Fast and Accurate Approximations for the Analysis of Energy Detection in Nakagami-m Channels

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***Abstract*—Previous research has identified several exact methods for the evaluation of the probability of detection for energy detectors operating on Nakagami-m faded channels. However, these methods rely on discrete summations of complicated functions, and so can take a prohibitively long time to evaluate. In this paper, three approximations for the probability of detection in Nakagami-m faded channels, having distinct regions of applicability, are derived. All have closed forms, and enable the fast and accurate computation of key performance metrics.**

***Index Terms*—Signal detection, cooperative systems, fading.**

I. Introduction

The analysis of energy detection in Nakagami-m faded channels is a well-researched topic with applications in wireless sensor networks, cognitive radio / spectrum sensing and classical detection theory. In the last decade, several exact methods for the evaluation of the probability of detection in such situations have been developed [1], [2], [3]. However, the practical use of such methods is limited due to their high computational complexity.

To address this issue, three approximations for the probability of detection in Nakagami-m channels are derived. As the Rayleigh channel is a special case of the Nakagami-m channel, the approximations can also be used to analyse performance in Rayleigh faded environments.

II. System Model

At the j^{th} energy detector node in a network of cooperating nodes, the received signal, $r_j(t)$, is defined as:

$$r_j(t) = \begin{cases} n_j(t) & H_0 \\ \alpha s_j(t) + n_j(t) & H_1, \end{cases} \tag{1}$$

where $n_j(t)$ is additive white Gaussian noise (AWGN) interference at the j^{th} node, α is the channel fading amplitude, $s_j(t)$ is the transmitted signal measured at the j^{th} node, and H_0 and H_1 are the null and alternative hypotheses, respectively [4].

The power of $n_j(t)$, with reference to a 1Ω resistor, σ_j^2, is defined as $\sigma_j^2 = 2N_{02}W$, where N_{02} is the two sided noise power spectral density and W is the bandwidth of the channel of interest [5]. It is assumed that an accurate estimate of the noise power is available at each node.

The instantaneous signal to noise ratio at the j^{th} node, γ_j, is defined as:

$$\gamma_j = \frac{1}{M}\sum_{i=1}^{M}\frac{\alpha^2 s_j^2\left(\frac{i}{2W}\right)}{\sigma_j^2}, \tag{2}$$

where M is the number of samples of the received signal, equal to twice the time bandwidth product, i.e. $M = 2TW$, where T is the length of the received signal in seconds, and the $\frac{i}{2W}$ term refers to the i^{th} discrete sample of the transmitted signal [5]. It is assumed that T is constant at each node.

The energy detector at node j computes the test statistic, V_j', from samples of the received signal $r_j(t)$:

$$V_j' = \sum_{i=1}^{M}\frac{r_j^2\left(\frac{i}{2W}\right)}{\sigma_j^2}. \tag{3}$$

In cooperative networks, each node compresses, and then transmits, its test statistic to a fusion center, which decides on the state of the channel. Under this scheme, known as hard decision fusion, the decision probabilities at the fusion center depend on the number of bits used to represent the test statistic [6]. Thus, the performance of hard decision fusion is upper bounded by the hypothetical case where infinite precision statistics are transmitted to the fusion center. This is known as soft decision fusion, and is equivalent in formulation to diversity reception with square law combining [1].

In soft decision fusion, the fusion center computes an overall test statistic according to:

$$V' = \sum_{j=1}^{n} V_j', \tag{4}$$

where V' is the fusion center test statistic and n is the number of nodes in the network.

Ma and Li [6] have shown that the fusion center test statistic can be closely approximated by a normal distribution with mean Mn and variance $2Mn$ when the channel is unoccupied, and by a normal distribution with mean $M(n+\gamma)$ and variance $2M(n+2\gamma)$ otherwise, where $\gamma = \sum_{j=1}^{n}\gamma_j$. The fusion center decision probabilities under soft decision fusion are therefore well approximated by:

$$P_f \approx Q\left(\frac{\lambda - Mn}{\sqrt{2Mn}}\right), \tag{5}$$

$$P_d(\gamma) \approx Q\left(\frac{\lambda - M(n+\gamma)}{\sqrt{2M(n+2\gamma)}}\right), \tag{6}$$

where P_f and $P_d(\gamma)$ are the probabilities of false alarm and detection at the fusion center, respectively, and λ is the fusion center threshold [6].

