# Asymptotically Optimal Codes Correcting Fixed-Length Duplication Errors in DNA Storage Systems 

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#### Abstract

A (tandem) duplication of length $k$ is an insertion of an exact copy of a substring of length $k$ next to its original position. This and related types of impairments are of relevance in modeling communication in the presence of synchronization errors, as well as in several information storage applications. We demonstrate that Levenshtein's construction of binary codes correcting insertions of zeros is, with minor modifications, applicable also to channels with arbitrary alphabets and with duplication errors of arbitrary (but fixed) length $k$. Furthermore, we derive bounds on the cardinality of optimal $q$-ary codes correcting up to $t$ duplications of length $k$, and establish the following corollaries in the asymptotic regime of growing block-length: 1) the presented family of codes is optimal for every $q, t, k$, in the sense of the asymptotic scaling of code redundancy; 2) the upper bound, when specialized to $q=2, k=1$, improves upon Levenshtein's bound for every $t \geq 3 ; 3$ ) the bounds coincide for $t=1$, thus yielding the exact asymptotic behavior of the size of optimal single-duplication-correcting codes.


Index Terms-Tandem duplication, sticky insertion, deletions of zeros, repetition error, synchronization error, bounds on codes, Sidon set, magnetic storage, DNA storage.

## I. Introduction and Preliminaries

THE EMERGING technology of DNA data storage [14], apart from having a multitude of applications, poses interesting new challenges to the traditional lines of research in information theory and error control coding. In particular, several channel models arise in this context that are typically not encountered in more conventional data transmission and storage systems. Motivated by one such model that was introduced recently, we address in this letter the problem of error correction in channels where the only impairments are duplications of substrings in the transmitted string of symbols. Although the main motivating examples are channels with binary or quaternary alphabets, in the interest of generality we will in fact study channels with arbitrary alphabets. In the following two subsections we describe precisely the channel model we have in mind and our contributions.

## A. The Channel Model

Throughout this letter, $\mathbb{Z}$ denotes the integers, $\mathbb{N}$ the positive integers, and $\mathbb{Z}_{q}:=\mathbb{Z} /(q \mathbb{Z})$ the integers modulo $q$.

We assume that the channel alphabet, both input and output, is $\mathbb{Z}_{q}$. The channel inputs are strings of length $n$ over $\mathbb{Z}_{q}$, i.e.,

[^0]elements of $\mathbb{Z}_{q}^{n}$. The channel acts on the transmitted strings by introducing multiple duplication errors of length $k$ in succession, where a duplication of length $k$ is defined as an insertion of an exact copy of a substring of length $k$ next to its original position; see Example 1 for an illustration. We refer to this channel as the $k$-duplication channel.
Example 1. Consider the following input string $\tilde{\boldsymbol{x}} \in \mathbb{Z}_{3}^{10}$ and the corresponding output string $\tilde{\boldsymbol{y}}$ obtained after the channel has introduced several duplication errors of length $k=3$ :
\[

\left.$$
\begin{array}{rl}
\tilde{\boldsymbol{x}} & =01120211002 \\
& \hookrightarrow  \tag{1}\\
& 011
\end{array}
$$\right)
\]

The inserted substrings at each step are underlined. The total number of duplications that occurred in the channel is 3. $\boldsymbol{\Delta}$

By using the transformation $\phi_{k}: \mathbb{Z}_{q}^{n} \rightarrow \mathbb{Z}_{q}^{n}, \tilde{\boldsymbol{x}} \mapsto \boldsymbol{x}$, defined by $x_{i}=\tilde{x}_{i}-\tilde{x}_{i-k}, 1 \leq i \leq n$, where subtraction is performed modulo $q$ and it is understood that $\tilde{x}_{i}=0$ for $i \leq 0$, one can show that duplication errors of length $k$ are essentially equivalent to insertions of blocks of $k$ zeros, denoted $0^{k}$ [5]. For example, for the strings in (1) and $k=3$ we would have:

$$
\begin{align*}
& \boldsymbol{x}=0112212011 \\
& \boldsymbol{y}=0112 \underline{0} 002 \underline{0} 001201 \underline{000} 1 . \tag{2}
\end{align*}
$$

