

Cooperative Spectrum Sensing based on the Limiting Eigenvalue Ratio Distribution in Wishart Matrices

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Abstract

Recent advances in random matrix theory have spurred the adoption of eigenvalue-based detection techniques for cooperative spectrum sensing in cognitive radio. These techniques use the ratio between the largest and the smallest eigenvalues of the received signal covariance matrix to infer the presence or absence of the primary signal. The results derived so far are based on asymptotical assumptions, due to the difficulties in characterizing the exact eigenvalues ratio distribution. By exploiting a recent result on the limiting distribution of the smallest eigenvalue in complex Wishart matrices, in this paper we derive an expression for the limiting eigenvalue ratio distribution, which turns out to be much more accurate than the previous approximations also in the non-asymptotical region. This result is then applied to calculate the decision sensing threshold as a function of a target probability of false alarm. Numerical simulations show that the proposed detection rule provides a substantial improvement compared to the other eigenvalue-based algorithms.

I. INTRODUCTION

Blind detection algorithms, relying on the received signal diversity achieved through multiple antennas, user cooperation, or oversampling, have been recently proposed for Cognitive Radio. Most of these methods [1], [2] are based on the properties of the eigenvalues of the received signal's covariance matrix and use results from random matrix theory (RMT).

Their main advantage, with respect to classical energy detection (ED) or cyclostationary feature detection (CFD) [3], is that they do not require any prior information on the primary signal or on

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the noise power. Among blind algorithms, the eigenvalue-based approach was shown to outperform ED, especially in case of noise uncertainty [2].

However, the decision rules of the eigenvalue-based detection schemes proposed so far are based on asymptotical approximations, that make them inaccurate in many practical scenarios. Using some recent RMT results, in this paper we first derive an analytical expression for the limiting distribution of the ratio between the largest and the smallest eigenvalues of the covariance matrix. Then, based on this result, we obtain a novel decision rule that outperforms the previously proposed eigenvalue-based detection schemes.

The rest of the paper is organized as follows: Sec. II reviews the eigenvalue-based algorithms proposed in the literature. Sec. III deals with the threshold optimization problem and presents the contribution of this paper. Numerical results are presented and discussed in Sec. IV. Sec. V contains the conclusions.

II. MATHEMATICAL BACKGROUND

A. System Model

Denote with K the number of collaborating receivers (or antennas) and with N the number of samples collected by each receiver during the sensing time; let $y_k(n)$ be the discrete baseband sample at receiver k ($k = 1, \dots, K$) and time instant n ($n = 1, \dots, N$). Two hypotheses exist: under $\mathcal{H}_0$ (no primary signal: the samples contain only noise) $y_k(n)|\mathcal{H}_0 = v(n)$, where $v(n)$ is circularly symmetric complex Gaussian (CSCG) noise with zero mean and variance σ_v^2 ; under $\mathcal{H}_1$ (presence of primary signal) $y_k(n)|\mathcal{H}_1 = h_k(n)s(n) + v(n)$, where $s(n)$ is the primary signal, with $E|s(n)|^2 = \sigma_s^2 \neq 0$, and $h_k(n)$ is the channel between primary source and receiver k at time n .

Let $\mathbf{y}(n) = [y_1(n) \dots y_K(n)]^T$ be a $K \times 1$ vector containing K received samples at time n and $\mathbf{Y} = [\mathbf{y}(1) \dots \mathbf{y}(N)]$ a $K \times N$ matrix containing all the samples received during the sensing period. The sample covariance matrix, $\mathbf{R}(N) = \frac{1}{N} \mathbf{Y} \mathbf{Y}^H$, converges to $\mathbf{R} = E[\mathbf{y} \mathbf{y}^H]$ for $N \rightarrow \infty$: from the eigenvalues of $\mathbf{R}(N)$ it is possible to infer the presence or absence of primary signal.

B. Previous Results

Let λ_{max} and λ_{min} be the largest and the smallest eigenvalues of $\mathbf{R}(N)$, and l_{max} and l_{min} those of the normalized covariance matrix, defined as $\mathbf{R}'(N) = \frac{N}{\sigma_v^2} \mathbf{R}(N)$. Under $\mathcal{H}_0$, $\mathbf{R}'(N)$ turns out to be a complex white Wishart matrix and, by the Marchenko-Pastur law, the eigenvalue support is finite [4]. Under $\mathcal{H}_1$, the covariance matrix belongs to the class of ‘spiked population models’ and its largest eigenvalue increases outside the Marchenko-Pastur support [5]. This property suggests to use $T = l_{max}/l_{min} = \lambda_{max}/\lambda_{min}$

as test statistic for signal detection. Denoting as γ the decision threshold, the detector decides for $\mathcal{H}_0$ if $T < \gamma$, for $\mathcal{H}_1$ otherwise. Two approaches to set γ are proposed in the literature.

