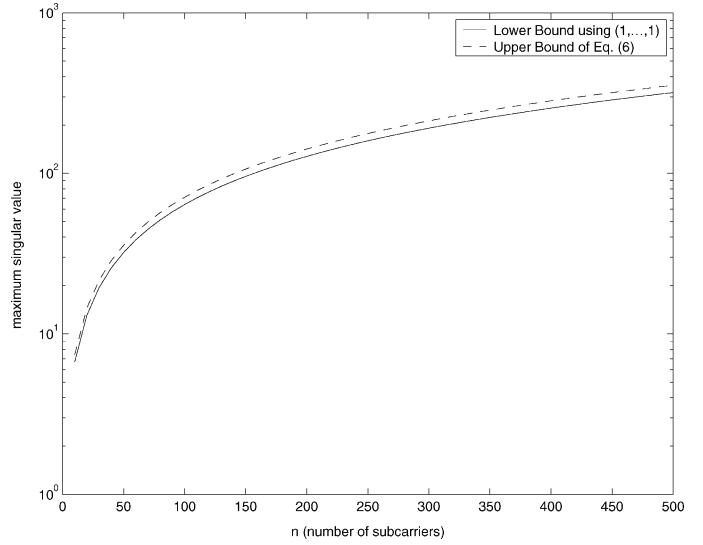


Fig. 1. Lower and upper bounds for the maximum  $\sigma(\mathcal{H}_T)$  over all the codewords when  $c_i$ 's are chosen from MPSK constellations.

the eigenvalues of a matrix is equal to the Frobenius norm of the matrix. This implies that

$$\begin{aligned} \max_{(c_1, \dots, c_n)} \sigma(\mathcal{H}_T) &\leq \max_{(c_1, \dots, c_n)} \left\{ \sum_{i=1}^n \lambda_i(\mathcal{H}_T^* \mathcal{H}_T) \right\}^{1/2} \\ &= \max_{(c_1, \dots, c_n)} \{ \text{tr}(\mathcal{H}_T^* \mathcal{H}_T) \}^{1/2} \\ &= \max_{(c_1, \dots, c_n)} \left\{ \sum_{i=1}^n \sum_{j=1}^i |c_i|^2 \right\}^{1/2} \\ &= \left\{ \sum_{i=1}^n i \right\}^{1/2} = \sqrt{\frac{n(n+1)}{2}} \end{aligned} \quad (6)$$

where  $\lambda_i(\cdot)$  and  $\text{tr}(\cdot)$  denote the  $i$ th eigenvalue and the trace of its argument, respectively. A lower bound on  $\sigma(\mathcal{H}_T)$  can be also obtained by considering the codeword  $(1, 1, \dots, 1)$ . The lower and upper bounds are shown in Fig. 1 for  $n$  from 10 to 500. Clearly the bounds are very close and therefore the worst case improvement is just from  $n$  to  $\sqrt{n(n+1)/2}$ .

However, we can also evaluate the average and the distribution of the maximum singular value of  $\mathcal{H}_T$  by generating sufficiently large number of random codewords. Fig. 2 shows the average of  $\max \sigma(\mathcal{H}_T)$  for BPSK evaluated by using  $10^4$  randomly generated codewords. It can be observed that the average behaves very close to  $\sqrt{2n \log \log n}$  that is much better than  $\sqrt{n \log n}$  which is the average of  $\|T(z)\|_\infty$  before using dummy tones asymptotically [8]. Fig. 3 also shows the distribution function of the maximum for alternative cases by truncating  $K_1(z)$  to a finite degree polynomial and also the distribution of  $\sigma(\mathcal{H}_T)$  which is the best reduction in the maximum that can be achieved by using the Nehari solution  $K_1(z)$  as in (5). Clearly, we have quite a lot of improvement with 20 reserved carriers and further increasing the number of dummy tones does not improve the distribution as much as before.

It is worth noting that the problem of finding the optimum values for a finite number of reserved carriers can be exactly

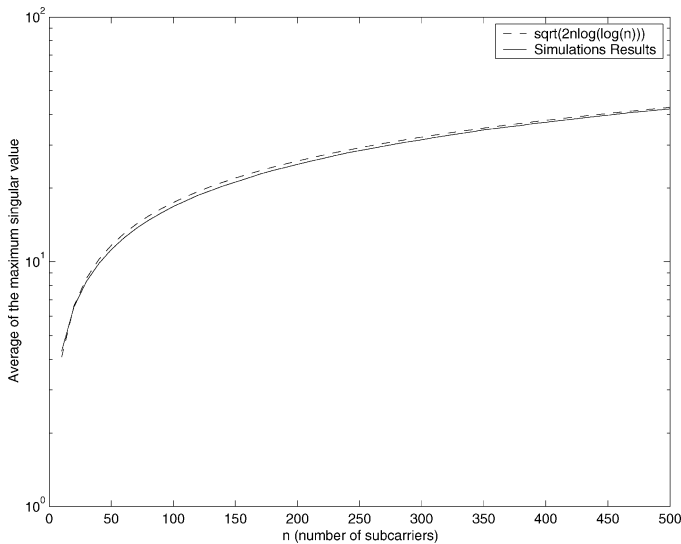


Fig. 2. Average of  $\sigma(\mathcal{H}_T)$  for BPSK evaluated by using  $10^4$  randomly generated codewords and  $\sqrt{2n \log \log n}$ .

