Generalized Karagiannidis–Lioumpas Approximations and Bounds to the Gaussian Q-Function With Optimized Coefficients

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***Abstract*— We develop extremely tight novel approximations, lower bounds and upper bounds for the Gaussian Q-function and offer multiple alternatives for the coefficient sets thereof, which are optimized in terms of the four most relevant criteria: minimax absolute/relative error and total absolute/relative error. To minimize error maximum, we modify the classic Remez algorithm to comply with the challenging nonlinearity that pertains to the proposed expression for approximations and bounds. On the other hand, we minimize the total error numerically using the quasi-Newton algorithm. The proposed approximations and bounds are so well matching to the actual Q-function that they can be regarded as virtually exact in many applications since absolute and relative errors of 10^{-9} and 10^{-5}, respectively, are reached with only ten terms. The significant advance in accuracy is shown by numerical comparisons with key reference cases.**

***Index Terms*— Gaussian Q-function, error probability.**

I. Introduction

THE Gaussian Q-function and the related complementary error function $\mathrm{erfc}(\cdot)$ are very important entities for communication theory (as well as in statistical sciences at large). They emerge often when noise, interference, or a signal is characterized by the normal distribution. Although the Q-function, which has no exact closed form, can be evaluated using many software packages, the literature is rich in several approximations and bounds [1]–[10] based on either the statistical definition [5, Eq. 1] or on the alternative representation proposed by Craig [11]. Their significant value is in facilitating closed-form calculations of error probabilities for different digital modulations and fading models [12]–[14], in which functions of Q-function usually appear in integrands.

The expression by Karagiannidis and Lioumpas in [1] is one of the most common tools to approximate the Q-function in the different problems of communication theory due to its tractability and accuracy compared to others. In particular, they approximate $\mathrm{erfc}(\cdot)$ by an inverse factorial series which is then truncated to a single term but the resulted expression is loose for small arguments. Therefore, they multiply it by a monotonically increasing function to tighten it there and, thus, to approximate accurately the Q-function for all $x \geq 0$ as

$$Q(x) \approx a\left(1-\exp\left(-c\,x\right)\right)\cdot\frac{\exp\left(-b\,x^2\right)}{x}, \tag{1}$$

where $a = \frac{1}{1.135\sqrt{2\pi}}$, $b = \frac{1}{2}$, and $c = \frac{1.98}{\sqrt{2}}$ originally, while [2] presents alternative coefficients for tailoring accuracy in different applications or transforming it into a bound.

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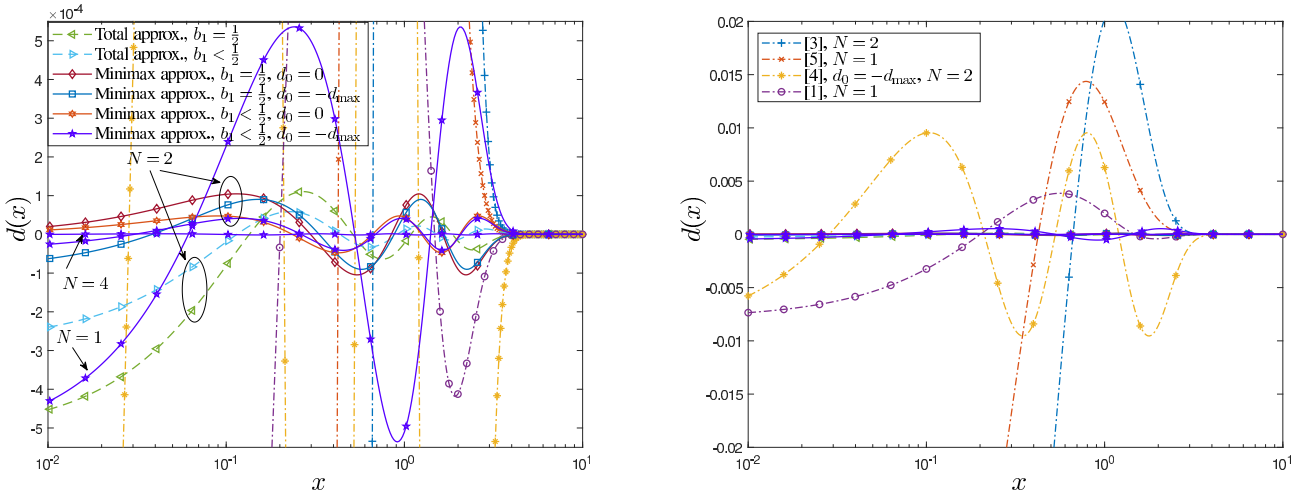
Inspired by the Karagiannidis–Lioumpas (KL) approximation, our first main contribution is to propose a new expression to approximate or bound the Q-function:

$$\tilde{Q}(x) \triangleq \underbrace{\frac{1-\exp\left(-c\,x\right)}{x}}_{\triangleq g(x)}\cdot\underbrace{\sum_{n=1}^{N} a_n \exp\left(-b_n\,x^2\right)}_{\triangleq h(x)}, \tag{2}$$

which is referred to as the generalized KL (GKL) expression since it is reduced to the original KL expression in the special case of $N = 1$ [1], [2] (but is novel herein for $N > 1$). Conceptually, an approach analogous to that in [1] is used by first approximating the Q-function with the sum of exponentials $h(x)$ as in [3, Eq. 8], which results in unbounded relative error and thus lower accuracy for the higher arguments; then it is multiplied by the term $g(x)$ to bound the relative error with $b_1 \triangleq \min\{b_n\}_{n=1}^{N} = \frac{1}{2}$. This yields the accurate GKL expression in (2) that is limited to the domain $x \geq 0$, not so unlike most related approximations, but the relation $Q(x) = 1 - Q(-x)$ extends it to $x < 0$.

As the second main contribution, we solve the research problem of optimizing the coefficients, $\{(a_n, b_n)\}_{n=1}^{N}$ and c, in order to minimize the global or total absolute/relative error of the corresponding approximation. Furthermore, the coefficients are optimized in the minimax sense to derive tight lower and upper bounds too. We show that the GKL approximations/bounds together with the optimized coefficients achieve very high, increasing accuracy so that using not-so-large number of terms they can become virtually exact, i.e., the error may not be notable in many applications in communications systems' analysis. By these main contributions, we provide researchers with accuracy-controllable approximations/bounds in terms of several optimization criteria, from which they can choose one that best suits their needs in order to ease expression manipulations with extremely high accuracy.

Two types of complexity are of relevance herein, namely the analytical and the computational. The former, which refers to the difficulty of the analytical form of (2) and the tractability thereof in symbolic calculations for mathematical operations, is kept the same as for (1) while significantly increasing the accuracy. On the other hand, the latter can refer herein either to the difficulty and processing time of the proposed optimization methodology or to those in using the approximation. The offline complexity of coefficient optimization is hardly relevant since it is already implemented by us and the coefficients are released to public domain[1] so that there is no need for redoing it later, whereas the online complexity of using the GKL expression is directly proportional to the number of terms N used in the approximation. Hence, (2) with the optimized coefficients can reliably substitute the Q-function



(a) zoom-in to the small error of the proposed approximations

(b) zoom-out to the large error of the reference approximations

Fig. 1. Comparison between our approximations and the reference ones for $N = 1$, $N = 2$ and $N = 4$ in terms of the absolute error.

in derivations of almost exact closed-form expressions for different performance measures with exactly the same analytical tractability as with the original KL approximation and with moderately increased computational complexity that is controllable with the choice of the number of terms.

The remainder of this letter is organized as follows. The next section presents our new approximations and bounds together with the optimization methodologies used for solving the sets of coefficients. The accuracy of the proposed approximations and bounds is validated in Section III by numerical results. After an overview of various applications of (2) in Section IV, the conclusion is given in Section V.

II. Novel Approximations and Bounds

This section finds the optimized coefficients, $\{(a_n^*, b_n^*)\}_{n=1}^N$ and c^*, for the proposed GKL expression that offer variety to tailor accuracy for some specific application or to use bounds. For this reason, several optimization criteria are considered and each of them requires more or less different approach. The first two minimize maximum absolute and relative errors, whereas the remaining two minimize total absolute and relative errors.