The average probability of detection under Nakagami-m fading, $P_{d_{Nak}}$, is given by:

$$P_{d_{Nak}} = \int_0^{\infty} P_d(\gamma) f_{Nak}(\gamma) d\gamma, \tag{7}$$

where $f_{Nak}(\gamma)$ is the probability density function of the instantaneous signal to noise ratio at the fusion center, γ, under

Manuscript received September 4, 2012. The associate editor coordinating the review of this letter and approving it for publication was Z. Z. Lei.

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Digital Object Identifier 10.1109/LCOMM.2012.12.121964

Nakagami-m fading, given by:

$$f_{Nak}(\gamma) = \frac{1}{\Gamma(mn)}\left(\frac{m}{\bar{\gamma}}\right)^{mn} \gamma^{mn-1} e^{-\frac{m\gamma}{\bar{\gamma}}}, \tag{8}$$

where $\Gamma(x)$ is the Gamma function, $\bar{\gamma}$ is the average instantaneous signal to noise ratio at the fusion center, and m is the Nakagami-m fading parameter which describes the severity of the fading [1]. When $m = 1$, Nakagami-m fading is equivalent to Rayleigh fading.

As (5) and (6) simplify to the single node case when $n = 1$, the decision probabilities of both hard and soft decision type networks in AWGN, Nakagami-m and Rayleigh channels are encapsulated by the general formulations of (5) - (7).

III. Proposed Approximations

A. Integer mn approximation

Letting $n + 2\gamma \approx n$, (6) simplifies[1] to:

$$P_d(\gamma) \approx Q\left(\frac{\lambda - M(n+\gamma)}{\sqrt{2Mn}}\right). \tag{9}$$

Substituting the error function for the Q-function in (9), and using the change of variables $x = \frac{m\gamma}{\bar{\gamma}}$, (7) becomes:

$$P_{d_{Nak}} \approx \frac{1}{2}\left[1 - \frac{1}{\Gamma(mn)}\int_0^{\infty} x^{mn-1} e^{-x} \text{erf}\left(A + Bx\right) dx\right], \tag{10}$$

where erf(x) is the error function, $A = \frac{\lambda - Mn}{2\sqrt{Mn}}$ and $B = -\frac{M\bar{\gamma}}{2m\sqrt{Mn}}$. Exploiting the fact that $\frac{\delta^n}{\delta t^n} e^{-tx} = (-1)^n x^n e^{-tx}$ for $n \in \mathbb{N}$, (10) can be rewritten as:

$$P_{d_{Nak}} \approx \frac{1}{2}\left[1 - \frac{(-1)^{mn-1}}{\Gamma(mn)} \times \frac{\delta^{mn-1}}{\delta t^{mn-1}}\left(\int_0^{\infty} e^{-tx} \text{erf}\left(A + Bx\right) dx\right)\Bigg|_{t=1}\right]. \tag{11}$$

Equation (11) is now of the form in [7, Equation 7.4.36] and can be integrated accordingly. Simplifying the resulting expression using (5), it can be shown that $P_{d_{Nak}}$ is given by:

$$P_{d_{Nak}} \approx P_f + \frac{1}{2}\frac{(-1)^{mn-1}}{\Gamma(mn)} \times \frac{\delta^{mn-1}}{\delta t^{mn-1}}\left[\frac{e^{\left(\frac{t}{2B}\right)^2 + \frac{At}{B}}\left[1 + \text{erf}\left(A + \frac{t}{2B}\right)\right]}{t}\right]\Bigg|_{t=1}. \tag{12}$$

As, $\frac{\delta^{mn-1}}{\delta t^{mn-1}}$ represents the derivative of order $mn - 1$, (12) is valid only for $mn \in \mathbb{N}^+$. Consequently, (12) shall henceforth be referred to as the integer mn approximation.

