# Bit-Metric Decoding of Non-Binary LDPC Codes with Probabilistic Amplitude Shaping

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*Abstract*—A new approach for combining non-binary lowdensity parity-check (NB-LDPC) codes with higher-order modulation and probabilistic amplitude shaping (PAS) is presented. Instead of symbol-metric decoding (SMD), a bit-metric decoder (BMD) is used so that matching the field order of the nonbinary code to the constellation size is not needed, which increases the flexibility of the coding scheme. Information rates, density evolution thresholds and finite-length simulations show that the flexibility comes at no loss of performance if PAS is used.

## I. INTRODUCTION

Higher-order modulation and advanced channel coding schemes play a central role for increasing the spectral efficiency (SE) in next-generation communication systems. For instance, the upcoming 5G standard extended the range of modulation formats from 64-quadrature amplitude modulation (QAM) to 256-QAM [1]. Non-binary codes are a natural candidate for forward error correction (FEC) schemes targeting higher-order modulation, such as *M*-amplitude shift keying (ASK) or *M*-QAM, as the codeword symbols in the finite field  $\mathbb{F}_{2^p}$  can be mapped directly to a sequence of constellation symbols, where  $M = 2^m$  and  $p = \ell \cdot m$ ,  $\ell \in \mathbb{N}$ . The receiver uses symbol-metric decoding (SMD) [2], [3] for decoding.

Most practical transponders use pragmatic schemes such as bit-interleaved coded modulation (BICM) [4] with binary codes and bit-metric decoding (BMD). BMD ignores the correlation between the bit-levels forming one higher-order constellation symbol, which results in a loss of 0.4 dB to 0.5 dB for low to medium SNR ranges in the additive white Gaussian noise (AWGN) channel (e.g., see the difference between the solid and dashed blue curves in Fig. 1). As an important benefit, BMD decouples the field size of the FEC code from the employed modulation order.

Probabilistic amplitude shaping (PAS) [5] was introduced as a layered coded modulation (CM) architecture that combines probabilistic shaping (PS) with binary FEC. PAS closes the gap to the Shannon limit and allows flexible rate adaptation. Numerical results show that PAS entails almost no loss even with BMD. Non-binary (NB) codes show excellent performance for short blocklengths and low error rates [6], which make them interesting candidates for ultra reliable communication scenarios. The authors of [7], [8] suggest extensions of PAS with NB codes that enforce a relation between the modulation order and the field size of the NB code. This property is not desired for flexible communication systems.

In this correspondance we show how non-binary low-density parity-check (NB-LDPC) codes can be operated with BMD and PAS to improve the flexibility for higher-order modulation and to avoid the BMD loss. BMD for NB-LDPC codes was

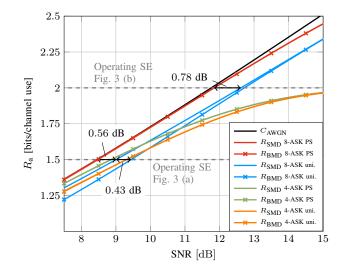


Fig. 1. Information rates for BMD and SMD for both uniform and PS constellations.

proposed in [9] to increase diversity in a fast Rayleigh fading scenario. In our work, the use of BMD with NB-LDPC codes is introduced in conjunction with PAS, as a mean to achieve flexibility from two viewpoints. First, the de-coupling of constellation size and finite field order allows using codes constructed over large order finite fields with constellations of arbitrary cardinality without placing any constraints on the matching of the two. Second, the use of PAS enables the achievement of large shaping gains and attaining a remarkable degree of flexibility with respect to transmission rates. This large flexibility comes at no visible performance loss with respect to SMD applied to NB-LDPC codes [7], [8]. Our findings are validated by Monte Carlo density evolution (DE) [10, Ch. 47.5] and finite length simulations with ultrasparse NB-LDPC codes.

