

# On the Message Dimensions of Vector Linearly Solvable Networks

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## Abstract

It is known that there exists a network which does not have a scalar linear solution over any finite field but has a vector linear solution when message dimension is 2 [3]. It is not known whether this result can be generalized for an arbitrary message dimension. In this paper, we show that there exists a network which admits an  $m$  dimensional vector linear solution, where  $m$  is a positive integer greater than or equal to 2, but does not have a vector linear solution over any finite field when the message dimension is less than  $m$ .

## I. INTRODUCTION

Network coding - which emerged as an improvement over routing - has been an intense area of research since its inception in the year 2000 [1]. It was shown in [1] that for the multicast networks, the min-cut bound can be achieved using network coding. Moreover, scalar linear network coding over a sufficiently large finite field was shown to be sufficient to achieve the capacity of a multicast network [2]. However, for the non-multicast networks, Médard *et al.* [3] presented an instance of a network coding problem which does not admit a scalar linear solution over any finite field but has a 2 dimensional vector linear solution over every finite field. This network is known as the M-network in the literature. It is a natural question whether, for any positive integer  $m \geq 2$ , there exists a network which admits an  $m$  dimensional vector linear solution but has no vector linear solution over any finite field when message dimension is less than  $m$ . In this paper, we show that indeed there exists such a network.

The prior works related to the problem addressed in this paper are as follows. [4], [5] studied the effect of message dimension on vector linear solvability. In [4], a network was presented which has an  $m$  dimensional vector linear solution over  $\mathbb{F}_2$  for an arbitrary positive  $m$ , but does not have an  $m$  dimensional vector linear solution over  $\mathbb{F}_q$  for odd  $q$  and any  $m$ . In [5], a multicast network was presented which has a vector linear solution over  $\mathbb{F}_2$  for message dimension 4 but not for message dimension 5. In [6], it was shown that the notion of scalar linear solvability of networks can be captured by matroids. Specifically, it was shown that a network is scalar linearly solvable only if it is a matroidal network associated with a representable matroid over a finite field. The converse of this result was proved in [7]. [8] generalized the results of [6], [7] for the vector linearly solvable networks and showed that the existence of an  $m$  dimensional vector linear network code solution implies the existence of a discrete polymatroid with certain properties and vice versa. To show that for any positive integer  $m \geq 2$ , there exists a network which has an  $m$  dimensional vector linear solution but has no vector linear solution for a message dimension less than  $m$ , either one has to give construction of such a network, or equivalently, one can construct a discrete polymatroid such that it is representable if the rank of its every element is allowed to be less than or equal to  $m$ , but not representable if the rank of its every element is strictly lesser than  $m$  [8]. To the best of our knowledge, neither such a network has been presented in the literature nor it has been shown that for every integer  $m \geq 2$ , there do not exist such networks; likewise in the area of matroid theory, no discrete polymatroid having the above mentioned property has been reported nor it has been shown that such a discrete polymatroid does not exist.

The organization of the rest of the paper is as follows. In Section II, we present the formal definitions related to network coding. In Section III, we present the main result of the paper. Section IV is devoted to the proof of the main theorem of the paper. We conclude the paper in Section V.

## II. PRELIMINARIES

The networks considered in this paper are directed acyclic networks. A directed acyclic network can be modelled by a directed acyclic graph  $G = (V, E)$ , where  $V$  is the set of nodes and  $E \subseteq V \times V$  is the set of edges. For an edge  $e = (u, v)$ ,  $u$  is denoted by  $\text{tail}(e)$  and  $v$  is denoted by  $\text{head}(e)$ . For a node  $v$ , the set of edges  $\{(x, v) | x \in V\}$  is denoted by  $\text{In}(v)$ . A set of nodes, denoted by  $S \subset V$ , will be called as sources and a set of nodes, denoted by  $T \subset V$ , will be called as terminals. Without loss of generality (w.l.o.g), we assume that a source does not have any incoming edge and a terminal does not have any outgoing edge. Associated with every source, there is an i.i.d. random process which is uniformly distributed over a finite field  $\mathbb{F}_q$ . Each source process is independent of all other source processes. The source process associated with a source  $s_i \in S$  is indicated by  $X_i$ , and the message carried over an edge  $e$  is indicated by  $Y_e$ . Each terminal demands a subset of source processes. All the edges in the network are assumed to be unit capacity edges. An  $(n, n)$  vector linear network code over a finite field  $\mathbb{F}_q$  can be described in terms of the message passing through every edge and the decoding function at every terminal. For an edge  $e$ , when  $\text{tail}(e) = s_i \in S$ ,  $Y_e = A_{\{s_i, e\}} X_i$ , where  $X_i, Y_e \in \mathbb{F}_q^n$ , and  $A_{\{s_i, e\}} \in \mathbb{F}_q^{n \times n}$ . When  $\text{tail}(e)$