B. Large SNR approximation

For moderate to large values of $\bar{\gamma}$ (e.g. $\bar{\gamma} \geq -10dB$), B is also large. As a result, (12) can be further simplified to:

$$P_{d_{Nak}} \approx P_f + (1 - P_f)\frac{\Gamma\left(mn, \dfrac{\lambda - Mn}{M(\frac{\bar{\gamma}}{m})}\right)}{\Gamma(mn)}, \tag{13}$$

[1]This approximation is valid for small γ only. However, Ma and Li's approximation requires that the number of samples be large ($M \geq 250$ is suggested). At such large numbers of samples, if γ is large, both the probability of detection and its approximation will converge towards unity, and so there is no significant increase in error.

TABLE I: Parameter values used in numerical simulations.

Parameter	Values
n	$1, 2, 3, 4, 5, 10, 20$
m	$0.5, 1, 1.5, 2$
M	$1000, 10000, 50000$
$\bar{\gamma}$ (dB)	$-21, -20, ..., -1, 0$
P_f	$0.49, 0.3, 0.2, 0.1, 0.05, 0.01, 0.005, 0.001, 0.0005, 0.0001, 0.00005, 0.00001$

where $\Gamma(x, y)$ is the upper incomplete Gamma function.

The simplified form of (13) clearly illustrates the relationship between detection probability and channel parameters when the average signal to noise ratio is moderate to large. Consequently, it shall be referred to as the large SNR approximation.

C. Large mn approximation

For large values of mn, (12) requires high order differentiation, which increases computation time. The $\frac{1}{\Gamma(mn)}$ term also becomes very small with increasing mn, and so precision may become a problem, depending on the implementation platform. This motivates a further approximation of $P_{d_{Nak}}$ for large values of mn.

When mn is large, the probability density function given in (8) can be well approximated by a normal distribution (see Figure 1) with mean $n\bar{\gamma}$ and variance $\frac{n\bar{\gamma}^2}{m}$:

$$\hat{f}_{Nak}(\gamma) \approx \frac{1}{\bar{\gamma}}\sqrt{\frac{m}{2\pi n}} e^{-\frac{m(\gamma - \bar{\gamma}n)^2}{2\bar{\gamma}^2 n}}, \tag{14}$$

where $\hat{f}_{Nak}(\gamma) \approx f_{Nak}(\gamma)$ for large mn.

Substituting (9) and (14) into (7) and using the approximation proposed by López-Benítez and Casadevall [8]:

$$erf(x) \approx 1 - 2e^{2ax^2 + \sqrt{2}bx + c} \tag{15}$$

where a, b and c are the min-MARE parameters defined in [8], it can readily be shown that $P_{d_{Nak}}$ can be written as in (16).

The integrals in (16) are of the form in [9, Equation 2.325-13] and can be integrated accordingly to yield (17), where $z = \frac{\lambda - Mn(1+\bar{\gamma})}{\sqrt{2Mn}}$ and $D = m - a\bar{\gamma}^2 M$. Equation (17) is valid for $mn \in \mathbb{R}^+$ and, henceforth, it shall be referred to as the large mn approximation.

IV. Numerical Results

A. Accuracy

Using the integer mn approximation for $mn < 10$ and the large mn approximation for $mn \geq 10$ (the selection of a "good" switching point between the two is discussed in Section IV-C), the probability mass function (PMF) of the relative error, ϵ_r, between the exact method [3, Equation (21)] and the approximations was calculated for each parameter set[2] in Table I. As can be seen in Figure 2a, the distribution is very tight, with a mean of -0.00301 and a standard deviation of 0.01515.