## II. PRELIMINARIES

#### A. System Model

Consider transmission over a real-valued AWGN channel

$$Y_i = X_i + Z_i \tag{1}$$

for i = 1, ..., n. The alphabet of the channel input  $X_i$  is a scaled  $M = 2^m$ -ary ASK constellation  $\mathcal{X} = \{\pm 1, \pm 3, ..., \pm (M-1)\}$  such that  $\mathbb{E}[X_i^2] = 1$ . The results extend directly to QAM by using ASK for the in-phase and quadrature transmission. The noise  $Z_i$  is a Gaussian random variable with zero mean and variance  $\sigma^2$ . The signal-to-noise ratio (SNR) is  $1/\sigma^2$ . As the channel is memoryless, we drop

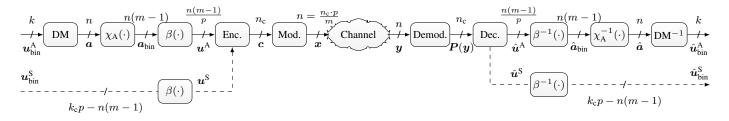


Fig. 2. Operating a rate  $R_c = k_c/n_c$  NB-LDPC code with PAS and BMD. The dashed lines are needed for code rates  $R_c > (m-1)/m$  [5, Sec. IV-D]. The functions  $\chi_A(\cdot)$  and  $\beta(\cdot)$  are applied to each amplitude in the vector  $\boldsymbol{a}$  and chunks of p consecutive bits in  $\boldsymbol{a}_{bin}$ , respectively. The decoder input is the matrix  $\boldsymbol{P}(\boldsymbol{y}) = (\boldsymbol{P}_1(\boldsymbol{y})^T, \boldsymbol{P}_2(\boldsymbol{y})^T, \dots, \boldsymbol{P}_{n_c}(\boldsymbol{y})^T)$ .

the index *i* and denote the channel density as  $p_{Y|X}$ . The mutual information maximizing distribution under an average power constraint is a zero mean Gaussian input X with unit variance, and it yields the capacity expression

$$\mathsf{C}_{\mathsf{AWGN}}(\mathsf{SNR}) = \frac{1}{2}\log_2(1 + \mathsf{SNR}). \tag{2}$$

In [11], it is shown that an achievable rate is

$$R_{\rm a} = \left[ \mathrm{H}(X) - \mathrm{E}\left[ -\log_2\left(\frac{q(X,Y)}{\sum_{x \in \mathcal{X}} q(x,Y)}\right) \right] \right]^+ \quad (3)$$

where H(X) is the entropy of the discrete random variable (RV) X,  $[\cdot]^+ = \max(0, \cdot)$  and  $q(x, y) : \mathcal{X} \times \mathbb{R} \to \mathbb{R}^+$  is the decoding metric. For SMD, the decoder uses the metric

$$q(x,y) \propto P_{X|Y}(x|y) \tag{4}$$

where  $P_{X|Y}(x|y)$  is the conditional probability of the event X = x when Y = y. The choice (4) reduces (3) to the mutual information I(X;Y) between the channel input X and channel output Y, i.e., we have

$$R_{\rm SMD}({\rm SNR}; P_X) = I(X; Y).$$
<sup>(5)</sup>

For BMD, we label each constellation point  $x \in \mathcal{X}$  with an *m*-bit binary label, i.e.,  $\chi : \mathcal{X} \to \{0,1\}^m$  and  $\chi(x) = b_1 b_2 \dots b_m = \mathbf{b}$ . Its inverse is  $\chi^{-1} : \{0,1\}^m \to \mathcal{X}$ . A binary reflected Gray code (BRGC) [12] usually performs well for BMD and the BMD decoder uses the metric

$$q(x,y) = \tilde{q}(\boldsymbol{b},y) \propto \prod_{i=1}^{m} P_{B_i|Y}(b_i|y).$$
(6)