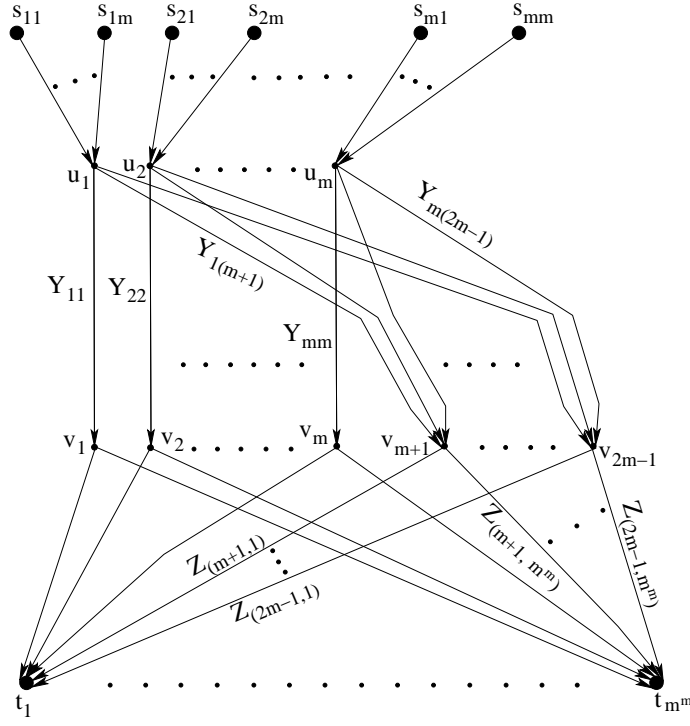


Fig. 1. A communication network  $\mathcal{N}$ , which we name as “generalized M-network”, which is vector linearly solvable only when the message dimension is positive integer multiple of  $m$

is an intermediate node (a node other than a source or a terminal),  $Y_e = \sum_{e' \in \text{In}(\text{tail}(e))} A_{\{e', e\}} Y_{e'}$ , where  $Y_e, Y_{e'} \in \mathbb{F}_q^n$  and  $A_{\{e', e\}} \in \mathbb{F}_q^{n \times n}$ . And for a terminal node  $t_i$ ,  $Y_{t_i} = \sum_{e' \in \text{In}(t_i)} A_{\{e', t_i\}} Y_{e'}$ , where  $Y_{e'}, Y_{t_i} \in \mathbb{F}_q^n$ , and  $A_{\{e', t_i\}} \in \mathbb{F}_q^{n \times n}$ .

A  $(1, 1)$  vector linear code is referred as a scalar linear code. In an  $(n, n)$  vector linear code,  $n$  is said to be the message dimension. An  $(n, n)$  vector linear code is termed as an  $n$  dimensional vector linear code. A network is said to have an  $(n, n)$  vector linear solution if there exists an  $(n, n)$  vector linear code for the network which enables all the terminals in the network to receive  $n$  source symbols (in  $n$  uses of every edge) from each of their respective demanded sources. If a network has an  $n$  dimensional vector linear solution then it is said to be vector linearly solvable (for  $n = 1$ , scalar linearly solvable).