In particular, if a code $\mathcal{C} \subseteq \mathbb{Z}_{q}^{n}$ can correct $t$ insertions of blocks $0^{k}$, then $\tilde{\mathcal{C}}=\phi_{k}^{-1}(\mathcal{C})$ can correct $t$ duplications of length $k$; furthermore, since $\phi_{k}$ is a bijection, we have $|\mathcal{C}|=|\tilde{\mathcal{C}}|$. For convenience, we will focus in the sequel on the $0^{k}$-insertion channel-the channel with insertions of blocks $0^{k}$ as the only type of noise. Due to the above-described equivalence, our main results can easily be translated to the corresponding results for the $k$-duplication channel: (1) asymptotic bounds on codes for the $0^{k}$-insertion channel are automatically valid for the $k$-duplication channel as well, and (2) a construction of codes for the $k$-duplication channel can be obtained from a construction of codes for the $0^{k}$-insertion channel by applying the transformation $\phi_{k}^{-1}$ on the latter.

## B. Previous Work and Main Results

The binary channel with insertions of zeros was first studied in [11], where a construction of codes correcting $t$ such errors was described and bounds on the cardinality of optimal codes derived. As mentioned in the previous subsection, these results are applicable to channels with duplication errors of length $k=1$ as well. Different constructions of codes for the binary 1-duplication channel were subsequently given in [3], [12].

A more general model, that is also studied here, with arbitrary alphabets and duplications of length $k$ was introduced in [5]. In that work, in particular, optimal codes correcting all patterns of duplications of length $k$ were found $(t=\infty)$. It was also shown in [5] that optimal codes correcting $t \in \mathbb{N}$ duplications of length $k$ can be obtained from optimal codes in the $\ell_{1}$ metric. However, constructions of optimal codes in the $\ell_{1}$ metric for general parameters are not known at this point, and hence no estimate of the cardinality of the resulting duplication-correcting codes was given in [5]. An explicit construction of codes for the special case $t=1$ was recently given in [10].

Our contributions can be summarized as follows:

- We show that $q$-ary codes correcting $t$ insertions of blocks $0^{k}$ can be constructed from Sidon sets, a notion borrowed from additive combinatorics (Theorem 2).
- We derive bounds on the cardinality of optimal codes of length $n \rightarrow \infty$ correcting $t$ insertions of blocks $0^{k}$ (Theorem 4). In particular, we obtain the exact asymptotic behavior of the size of optimal single-duplicationcorrecting codes $(t=1)$, for arbitrary $q, k$.
- Specializing the bounds to $q=2, k=1$, we obtain an improvement over the best known upper bound from [11] (Remark 2).
While this paper was under review, another work appeared [9] addressing very similar problems-constructions and bounds on $q$-ary codes correcting $t$ duplications of length $k$. The asymptotic lower bounds obtained here and in [9] are the same, whereas our upper bound is strictly better than the one in [9], for every $q, k, t$.

Apart from error correction, various other problems concerning duplications in strings were studied in the literature; see, e.g., the references in [5], [15].

## II. Codes Correcting Insertions and Deletions of Blocks of Zeros

## A. General Properties

The $0^{k}$-insertion channel, by its definition, affects only the lengths of runs of zeros in the transmitted strings, it does not alter the non-zero symbols. In particular, the Hamming weight of the transmitted string is always preserved. This fact simplifies the analysis considerably and enables one to focus on studying constant-weight codes without loss of generality.

We say that a code $\mathcal{C} \in \mathbb{Z}_{q}^{n}$ can correct $t$ insertions (resp. deletions) of blocks $0^{k}$ if every codeword $\boldsymbol{x} \in \mathcal{C}$ can be reconstructed uniquely after inserting (resp. deleting) up to $t$ blocks $0^{k}$. We say that $\mathcal{C} \in \mathbb{Z}_{q}^{n}$ can correct $t$ insertions and deletions of blocks $0^{k}$ if every codeword $\boldsymbol{x} \in \mathcal{C}$ can be reconstructed uniquely after inserting $t_{\text {ins }}$ and deleting $t_{\text {del }}$ blocks $0^{k}$, for any $t_{\text {ins }}, t_{\text {del }}$ with $t_{\text {ins }}+t_{\text {del }} \leq t$. The following claim is a straightforward generalization of [11, Lem. 1] to arbitrary $q, k$, so the proof is omitted.