1) *Asymptotic Approach [1]*: Thanks to the asymptotical properties of Wishart matrices [4], the smallest and the largest eigenvalues of $\mathbf{R}'(N)$ under $\mathcal{H}_0$) converge almost surely to

$$l_{min} \rightarrow a = \left(N^{1/2} - K^{1/2}\right)^2 \quad (1)$$

$$l_{max} \rightarrow b = \left(N^{1/2} + K^{1/2}\right)^2 \quad (2)$$

in the limit

$$N, K \rightarrow \infty \text{ with } K/N \rightarrow \bar{c} \quad (3)$$

where $\bar{c} \in (0, 1)$ is a constant. Under $\mathcal{H}_1$, according to the theory of spiked models, the largest eigenvalue converges almost surely to a value $b' > b$ [5]. Based on these results, an asymptotic detection rule was proposed in [1] with decision threshold

$$\gamma_{as} = \frac{b}{a} \quad (4)$$

2) *Semi-asymptotic Approach [2]*: This approach is based on the use of the recently-found limiting distribution of l_{max} instead of its asymptotical value (2). Results from [6] state that under the same assumptions (3) the random variable

$$L_{max} = \frac{l_{max} - b}{\nu} \quad (5)$$

with

$$\nu = \left(N^{1/2} + K^{1/2}\right) \left(N^{-1/2} + K^{-1/2}\right)^{1/3} \quad (6)$$

converges in distribution to the Tracy-Widom law¹ of order 2. The authors of [2] exploit this result to link the decision threshold to the probability of false alarm, defined as

$$P_{fa} = P(T > \gamma | \mathcal{H}_0) \quad (7)$$

by using the asymptotical limit (1) for the smallest eigenvalue and the Tracy-Widom cumulative distribution function (CDF) for the largest one. The threshold can be written as:

$$\gamma_{sa} = \gamma_{as} \cdot \left(1 + \frac{(\sqrt{N} + \sqrt{K})^{-2/3}}{(NK)^{1/6}} F_{TW2}^{-1}(1 - P_{fa})\right) \quad (8)$$

where $F_{TW2}^{-1}(y)$ is the inverse Tracy-Widom CDF of order 2.

¹The Tracy-Widom distribution was defined in [7] as:

$F_{TW2}(s) = \exp\left(-\int_s^{+\infty} (x-s)q^2(x)dx\right)$, where $q(s)$ is the solution of the Painlevé II differential equation $q''(s) = sq(s) + 2q^3(s)$ satisfying the condition $q(s) \sim -Ai(s)$ (the Airy function) for $s \rightarrow +\infty$. For its importance in RMT this distribution has been extensively studied and tabulated; a Matlab routine to compute is available at [8].

III. EIGENVALUE RATIO DISTRIBUTION AND NEW DETECTION THRESHOLD

The asymptotic approach (Sec. II-B1) uses limiting approximations, valid for very large N and K . In practical conditions, that may be characterized by small number of observations due to time-varying channel and/or detection in the shortest possible time, the asymptotic threshold turns out to be very unbalanced with respect to the actual eigenvalue ratio distribution (see next section, Fig. 1). In addition, this approach does not allow to tune the threshold as a function of a target P_{fa} . The semi-asymptotic approach (Sec. II-B2) allows such a control, but it is still based on the asymptotical limit for the smallest eigenvalue and it becomes inaccurate when N decreases.

Recently, Feldheim and Sodin [9] found that the smallest eigenvalue also converges to the Tracy-Widom distribution as $K, N \rightarrow \infty$, up to a proper rescaling factor. Thus, the random variable:

$$L_{min} = \frac{l_{min} - a}{\mu} \quad (9)$$

converges in distribution to the Tracy-Widom law of order 2, with:

$$\mu = \left(K^{1/2} - N^{1/2} \right) \left(K^{-1/2} - N^{-1/2} \right)^{1/3} \quad (10)$$

As a consequence of (3), μ is always negative in the considered range of $\bar{c}$. Now, the test statistic T may be written as:

$$T = \frac{l_{max}}{l_{min}} = \frac{\nu L_{max} + b}{\mu L_{min} + a} \quad (11)$$

Denote with $\bar{f}_{l_{max}}(z)$ and $\bar{f}_{l_{min}}(z)$, respectively, the limiting probability density functions (PDFs) of the numerator and the denominator of T for $K, N \rightarrow \infty$. From (5) and (9), these PDFs may be expressed through a linear random variable transformation of the second-order Tracy-Widom PDF, $f_{TW2}(x)$:

$$\bar{f}_{l_{max}}(z) = \frac{1}{\nu} f_{TW2} \left(\frac{z - b}{\nu} \right) \quad (12)$$

and, recalling that $\mu < 0$:

$$\bar{f}_{l_{min}}(z) = \frac{1}{|\mu|} f_{TW2} \left(\frac{a - z}{|\mu|} \right) = -\frac{1}{\mu} f_{TW2} \left(\frac{z - a}{\mu} \right) \quad (13)$$

