# Twisted Reed-Solomon Codes With One-dimensional Hull 

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#### Abstract

The hull of a linear code is defined to be the intersection of the code and its dual. When the size of the hull is small, it has been proved that some algorithms for checking permutation equivalence of two linear codes and computing the automorphism group of a linear code are very effective in general. Maximum distance separable (MDS) codes are codes meeting the Singleton bound. Twisted Reed-Solomon codes is a generalization of Reed-Solomon codes, which is also a nice construction for MDS codes. In this short letter, we obtain some twisted Reed-Solomon MDS codes with one-dimensional hull. Moreover, these codes are not monomially equivalent to Reed-Solomon codes.


Index Terms-twisted Reed-Solomon codes, one-dimensional hull, monomially equivalent.

## I. Introduction

GIVEN a linear code $\mathcal{C}$ of length $n$ over the finite field $\mathbb{F}_{q}$, the dual code of $\mathcal{C}$ is defined by

$$
\mathcal{C}^{\perp}=\left\{\mathbf{x} \in \mathbb{F}_{q}^{n} \mid \mathbf{x y}^{T}=0 \text { for all } \mathbf{y} \in \mathcal{C}\right\}
$$

where $\mathbf{x y}{ }^{T}$ denotes the standard inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$. The hull of the linear code $\mathcal{C}$ is defined to be

$$
\operatorname{Hull}(\mathcal{C}):=\mathcal{C} \cap \mathcal{C}^{\perp}
$$

It is clear that $\operatorname{Hull}(\mathcal{C})$ is also a linear code over $\mathbb{F}_{q}$. The hull was originally introduced in 1990 by Assmus, Jr. and Key [1] to classify finite projective planes. It had been shown that the hull plays an important role in determining the complexity of algorithms for checking permutation equivalence of two linear codes and computing the automorphism group of a linear code (see [10], [11], [22]-[24]), which are very effective in general when the dimension of the hull is small.

It is worth mentioning that the special case of the hulls of linear codes is of much interest. Namely the codes with trivial intersection with its dual, which is also named linear complementary dual (LCD) codes. Massey [18] first introduced this class of codes and proved that there exist asymptotically good LCD codes. A practical application of binary LCD codes against side-channel attacks (SCAs) and fault injection attacks (FIAs) was investigated by Carlet et al. [3] and Carlet and Guilley [4]. Since then, the study of LCD codes is thus becoming a hot research topic in coding theory ([6]-[9], [12],

[^0][13], [15], [16],[26]-[29]). Some nice progress on linear codes with small hulls has been made, for examples ([5], [14]).

A maximum distance separable (MDS) code has the greatest error correcting capability when its length and dimension is fixed. MDS codes are extensively used in communications (for example, Reed-Solomon codes are all MDS codes), and they have good applications in minimum storage codes and quantum codes. There are many known constructions for MDS codes; for instance, Generalized Reed-Solomon (GRS) codes [19], based on the equivalent problem of finding $n$-arcs in projective geometry [17], circulant matrices [20], Hankel matrices [21], or extending GRS codes.
Recently the authors in [8], [16] investigated the hull of MDS codes via generalized Reed-Solomon codes over finite fields. Beelen et al. [2] first gave the definition of twisted Reed-Solomon codes, which is a generalization of the Reed-Solomon codes, and they proved under some conditions twisted Reed-Solomon codes could be not monomially equivalent to the Reed-Solomon codes. However, the hull of twisted Reed-Solomon codes have not been studied in that paper. Recently, Wu, Hyun and Lee [25] constructed some LCD twisted Reed-Solomon codes.

In this letter, as a follow-up work we will focus on the hull of twisted Reed-Solomon codes. In particular, we will consider to construct some twisted Reed-Solomon MDS codes with one-dimensional hull, which are not monomially equivalent to Reed-Solomon codes. The rest of this letter is organized as follows. In Section II, we introduce basic concepts on the hull of linear codes and twisted Reed-Solomon codes. In Sections III, we present our main results and give some examples. We conclude the letter in Section IV.