A. Minimax Approximations and Bounds

The GKL expression in (2) is optimized herein in the minimax sense by solving its corresponding coefficients as

$$\{(a_n^*, b_n^*)\}_{n=1}^N,\, c^* \triangleq \underset{\{(a_n,b_n)\}_{n=1}^N,\, c}{\arg\min} \quad e_{\max}, \tag{3}$$

where

$$e_{\max} \triangleq \max_{x \geq 0} |e(x)|, \tag{4}$$

and the shorthand $e \in \{d, r\}$ collectively represents both the absolute and relative error functions which are defined respectively as $d(x) \triangleq \tilde{Q}(x) - Q(x)$ and $r(x) \triangleq \frac{Q(x)}{\tilde{Q}(x)} - 1$.

The minimax optimization results in uniform error functions that oscillate between local maximum and minimum values of equal magnitude and alternating signs as illustrated by the minimax approximations in Fig. 1(a). The absolute and relative error functions' derivatives vanish at these extrema points and are given respectively by $d'(x) = \tilde{Q}'(x) - Q'(x)$ and $r'(x) = (\tilde{Q}'(x)Q(x) - \tilde{Q}(x)Q'(x))/[Q(x)]^2$ where $Q'(x) = -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$ and $\tilde{Q}'(x) = -\frac{1}{x^2} \sum_{n=1}^N a_n \big((2b_n x^2 + 1) \cdot \exp(c\,x) - 2b_n x^2 - c\,x - 1\big) \exp(-b_n x^2 - c\,x)$.

We will shortly use the fact that the error functions converge to explicit values, which may be local extrema, at both ends of the non-negative real axis as follows:

$$d_0 \triangleq \lim_{x \to 0} d(x) = c \sum_{n=1}^N a_n - \tfrac{1}{2}, \qquad \lim_{x \to \infty} d(x) = 0,$$

$$r_0 \triangleq \lim_{x \to 0} r(x) = 2\,c \sum_{n=1}^N a_n - 1,$$

$$\lim_{x \to \infty} r(x) = \begin{cases} \infty, & \text{if } b_1 < \frac{1}{2}, \\ \sqrt{2\pi}\, a_1 - 1, & \text{if } b_1 = \frac{1}{2}, \\ -1, & \text{if } b_1 > \frac{1}{2}, \end{cases} \tag{5}$$

where a_1 is the counterpart of $b_1 \triangleq \min\{b_n\}_{n=1}^N$.

It can be concluded from the above limit that global approximations and bounds exist in terms of the relative error if and only if $b_1 = \frac{1}{2}$, opposing to the absolute error function which is always bounded regardless of b_1's value. Nevertheless, this study shows that the absolute and total errors can be reduced by allowing $b_1 < \frac{1}{2}$. Thus, we consider herein two variations of approximations w.r.t. absolute and total errors, namely, first variation with $b_1 < \frac{1}{2}$ and second variation with $b_1 = \frac{1}{2}$.

1) Approximations: The optimized coefficients can be found by solving the following set of equations which describes the shape of the corresponding error function, for which x_k refers to the location of the error function's extrema and K refers to their number excluding the endpoints:

$$\begin{cases} f_0(\mathbf{v}) = e_0 + e_{\max} = 0 & \\ f_k(\mathbf{v}) = e(x_k) + (-1)^k\, e_{\max} = 0, & \text{for } k = 1, 2, \ldots, K, \\ f_k'(\mathbf{v}) = e'(x_k) = 0, & \text{for } k = 1, 2, \ldots, K, \\ f_{K+1}(\mathbf{v}) = a_1 + \frac{r_{\max} - 1}{\sqrt{2\pi}} = 0, & \text{only when } e = r. \end{cases} \tag{6}$$

Above, $\mathbf{v}$ is a vector of the approximation's coefficients with $e_{\max}$ which are to be optimized. More specifically,

$\mathbf{v} = [a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, c, e_{\max}]$ with excluding b_1 for the second variation of the absolute error and for the relative error since then $b_1 = \frac{1}{2}$. In addition, $f_k(\mathbf{v})$ and $f'_k(\mathbf{v})$ are two equations that report the error function's value and zero-derivative at each of the extrema points, and $f_0(\mathbf{v})$ and $f_{K+1}(\mathbf{v})$ result from evaluating the limits at both ends of the range $[0, \infty)$ as in (5) to give one equation for the absolute error and two equations for the relative error that converges to $-r_{\max}$ as x tends to infinity. For both error measures, the error function is assumed to start from $e_0 = -e_{\max}$.

