Rate-Compatible Protograph-based LDPC Codes for Inter-Symbol Interference Channels

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Abstract— This letter produces a family of rate-compatible protograph-based LDPC codes approaching the independent and uniformly distributed (i.u.d.) capacity of inter-symbol interference (ISI) channels. This problem is highly nontrivial due to the joint design of structured (protograph-based) LDPC codes and the state structure of ISI channels. We describe a method to design nested high-rate protograph codes by adding variable nodes to the protograph of a lower rate code. We then design a family of rate-compatible protograph codes using the extension method. The resulting protograph codes have iterative decoding thresholds close to the i.u.d. capacity. Our results are supported by numerical simulations.

Index Terms—LPDC, rate-compatible, protograph, intersymbol interference

I. INTRODUCTION

Rate-compatibility is a desirable feature that allows a single encoder/decoder operating over a variety of code rates, which can be selected to match channel conditions. While rate-compatible LDPC codes have been studied for memoryless AWGN channels, in channels with memory, especially inter-symbol-interference (ISI) channels, the problem of ratecompatible LDPC coding has been open until now.

The capacity of ISI channels with finite-alphabet inputs remains open. If the source is restricted to be independent and uniformly distributed (i.u.d.), the capacity of binary-input ISI channels, also called partial response channels, is known as the i.u.d. capacity [1] (also known as symmetric information rate). Irregular LDPC codes have been designed to approach the i.u.d. capacity of partial response channels [2], [3], [4], [5], but they generally lack a structure to enable easy encoding and fast decoding. In this letter, we address the problem of designing rate-compatible structured LDPC codes that are capacity-approaching for ISI channels.

Our structured LDPC codes are built from a small graph, called a protograph [6]. Protograph codes have demonstrated very good performance in terms of iterative decoding thresholds as well as finite-length performance with low encoding and decoding complexity [7], [8] in the AWGN channels. Protograph codes have been designed for several partial response channels [9], [10]. The authors of the present paper in [9] proposed a method to design a rate-1/2 capacity-approaching protograph code in ISI channels. Fang et al. [10] designed a nested protograph family that is good for the dicode and

EPR4 channels based on finite-length EXIT analysis. However, the prior activity in the open literature did not produce *rate-compatible* LDPC codes for the ISI channel.

The main contribution of this paper is a set of ratecompatible structured codes based on protographs that are i.u.d. capacity approaching. We design two families of protograph codes: the nested high-rate protographs where high-rate codes are built from a low-rate protograph by adding more variable nodes; and the rate-compatible protographs where information payload of all members are identical. All proposed codes have thresholds that are within a gap of 0.5 dB to i.u.d. capacity. The performance of the proposed codes with 16k data over the dicode and EPR4 channels is reported. The performance of our codes exhibits a frame error rate of 3×10^{-6} at a gap of 1.1 dB from the i.u.d. capacity limits.

II. PROTOGRAPH DESIGN FOR ISI CHANNELS

A protograph [6] is a Tanner graph with a relatively small number of nodes. A protograph code (an equivalent LDPC code) is a larger derived graph constructed by applying a "copy-and-permutation" operation on a protograph, a process known as *lifting*. In this lifting, the protograph is copied Ntimes, then a large LDPC code graph is obtained by permuting N variable-to-check pairs (edges), corresponding to each of the edge types of the original protograph.

Designing good protographs for the AWGN channel has been addressed in [8], [11]. Its extension to ISI channels is not easy due to its concatenation with a BCJR equalizer. We proposed in [9] a direct method to design a rate-1/2 protograph in partial response channels. However, practical systems, e.g., in magnetic recoding, require LDPC codes with rates up to 0.9. Applying the method of [9] to design high-rate codes is impractical due to the high dimensionality of protographs. Another approach to avoid this problem is to exploit a nested structure which allows building a high-rate code from a lowrate code. In this letter, we consider the same system model and receiver structure as in [9] where two popular channels, the dicode channel $h(D) = (1-D)/\sqrt{2}$ and the EPR4 channel $h(D) = (1 + D - D^2 - D^3)/2$, are studied.

Let us first focus on designing a rate-1/2 code. We do not use the rate-1/2 protograph reported in [9] because it is too large¹ for our goal of building nested codes with rates up to 0.9. We apply the method in [9] to design a smaller rate-1/2graph, thus making it easier to design good high-rate, ratecompatible codes in the next section.

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¹Large protographs may require many intermediate steps to get a rate of 0.9.