The large SNR approximation was also compared with the exact method. For $\bar{\gamma} \geq -10dB$, and all other parameters as in Table I, no significant increase in error was found. When the average signal to noise ratio was less than $-10dB$, however, the error increased rapidly and the approximation was no

[2]All possible combinations of parameters were used, subject to the condition that $mn \in \mathbb{N}^+$, giving a total of 17424 sets of input parameters.

$$\begin{aligned} P_{d_{Nak}} \approx & \frac{1}{\bar{\gamma}}\sqrt{\frac{m}{2\pi n}}\left[\int_0^{\frac{\lambda-Mn}{M}} e^{-\frac{m(\gamma-\bar{\gamma}n)^2}{2\bar{\gamma}^2 n}} \left(1-e^{a\left(\frac{\lambda-M(n+\gamma)}{\sqrt{2Mn}}\right)^2+b\left(\frac{\lambda-M(n+\gamma)}{\sqrt{2Mn}}\right)+c}\right)d\gamma\right. \\ & \left. +\int_{\frac{\lambda-Mn}{M}}^{\infty} e^{-\frac{m(\gamma-\bar{\gamma}n)^2}{2\bar{\gamma}^2 n}} e^{a\left(\frac{\lambda-M(n+\gamma)}{\sqrt{2Mn}}\right)^2-b\left(\frac{\lambda-M(n+\gamma)}{\sqrt{2Mn}}\right)+c} d\gamma\right]. \end{aligned} \tag{16}$$

$$\begin{aligned} P_{d_{Nak}} \approx & \frac{1}{2}\left(1+\mathrm{erf}\left[-\frac{1}{\bar{\gamma}}\sqrt{\frac{m}{M}}z\right]+\sqrt{\frac{m}{D}}e^{\frac{am}{D}z^2+\frac{b^2\bar{\gamma}^2M}{4D}+c}\left[e^{-\frac{bm}{D}z}\left(\mathrm{erfc}\left[\bar{\gamma}\sqrt{\frac{M}{D}}\left(\frac{b}{2}-\frac{az}{1-\frac{D}{m}}\right)\right]-2\right)\right.\right. \\ & \left.\left.+e^{\frac{bm}{D}z}\left(\mathrm{erf}\left[\bar{\gamma}\sqrt{\frac{M}{D}}\left(\frac{b}{2}+\frac{az}{1-\frac{D}{m}}\right)\right]-\mathrm{erf}\left[\bar{\gamma}\sqrt{\frac{M}{D}}\left(\frac{b}{2}+az\right)-\sqrt{\frac{nD}{2}}\right]\right)\right]\right). \end{aligned} \tag{17}$$

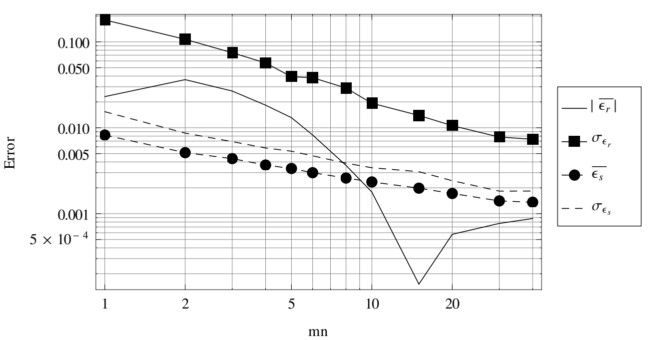
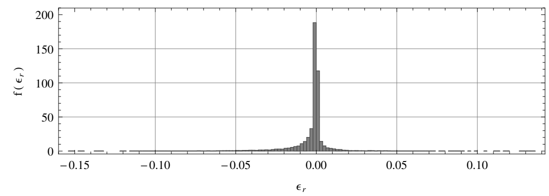
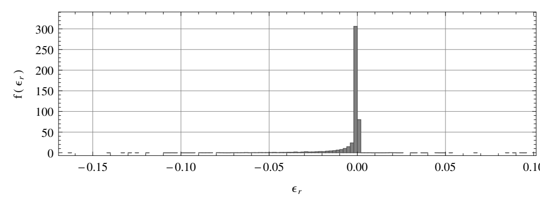


Fig. 1: Log-log plot of the absolute value of the mean, $|\bar{\epsilon}_r|$, and standard deviation, σ_{ϵ_r}, of the relative error PMF and the mean and standard deviation of the squared error, ϵ_s, between f_{Nak} and $\hat{f}_{Nak}$ versus mn.