When considering the absolute error, $K = 2\,N + 1$ for the first variation and $K = 2\,N$ for the second variation. A total of $2K + 1$ equations including that at $x = 0$ are formulated. On the other hand, for the bounded relative error, a total of $2K + 2$ equations including those at the endpoint limits are formulated with $K = 2\,N - 1$. Generally, the number of equations for both error measures are equal to the number of unknowns, namely, $\mathbf{v}$ and $\{x_k\}_{k=1}^{K}$. It is worth mentioning that the error function can also start from $e_0 = 0$ to achieve continuity at the origin when extended to the negative values of x like for $Q(x)$, but at the expense of slightly less accuracy.

2) Bounds: Here we need to find the optimized sets of coefficients which, when substituted in (2), give uniform lower and upper bounds for which $e(x) \leq 0$ and $e(x) \geq 0$, respectively. Error of a lower bound oscillates between zero and $-e_{\max}$, must have $b_1 = \frac{1}{2}$ and start from $e_0 = -e_{\max}$ for both error types. In addition, when it is optimized in terms of absolute error, $K = 2N$ and its corresponding error function converges to zero as x tends to infinity, whereas when it is optimized in terms of relative error, $K = 2N - 1$ and its error function converges to $-r_{\max}$ as x tends to infinity.

On the other hand, the upper bound oscillates between zero and $e_{\max}$ and must always start from $e_0 = 0$ and converge to zero as x tends to infinity for both error types. In particular, for its optimization in terms of absolute error, $K = 2N + 1$, whereas $K = 2N - 1$ for its optimization in terms of relative error. Using the aforementioned description, the optimization problem can be easily formulated in the same way as in (6).

3) Implementation of the Minimax Optimization and the Remez Exchange Algorithm: The sets of equations formulated for each of the proposed approximations and bounds can be straightforwardly solved using any numerical tool. However, good initial guesses for the unknowns are required in order for their values to converge to the optimized ones. The initial guesses used herein are obtained heuristically and it was quite a challenge to get good ones for $N > 5$. We have solved this problem by proposing a variation of the Remez exchange algorithm for acquiring the optimized coefficients for $N > 5$ and establishing the same uniform minimax error function but with K equations ($\{f'_k\}_{k=1}^{K}$) less than the approach introduced in Section II-A1. The absence of the derivative equations makes it less sensitive to the right choice of the initial guesses.

In particular, we construct a system of nonlinear equations describing the values of the extrema points of the corresponding error function, which alternate exactly $L = K$ times for the absolute error and $L = K + 1$ times for the relative error, as $\mathbf{f}(\mathbf{v}) \triangleq [f_0(\mathbf{v}), f_1(\mathbf{v}), \ldots, f_L(\mathbf{v})]^T$ for which $f_l, l = 0, 1, \ldots, L$ and $\mathbf{v}$ are defined in (6), and $\mathbf{f}$ and $\mathbf{v}$ have equal lengths. We set up the Remez algorithm by initializing the locations of the K extrema while taking into consideration both endpoints, which might be local extrema.

Next, we start the first iteration by solving $\mathbf{f}$ for $\mathbf{v}$ using the iterative Newton–Raphson method whose iterations also require initial guesses for $\mathbf{v}$ and are performed as

$$\mathbf{v}^{(t+1)} = \mathbf{v}^{(t)} - \left[\mathbf{J}^{(t)}\left(\mathbf{v}^{(t)}\right)\right]^{-1} \mathbf{f}\left(\mathbf{v}^{(t)}\right), \tag{7}$$