TABLE I THRESHOLD ($E_b/N_0~{
m dB}$) of New Codes

	dicode		EPR4	
Code	Code	Gap to	Code	Gap to
Rate	thres.	cap.	thres.	cap.
The nested family				
1/2	1.3	0.5	1.7	0.5
2/3	2.2	0.4	2.6	0.4
3/4	2.7	0.3	3.2	0.4
4/5	3.1	0.3	3.6	0.4
5/6	3.3	0.2	3.9	0.3
6/7	3.6	0.2	4.1	0.3
7/8	3.8	0.2	4.3	0.3
8/9	4.0	0.3	4.2	0.3
9/10	4.2	0.3	4.7	0.4
The rate-compatible family				
9/10	4.2	0.3	4.7	0.4
27/31	4.0	0.4	4.3	0.3
27/32	3.6	0.3	4.0	0.3
27/33	3.3	0.3	3.7	0.3
27/34	2.9	0.2	3.5	0.3
27/35	2.9	0.3	3.3	0.3
27/37	2.6	0.3	3.0	0.3
27/39	2.3	0.3	2.7	0.3
27/41	2.1	0.3	2.5	0.3

To demonstrate, let us search for a rate-1/2 protograph that contains 3 check nodes and 6 variable nodes without any punctured node, as pointed out in [9]. A good protograph should contain nodes of both degree 1 and 2. Thus, to greatly reduce the search space, we start by a search structure with one degree-1 and one degree-2 variable node in the form of the protomatrix

$$H_{search}^{1/2} = \begin{pmatrix} 1 & 0 & x_1 & x_4 & x_7 & y_1 \\ 0 & 1 & x_2 & x_5 & x_8 & y_2 \\ 0 & 1 & x_3 & x_6 & x_9 & y_3 \end{pmatrix} , \qquad (1)$$

where x_i , $i = 1, \ldots, 9$ and y_j , $j = 1, \ldots, 3$, are the number of edges connecting their associated row (check node) and column (variable node) in which y_1, y_2, y_3 correspond to the highest degree variable node. In order for the code to have the linear minimum distance growth property, the edge summation over the last two rows within the last 4 columns should be 3 or higher [12], i.e., $x_2 + x_3 \ge 3$, $x_5 + x_6 \ge 3$, $x_8 + x_9 \ge 3$ and $y_2 + y_3 \ge 3$. We can further simplify the problem by limiting $x_i \in \{0, 1, 2\}$ and $y_j \in \{1, 2, 3, 4\}$. Our objective in this specific example is to find a protograph that has the lowest iterative decoding threshold over the dicode channel.

After a simple search, the resulting protograph, called the ISI code, is in the form of

$$H_{ISI}^{1/2} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 2 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 1 & 1 \end{pmatrix} .$$
(2)

The search is performed using PEXIT (protograph EXIT) analysis [13]. The designed protograph is shown in Fig. 1, and the corresponding thresholds in a variety of channels are given in Table I. This code has a threshold within 0.5 dB of the i.u.d. capacity limit of the dicode channel.

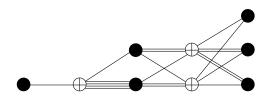


Fig. 1. The rate-1/2 protograph optimized for the dicode channel

III. RATE-COMPATIBLE CODES FOR ISI CHANNELS

There are two popular methods of building rate-compatible codes. One of them is via puncturing, i.e., starting from a low-rate code and selectively deleting parity bits to build highrate codes. The other method involves extension, i.e., starting from a high-rate code, low-rate codes are built by adding the same number of variable and check nodes. The former method in general is not good for channels with memory, as pointed out by [9] except when only degree-1 variable nodes are punctured [10]. The latter method will be applied to our rate-compatible code design in this section.

A. Nested High-rate Codes

In order to use the extension method to design ratecompatible codes, we first need to design a high-rate code. We use a nested lengthening structure as follows

$$H_h = \begin{bmatrix} H_l & H_e \end{bmatrix},\tag{3}$$

where H_h and H_l are protomatrices of a high-rate and lowrate protograph, respectively, H_e is an extension matrix whose columns are the newly added variable nodes and its elements are the number of edges connecting a new variable node to an existing check node.

In this example, we start from the rate-1/2 protograph designed in Section II. We then design a family of nested codes with rates in the form of $R = \frac{n}{n+1}$, n = 1, 2, ... Within this nested family, successively higher rate protographs are constructed by adding 3 new variable nodes with each step.² We then apply the same search method used in Section II to find the protographs with the lowest threshold. We then are able to design the code with rate up to 9/10. Due to space limitation, we only present the result of the rate 0.9 code in which the other eight codes are deduced. The rate-0.9 code protomatrix is

In the above protomatrix, we separate each code rate level by a line as shown in (4). The thresholds of these nine nested codes are shown in Table I where the high-rate codes have their iterative decoding thresholds within 0.4 dB of the i.u.d. capacity of dicode and EPR4 channels. Considering this small gap to i.u.d. capacity of ISI channels, we can say that nested

²If we use the rate-1/2 code in [9], 4 new variable nodes are needed instead.

high-rate codes are good enough and there is no need to design a high-rate code directly as described in Section II.