Lemma 1. The following statements are equivalent for every $q, n, t, k \in \mathbb{N}, q \geq 2$, and every code $\mathcal{C} \subseteq \mathbb{Z}_{q}^{n}$ :

- $\mathcal{C}$ can correct $t$ insertions of blocks $0^{k}$.
- $\mathcal{C}$ can correct $t$ deletions of blocks $0^{k}$.
- $\mathcal{C}$ can correct $t$ insertions and deletions of blocks $0^{k}$.

The third point of Lemma 1 in particular, will be used in the proof of Theorem 4 to optimize the upper bound on the cardinality of codes correcting insertions of blocks $0^{k}$.

## B. Construction

Let $G$ be a finite Abelian group, written additively. A set $B=\left\{b_{1}, \ldots, b_{w}\right\} \subseteq G$ is said to be a Sidon set of order $t$ (or $B_{t}$ set) if the sums $b_{i_{1}}+\cdots+b_{i_{u}}$ have different values for every choice of $u \in\{0,1, \ldots, t\}$ and $1 \leq i_{1} \leq \cdots \leq i_{u} \leq w$. Put another way, the sums $\sum_{i=1}^{w} u_{i} b_{i}$ are required to be different for all $u_{1}, \ldots, u_{w} \in \mathbb{Z}$ with $u_{i} \geq 0, \sum_{i=1}^{w} u_{i} \leq t$ (here $u_{i} b_{i}$ denotes the sum of $u_{i}$ copies of the element $b_{i} \in G$ ). These and related objects have been studied quite extensively in combinatorics and additive number theory; see [13] for references. We next describe a code construction based on the notion of Sidon sets. The construction is a generalization of the one given in 11$]^{1}$ for $q=2, k=1$.

Let $\mathrm{wt}_{\mathrm{H}}(\boldsymbol{x})$ denote the Hamming weight of the string $\boldsymbol{x} \in \mathbb{Z}_{q}^{n}$. Let also $r_{i}(\boldsymbol{x})$ denote the length of the $i$ 'th run of zeros in $\boldsymbol{x}$. In other words, if $\mathrm{wt}_{\mathrm{H}}(\boldsymbol{x})=w$, we have $\boldsymbol{x}=$ $0^{r_{0}(\boldsymbol{x})} \alpha_{1} 0^{r_{1}(\boldsymbol{x})} \alpha_{2} \cdots 0^{r_{w-1}(\boldsymbol{x})} \alpha_{w} 0^{r_{w}(\boldsymbol{x})}$, where $\alpha_{i} \in \mathbb{Z}_{q} \backslash\{0\}$.

Theorem 2. Fix $q, n, w, t, k \in \mathbb{N}, q \geq 2$, an Abelian group $G$, a subset $B=\left\{b_{1}, \ldots, b_{w}\right\} \subseteq G$, an element $b \in G$, and define the code:

$$
\begin{equation*}
\left\{\boldsymbol{x} \in \mathbb{Z}_{q}^{n}: \mathrm{wt}_{\mathrm{H}}(\boldsymbol{x})=w, \sum_{i=1}^{w}\left\lfloor\frac{r_{i}(\boldsymbol{x})}{k}\right\rfloor b_{i}=b\right\} \tag{3}
\end{equation*}
$$

If $B$ is a Sidon set of order $t$, then the code (3) can correct $t$ insertions of blocks $0^{k}$.

Proof: Let $\boldsymbol{x}$ be the transmitted codeword and suppose that, after $u$ insertions of blocks $0^{k}$ in the channel, the string $\boldsymbol{y}$ was produced at the output. If $u_{i}$ blocks $0^{k}$ were inserted in the $i$ 'th run of zeros in $\boldsymbol{x}, i=0,1, \ldots, w$, then $r_{i}(\boldsymbol{y})-r_{i}(\boldsymbol{x})=$ $u_{i} k$ and $\sum_{i=0}^{w} u_{i}=u$, where $w=\mathrm{wt}_{\mathrm{H}}(\boldsymbol{x})=\mathrm{wt}_{\mathrm{H}}(\boldsymbol{y})$. Given $\boldsymbol{y}$, the receiver computes the following check-sum:

$$
\begin{equation*}
\sum_{i=1}^{w}\left\lfloor\frac{r_{i}(\boldsymbol{y})}{k}\right\rfloor b_{i}=\sum_{i=1}^{w}\left(\left\lfloor\frac{r_{i}(\boldsymbol{x})}{k}\right\rfloor+u_{i}\right) b_{i}=b+\sum_{i=1}^{w} u_{i} b_{i} \tag{4}
\end{equation*}
$$