where t is its counter and $\mathbf{J}(\cdot)$ is the Jacobian matrix defined as $\mathbf{J}(\mathbf{v}) = \left[\frac{\partial \mathbf{f}}{\partial v_0}, \frac{\partial \mathbf{f}}{\partial v_1}, \ldots, \frac{\partial \mathbf{f}}{\partial v_L}\right]$. For the absolute error, $\frac{\partial f_0}{\partial a_n} = c$, $\frac{\partial f_0}{\partial b_n} = 0$, $\frac{\partial f_0}{\partial c} = \sum_{n=1}^{N} a_n$, $\frac{\partial f_k}{\partial a_n} = \frac{(1-\exp(-c\,x_k))}{x_k} \exp(-b_n\,x_k^2)$, $\frac{\partial f_k}{\partial b_n} = -a_n\,x_k\,(1 - \exp(-c\,x_k)) \exp(-b_n\,x_k^2)$, $\frac{\partial f_k}{\partial c} = \exp(-c\,x_k) \sum_{n=1}^{N} a_n \exp(-b_n\,x_k^2)$, whereas for the relative error, we multiply the above relations $\frac{\partial f_0}{\partial a_n}$ and $\frac{\partial f_0}{\partial c}$ by two and divide $\frac{\partial f_k}{\partial a_n}$, $\frac{\partial f_k}{\partial b_n}$ and $\frac{\partial f_k}{\partial c}$ by $Q(x_k)$. Also, for the relative error only, $\frac{\partial f_{K+1}}{\partial a_1} = 1$, $\frac{\partial f_{K+1}}{\partial a_n}|_{n \neq 1} = \frac{\partial f_{K+1}}{\partial b_n} = \frac{\partial f_{K+1}}{\partial c} = 0$, and $\frac{\partial f_{K+1}}{\partial r_{\max}} = \frac{1}{\sqrt{2\,\pi}}$. In addition, $\frac{\partial f_0}{\partial e_{\max}} = 1$ and $\frac{\partial f_k}{\partial e_{\max}} = (-1)^k$ for both error measures. The Newton–Raphson iterations are repeated until $\Delta\mathbf{v} = \mathbf{v}^{(t+1)} - \mathbf{v}^{(t)}$ is less than a threshold value.

Then, we locate the new extrema of the resulting error function and use them for the following Remez iteration which we repeat until the difference between the old and new K extrema lies below a threshold value. Note that the Newton–Raphson method is implemented in every iteration of the Remez algorithm. Although the Remez algorithm still requires initial guesses for the unknowns like the approach in Section II-A1, it is much more robust against the accuracy of the initial guesses and converges very rapidly to the optimal solution. The optimized coefficients of minimax GKL approximations and bounds are solved herein up to $N = 10$ for the two variations of the absolute error and for the relative error and released to public domain as a supplementary dataset.[1]

B. Numerical Optimization in Terms of Total Error

The coefficients of the GKL expression can also be optimized in terms of the total integrated error as

$$\{(a_n^*, b_n^*)\}_{n=1}^{N},\ c^* \triangleq \underset{\{(a_n, b_n)\}_{n=1}^{N},\, c}{\arg\min}\ e_{\text{tot}}, \tag{8}$$