### III. MAIN RESULT

In Fig. 1, we present a network  $\mathcal{N}$  which has a vector linear solution in  $m \geq 2$  message dimension but has no vector linear solution if the message dimension is less than  $m$ .  $\mathcal{N}$  has  $m^2$  sources and  $m^m$  terminals. The sources are partitioned into  $m$  sets. The  $j^{\text{th}}$  source in the  $i^{\text{th}}$  set is denoted by  $s_{ij}$ . The source  $s_{ij}$  generates the message  $X_{ij}$ . Below we list the edges in the network:

- 1) An edge  $(s_{ij}, u_i)$  for  $1 \leq i, j \leq m$
- 2) An edge  $e_{ii} = (u_i, v_i)$  and an edge  $e_{ij} = (u_i, v_j)$  for  $1 \leq i \leq m$  and  $m+1 \leq j \leq 2m-1$
- 3) An edge  $(v_i, t_j)$  for  $1 \leq i \leq 2m-1$  and  $1 \leq j \leq m^m$

The message transmitted over the edge  $(s_{ij}, u_i)$  is denoted by  $X_{ij}$ . The message transmitted over the edge  $e_{ij}$  is denoted by  $Y_{ij}$ . The message transmitted over the edge  $(v_i, t_j)$  is denoted by  $Z_{ij}$  for  $m+1 \leq i \leq 2m-1$  and  $1 \leq j \leq m^m$ . Each terminal demands a unique tuple of  $m$  source messages where the  $i^{\text{th}}$  element of the tuple is  $X_{ij}$  for any  $j$  in the range:  $1 \leq j \leq m$ . W.l.o.g, we assume that  $t_1$  demands the source messages:  $(X_{11}, X_{21}, \dots, X_{m1})$ .

We note that for  $m = 2$ ,  $\mathcal{N}$  is the M-network. Therefore, the presented network  $\mathcal{N}$  is a generalisation of the M-network.

**Theorem 1.** For an arbitrary finite field  $\mathbb{F}_q$ , the network  $\mathcal{N}$  has a  $d$  dimensional vector linear solution over  $\mathbb{F}_q$  if and only if  $d$  is a multiple of  $m$ .

As a consequence, we have the following corollary.

**Corollary 2.** The network  $\mathcal{N}$  is not vector linearly solvable for any message dimension less than  $m$ , but has an  $m$  dimensional vector linear solution.

**Remark 1.** It was shown in [6] that the M-network does not have an odd dimensional vector linear solution. This result is a special case of Theorem 1 for the case  $m = 2$ .

#### IV. PROOF OF THEOREM 1

Our proof of Theorem 1 relies on a result related to discrete polymatroids [8]. Therefore to make the proof accessible, in the following, the definitions of discrete polymatroid, representable discrete polymatroid and discrete polymatroidal network are given [8]–[10]. The definitions are borrowed from [8]. We first fix some notations. Let  $N = \{1, 2, \dots, n\}$ . The set of all non-negative integers are denoted by  $\mathbb{Z}_{\geq 0}$  and the set of positive integers are denoted by  $\mathbb{Z}_{> 0}$ . The notation  $\mathbb{Z}_{\geq 0}^n$  indicates the set of all  $n$  length vector whose elements are in  $\mathbb{Z}_{\geq 0}$ . For an  $n$  length vector  $v$ , and  $A \subseteq N$ ,  $v(A)$  is the vector having only the components indexed by the elements of  $A$ , and  $|v(A)|$  is the sum of the components of  $v(A)$ .

**Definition 1.** Let  $\rho$  be a function that maps  $2^N$  into  $\mathbb{Z}_{\geq 0}$  and satisfies the following three conditions:

- (1)  $\rho(\emptyset) = 0$ .
- (2) If  $X \subseteq Y \subseteq N$  then  $\rho(X) \leq \rho(Y)$ .
- (3) If  $X, Y \subseteq N$ , then  $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$ .

Let  $\mathbb{D}$  be the collection of all elements  $v \in \mathbb{Z}_{\geq 0}^n$  such that  $|v(A)| \leq \rho(A)$  for  $\forall A \subseteq N$ . Then  $\mathbb{D}$  is a discrete polymatroid having rank function  $\rho$  and ground set  $N$ .