and also infers the total number of insertions $u$ from the length of $\boldsymbol{y}$. Since $B$ is a Sidon set of order $t$, the check-sums $b+\sum_{i=1}^{w} u_{i} b_{i}$ are different for all $u_{1}, \ldots, u_{w}$ satisfying $u_{i} \geq 0$, $\sum_{i=1}^{w} u_{i} \leq t$. Therefore, given $\boldsymbol{y}$ and assuming that $u \leq t$, the decoder can uniquely recover the pattern of insertions $u_{0}, u_{1}, \ldots, u_{w}$ by computing (4), inferring $u_{1}, \ldots, u_{w}$ from the result, and concluding that $u_{0}=u-\sum_{i=1}^{w} u_{i}$.
Note that the construction (3) is not explicit. For it to be made "practical", one would need to describe efficient constructions of Sidon sets, optimal ways of choosing the element $b$, and explicit mappings of information sequences to codewords. Describing explicit and efficient constructions for this and related channel models is an important problem that we shall have to leave for future investigation.

[^1]
## C. Bounds

The following notation is used in the rest of this section: given two non-negative real sequences $\left(a_{n}\right)$ and $\left(b_{n}\right), a_{n} \sim b_{n}$ stands for $\lim _{n \rightarrow \infty} \frac{a_{n}}{b_{n}}=1, a_{n} \lesssim b_{n}$ for $\lim \sup _{n \rightarrow \infty} \frac{a_{n}}{b_{n}} \leq 1$, and $a_{n}=o\left(b_{n}\right)$ for $\lim _{n \rightarrow \infty} \frac{a_{n}}{b_{n}}=0$. The base- 2 logarithm is denoted by log.
We first give one auxiliary result that will be needed in the derivation of the bounds in Theorem4 Informally, it states that the "typical" values of the Hamming weight and the number of runs of zeros of length $\geq k$ in $q$-ary strings of length $n \rightarrow \infty$ are $\frac{q-1}{q} n$ and $\frac{q-1}{q^{k+1}} n$, respectively. To state the lemma precisely, let us denote by $S_{q}^{(\geq k)}(n, w, m)$ the number of $q$-ary strings of length $n$, Hamming weight $w$, and having exactly $m$ runs of zeros of length $\geq k$.

Lemma 3. Fix $q, t, k \in \mathbb{N}, q \geq 2$, and define $\omega_{q}:=(q-1) / q$ and $\mu_{q, k}:=\omega_{q}\left(1-\omega_{q}\right)^{k}=(q-1) / q^{k+1}$. There exists a sub-linear function ${ }^{2} f(n)=o(n)$ such that, for all $n \geq 1$,

$$
\begin{equation*}
q^{n}-\sum_{\substack{w, m:\left|w-\omega_{q} n\right| \leq f(n),\left|m-\mu_{q, k} n\right| \leq f(n)}} S_{q}^{(\geq k)}(n, w, m)<\frac{q^{n}}{n^{\log n}} \tag{5}
\end{equation*}
$$

Proof: The analysis parallels that in [8, Sec. II.B], the main difference being that the alphabet is $q$-ary in our case, so we only give an outline. Denote by $S_{q}^{(j)}(n, w, \ell)$ the number of $q$-ary strings of length $n$, Hamming weight $w$, and having exactly $\ell$ runs of zeros of length $j$. In the asymptotic regime $n \rightarrow \infty, w \sim \omega n, \ell \sim \lambda n$, for fixed $\omega \in(0,1), \lambda \in(0, \omega)$, this quantity grows exponentially with the exponent [8]

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \frac{1}{n} \log S_{q}^{(j)}(n, \omega n, \lambda n)= \\
& \omega \log (q-1)+\omega H\left(\frac{\lambda}{\omega}\right)+(\omega-\lambda) \log \sum_{\substack{i=1 \\
i \neq j}}^{\infty} \rho_{\omega, \lambda}^{i-\frac{1-\lambda(j+1)}{\omega-\lambda}} \tag{6}
\end{align*}
$$

where $H(\cdot)$ is the binary entropy function, and $\rho_{\omega, \lambda}$ is the unique positive solution to the equation:

$$
\begin{equation*}
\sum_{\substack{i=1 \\ i \neq j}}^{\infty}\left(i-\frac{1-\lambda(j+1)}{\omega-\lambda}\right) z^{i}=0 \tag{7}
\end{equation*}
$$