(a) Integer mn / large mn, with switching point $mn = 10$.



(b) Large SNR, with $\bar{\gamma} \geq -10dB$.

Fig. 2: Probability mass function of the relative error between the exact method and the proposed approximations.

longer useful. The PMF of the relative error between the large SNR approximation and the exact solution is also shown in Figure 2b. As can be seen, the distribution is very tight, with a mean of -0.00372 and a standard deviation of 0.01179.

B. Computational complexity

It is insightful to compare the computational complexities of the derived approximations with existing methods. Digham et al. [1, Equation (7)] and Herath et al. [2, Equation (24)] developed methods based on summations of confluent hypergeometric functions. Borwein and Borwein state that such hypergeometric functions have a computational complexity of order $O(log^2(d)M(d))$, where d is the number of digits of precision to which the function is evaluated and $M(d)$ is the computational complexity of the chosen multiplication algorithm (there exist several multiplication algorithms with differing regions of applicability) [10]. For Digham's method, $\frac{Mn}{2}-1$ confluent hypergeometric functions must be evaluated to compute the summation; for Herath's method, $\frac{Mn}{2}$ confluent hypergeometric function evaluations are required. Annamalai et al. [3, Equation (21)] developed an infinite series method, based on incomplete Gamma functions, which are known to have lower computational complexity than confluent hypergeometric functions ($O(\sqrt{d}M(d))$ according to [10]). The number of Gamma function evaluations required increases as M increases, ranging from nine for small M to as large as several thousand for large M. In contrast, the proposed approximations consist of error functions, which are known to have lower computational complexity than both confluent hypergeometric and Gamma functions [11]. The integer mn approximation requires only two error function evaluations, the large SNR approximation requires one error function evaluation and one Gamma function evaluation, while the large mn approximation requires four error function evaluations. For the integer mn approximation, the high order differentiation required for large mn consumes most of the processing time.

These differences can be seen quite clearly in Table II, where the computation time for each method was measured[3] using Mathematica 8. The integer mn method computes the probability of detection to a high degree of accuracy in a much faster time than any of the exact methods, although more time is required for large mn than for small mn. As expected, the large SNR approximation is only accurate for $SNR \geq -10dB$, while the large mn approximation has a consistently short computation time in all cases but is only accurate for large mn products.

C. Switching between approximations

There is clearly a question as to how large mn should be before (17) can be used without a significant loss in accuracy. To answer this, consider Figure 1 where the mean and standard deviation of the relative error PMF are plotted as functions of

[3]Each calculation was performed a thousand times or until more than ten minutes of CPU time had been used - whichever came first - and the average CPU time per calculation, $\bar{t}_{cpu}$, was then computed. To ensure a fair comparison, the system cache was cleared before each iteration of each calculation. The truncation points for Annamalai's method were calculated separately, and so have no impact on the results. Where no probability of detection is given, the calculation was aborted after using more than ten minutes of CPU time.

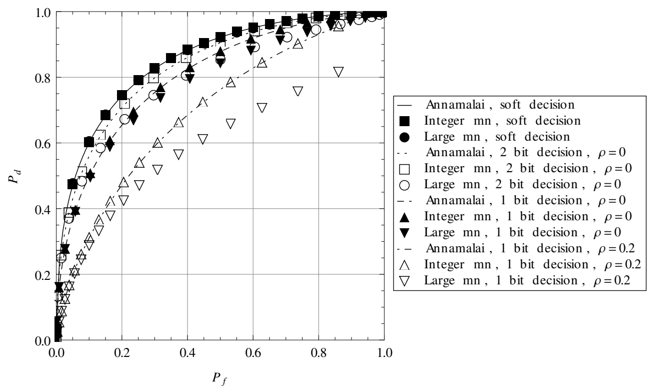


Fig. 3: Receiver operating characteristics for various network types with $n = 5$, $m = 2$, $M = 10000$ and $\bar{\gamma} = -20dB$. For the hard decision fusion schemes, ρ indicates the average decision correlation between nodes, as in [12], and the majority voting rule was applied at the fusion center.