## II. Preliminaries

Let $\mathbb{F}_{q}$ be the finite field of order $q$, where $q$ is a prime power. An $[n, k]_{q}$ linear code $\mathcal{C}$ over $\mathbb{F}_{q}$ is a $k$-dimensional subspace of $\mathbb{F}_{q}^{n}$. The minimum distance d of a linear code $\mathcal{C}$ is bounded by the so-called Singleton bound : $d \leq n-k+1$. If $d=n-k+1$, then the $\operatorname{code} \mathcal{C}$ is called a maximum distance separable (MDS) code.

The following lemma on the hull of linear codes, which is very important for obtaining our main results.

Lemma 2.1: [14, Proposition 1] Let $\mathcal{C}$ be an $[n, k]$ linear code over $\mathbb{F}_{q}$ with generator matrix $G$. Then the code $\mathcal{C}$ has one-dimensional hull if and only if the rank of the matrix $G G^{T}$ is $k-1$, where $G^{T}$ denotes the transpose of $G$.

Recall that a monomial matrix is a square matrix which has exactly one nonzero entry in each row and each column.

Definition 2.2: Let $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ be two linear codes of the same length over $\mathbb{F}_{q}$, and let $G_{1}$ be a generator matrix of $\mathcal{C}_{1}$. Then $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ are monomially equivalent if there is a monomial matrix $M$ such that $G_{1} M$ is a generator matrix of $\mathcal{C}_{2}$.

Next we will recall some constructions of MDS codes. We begin with the well-known generalized Reed-Solomon codes.

Definition 2.3: Let $\alpha_{1}, \ldots, \alpha_{n}$ be distinct elements in $\mathbb{F}_{q} \cup$ $\{\infty\}$ and $v_{1}, \ldots, v_{n}$ be nonzero elements in $\mathbb{F}_{q}$. For $1 \leq k \leq$ $n$, the corresponding generalized Reed-Solomon (GRS) code over $\mathbb{F}_{q}$ is defined by
$G R S_{k}(\boldsymbol{\alpha}, \mathbf{v}):=\left\{\left(v_{1} f\left(\alpha_{1}\right), \ldots, v_{n} f\left(\alpha_{n}\right)\right) \mid f(x) \in \mathbb{F}_{q}[x], \operatorname{deg}(f(x))<k\right\}$, where $\boldsymbol{\alpha}=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right) \in\left(\mathbb{F}_{q} \cup\{\infty\}\right)^{n}$ and $\mathbf{v}=$ $\left(v_{1}, v_{2}, \ldots, v_{n}\right)$, and the quantity $f(\infty)$ is defined as the coefficient of $x^{k-1}$ in the polynomial $f$.

If $v_{i}=1$ for every $i=1, \ldots, n$, then $G R S_{k}(\boldsymbol{\alpha}, \mathbf{v})$ is called a Reed-Solomon code. In fact, $G R S_{k}(\boldsymbol{\alpha}, \mathbf{v})$ has a generator matrix as follows:
$\left(\begin{array}{cccc}v_{1} & v_{2} & \ldots & v_{n} \\ v_{1} \alpha_{1} & v_{2} \alpha_{2} & \ldots & v_{n} \alpha_{n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1} \alpha_{1}^{k-1} & v_{2} \alpha_{2}^{k-1} & \ldots & v_{n} \alpha_{n}^{k-1}\end{array}\right)=\left(\begin{array}{cccc}1 & 1 & \ldots & 1 \\ \alpha_{1} & \alpha_{2} & \ldots & \alpha_{n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1}^{k-1} & \alpha_{2}^{k-1} & \ldots & \alpha_{n}^{k-1}\end{array}\right)\left(\begin{array}{cccc}v_{1} & 0 & \ldots & 0 \\ 0 & v_{2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & v_{n}\end{array}\right)$
It is well-known that a generalized Reed-Solomon code $G R S_{k}(\boldsymbol{\alpha}, \mathbf{v})$ is an $[n, k, n-k+1]$ MDS code and it is monomially equivalent to a Reed-Solomon code.

In 2007, Beelen et al. [2] presented a generalization of Reed-Solomon codes, so-called twisted Reed-Solomon codes.