The choice (6) reduces (3) to

$$R_{\text{BMD}}(\text{SNR}; P_X) = \left[ \mathrm{H}(\boldsymbol{B}) - \sum_{i=1}^m \mathrm{H}(B_i | Y) \right]^{+}.$$
 (7)

## B. Non-Binary LDPC Codes

A NB-LDPC code C is defined as the nullspace of the sparse parity-check matrix H of dimension  $m_c \times n_c$  where the non-zero entries of H are taken from a finite field  $\mathbb{F}_q$ , i.e.,  $C = \{c \in \mathbb{F}_q^{n_c} : cH^T = 0\}$ . In the following, we consider only extension fields of the Galois field  $\mathbb{F}_2$ , i.e., we consider  $\mathbb{F}_q$  where  $q = 2^p$ . The primitive element of  $\mathbb{F}_q$  is referred to as  $\alpha$ . The number of non-zero elements in each column  $i \in \{1, \ldots, n_c\}$  (row  $j \in \{1, \ldots, m_c\}$ ) is referred to as the corresponding variable node degree  $d_{v,i}$  (check node degree

 $d_{c,j}$ ). In the following, we use a special class of NB-LDPC codes, namely ultra-sparse regular LDPC codes [13], which have a constant variable node degree of  $d_{v,i} = d_v = 2$  and a constant check node degree  $d_c$ . Their design rate  $R_c$  is therefore  $1-2/d_c$ . We consider a full rank H in the following and perform probability-domain based decoding [14]. Decoding approaches for NB-LDPC codes with lower complexity are discussed in, e.g., [15]. They are also applicable for the proposed BMD. We also introduce the mapping  $\beta(\cdot)$  which relates a length p binary string to a field element in  $\mathbb{F}_q$ , i.e.,

$$\beta: \{0,1\}^p \to \mathbb{F}_q. \tag{8}$$

Its inverse  $\beta^{-1}(c)$  for  $c \in \mathbb{F}_q$  is the binary image of c.

#### C. Probabilistic Amplitude Shaping (PAS)

PAS is a CM scheme that combines PS with FEC [5]. It builds upon two important properties. First, the capacity achieving distribution  $P_X^*$  for the AWGN channel is symmetric. We therefore factor the input distribution into an amplitude and sign part as  $P_X(x) = P_A(|x|) \cdot P_S(\operatorname{sign}(x))$ , where  $P_A$  is non-uniform on  $\mathcal{A} = \{|x|, x \in \mathcal{X}\}$  and S is uniform on  $\{-1, +1\}$ . Second, the scheme exploits systematic encoding to preserve the non-uniform  $P_A$ . It copies the amplitudes (or a representation thereof) into the information part of the codeword and uses the approximately uniform distributed parity bits as signs. As a result, PAS requires FEC code rates with  $R_c \geq (m-1)/m$  [5, Sec. IV-B, IV-D].

In the following, we distinguish between sign and amplitude bit labels. For this, we introduce an amplitude labeling function  $\chi_A : \mathcal{A} \to \{0, 1\}^{m-1}$  such that  $\chi(x) = (b_1, b_2, \dots, b_m) = (b_1, \chi_A(|x|))$ , i.e., the sign bit is placed in the first bit-level.

The distribution matcher (DM) [16] realizes the nonuniform distribution  $P_A$  on the amplitude symbols. It takes k uniformly distributed input bits and maps them to a length n sequence of symbols with a specified empirical distribution. For PAS, the output set is the set of amplitude values  $\mathcal{A} = \{1, 3, \dots, M-1\}$ . The DM rate is  $R_{dm} = k/n$ . The transmission rate is [5, Sec. IV-D]

$$\eta = R_{\rm dm} + 1 - (1 - R_{\rm c}) \cdot m. \tag{9}$$

## III. SYMBOL-METRIC DECODING OF NB-LDPC CODES A. SMD for Uniform Signaling

For a given  $M = 2^m$ -ASK signalling constellation, we choose a  $\mathbb{F}_q$  code with  $q = \ell \cdot m, \ell \in \mathbb{N}$ , such that a length