Now, since the total number of $q$-ary strings of length $n$ is $q^{n}$, and since there are only linearly (in $n$ ) many possible weights $w$ and numbers of runs $\ell$, there must exist values of $\omega$ and $\lambda$ for which the right-hand side of (6) (the exponent) equals $\log q$. Differentiating this exponent with respect to $\omega$ and $\lambda$, one finds that it is uniquely maximized for $\omega=\omega_{q}=\frac{q-1}{q}$ and $\lambda=\omega_{q}^{2}\left(1-\omega_{q}\right)^{j}=: \lambda_{q, j}$. This implies that, for any given $\epsilon>0$, if we exclude the strings of weight $w \in\left(\left(\omega_{q}-\epsilon\right) n,\left(\omega_{q}+\epsilon\right) n\right)$ having $\ell \in$ $\left(\left(\lambda_{q, j}-\epsilon\right) n,\left(\lambda_{q, j}+\epsilon\right) n\right)$ runs of zeros of length $j$, the number of the remaining strings is exponential with an exponent strictly smaller than $\log q$. In other words, for every $\epsilon>0$ there exists a (sufficiently small) $\delta(\epsilon)>0$ such that, as $n \rightarrow \infty$,

$$
\begin{equation*}
q^{n}-\sum_{\substack{w, \ell:\left|w-\omega_{q} n\right| \leq \epsilon n \\\left|\ell-\lambda_{q, j} n\right| \leq \epsilon n}} S_{q}^{(j)}(n, w, \ell) \lesssim q^{(1-\delta(\epsilon)) n} \tag{8}
\end{equation*}
$$

[^2]This further implies that, for every $\epsilon>0$ and large enough $n$,

$$
\begin{equation*}
q^{n}-\sum_{\substack{w, \ell:\left|w-\omega_{q} n\right| \leq \epsilon n,\left|\ell-\lambda_{q, j} n\right| \leq \epsilon n}} S_{q}^{(j)}(n, w, \ell)<\frac{q^{n}}{n^{\log n}} . \tag{9}
\end{equation*}
$$

Let $n_{0}(\epsilon)$ be the smallest positive integer such that (9) holds for all $n \geq n_{0}(\epsilon)$. Take an arbitrary sequence $\left(\epsilon_{i}\right)$ satisfying $1=\epsilon_{0}>\epsilon_{1}>\epsilon_{2}>\ldots$ and $\lim _{i \rightarrow \infty} \epsilon_{i}=0$, and define the function:

$$
\begin{equation*}
f^{\prime}(n):=\epsilon_{i} n, \quad n_{0}\left(\epsilon_{i}\right) \leq n<n_{0}\left(\epsilon_{i+1}\right) . \tag{10}
\end{equation*}
$$

Clearly, $f^{\prime}(n)=o(n)$. Furthermore, from (9) and (10) we conclude that, for all $n \geq n_{0}(1)=1$,

$$
\begin{equation*}
q^{n}-\sum_{\substack{w, \ell:\left|w-\omega_{q} n\right| \leq f^{\prime}(n),\left|\ell-\lambda_{q, j} n\right| \leq f^{\prime}(n)}} S_{q}^{(j)}(n, w, \ell)<\frac{q^{n}}{n^{\log n}} \tag{11}
\end{equation*}
$$

which essentially completes the proof. It is now not difficult to conclude that the relation (5) holds as well (with a possibly different sub-linear function, $f$ ). The typical value of the number of runs of length $\geq k$ is obtained simply by adding up the typical values of the numbers of runs of length $j$ : $\sum_{j=k}^{\infty} \lambda_{q, j}=\omega_{q}\left(1-\omega_{q}\right)^{k}=\mu_{q, k}$.