Definition 2.4: Let $\eta$ be a nonzero element in the finite field $\mathbb{F}_{q}$. Let $k, t$ and $h$ be nonnegative integers such that $0 \leq h<$ $k \leq q, k<n$, and $0<t \leq n-k$. Let $\alpha_{1}, \ldots, \alpha_{n}$ be distinct elements in $\mathbb{F}_{q} \cup\{\infty\}$, and we write $\boldsymbol{\alpha}=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right)$. Then the corresponding twisted Reed-Solomon code over $\mathbb{F}_{q}$ of length $n$ and dimension $k$ is given by
$\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)=\left\{\left(f\left(\alpha_{1}\right), \cdots, f\left(\alpha_{n}\right)\right): f(x)=\sum_{i=0}^{k-1} a_{i} x^{i}+\eta a_{h} x^{k-1+t} \in \mathbb{F}_{q}[x]\right\}$.
In fact,

$$
G=\left(\begin{array}{cccc}
1 & 1 & \ldots & 1  \tag{1}\\
\alpha_{1} & \alpha_{2} & \ldots & \alpha_{n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{1}^{h-1} & \alpha_{2}^{h-1} & \ldots & \alpha_{n}^{h-1} \\
\alpha_{1}^{h}+\eta \alpha_{1}^{k-1+t} & \alpha_{2}^{h}+\eta \alpha_{2}^{k-1+t} & \ldots & \alpha_{n}^{h}+\eta \alpha_{n}^{k-1+t} \\
\alpha_{1}^{h+1} & \alpha_{2}^{h+1} & \ldots & \alpha_{n}^{h+1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{1}^{k-1} & \alpha_{2}^{k-1} & \ldots & \alpha_{n}^{k-1}
\end{array}\right)
$$

is the generator matrix of the twisted Reed-Solomon code $\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)$.

Note that in general, the twisted Reed-Solomon codes are not MDS. Beelen et al. got some results on the twisted ReedSolomon codes as follows:

Lemma 2.5: [2, Theorem 17] Let $\mathbb{F}_{s} \subset \mathbb{F}_{q}$ be a proper subfield and $\alpha_{1}, \ldots, \alpha_{n} \in \mathbb{F}_{s}$. If $\eta \in \mathbb{F}_{q} \backslash \mathbb{F}_{s}$, then the twisted Reed-Solomon code $\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)$ is MDS.

Lemma 2.6: [2, Theorem 18] Let $\alpha_{1}, \ldots, \alpha_{n} \in \mathbb{F}_{q}$ and $2<k<n-2$. Furthermore, let $H \subseteq \mathbb{F}_{q}$ satisfy that the twisted Reed-Solomon code $\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)$ is MDS for every $\eta \in H$. Then there are at most 6 choices of $\eta \in H$ such that
$\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)$ is monomially equivalent to a Reed-Solomon code.

Lemma 2.7: [2, Corollary 20] Let $\mathbb{F}_{s} \subset \mathbb{F}_{q}$ with $\left|\mathbb{F}_{q} \backslash \mathbb{F}_{s}\right|>$ 6 . Let $2<k<n-2$ and $n \leq s$. Then there exists $\eta \in \mathbb{F}_{q} \backslash \mathbb{F}_{s}$ such that $\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)$ is MDS but not monomially equivalent to a Reed-Solomon code.

Throughout the paper, if a code is not monomially equivalent to a Reed-Solomon code, then we call it a code of non-Reed-Solomon type or a non-Reed-Solomon code.