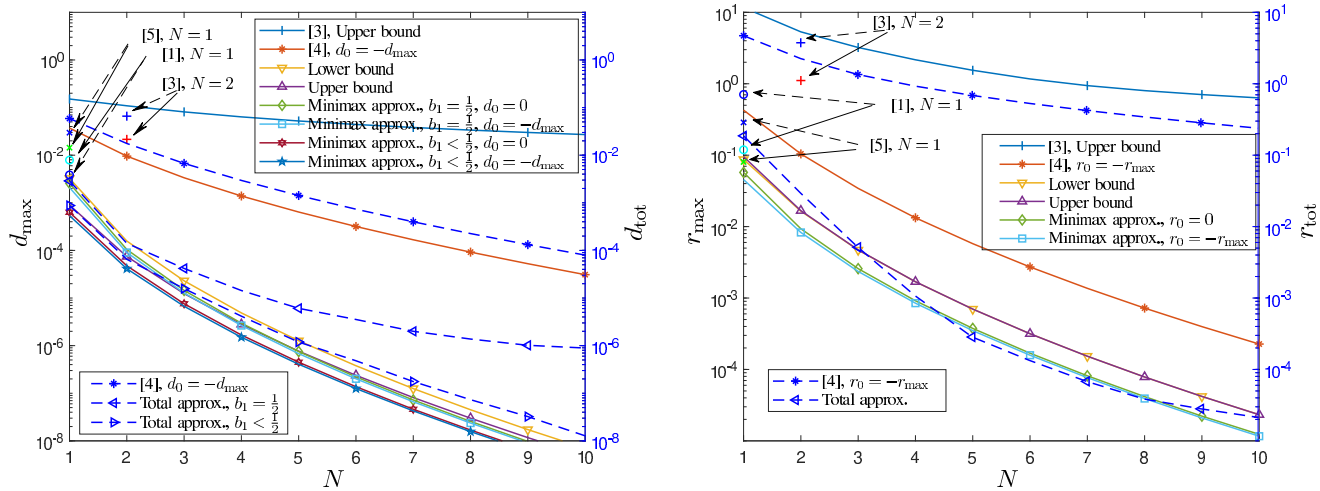
where

$$e_{\text{tot}} \triangleq \int_0^R |e(x)|\,\mathrm{d}x. \tag{9}$$

For $e = d$, $R \to \infty$ in order to obtain globally optimized approximations since $d(x)$ converges to zero when x tends to infinity, whereas R is some constant for $e = r$ which converges to a constant value when x tends to infinity for $b_1 = \frac{1}{2}$. We apply the quasi-Newton algorithm to perform the optimization herein. In particular, we used the `fminunc` command in Matlab with setting its `'Algorithm'` to `'quasi-newton'` in order to minimize the target function e_{tot}. The error function can also be forced to start from zero by adding the constraint $\sum_{n=1}^{N} a_n = \frac{1}{2c}$, which results from the limit at zero, and we then used `fmincon` command instead.

We start with heuristic initial guesses for the unknowns that converge eventually to the optimized values. In fact,

[1] Available at https://doi.org/10.5281/zenodo.5806271 for download.



(a) absolute error of GKL approximations and bounds with reference cases (b) relative error of GKL approximations and bounds with reference cases

Fig. 2. The proposed GKL approximations and bounds compared to existing ones from the literature including the original KL approximation.

we were able to use the minimax-optimized sets as mean values around which small random variance is introduced to work as initial guesses for their equivalent cases herein. Note that the `fminunc` command finds the local minimum of the target function. Therefore, we need to repeatedly run the local solver to locate a solution that has the lowest target function value. The optimized coefficients to the GKL approximations are also solved herein for the two variations of the absolute error and for the relative error with $R = 10$ in terms of the total error.[1]

III. Numerical Results

This section demonstrates how excellent the GKL approximations and bounds perform, were they achieve world-record low error levels as will be seen next. Figure 1 illustrates the absolute error functions resulted from applying our approximations and key existing ones. Obviously, the proposed approximations are extremely tight and even more interestingly, they substantially outperform all the reference cases for the whole non-negative real axis even with only one term as can be noted from the huge displacement in the corresponding curves in Fig. 1(b). The accuracy increases considerably further when increasing N as seen by the comparison between $N = 1$, $N = 2$ and $N = 4$ for the minimax approximation.

Figure 2 plots the global error of the minimax approximations and bounds proposed in Section II-A together with the reference cases (solid lines and markers with solid arrows), in addition to the total error of the approximations proposed in Section II-B and the reference cases (dashed lines and markers with dashed arrows), both for $N = 1, 2, \ldots, 10$ and in terms of both error measures. With small $N = 1, 2, 3$, they already significantly outperform the reference ones and their accuracy increases considerably by increasing N. Ultimately, the proposed GKL expression with optimized coefficients reaches extremely low levels in the order of 10^{-9} and 10^{-5} for absolute and relative errors, respectively, with $N = 10$. It should be noted that the proposed approximations and bounds in the special case of $N = 1$ are the same as those in [2].

IV. Overview of Applications

The applications of the original KL approximation and the newly proposed GKL approximation (2)—both have the same analytical complexity—are about the same and span different areas of communication theory. A popular application example would be evaluating the average symbol error probability for coherent detection, which results in linear combinations of the following integral with different integer values of P:

$$I_P(\bar{\gamma}) \triangleq \int_0^\infty Q^P(\sqrt{\gamma})\, \phi_\gamma(\gamma)\, \mathrm{d}\gamma, \tag{10}$$

where $\phi_\gamma(\gamma)$ is the fading probability density function of the instantaneous signal-to-noise ratio γ with average $\bar{\gamma}$.