B. Rate-Compatibility via Extension

Using the extension method, the low rate protomatrix is in the following form

$$H_l = \begin{pmatrix} H_1 & 0\\ A & B \end{pmatrix} , \tag{5}$$

where H_l and H_1 are the low-rate and high-rate protomatrices, respectively, 0 is the zero matrix, A is the matrix whose elements are number of edges connecting new check nodes to the existing variable nodes, and B is the matrix whose elements are the number of edges connecting between new check nodes to new variable nodes. To simplify the problem, we assume that B is identity.

Starting with the rate-9/10 code, we design a family of ratecompatible codes as follows. Each time, we add one variable node and one check node to the protomatrix of the high-rate code. The new codes have the rates in the form of $R = \frac{27}{30+m}$, where *m* is the number of variable and check nodes added into the rate-9/10 code. Due to space limitation, we only design nine rate-compatible codes with rates from 27/41 to 9/10, where the biggest graph/code has the lowest rate of R = 27/41. Equation (6) shows its protomatrix which contains the other eight rate-compatible code protomatrices. Iterative decoding thresholds of these rate-compatible codes over the dicode and EPR4 channels are shown in Table I. Again, all these codes can operate closely to capacity with thresholds gaps of 0.4 dB to i.u.d. capacity limits.

IV. NUMERICAL RESULTS

Our protograph codes are derived from protomatrices (protographs) designed in Section III in two lifting steps. First, the protograph is lifted by a factor of 4 using the progressive edge growth (PEG) algorithm [14] in order to remove all parallel edges. Then, a second lifting using the PEG algorithm was performed to determine a circulant permutation of each edge class that would yield the desired code block length. A circulant lifting of the protograph results in an overall code that is quasi-cyclic, which is known [15] to have an upper bound on minimum distance that is independent of blocklength. However, experience shows that even with circulant lifting it is preferable to use protographs that are designed with linear distance properties, especially with a two-step lifting where the first step removes parallel edges.

In this section, protograph codes are simulated with the information payload of 16k bits. For nested codes, lifting factors of codes with rates 1/2, 2/3, 3/4, 4/5, 5/6, 6/7, 7/8, 9/10 are 4×1364 , 4×683 , 4×455 , 4×342 , 4×273 , 4×227 , 4×195 , and 4×153 , respectively. For rate-compatible codes, only the lowest rate code whose protomatrix is shown in (6) is constructed with the lifting factor of 4×153 . Other codes are obtained by removing coded bits and check equations.

The FER performance of the nested and rate-compatible codes over the dicode and EPR4 channels are shown in Figs. 2, 3, 4 and 5, respectively. No error floors are observed down

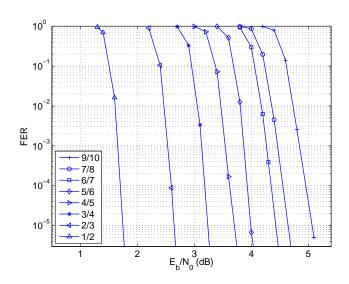


Fig. 2. Nested protograph family over the dicode channel, payload 16k

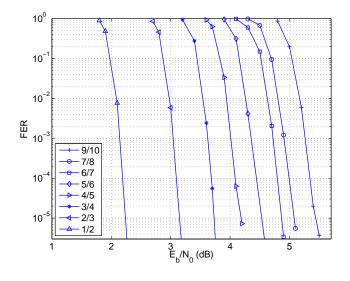


Fig. 3. Nested protograph family over the EPR4 channel, payload 16k

to $FER = 3 \times 10^{-6}$, with a gap of 1.1 dB from the i.u.d. capacity.

V. CONCLUSION

This letter presents a design for nested and rate-compatible protograph-based LDPC codes for ISI channels. Iterative decoding thresholds and finite length performances of the codes are reported. Analysis and simulation results show that our codes, which allow easy encoding and fast decoding, can perform closely to i.u.d. capacity limits.

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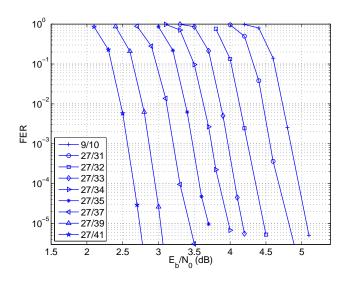


Fig. 4. Rate-compatible protograph family over the dicode channel

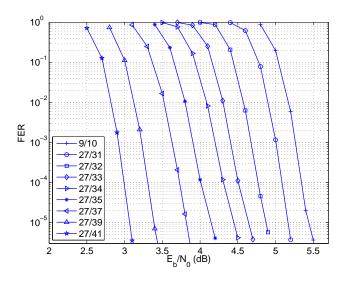


Fig. 5. Rate-compatible protograph family over the EPR4 channel

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