## III. Twisted Reed-Solomon codes with ONE-DIMENSIONAL HULL

Let $\gamma$ be a primitive element of $\mathbb{F}_{q}$ and $k \mid(q-1)$. Then $\gamma^{\frac{q-1}{k}}$ generates a subgroup of $\mathbb{F}_{q}^{*}$ of order $k$. Let $\alpha_{i}=\gamma^{\frac{q-1}{k} i}$ for $1 \leq i \leq k$. One can easily check that

$$
\theta_{f}=\alpha_{1}^{f}+\cdots+\alpha_{k}^{f}=\left\{\begin{array}{cl}
k & \text { if } f \equiv 0 \quad(\bmod k)  \tag{2}\\
0 & \text { otherwise }
\end{array}\right.
$$

Lemma 3.1: Let $q$ be a power of two. If $k$ is a positive integer with $k \mid(q-1), k<(q-1)$, and $h>$ 0 , then there exists a $[2 k, k]_{q}$ twisted Reed-Solomon code $\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)$ over $\mathbb{F}_{q}$ with one-dimensional hull for $\boldsymbol{\alpha}=$ $\left(\alpha_{1}, \ldots, \alpha_{k}, \gamma \alpha_{1}, \ldots, \gamma \alpha_{k}\right)$, where $\gamma$ is a primitive element of $\mathbb{F}_{q}$ and $\alpha_{i}=\gamma^{\frac{q-1}{k} i}$ for $1 \leq i \leq k$.

Proof By Definition 2.4, to make sure that $\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)$ is a twisted Reed-Solomon code, we need $k \neq q-1$. From (1), we recall that $G$ is a generator matrix of the twisted ReedSolomon code $\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)$ over $\mathbb{F}_{q}$.
Let
$A_{\beta}=\left(\begin{array}{ccccc}1 & 1 & \cdots & 1 & 1 \\ \beta \alpha_{1} & \beta \alpha_{2} & \cdots & \beta \alpha_{k-1} & \beta \alpha_{k} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \left(\beta \alpha_{1}\right)^{k-1} & \left(\beta \alpha_{2}\right)^{k-1} & \cdots & \left(\beta \alpha_{k-1}\right)^{k-1} & \left(\beta \alpha_{k}\right)^{k-1}\end{array}\right)$.
By (2), we have

$$
A_{\beta} A_{\beta}^{T}=\left(\begin{array}{cccccc}
k & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \beta^{k} k \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
0 & 0 & \beta^{k} k & \cdots & 0 & 0 \\
0 & \beta^{k} k & 0 & \cdots & 0 & 0
\end{array}\right)
$$

Let $C_{\beta}=A_{\beta}+B_{\beta}$, where
$B_{\beta}=\left(\begin{array}{ccccc}0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \eta\left(\beta \alpha_{1}\right)^{k-1+t} & \eta\left(\beta \alpha_{2}\right)^{k-1+t} & \cdots & \eta\left(\beta \alpha_{k-1}\right)^{k-1+t} & \eta\left(\beta \alpha_{k}\right)^{k-1+t} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0\end{array}\right) \leftarrow(h+1) t h$.
Let $\theta_{j}=\sum_{i=1}^{n} \alpha_{i}^{j}$ and $l=k-1+t$, then we have

$$
\begin{aligned}
& C_{\beta} C_{\beta}^{T}=\left(\begin{array}{ccccc}
k & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \beta^{k} k \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & \beta^{k} k & \cdots & 0 & 0
\end{array}\right) \\
& +\left(\begin{array}{cccccccc}
0 & 0 & \ldots & 0 & \eta \beta^{l} \theta_{l} & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & \eta \beta^{l+1} \theta_{l+1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & \ldots & 0 & \eta \beta^{l+h-1} \theta_{l+h-1} & 0 & \ldots & 0 \\
\eta \beta^{l} \theta_{l} & \eta \beta^{l+1} \theta_{l+1} & \ldots & \eta \beta^{l+h-1} \theta_{l+h-1} & 2 \eta \beta^{l+h} \theta_{l+h}+\eta^{2} \beta^{2 l} \theta_{2 l} & \eta \beta^{l+h+1} \theta_{l+h+1} & \ldots & \eta \beta^{l+k-1} \theta_{l+k-1} \\
0 & 0 & \ldots & 0 & \eta \beta^{l+h+1} \theta_{l+h+1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \ldots & 0 & \eta \beta^{l+k-1} \theta_{l+k-1} & 0 & \cdots & 0
\end{array}\right) .
\end{aligned}
$$