A discrete polymatroid  $\mathbb{D}$  has  $\rho_{max} = d$  if  $\rho$  follows this additional rule: for  $\forall A \subseteq N$ ,  $\rho(A) \leq d|A|$ .

By setting  $\rho_{max} = 1$ , one can completely describe a matroid with a discrete polymatroid [8]. So the discrete polymatroids can be viewed as the generalized version of matroids. Conditions (1)-(3) are known as the *polymatroidal axioms*. It is known that if the entropy function  $H(\cdot)$ , in a Shannon-type inequality, is replaced by a function that obeys the polymatroidal axioms then the inequality still remains valid [6]. We will use this fact in our proof.

**Definition 2.** A discrete polymatroid  $\mathbb{D}$  with rank function  $\rho$  and ground set  $N$ , is said to be representable over  $\mathbb{F}_q$  if there exist vector subspaces  $V_1, V_2, \dots, V_n$  of a vector space  $V$  over  $\mathbb{F}_q$  such that  $\dim(\sum_{i \in X} V_i) = \rho(X)$  for  $\forall X \subseteq N$ . The set of vector spaces  $V_i, i \in N$  is said to form a representation of  $\mathbb{D}$ . A discrete polymatroid is said to be representable if it is representable for some field.

Let  $\epsilon_{in}$  be an  $n$  length vector whose  $i^{\text{th}}$  component is one and all other components are zero. Consider a network in which the set of source messages, the set of nodes and the set of messages carried over by the edges are denoted by  $\mu$ ,  $V$  and  $\delta$  respectively. For a non-source node  $v \in V$ , let  $I(v)$  be the set of messages carried by the edges in  $In(v)$ . When  $v$  is a source, let  $I(v)$  denote the message generated at  $v$ . Also, let  $O(v)$  be the set of messages carried by the edges in  $Out(v)$ . Let  $\mathbb{D}$  be a discrete polymatroid with ground set  $N$  and the rank function  $\rho$  where  $\rho_{max} = d$ .

**Definition 3.** A network is a discrete polymatroidal network associated with  $\mathbb{D}$  if there exists a function  $f : \mu \cup \delta \rightarrow N$  such that the following conditions are satisfied:

- (1)  $f$  is one-to-one on  $\mu$ .
- (2)  $\sum_{i \in f(\mu)} d\epsilon_{in} \in \mathbb{D}$ .
- (3)  $\rho(f(I(x))) = \rho(f(I(x) \cup O(x)))$ ,  $\forall x \in V$ .

Note that if a network is matroidal (defined in [6]) then it is also discrete polymatroidal for  $\rho_{max} = 1$ . The following theorem is reproduction of *Theorem 1* from [8].

**Theorem 3.** A network has a  $k$  dimensional vector linear solution over  $\mathbb{F}_q$  if and only if it is discrete polymatroidal with respect to a discrete polymatroid  $\mathbb{D}$  representable over  $\mathbb{F}_q$  with  $\rho_{max} = k$ .

*Proof of Theorem 1:* The proof is similar to that used in [6] to show that the M-network is not matroidal. Say the function  $f$  maps the network  $\mathcal{N}$  to a discrete polymatroid  $\mathbb{D}$  conforming to the rules of mapping presented in Definition 3. Also let  $g = \rho \circ f$  where  $\rho$  is the rank function of  $\mathbb{D}$ . Assume  $\rho_{max} = d$ . Our proof depends on the following two sets of inequalities.

**Set I:**

$$g(Y_{11}, X_{1j_1}) + g(Y_{22}, X_{2j_2}) + \dots + g(Y_{mm}, X_{mj_m}) \leq (2m-1)d \text{ for } j_i \in \{1, 2, \dots, m\}, 1 \leq i \leq m \quad (1)$$

**Set II:**

$$g(Y_{ii}, X_{i1}) + g(Y_{ii}, X_{i2}) + \dots + g(Y_{ii}, X_{im}) \geq (2m-1)d \text{ for } 1 \leq i \leq m \quad (2)$$