mn. As mn grows large, $\hat{f}_{Nak}(\gamma) \to f_{Nak}(\gamma)$. However, as a result of the central limit theorem, $\hat{f}_{Nak}(\gamma)$ converges towards $f_{Nak}(\gamma)$, but never equals it [13]. Thus, there is no globally optimum value of mn at which to switch to (17) from (12) but, rather, it is a matter of preference, depending on how large a relative error the designer is willing to tolerate.

If mn is small and non-integer, and the error from the use of the large mn approximation is unacceptable then, currently, a more computationally expensive exact method must be used. To this end, the method developed by Annamalai et al. appears to be the least computationally expensive.

V. Application

Using the proposed approximations, performance metrics for both hard and soft decision fusion schemes can be generated quickly and accurately. Receiver operating characteristics for a network with five nodes, operating in a Nakagami-m faded channel with $m = 2$, are shown in Figure 3. Soft decisions and 1-bit and 2-bit hard decisions are considered. In the case of 1-bit hard decisions, the effect of correlated decisions is also considered by combining the approximations with the work of Drakopoulos and Lee [12]. In all cases, the integer mn approximation closely matches Annamalai's exact method. For the soft decision fusion scheme, as $mn = 10$, the large mn approximation is accurate. However, for the hard decision schemes, the fusion center probability of detection depends on the probability of detection at each node [6], i.e. $n = 1$ in (17). Thus, the large mn approximation performs poorly. On average, the large mn approximation was 200 times faster than Annamalai's exact method for soft decision fusion and the integer mn approximation was 100-200 times faster for hard decision fusion.

VI. Conclusion

Three approximations for the calculation of the probability of detection for cooperative networks of energy detectors in Nakagami-m faded channels were derived. These methods allow the probability of detection to be computed quickly, with a very small loss in accuracy. When mn is small and $mn \in \mathbb{N}^+$, (12) can be used to calculate the probability of detection, and (13) may be used when $SNR \geq -10dB$; when mn is large, then (17) may be used for $mn \in \mathbb{R}^+$. The choice of whether to use the integer mn or large mn approximations for a given mn depends on the error the designer is willing to accept.

TABLE II: Average CPU time required for $P_f = 0.1$.

Method	n	m	M	$\bar{\gamma}$ (dB)	P_d	$\bar{t}_{cpu}$ (s)
Digham	1	1	1000	0	0.941015	2.85506
Herath	1	1	1000	0	0.941016	1.59873
Annamalai	1	1	1000	0	0.941015	1.86388
Integer mn	1	1	1000	0	0.943086	0.02036
Large SNR	1	1	1000	0	0.949869	0.00045
Large mn	1	1	1000	0	0.826366	0.00638
Digham	2	0.5	10000	-10	-	600
Herath	2	0.5	10000	-10	0.878277	7.62848
Annamalai	2	0.5	10000	-10	0.878276	3.15383
Integer mn	2	0.5	10000	-10	0.879188	0.00082
Large SNR	2	0.5	10000	-10	0.891745	0.00088
Large mn	2	0.5	10000	-10	0.806058	0.00588
Digham	10	0.5	10000	-20	-	600
Herath	10	0.5	10000	-20	-	600
Annamalai	10	0.5	10000	-20	0.743524	7.59462
Integer mn	10	0.5	10000	-20	0.745164	0.00166
Large SNR	10	0.5	10000	-20	0.853581	0.00092
Large mn	10	0.5	10000	-20	0.749305	0.00647
Digham	20	1	10000	-20	-	600
Herath	20	1	10000	-20	-	600
Annamalai	20	1	10000	-20	0.939119	129.036
Integer mn	20	1	10000	-20	0.940245	0.22524
Large SNR	20	1	10000	-20	0.999732	0.00045
Large mn	20	1	10000	-20	0.937594	0.00654

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