Since every $\theta_{t}$ for $l \leq t \leq l+k-1$ is zero except exactly one $\theta_{t^{\prime}}$, we can rewrite

$$
\begin{gathered}
C_{\beta} C_{\beta}^{T}=\left(\begin{array}{ccccc}
k & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \beta^{k} k \\
\vdots & \vdots & \ldots & \vdots & \vdots \\
0 & \beta^{k} k & \ldots & 0 & 0
\end{array}\right) \\
+\left(\begin{array}{ccccccc}
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & 0 & \ldots & *_{\beta} & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & *_{\beta} & \ldots & \Delta_{\beta} & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0
\end{array}\right),
\end{gathered}
$$

where $*_{\beta}$ and $\Delta_{\beta}$ are all elements in $\mathbb{F}_{q}$, the $*_{\beta}$ and $\Delta_{\beta}$ are respectively entries located in the $(i+1, h+1) t h,(h+1, i+$ $1)$ th and $(h+1, h+1)$ th positions, and the other elements are all zero.

Let $G=\left[C_{1}: C_{\gamma}\right]$ and $h>0$. Then

$$
\begin{aligned}
& G G^{T}=C_{1} C_{1}^{T}+C_{\gamma} C_{\gamma}^{T} \\
& =\left(\begin{array}{cccccc}
2 k & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \left(1+\gamma^{k}\right) k \\
\vdots & \vdots & \ldots & \vdots & \vdots \\
0 & \left(1+\gamma^{k}\right) k & \ldots & 0 & 0
\end{array}\right) \\
& +\left(\begin{array}{ccccccc}
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & 0 & \ldots & *_{1}+*_{\gamma} & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & *_{1}+*_{\gamma} & \ldots & \Delta_{1}+\Delta_{\gamma} & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0
\end{array}\right) .
\end{aligned}
$$

It is easy to find an elementary matrix $P$ such that

$$
\begin{aligned}
& P G G^{T} P^{T}=\left(\begin{array}{cccccc}
2 k & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \left(1+\gamma^{k}\right) k \\
\vdots & \vdots & \ldots & \vdots & \vdots \\
0 & \left(1+\gamma^{k}\right) k & \ldots & 0 & 0
\end{array}\right) \\
& +\left(\begin{array}{ccccccc}
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & 0 & \ldots & \Delta_{1}+\Delta_{\gamma} & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0
\end{array}\right)
\end{aligned}
$$

By the given conditions, we have $k \mid(q-1), k<(q-1), q$ is even, and $\gamma$ is a primitive element of $\mathbb{F}_{q}$. Hence, $\gamma^{k}+1 \neq 0$ and $2 k=0$. Then we have $\operatorname{rank}\left(G G^{T}\right)=k-1$. The result follows from Lemma 2.1.

Lemma 3.2: Let $\mathbb{F}_{q}$ be a finite field of odd order $q$ and $k$ be a positive integer with $k \mid(q-1)$ and $2<k<(q-1) / 2$. If $h>$

1, then there exists a $[2 k, k-1]_{q}$ twisted Reed-Solomon code $\mathcal{C}_{k-1}(\boldsymbol{\alpha}, t, h, \eta)$ over $\mathbb{F}_{q}$ with one-dimensional hull for $\boldsymbol{\alpha}=$ $\left(\alpha_{1}, \ldots, \alpha_{k}, \gamma \alpha_{1}, \ldots, \gamma \alpha_{k}\right)$, where $\gamma$ is a primitive element of $\mathbb{F}_{q}$ and $\alpha_{i}=\gamma^{\frac{q-1}{k} i}$ for $1 \leq i \leq k$.

Proof Let

$$
D_{\beta}=\left(\begin{array}{ccccc}
1 & 1 & \cdots & 1 & 1  \tag{4}\\
\beta \alpha_{1} & \beta \alpha_{2} & \cdots & \beta \alpha_{k-1} & \beta \alpha_{k} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\left(\beta \alpha_{1}\right)^{k-2} & \left(\beta \alpha_{2}\right)^{k-2} & \cdots & \left(\beta \alpha_{k-1}\right)^{k-2} & \left(\beta \alpha_{k}\right)^{k-2}
\end{array}\right)
$$