**Claim 1.** The inequalities in Set I hold true.

*Proof:* We give the proof for  $j_i = 1$ , for  $1 \leq i \leq m$ . The rest of the inequalities can be proved in a similar way. To prove our claim we use the following fact about the entropy function: for any set of random variables  $A_j$ ,  $1 \leq j \leq k$ ,

$$H(A_1) + H(A_2) + \dots + H(A_k) = H(A_1, A_2, \dots, A_k) \text{ iff } H(A_i | A_1, A_2, \dots, A_{i-1}) = H(A_i) \text{ for } 2 \leq i \leq k \quad (3)$$

Now in  $\mathcal{N}_1$ ,  $H(Y_{22}, X_{21} | Y_{11}, X_{11}) = H(Y_{22}, X_{21})$ .

Similarly, for  $1 \leq i, j, l, k \leq m$ ,  $H(Y_{ii}, X_{ij} | \cup_{k \neq i} Y_{kk}, \cup_{k \neq i} X_{kl}) = H(Y_{ii}, X_{ij})$ . Therefore,

$$H(Y_{11}, X_{11}) + H(Y_{22}, X_{21}) + \dots + H(Y_{mm}, X_{m1}) = H(Y_{11}, X_{11}, Y_{22}, X_{21}, \dots, Y_{mm}, X_{m1}) \quad (4)$$

Since  $g$  obeys polymatroid axioms, it also obeys (4) if  $H$  is replaced by  $g$ . Thus,

$$\begin{aligned}
& g(Y_{11}, X_{11}) + g(Y_{22}, X_{21}) + \cdots + g(Y_{mm}, X_{m1}) \\
&= g(Y_{11}, X_{11}, Y_{22}, X_{21}, \dots, Y_{mm}, X_{m1}) \\
&\leq g(Y_{11}, X_{11}, Y_{22}, X_{21}, \dots, Y_{mm}, X_{m1}, Z_{(m+1,1)}, Z_{(m+2,1)}, \dots, Z_{(2m-1,1)}) \\
&= g(Y_{11}, Y_{22}, \dots, Y_{mm}, Z_{(m+1,1)}, \dots, Z_{(2m-1,1)}) \\
&\leq (2m-1)d
\end{aligned} \tag{5}$$

The equation (5) is true because the terminal  $t_1$  computes  $(X_{11}, X_{21}, \dots, X_{m1})$  from the messages  $\{Y_{11}, Y_{22}, \dots, Y_{mm}, Z_{(m+1,1)}, Z_{(m+2,1)}, \dots, Z_{(2m-1,1)}\}$ . Equation (6) is true because each element can have rank maximum of  $d$  and there are  $(2m-1)$  elements. This concludes the proof of Claim 1. ■

**Claim 2.** *The inequalities in Set II hold true.*

*Proof:* We will give the proof of the inequality for  $i = 1$ . The rest can be proved similarly. First we show that  $g(Y_{ii}) = g(Y_{ij}) = d$  for  $1 \leq i \leq m, m+1 \leq j \leq 2m-1$ . Since all source messages are independent,

$$\begin{aligned}
m^2d &= g(X_{11}, X_{12}, \dots, X_{mm}) \\
&\leq g(X_{11}, \dots, X_{mm}, Y_{11}, \dots, Y_{mm}, Y_{1(m+1)}, \dots, Y_{m(2m-1)}) \\
&= g(Y_{11}, Y_{22}, \dots, Y_{mm}, Y_{1(m+1)}, Y_{1(m+2)}, \dots, Y_{m(2m-1)}) \\
&\leq g(Y_{11}) + g(Y_{22}) + \cdots + g(Y_{mm}) + g(Y_{1(m+1)}) + g(Y_{1(m+2)}) + \cdots + g(Y_{m(2m-1)}) \\
&\leq md + m(m-1)d = m^2d
\end{aligned} \tag{7}$$

Equality in (7) follows because every symbol is demanded by some terminal. Hence,

$$g(Y_{11}) + g(Y_{22}) + \cdots + g(Y_{mm}) + g(Y_{1(m+1)}) + g(Y_{1(m+2)}) + \cdots + g(Y_{m(2m-1)}) = m^2d \tag{8}$$