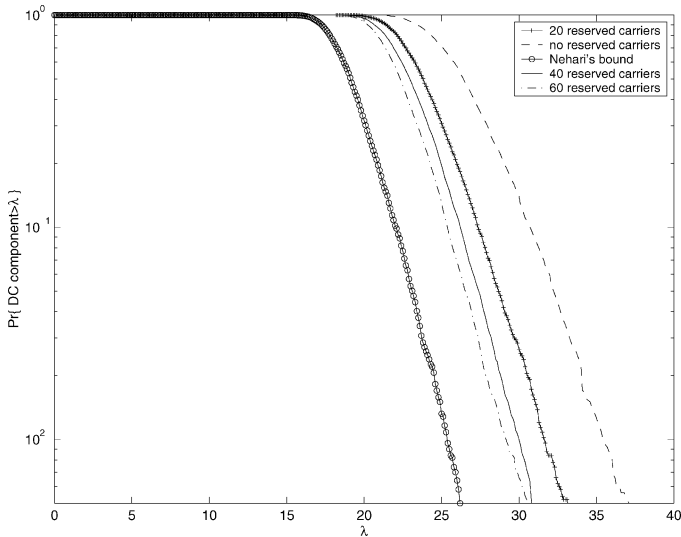


Fig. 3. Distribution function of the required dc component with and without using reserved carriers and the effect of truncation of  $K_1(z)$  for  $n = 128$  using  $10^3$  randomly generated codewords.

solved by invoking the Bounded Real Lemma and using the Linear Matrix Inequality (LMI) optimization [6], [9, Th.

12.6.6]. However, it can be shown that the complexity of the LMI is  $O((n + L)^6)$  where  $L$  is the number of reserved tones. On the other hand, all that the computation of  $K_1(z)$  requires is a matrix inversion and a maximum singular value computation which has  $O(n^3)$  complexity [7]. Therefore truncating the Nehari solution to finite length  $L$  seems to be a more practical way of finding good values for reserved carriers.

#### IV. CONCLUSIONS

In this letter we consider the achievable average power reduction for multiple subcarrier IM/DD optical signals by using dummy tones with optimum value at the end of the signal. Several problems remain to be answered. We considered the problem of reserved carriers when they are placed at frequencies larger than  $n$ . It would be also interesting to investigate the effect of changing their place on the average power reduction. As the numerical results in Fig. 2 suggest, it is intriguing to show that the average of the maximum singular value of random BPSK Hankel matrices is equal to  $\sqrt{2n \log \log n}$  with high probability.

#### REFERENCES

- [1] T. Ohtsuki, "Multiple-subcarrier modulation in optical wireless communications," *IEEE Commun. Mag.*, vol. 41, no. 3, pp. 74–79, Mar. 2003.
- [2] R. You and J. M. Kahn, "Average power reduction techniques for multiple subcarrier intensity modulated optical signals," *IEEE Trans. Commun.*, vol. 49, pp. 2164–2171, Dec. 2001.
- [3] Q. Shin, "Error performance of OFDM-QAM in subcarrier multiplexed fiber-optic transmission," *IEEE Photon. Technol. Lett.*, vol. 9, pp. 845–847, June 1997.
- [4] J. Tellado and J. M. Cioffi, "Efficient algorithms for reducing PAR in multicarrier systems," in *Proc. IEEE Int. Symp. Information Theory*, Aug. 1998, p. 191.
- [5] Z. Nehari, "On bounded bilinear forms," *Ann. Math.*, vol. 65, no. 1, pp. 153–162, 1957.
- [6] B. Hassibi, A. H. Sayed, and T. Kailath, *Indefinite-Quadratic Estimation and Control: A Unified Approach to  $H^2$  and  $H^\infty$  Theories*. SIAM, studies in applied and numerical mathematics, 1999.
- [7] M. Sharif and B. Hassibi, "On the average power of multiple subcarrier intensity modulated optical signals: Nehari's problem and coding bounds," in *Proc. IEEE Int. Conf. on Communications*, May 11–14, 2003, pp. 2969–2973.
- [8] —, "Existence of codes with constant PMEPR and related design," *IEEE Trans. Signal Processing*, to be published.
- [9] S. p. Wu, S. Boyd, and L. Vandenberghe, "FIR filter design via semidefinite programming and spectral factorization," in *Proc. 35th IEEE Conf. on Decision and Control*, 1996, pp. 271–276.