It follows from the above proof that Lemma 3 continues to hold if $n^{\log n}$ is replaced with an arbitrary sub-exponential function, but this choice is sufficient for our purposes. In particular, since $\frac{q^{n}}{n^{\log n}}=o\left(\frac{q^{n}}{n^{t}}\right)$ for any fixed $t$, Lemma 3 will enable us to disregard the non-typical input strings in the asymptotic analysis of the size of optimal codes.
Let $M_{q}(n ; t ; k)$ denote the size of an optimal code in $\mathbb{Z}_{q}^{n}$ correcting $t$ insertions of blocks $0^{k}$ (or, equivalently, $t$ insertions and deletions of blocks $0^{k}$; see Lemma 11, and $M_{q}(n, w ; t ; k)$ the size of an optimal constant-weight code with the same properties and weight $w$.
Theorem 4. For any fixed $q, t, k \in \mathbb{N}, q \geq 2$, the following bounds hold as $n \rightarrow \infty$ :

$$
\begin{equation*}
\frac{q^{n}}{n^{t}}\left(\frac{q}{q-1}\right)^{t} \lesssim M_{q}(n ; t ; k) \lesssim \frac{q^{n}}{n^{t}}\left(\frac{q}{q-1}\right)^{t} q^{k s} s!(t-s)!, \tag{12}
\end{equation*}
$$

where $s=\left\lfloor\frac{t+1}{q^{k}+1}\right\rfloor$. In particular, for $t=1$,

$$
\begin{equation*}
M_{q}(n ; 1 ; k) \sim \frac{q^{n}}{n} \cdot \frac{q}{q-1} \tag{13}
\end{equation*}
$$

Proof: The lower bound in (12) is a consequence of the construction in Theorem 2 For fixed $q, n, w, t, k$, and a Sidon set $B \subseteq G$ of order $t$, the only parameter that is left to be specified in (3) is $b \in G$. Since the choice of $b$ can be made in $|G|$ ways, resulting in at most $|G|$ (disjoint) codes, and since the total number of $q$-ary strings of length $n$ and weight $w$ is $\binom{n}{w}(q-1)^{w}=: S_{q}(n, w)$, we conclude from Theorem 2 that $M_{q}(n, w ; t ; k) \geq S_{q}(n, w) /|G|$. By the result of Bose and Chowla [1], the cardinality of the smallest Abelian group containing a Sidon set of order $t$ and size $w$ can be upper bounded as $|G| \lesssim w^{t}$, for any fixed $t$ and $w \rightarrow \infty$. This implies that, as $n \rightarrow \infty$ and $w \sim \omega n$,

$$
\begin{equation*}
M_{q}(n, w ; t ; k) \gtrsim \frac{S_{q}(n, w)}{w^{t}} \tag{14}
\end{equation*}
$$

Now, to obtain the lower bound in (12), write:

$$
\begin{align*}
M_{q}(n ; t ; k) & =\sum_{w=0}^{n} M_{q}(n, w ; t ; k)  \tag{15}\\
& \geq \sum_{w=\omega_{q} n-f(n)}^{\omega_{q} n+f(n)} M_{q}(n, w ; t ; k)  \tag{16}\\
& \gtrsim \frac{1}{\left(\omega_{q} n+f(n)\right)^{t}} \sum_{w=\omega_{q} n-f(n)}^{\omega_{q} n+f(n)} S_{q}(n, w)  \tag{17}\\
& \sim \frac{q^{n}}{\left(\omega_{q} n\right)^{t}} \tag{18}
\end{align*}
$$

where (15) holds because the channel does not affect the Hamming weight of the transmitted string, (17) follows from (14), and (18) follows from Lemma 3 and the fact that $f(n)=o(n)$.

We now turn to the upper bound in (12). Let $\mathcal{C}^{*} \subseteq \mathbb{Z}_{q}^{n}$ be an optimal code correcting $t$ insertions and deletions of blocks $0^{k},\left|\mathcal{C}^{*}\right|=M_{q}(n ; t ; k)$. Consider a codeword $\boldsymbol{x} \in \mathcal{C}^{*}$ of weight $w$ and having $m$ runs of zeros of length $\geq k$. We first observe that the number of strings that can be produced after $\boldsymbol{x}$ is impaired by $s$ insertions and $t-s$ deletions of blocks $0^{k}$ is at least

$$
\begin{equation*}
\binom{w+s}{s}\binom{m-s}{t-s} \tag{19}
\end{equation*}
$$

and that all such strings are of length $n+k(2 s-t)$. Namely, since $\mathrm{wt}_{\mathrm{H}}(\boldsymbol{x})=w$, there are $w+1$ "bins" in which blocks can be inserted, so inserting $s$ blocks can be done in exactly $\binom{w+s}{s}$ ways. On the other hand, deleting $t-s$ blocks can be done in at least $\binom{m-s}{t-s}$ ways (we choose $t-s$ out of $m$ runs of length $\geq k$ and delete one block from each of them; however, we first exclude from these $m$ runs those runs into which a block has been inserted in the first step, because otherwise we could potentially get the same string we started with). In the asymptotic regime $n \rightarrow \infty, w \sim \omega n, m \sim \mu n$, the quantity in (19) scales as