Finally, assuming $\bar{f}_{l_{max}}(z)$ and $\bar{f}_{l_{min}}(z)$ as independent (which is reasonable for limiting distributions, with the size of $\mathbf{R}'(N)$ tending to infinity) and applying the ratio distribution formula [10], we can write the PDF of T as:

$$\begin{aligned} \bar{f}_{T|\mathcal{H}_0}(t) &= \left[\int_{-\infty}^{+\infty} |x| \bar{f}_{l_1, l_K}(tx, x) dx \right] \cdot I_{\{t > 1\}} \\ &= \left[\int_0^{+\infty} x \bar{f}_{l_1}(tx) \bar{f}_{l_K}(x) dx \right] \cdot I_{\{t > 1\}} \end{aligned} \quad (14)$$

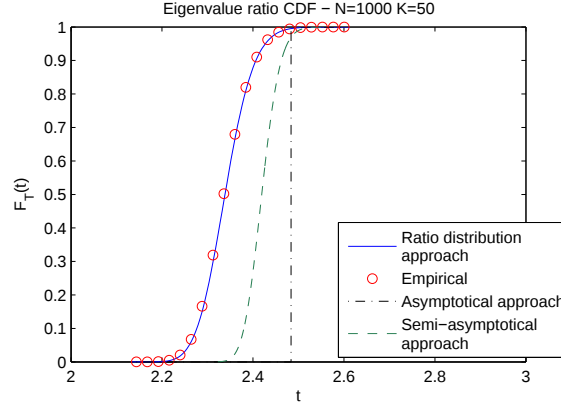


Fig. 1. Eigenvalue ratio CDF obtained using the novel ratio-based approach vs. empirical CDF, asymptotical approach, and semi-asymptotic approach. $N = 1000$, $K = 50$.

where the lower integration limit has been changed to 0 instead of $-\infty$, since the covariance matrix is positive-semidefinite therefore all the eigenvalues are non-negative; $I_{\{\cdot\}}$ is an indicator function, with the condition $t > 1$ to preserve the order of the eigenvalues ($l_1 > l_K$).

Given this new result, we can now introduce a sensing algorithm based on this limiting eigenvalue ratio distribution. Let $\overline{F}_T(t)$ be the cumulative density function (CDF) corresponding to (14). From (7), the false alarm probability is $P_{fa} = 1 - \overline{F}_T(\gamma)$, for large N and K ; hence, we derive the the novel decision threshold as a function of the false-alarm probability:

$$\gamma_{rd} = \overline{F}_T^{-1}(1 - P_{fa}) \quad (15)$$

In practical applications the values of $\overline{F}_T^{-1}(\cdot)$, evaluated numerically off-line, can be stored in a look-up table and then used by the receiver to set the proper threshold as a function of N , K , and the target P_{fa} . (Note that a look-up table or a similar approach is also needed for implementing (8), since F_{TW2}^{-1} does not have a closed-form expression).

IV. NUMERICAL RESULTS

Fig. 1 represents the eigenvalue ratio CDF resulting from the novel analytical approach and compares it to the empirical distribution, computed by Monte-Carlo simulation, and to those obtained from the two approaches of Sec. II-B. The number of samples was set to $N = 1000$ and the number of cooperating receivers to $K = 50$. The novel analytical CDF matches with the empirical data, whereas the asymptotic one (which is simply a step function) and the semi-asymptotic one are very unbalanced because the considered parameters ($N = 1000$ samples and $K = 50$ receivers), although large, are still far from the

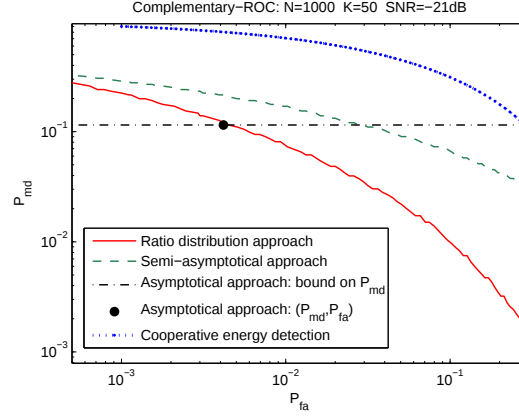


Fig. 2. Complementary ROC: the novel ratio-based approach vs. asymptotical approach, semi-asymptotic approach, and energy detection. $N = 1000$, $K = 50$, $SNR = -21dB$.

asymptotical region. From the detector's point of view, this means that neither the asymptotic nor the semi-asymptotic approach allow to set the decision threshold correctly according to the target P_{fa} .