Then

$$
D_{\beta} D_{\beta}^{T}=\left(\begin{array}{cccccc}
k & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & \beta^{k} k \\
\vdots & \vdots & \ldots & \vdots & \vdots & \\
0 & 0 & \beta^{k} k & \cdots & 0 & 0
\end{array}\right)
$$

Let $H_{\beta}=D_{\beta}+E_{\beta}$, where
$E_{\beta}=\left(\begin{array}{ccccc}0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \eta\left(\beta \alpha_{1}\right)^{k-2+t} & \eta\left(\beta \alpha_{2}\right)^{k-2+t} & \cdots & \eta\left(\beta \alpha_{k-1}\right)^{k-2+t} & \eta\left(\beta \alpha_{k}\right)^{k-2+t} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0\end{array}\right) \leftarrow(h+1) t h$.
Let $G=\left[H_{1}: H_{\gamma}\right]$. By the proof of Lemma 3.1,

$$
\begin{aligned}
& G G^{T}=H_{1} H_{1}^{T}+H_{\gamma} H_{\gamma}^{T} \\
& =\left(\begin{array}{ccccccc}
2 k & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & \left(1+\gamma^{k}\right) k & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & & \\
0 & 0 & \left(1+\gamma^{k}\right) k & \ldots & 0 & 0 & 0
\end{array}\right) \\
& +\left(\begin{array}{ccccccc}
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & 0 & \ldots & *_{1}+*_{\gamma} & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & *_{1}+*_{\gamma} & \ldots & \Delta_{1}+\Delta_{\gamma} & \ldots & 0 \\
\vdots & & \vdots & \ldots & \vdots & & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0
\end{array}\right)
\end{aligned}
$$

Note that $q$ is odd and $k<\frac{q-1}{2}$. We have $2 k \neq 0$ and $1+\gamma^{k} \neq$ 0 . By the same process of the proof of Lemma 3.1, the rank of $G G^{T}$ is $k-1$ and result follows from Lemma 2.1.

Remark 3.3: By the process of Lemma 3.2, we can also construct some twisted Reed-Solomon codes with small hulls.

An effective method for construction of twisted ReedSolomon codes with MDS property is to use the lifting of the finite field (refer to [2]). Hence, we obtain the following theorem by Lemmas 2.5, 2.7, 3.1 and 3.2.

Theorem 3.4: Let $q$ be a power of a prime and $\mathbb{F}_{s} \subset \mathbb{F}_{q}$ with $\left|\mathbb{F}_{q} \backslash \mathbb{F}_{s}\right|>6$. Suppose that $k$ is a positive integer with $k \mid(q-1)$.
(1) If $q$ is even and $2<k<(s-1)$, then there exists a $[2 k, k]_{q}$ MDS non-Reed-Solomon code with one-dimensional hull.
(2) If $q$ is odd and $2<k<(s-1) / 2$, then there exists a $[2 k, k-1]_{q}$ MDS non-Reed-Solomon code with onedimensional hull.

In the following, we will present some examples to show our main results.

Example 3.5: (1) Let $q=2^{4}=16, k=$ 5 , and $\gamma$ be a primitive element of $\mathbb{F}_{q}$. Consider a twisted Reed-Solomon code $\mathcal{C}_{5}(\boldsymbol{\alpha}, 1,3, \eta)$, when $\boldsymbol{\alpha}=$ $\left(1, \gamma^{3}, \gamma^{6}, \gamma^{9}, \gamma^{12}, \gamma, \gamma \gamma^{3}, \gamma \gamma^{6}, \gamma \gamma^{9}, \gamma \gamma^{12}\right)$ and $\eta=\gamma^{i} \in \mathbb{F}_{16}$. By Lemma 3.1, $\mathcal{C}_{5}\left(\boldsymbol{\alpha}, 1,3, \gamma^{i}\right)$ has one-dimensional hull for all $i$. By Magma, it follows that the codes $\mathcal{C}_{5}(\boldsymbol{\alpha}, 1,3, \eta)$ are not MDS codes, which have parameters $[10,5,5]_{16}$.
(2) Let $q=2^{8}=256, k=5$, and $w$ be a primitive element of $\mathbb{F}_{q}$ and $\gamma=w^{17} \in \mathbb{F}_{16}$. Consider a twisted Reed-Solomon code $\mathcal{C}_{5}(\boldsymbol{\alpha}, 1,3, \eta)$, when $\boldsymbol{\alpha}=$ $\left(1, \gamma^{3}, \gamma^{6}, \gamma^{9}, \gamma^{12}, \gamma, \gamma \gamma^{3}, \gamma \gamma^{6}, \gamma \gamma^{9}, \gamma \gamma^{12}\right)$ and $\eta=w^{i} \in$ $\mathbb{F}_{256}$. By Lemma 3.1, $\mathcal{C}_{5}\left(\boldsymbol{\alpha}, 1,3, w^{i}\right)$ has one-dimensional hull for all $i$. By Magma and Lemmas 2.5 and 2.7, there exists an integer $i$ with $17 \nmid i$ such that the code $\mathcal{C}_{5}(\boldsymbol{\alpha}, 1,3, \eta)$ is an MDS non-Reed-Solomon code with parameters $[10,5,6]_{256}$.