$$
\begin{equation*}
\sim\binom{\omega n}{s}\binom{\mu n}{t-s} \sim n^{t} \frac{\omega^{s}}{s!} \frac{\mu^{t-s}}{(t-s)!} \tag{20}
\end{equation*}
$$

Now, since $\mathcal{C}^{*}$ is assumed to correct $t$ insertions and deletions of blocks $0^{k}$, the sets of output strings that can be obtained in the above-described way from any two distinct codewords have to be disjoint. Since these outputs live in $\mathbb{Z}_{q}^{n+k(2 s-t)}$, and since, in the asymptotic regime of interest, we can assume that $\omega$ and $\mu$ take on their typical values $\omega_{q}$ and $\mu_{q, k}$ (see Lemma 3), we conclude that

$$
\begin{align*}
& M_{q}(n ; t ; k) \cdot n^{t} \frac{\omega_{q}^{s}}{s!} \frac{\mu_{q, k}^{t-s}}{(t-s)!} \lesssim q^{n+k(2 s-t)} \\
\Leftrightarrow \quad & M_{q}(n ; t ; k) \lesssim \frac{q^{n}}{n^{t}}\left(\frac{q}{q-1}\right)^{t} q^{k s} s!(t-s)! \tag{21}
\end{align*}
$$

It is left to optimize the bound over the possible choices of $s \in\{0,1, \ldots, t\}$. To that end note that the sequence $a_{s}:=$ $q^{k s} s!(t-s)!$ is convex since $a_{s}<\sqrt{a_{s-1} a_{s+1}} \leq \frac{1}{2}\left(a_{s-1}+\right.$ $\left.a_{s+1}\right)$. This implies that $a_{s}$ is minimized at the value of $s$ for which $a_{s} \leq a_{s-1}$ and $a_{s}<a_{s+1}$. By checking these conditions directly, we find this value to be $s=\left\lfloor\frac{t+1}{q^{k}+1}\right\rfloor$.

Remark 1. Note that the lower bound in (12) is independent of the duplication length $k$. An upper bound independent of $k$ can also be obtained by choosing a suboptimal value $s=0$ in (21), which gives $M_{q}(n ; t ; k) \lesssim \frac{q^{n}}{n^{t}}\left(\frac{q}{q-1}\right)^{t} t!$. Therefore, the duplication length does not seem to have a significant bearing on the problem addressed here (see also (13)).
Remark 2 (Binary channel with insertions/deletions of zeros). Specializing the bounds (12) to $q=2, k=1$, we get:

$$
\begin{equation*}
\frac{2^{n}}{n^{t}} 2^{t} \lesssim M_{2}(n ; t ; 1) \lesssim \frac{2^{n}}{n^{t}} 2^{t+s} s!(t-s)! \tag{22}
\end{equation*}
$$

where $s=\left\lfloor\frac{t+1}{3}\right\rfloor$. The lower bound in (22) was obtained 3 in [11, Lem. 3]. The upper bound in (22) strictly improves upon the bound 4 from [11] Lem. 2] for all $t \geq 3$.

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[^1]:    ${ }^{1}$ Similar constructions of codes based on Sidon sets appear in various contexts in coding theory; see, e.g., [2], [4], [6], (7]. The algebraic version of the construction given here and in the mentioned works can also be stated geometrically using the language of lattices; see [7], [8].

[^2]:    ${ }^{2}$ The function $f$ in general depends on the constants $q, k$ as well; this is suppressed for notational simplicity.

[^3]:    ${ }^{3}$ Actually, this bound was not stated explicitly in [11] because Levenshtein was unaware of the work [1] and the construction of Sidon sets therein. Consequently, he stated in 11] an explicit lower bound which is worse than what his code construction actually implies.
    ${ }^{4}$ The upper bound in [11] is of the same form as the one in (22), but with a suboptimal choice of $s: s=0$ for $t$ odd, and $s=t / 2$ for $t$ even.