 $\ell$  sequence of constellations points can be mapped exactly to one  $\mathbb{F}_q$  symbol using a mapping function  $\beta_{\mathcal{X}} : \mathcal{X}^{\ell} \to \mathbb{F}_q$ . The soft information for the decoder is given by a length-q vector at the *i*-th variable node by

$$\boldsymbol{P}_{i}(\boldsymbol{y}) = \left(P_{i}(\boldsymbol{y},0), P_{i}(\boldsymbol{y},1), \dots, P_{i}(\boldsymbol{y},\alpha^{q-2})\right).$$
(10)

where  $y = (y_1, \ldots, y_\ell)$ . The vector entries  $P_i(y, c)$  denote the probability that the *i*-th codeword symbol is *c* given that the associated receive sequence is y. It is calculated as

$$P_i(\boldsymbol{y}, c) \propto \prod_{j=1}^{\ell} p_{Y|X}(y_j | [\beta_{\mathcal{X}}^{-1}(c)]_j).$$
(11)

B. SMD for PAS

For PAS, a scheme with NB-LDPC codes and SMD was introduced in [8], which ensures that the desired amplitude distribution is not changed after encoding. This property can be achieved by mapping a length  $\ell \in \mathbb{N}$  sequence of amplitudes (each amplitude is represented by (m - 1) bits) to one  $\mathbb{F}_q$ symbol

and encoding them systematically. The bits forming the parity symbols (as well as potentially additional ones from the information part for  $R_c > (m-1)/m$ ) are used as signs for the amplitudes to form the channel inputs.

The soft information for the NB-LDPC decoder with SMD is calculated as shown in [8, Eqs. (9) and (10)]. This approach enforces the condition  $p = \ell \cdot (m - 1)$  between the NB code and the underlying constellation size.

#### IV. BIT-METRIC DECODING OF NB-LDPC CODES

## A. BMD for Uniform Constellations

We now describe how a NB-LDPC code can be operated with BMD. The blockwise application of (8) maps a length  $k_c \cdot p$  vector of uniformly distributed bits to  $k_c$  symbols of  $\mathbb{F}_q$ . This sequence is encoded into a length  $n_c$  symbols codeword c with binary representation  $c_{\text{bin}}$ . Eventually, the modulation mapper maps blocks of m bits to one  $2^m$ -ASK symbol

$$x_i = \chi^{-1}(c_{\text{bin},(i-1)\cdot m+1}, \dots, c_{\text{bin},i\cdot m}), \quad i = 1, \dots, n.$$

At the receiver side, the received sequence is demodulated by calculating the entries  $l_{i,j}$  of the soft information vector l

$$l_{i,j} = \log\left(\frac{P_{B_j|Y}(0|y_i)}{P_{B_j|Y}(1|y_i)}\right)$$
(12)

for i = 1, ..., n and j = 1, ..., m. The distribution  $P_{B_j|Y}(b|y)$  is

$$P_{B_j|Y}(b|y) \propto \sum_{x \in \mathcal{X}_j^b} p_{Y|X}(y|x) P_X(x)$$

where  $\mathcal{X}_{j}^{b} = \{x \in \mathcal{X} : [\chi(x)]_{j} = b\}$ . The input (10) to the NB-LDPC decoder is calculated as

$$P_i(c) = \frac{\tilde{P}_i(c)}{\sum_{c' \in \mathbb{F}_q} \tilde{P}_i(c')} \quad \text{with} \quad \tilde{P}_i(c) = \prod_{j=1}^p \tilde{P}_{ij} \qquad (13)$$

for  $i = 1, ..., n_c$  and j = 1, ..., p, where

$$\tilde{P}_{i,j} = \begin{cases} \frac{\exp(l_{i,j})}{1 + \exp(l_{i,j})}, & \text{if } [\beta_{\mathbb{F}_q}^{-1}(c)]_j = 0, \\ \frac{1}{1 + \exp(l_{i,j})}, & \text{if } [\beta_{\mathbb{F}_q}^{-1}(c)]_j = 1. \end{cases}$$
(14)