Example 3.6: (1) Let $q=3^{4}=81, k=$ 5 , and $\gamma$ be a primitive element of $\mathbb{F}_{q}$. Consider a twisted Reed-Solomon code $\mathcal{C}_{4}(\boldsymbol{\alpha}, 2,2, \eta)$, when $\boldsymbol{\alpha}=$ $\left(1, w, w^{2}, w^{3}, w^{4}, \gamma, \gamma w, \gamma w^{2}, \gamma w^{3}, \gamma w^{4}\right)$ and $\eta=\gamma^{i} \in \mathbb{F}_{81}$ and $w=\gamma^{16}$. By Lemma 3.2, $\mathcal{C}_{4}\left(\boldsymbol{\alpha}, 2,2, \gamma^{i}\right)$ has onedimensional hull for all $i$. By Magma, it follows that the codes $\mathcal{C}_{4}\left(\boldsymbol{\alpha}, 2,2, \gamma^{i}\right)$ are MDS with parameters $[10,4]_{81}$ when $\eta$ belongs to $H=\left\{\gamma^{j}: j=0,6,16,22,32,38,48,54,64,70\right\}$. Since $|H|>6$, by Lemma 2.6, there exists $\eta \in H$ such that $\mathcal{C}_{4}(\boldsymbol{\alpha}, 2,2, \eta)$ is a non-Reed-Solomon code. As a result, there exists a $[10,4]_{81}$ MDS non-Reed-Solomon code with one-dimensional hull.
(2) Let $q=3^{8}=6561, k=5$, and $\theta$ be a primitive element of $\mathbb{F}_{q}, \gamma=\theta^{82} \in \mathbb{F}_{81}$, and $w=\gamma^{16}$. Consider a twisted Reed-Solomon code $\mathcal{C}_{4}(\boldsymbol{\alpha}, 2,2, \eta)$, when $\boldsymbol{\alpha}=$ $\left(1, w, w^{2}, w^{3}, w^{4}, \gamma, \gamma w, \gamma w^{2}, \gamma w^{3}, \gamma w^{4}\right)$ and $\eta=\theta^{i} \in \mathbb{F}_{6561}$. By Lemma 3.2, $\mathcal{C}_{4}\left(\boldsymbol{\alpha}, 2,2, \gamma^{i}\right)$ has one-dimensional hull for all i. By Magma and Lemmas 2.5 and 2.7, there exists an integer $i$ with $82 \nmid i$ such that the code $\mathcal{C}_{4}\left(\boldsymbol{\alpha}, 2,2, \gamma^{i}\right)$ is an MDS non-Reed-Solomon code with parameters $[10,4,7]_{6561}$.

## IV. CONCLUDING REMARKS

For a given linear code, in general case it is hard to show that if the code is monomially equivalent to a Reed-Solomon code with the same parameters. In this paper, we applied twisted Reed-Solomon codes to construct some MDS codes, which have one-dimensional hull and are not monomially equivalent to Reed-Solomon codes. We also presented some examples by using Magma.

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