Since, there are  $m^2$  terms and each term can take a maximum value of  $d$ ,  $g(Y_{ii}) = g(Y_{ij}) = d$  for  $1 \leq i \leq m$  and  $m+1 \leq j \leq 2m-1$ . Now we prove the inequality:

$$\begin{aligned}
& g(Y_{11}, X_{11}) + g(Y_{11}, X_{12}) + \cdots + g(Y_{11}, X_{1m}) \\
&\geq g(Y_{11}, X_{11}, X_{12}) + g(Y_{11}) + \cdots + g(Y_{11}, X_{1m}) \\
&\geq g(Y_{11}, X_{11}, X_{12}, X_{13}) + 2g(Y_{11}) + \cdots + g(Y_{11}, X_{1m}) \\
&\quad \vdots \quad \quad \quad \vdots \\
&\geq g(Y_{11}, X_{11}, X_{12}, \dots, X_{1m}) + (m-1)g(Y_{11}) \\
&= md + (m-1)d = (2m-1)d
\end{aligned}$$

Here we have used condition (3) from definition 1 repeatedly. This concludes the proof of Claim 2. ■

We prove the theorem by finding a constraint on the rank function using the inequalities in Set I and Set II. We show that  $g(Y_{ii}, X_{ij}) = \frac{(2m-1)d}{m}$  for  $1 \leq i, j \leq m$ . We will give the proof only for  $g(Y_{mm}, X_{m1}) = \frac{(2m-1)d}{m}$ . The rest can be proved similarly. To prove that  $g(Y_{mm}, X_{m1}) = \frac{(2m-1)d}{m}$ , we consider an inequality from Set I which has  $g(Y_{mm}, X_{m1})$  on the left hand side. We then eliminate (one by one) all the rest terms except  $g(Y_{mm}, X_{m1})$  from left hand side using other inequalities from the Set I and inequalities from Set II. Consider the inequality:

$$g(Y_{11}, X_{11}) + g(Y_{22}, X_{21}) + \cdots + g(Y_{mm}, X_{m1}) \leq (2m-1)d \tag{9}$$

Now consider all other inequalities from Set I which differ only at the first term than the above inequality. There are exactly  $m-1$  such inequalities. These inequalities are written below:

$$\begin{aligned}
& g(Y_{11}, X_{12}) + g(Y_{22}, X_{21}) + \cdots + g(Y_{mm}, X_{m1}) \leq (2m-1)d \\
& g(Y_{11}, X_{13}) + g(Y_{22}, X_{21}) + \cdots + g(Y_{mm}, X_{m1}) \leq (2m-1)d \\
& \quad \vdots \quad \quad \quad \vdots \\
& g(Y_{11}, X_{1m}) + g(Y_{22}, X_{21}) + \cdots + g(Y_{mm}, X_{m1}) \leq (2m-1)d
\end{aligned}$$

Summing up all of the above  $m-1$  inequalities and the inequality in the equation (9), we get:

$$g(Y_{11}, X_{11}) + g(Y_{11}, X_{12}) + \cdots + g(Y_{11}, X_{1m}) + m\{g(Y_{22}, X_{21}) + \cdots + g(Y_{mm}, X_{m1})\} \leq m(2m-1)d$$

From Set II, we know that  $g(Y_{11}, X_{11}) + g(Y_{11}, X_{12}) + \dots + g(Y_{11}, X_{1m}) \geq (2m-1)d$ . Substituting this in the above equation we get:

$$m\{g(Y_{22}, X_{21}) + g(Y_{33}, X_{31}) + \dots + g(Y_{mm}, X_{m1})\} \leq m(2m-1)d - (2m-1)d \quad (10)$$

Note that the term  $g(Y_{11}, X_{11})$  has been eliminated in the equation (9). In the similar manner as above, we can show that

$$m\{g(Y_{22}, X_{2j}) + g(Y_{33}, X_{31}) + \dots + g(Y_{mm}, X_{m1})\} \leq m(2m-1)d - (2m-1)d \text{ for } 2 \leq j \leq m \quad (11)$$