## B. BMD for PAS

The same principle as shown in Sec. IV-A can also be applied to PAS and is shown in Fig. 2. A number of kuniformly distributed information bits are matched to n amplitudes following a specified distribution. Using the amplitude mapping  $\chi_A$  the amplitudes are mapped to a length  $n \cdot (m-1)$ bit string, mapped to  $\mathbb{F}_q$  symbols and encoded into the codeword c. A modulator then maps the binary image of cto channel inputs  $x \in \mathcal{X}$  via a consecutive application of  $\chi^{-1}$ .

At the receiver side, the demapper calculates a soft information vector as shown in (12), (13) and (14) for the uniform scenario.

*Example.* Consider a length  $n_c = 3$ , rate  $R_c = 2/3$  code over  $\mathbb{F}_{32}$  (p = 5), while using an 8-ASK constellation (m = 3)such that the channel is used  $n = (n_c \cdot p)/m = 5$  times with constellation symbols  $x_1, x_2, x_3, x_4, x_5$ . The length *m* binary label of the *i*-th channel symbol is referred to as  $b_{i,1} \dots b_{i,m}$ . That is, for the given scenario, we have  $\chi(x_i) = b_{i,1}b_{i,2}b_{i,3}$ . Conventional PAS with NB codes and SMD [8] is not possible for these parameters, as p = 5 is not an integer multiple of m - 1 = 2. After encoding, the binary image of the codeword  $c = (c_1, c_2, c_3)$  is

$$c_{\mathsf{bin}} = (\underbrace{b_{1,2}b_{1,3}b_{2,2}b_{2,3}b_{3,2}}_{\beta_{\mathbb{F}_{32}}(c_1)}, \underbrace{b_{3,3}b_{4,2}b_{4,3}b_{5,2}b_{5,3}}_{\beta_{\mathbb{F}_{32}}(c_2)}, \underbrace{b_{1,1}b_{2,1}b_{3,1}b_{4,1}b_{5,1}}_{\beta_{\mathbb{F}_{32}}(c_3)})$$

The binary image of the parity symbol  $c_3 \in \mathbb{F}_{32}$  provide the signs for the five channel uses and the soft information vector reads as

$$\boldsymbol{l} = (l_{1,2}l_{1,3}l_{2,2}l_{2,3}l_{3,2}l_{3,3}l_{4,2}l_{4,3}l_{5,2}l_{5,3}l_{1,1}l_{2,1}l_{3,1}l_{4,1}l_{5,1}).$$

Eventually, the vector l is combined as shown in (13) and (14) to form the decoder a-priori soft-information.

## V. SIMULATION RESULTS

We now present numerical simulation results that target SEs of 1.5 bits per channel use (bpcu), 2.0 bpcu for 8-ASK and 3.0 bpcu for 16-ASK, respectively. As benchmark curves, we plot Shannon's sphere packing (SP) bound [17] and Gallager's Random Coding bound (RCB) [18, Theorem 5.6.2]. We evaluate the later for the shaped distributions which are also used by the demapper. The considered modes and the employed FEC code rates are summarized in the first rows of Table I.

The transmission rate is  $\eta = R_c m$  for uniform signaling and (9) for PAS. For a given code rate, we can adjust the matcher rate  $R_{dm}$  to achieve a desired transmission rate. We use Maxwell-Boltzmann (MB) distributions [19] of the form  $P_X(x) \propto \exp(-\nu x^2)$ . Numerical results indicate that MB distributions also perform well for BMD, see, e.g., [5, Table 3].