Fig. 2 provides a performance comparison of the considered eigenvalue-based detectors, plus the traditional energy detector using a cooperative equal gain combining scheme [11]. This type of graph, commonly used for signal detection and called Complementary-ROC (Receiver Operating Characteristics), represents the achievable probability of missed detection $P_{md} = P(T < \gamma | \mathcal{H}_1)$ vs. the target P_{fa} . The simulation parameters are again $N = 1000$ and $K = 50$; the average signal-to-noise ratio under $\mathcal{H}_1$, defined as $SNR = \frac{\|\mathbf{h}\|^2 \sigma_s^2}{K \sigma_v^2}$ with $\|\mathbf{h}\|^2 = \sum_{k=1}^K |h_k|^2$, is equal to -21 dB. Such low values of SNR are typically used to evaluate detectors in critical conditions (e.g., in the case of “hidden node”). For energy detection, a noise uncertainty of 0.25 dB is assumed, whereas the eigenvalue-based algorithms are insensitive to the noise power uncertainty. The ROC plot shows that the novel ratio-distribution threshold provides lower probabilities of missed detection than the other approaches for any given probability of false alarm. Since the new algorithm uses a nearly-exact distribution, it allows to choose the lowest possible threshold for a given target P_{fa} , i.e., to obtain the minimum value of P_{md} .

For instance, given a target P_{fa} of 10^{-1} , the novel approach provides a P_{md} of $1.0 \cdot 10^{-2}$, while the semi-asymptotic approach would give $6.5 \cdot 10^{-2}$. The asymptotical approach, as previously mentioned, does not allow any control of P_{md} vs. P_{fa} since the threshold is fixed. The pair of (P_{fa}, P_{md}) it achieves is represented by a dot in the figure, at $(4 \cdot 10^{-3}, 1.15 \cdot 10^{-1})$; this value of $P_{md} = 1.15 \cdot 10^{-1}$ is a lower bound that cannot be improved regardless of the target P_{fa} , as highlighted by the straight dashed line.

V. CONCLUSION

In this paper an expression for the limiting eigenvalue ratio distribution in Wishart matrices has been derived and it has been applied to the problem of signal detection in cognitive radio. The analytical distribution has been shown to be consistent with the empirical data and, for this reason, the novel detection rule clearly outperforms the previously proposed ones especially for realistic numbers of sensing samples and cooperative receivers.

REFERENCES

- [1] L.S Cardoso, M. Debbah, P. Bianchi, J. Najim, "Cooperative spectrum sensing using random matrix theory", *3rd International Symposium on Wireless Pervasive Computing (ISWPC) 2008*, pp.334-338, 7-9 May 2008.
- [2] Y. Zeng, Y.-C. Liang, "Maximum-Minimum Eigenvalue Detection for Cognitive Radio", *18th Annual IEEE International Symposium on Personal, Indoor and Mobile Radio Communication (PIMRC) '07*, 2007.
- [3] I. F. Akyildiz, W.-Y. Lee, M. C. Vuran, S. Mohanty, "Next Generation / Dynamic Spectrum Access / Cognitive Radio Wireless Networks: A Survey", *ELSEVIER ComNet*, vol.50, no.13, pp.2127-2159, May 2006.
- [4] Z. D. Bai, "Methodologies in spectral analysis of large-dimensional random matrices, a review", *Statistica Sinica*, vol.9, pp.611-677, 1999.
- [5] J. Baik, J. W. Silverstein, "Eigenvalues of large sample covariance matrices of spiked population models", *J. Multivar. Anal.*, vol.97, no.6, pp.1382-1408, 2006.
- [6] I. M. Johnstone, "On the distribution of the largest eigenvalue in principal component analysis", *Annals of Statistics*, vol.29, no.2, pp.295-327, 2001
- [7] C. Tracy and H. Widom, "On orthogonal and symplectic matrix ensembles", *Comm. Math. Phys.*, vol.177, pp.727-754, 1996.
- [8] M. Dieng, *RMLab ver. 0.02*: <http://math.arizona.edu/~momar/>.
- [9] O. N. Feldheim, S. Sodin, "A universality result for the smallest eigenvalues of certain sample covariance matrices", online: http://arxiv.org/PS_cache/arxiv/pdf/0812/0812.1961v1.pdf
- [10] J. H. Curtiss, "On the Distribution of the Quotient of Two Chance Variables", *The Annals of Mathematical Statistics* 12(4):409-421, 1941.
- [11] J. Ma, G. Zhao, Y. Li, "Soft Combination and Detection for Cooperative Spectrum Sensing in Cognitive Radio Networks", *IEEE Transactions on Wireless Communications*, vol.7, no.11, pp.4502-4507, Nov. 2008