Summing up the above  $m-1$  inequalities and the inequality in the equation (10), we get:

$$\begin{aligned} mg(Y_{22}, X_{21}) + mg(Y_{22}, X_{22}) + \dots + mg(Y_{22}, X_{2m}) + m^2g(Y_{33}, X_{31}) + \dots + m^2g(Y_{mm}, X_{m1}) \\ \leq m^2(2m-1)d - m(2m-1)d \end{aligned} \quad (12)$$

From Set II, we have  $g(Y_{22}, X_{21}) + g(Y_{22}, X_{22}) + \dots + g(Y_{11}, X_{2m}) \geq (2m-1)d$ . Using this inequality in the equation (12), we have:

$$m^2g(Y_{33}, X_{31}) + \dots + m^2g(Y_{mm}, X_{m1}) \leq m^2(2m-1)d - 2m(2m-1)d \quad (13)$$

Note that, in the above inequality, the term  $g(Y_{22}, X_{21})$  from the equation (10) has been eliminated and thereby the terms  $g(Y_{11}, X_{11})$  and  $g(Y_{22}, X_{21})$  from the equation (9) have been eliminated. In this way, eliminating term after term from the left hand side of the equation (9), we get

$$m^{m-1}g(Y_{mm}, X_{m1}) \leq m^{m-1}(2m-1)d - (m-1)m^{m-2}(2m-1)d \quad (14)$$

$$\text{And hence, } g(Y_{mm}, X_{m1}) \leq \frac{(2m-1)d}{m} \quad (15)$$

Similarly it can be shown that

$$g(Y_{mm}, X_{mj}) \leq \frac{(2m-1)d}{m} \text{ for } 2 \leq j \leq m \quad (16)$$

From Set II, we have that  $g(Y_{mm}, X_{m1}) + g(Y_{mm}, X_{m2}) + \dots + g(Y_{mm}, X_{mm}) \geq (2m-1)d$ . Hence, it must be that

$$g(Y_{mm}, X_{m1}) = \frac{(2m-1)d}{m} \quad (17)$$

Note that  $\gcd(2m-1, m) = 1$ . Also, by definition, the rank function is integer valued. Therefore, for  $g(Y_{ii}, X_{ij})$  to be a positive integer,  $d$  has to be a positive integer multiple of  $m$ . Thus, by Theorem 3, for  $\mathcal{N}$  to be vector linearly solvable, it is necessary that the message dimension is a positive integer multiple of  $m$ .

Now we describe a coding scheme for the network achieving an  $m$  dimensional vector linear solution. In fact, our coding scheme is a routing scheme. Let the  $k^{th}$  symbol of the source  $s_{ij}$  is denoted by  $X_{ijk}$  where  $1 \leq k \leq m$ . The edge  $e_{ii}$  for  $1 \leq i \leq m$  carries the following  $m$  length vector:  $[X_{i11}, X_{i21}, \dots, X_{im1}]$ . And the edge  $e_{ij}$  for  $1 \leq i \leq m$  and  $m+1 \leq j \leq 2m-1$ , carries the vector  $[X_{i1(j-m+1)}, X_{i2(j-m+1)}, \dots, X_{im(j-m+1)}]$ . Now, for any terminal it can be seen that the demands can be satisfied just by routing the required symbols from  $v_i$  for  $1 \leq i \leq 2m-1$  to the terminals. For example, the demands of the terminal  $t_1$  is met in the following way: the terminal  $t_1$  gets  $X_{i11}$  from the message coming from  $(v_i, t_1)$  for  $1 \leq i \leq m$ ; and  $[X_{11(j+1)}, X_{21(j+1)}, \dots, X_{m1(j+1)}]$  for  $1 \leq j \leq m-1$  from the edge  $(v_{m+j}, t_1)$ . ■

## V. CONCLUSION

In this paper, we have shown that for any integer  $m \geq 2$ , there exists a network which has a  $m$  dimensional vector linear solution, but has no vector linear solution when message dimension is less than  $m$ . Our future research objective is to investigate the necessary and sufficient conditions for a network to have a 2-dimensional vector linear solution, but no scalar linear solution.

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