All codes are ultra-sparse NB-LDPC codes over  $\mathbb{F}_{64}$  or  $\mathbb{F}_{256}$  with a regular variable node degree of  $d_v = 2$ . The non-zero entries of the  $\mathbb{F}_{64}$  codes have been optimized rowwise via the binary-image method of [13], while the entries of the  $\mathbb{F}_{256}$  have been chosen randomly. The error floor for some  $\mathbb{F}_{64}$  codes is caused by low weight codewords and can be mitigated by ensuring the full rank condition of [13]. Observe that the finite length SMD and BMD frame error rate (FER) performance (after the inverse DM) of the PAS

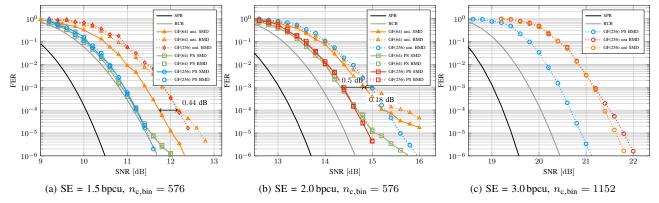


Fig. 3. Performance comparison of NB-LDPC codes for different SEs and decoding metrics.

 TABLE I

 Density Evolution Thresholds and required asymptotic SNR

 values for (5) and (7) in dB

	8-A		ASK		16-ASK	
	SE = 1.5 bpcu		SE = 2 bpcu		SE = 3 bpcu	
R <sub>c</sub>	1/2	3/4	2/3	3/4	3/4	5/6
Mode	uni.	PAS	uni.	PAS	uni.	PAS
$ \begin{array}{c} R_{\rm BMD}^{-1} ~ [\rm dB] \\ R_{\rm SMD}^{-1} ~ [\rm dB] \end{array} $	9.44	8.48	12.72	11.89	19.25	18.11
	9.00	8.46	12.61	11.87	19.17	18.10
$\mathbb{F}_{64}$ , BMD [dB] $\mathbb{F}_{64}$ , SMD [dB]	9.93 9.53	8.90 8.92	13.20 13.10	12.31 12.29	-	
$\mathbb{F}_{256}$ , BMD [dB]	9.91	8.93	13.20	12.31	19.79	18.54
$\mathbb{F}_{256}$ , SMD [dB]	-	8.93	-	12.31	19.85	-

schemes coincide in Fig. 3 (a) and (b) for all considered codes. This is also reflected in the Monte Carlo DE thresholds [10, Ch. 47.5] of Table I. Fig. 3 (c) shows a setting where BMD improves the flexibility of the modulation setup, as PAS with SMD can not be operated with 16-ASK and a NB code over  $\mathbb{F}_{256}$ . Using BMD circumvents this restriction. As expected from the DE thresholds, the performance of BMD for the uniform cases is degraded compared to SMD (compare Fig. 3 (a), (b) orange solid and dotted) for low code rates, but become similar for higher ones. In particular, for an information rate of 1.5 bpcu in Fig. 1, uniform 8-ASK has a higher BMD loss than shaped 8-ASK. For information rates above 1.5 bpcu the BMD loss of uniform 8-ASK decreases, while the gap to capacity becomes larger. In the PAS implementation, shaped 8-ASK uses a higher linear FEC code rate than uniform 8-ASK for the same information rate. In this sense, PAS allows to combine a small gap to capacity with a low BMD loss by using high rate FEC code rates.

## VI. CONCLUSION

We have shown that BMD of NB-LDPC codes with PAS achieves the same performance as SMD. Numerical simulation results confirm the information rate and DE threshold analysis. BMD of NB-LDPC codes with PAS increases the flexibility in code design as any field order can be combined with any modulation size. This is particularly important if NB codes over smaller field orders are designed for PAS, e.g., for  $\mathbb{F}_{32}$ , which could only be operated with 64-ASK in case of SMD.

This would decrease the decoding complexity significantly, while a careful code design is expected to provide similar performance as codes over high order fields.

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