When assuming generalized $\kappa - \mu$ distribution, (10) can be evaluated using [15, Eqs. 3.351.3, 3.462.1, and 8.445] after applying (2) to express tight approximations for the Pth integer power of the Gaussian Q-function using the multinomial expansion. This yields

$$I_P(\bar{\gamma}) \approx \sum_{\tau=0}^{\infty} \frac{\mu^{\mu+2\tau}}{\exp(-\kappa\,\mu)} \frac{\kappa^\tau (1+\kappa)^{\mu+\tau}}{\Gamma(\mu+\tau)\tau!} \Psi_P(\bar{\gamma}) \tag{11}$$

and

$$I_P(\bar{\gamma}) \approx \frac{m^m}{\Gamma(m)} \Psi_P(\bar{\gamma}) \tag{12}$$

in the general case and in the special case of Nakagami-m fading (that occurs at $\kappa = 0$ and $\mu = m$), respectively. The convergent infinite series in (11) can be truncated to the desired accuracy. For the above expressions,

$$\begin{aligned}\Psi_P(\bar{\gamma}) &= \sum_{p_1+p_2+\cdots+p_N=P} \sum_{j=0}^{P} \binom{P}{j} (-1)^j \frac{1}{\bar{\gamma}^\mu}\, G\, H\Bigg[C^{-A-1} \\ &\quad \times \Gamma(A+1) - C^{-A-\frac{3}{2}} \Bigg[\sqrt{C}\,\Gamma(A+1)\,{}_1F_1\left(A+1; \frac{1}{2}; \frac{B^2}{4C}\right) \\ &\quad - B\,\Gamma\left(A+\frac{3}{2}\right) {}_1F_1\left(A+\frac{3}{2}; \frac{3}{2}; \frac{B^2}{4C}\right)\Bigg]\Bigg],\end{aligned} \tag{13}$$

in which the first summation is taken over all combinations of non-negative integer indices p_1 through p_N such that the

sum of all p_n is P. The parameters are $G = \binom{P}{p_1, p_2, \ldots, p_N}$, $H = a_1^{p_1}\, a_2^{p_2} \cdots a_N^{p_N}$, $A = \frac{2\mu+2\tau-2-P}{2} > -1$, $B = c\,j$, $C = \Lambda + \frac{\mu(1+\kappa)}{\bar{\gamma}}$, and $\Lambda = b_1\, p_1 + b_2\, p_2 + \ldots + b_N\, p_N$; moreover, the parameters A and C reduce to $A = \frac{2m-2-P}{2} > -1$ and $C = \Lambda + \frac{m}{\bar{\gamma}}$ for the special case of Nakagami-m fading.

Let us then overview a few examples [16]–[20] from the wide range of applications available in the literature for which the proposed GKL expression is applicable as a substitute for the KL expression that was originally used in those publications. In particular, the GKL approximation/bound can be used to calculate the sampling bit error probability of binary phase shift keying [16], to approximate the phase noise probability density function in the system considered in [17], and to derive the coherent LoRa® symbol error rate under additive white Gaussian noise [18]. Beyond communications, it allows to approximate the distribution functions of particles experiencing compound subdiffusion [19] and to derive the predictive error of the probability of failure [20], for instance.

Furthermore, the simplified series expansion of the original KL expression proposed in [21] can be applied likewise to (2) with the optimized coefficients, which results in

$$Q(x) \approx \sum_{l=1}^{L} \sum_{n=1}^{N} \frac{(-1)^{l+1}\, a_n\, c^l}{l!} \exp\left(-b_n\, x^2\right) x^{l-1}. \quad (14)$$

Since (14) can be used as a direct substitute for [21, Eq. 3], the proposed GKL approximations are also useful for the applications considered in [22]–[25] (and many others that cite [21]) and improve the accuracy of the analysis thereof.

V. Conclusion

This Letter presented a new tractable expression for approximating the Gaussian Q-function together with multiple alternatives of coefficient sets for it[1] that are optimized to minimize either the global or total absolute/relative errors, from which the best suitable set is chosen for any application at hand. The extremely low error levels allow for their usage as highly reliable substitutions to the Q-function in order to derive virtually exact analytical expressions for different performance metrics in communication theory. Moreover, we extended the proposed expression to minimax bounds (with comparable accuracy to that of the approximations) that are useful when the worst/best case scenarios are of interest.

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