Distributed Economic MPC with Separable Control Contraction Metrics

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***Abstract*—In this paper, a flexible and scalable distributed economic model predictive control (DEMPC) scheme is proposed for large-scale networked systems consisting of nodes with local self-interest economic objectives but a shared responsibility for stability of the interconnected network. In this paper, subsystems are controlled to track reference trajectories while optimizing their own economic costs during the transitions. A reference-independent stability constraint is introduced using a sum-separable control contraction metric (CCM). The proposed control design approach is thereby independent of both reference trajectories and the economic cost functions and as such suitable for flexible manufacturing. We observe that self-interested distributed optimization with shared responsibility for stability results in the same general problem formulation as network utility maximization (NUM), and make use of the alternating direction method of multipliers (ADMM) to coordinate the optimization. The proposed approach is illustrated with an example of distributed control of a network with 120 nodes, each of which is a three-state nonlinear system.**

***Index Terms*—Distributed control, control of networks, predictive control for nonlinear systems**

I. Introduction

ADVANCES in information and network technology enable the development of large scale "smart infrastructure" (e.g. power, transport) and "smart manufacturing" systems, which are more agile, cost-effective and continuously operated close to the market. For example, in the process industry, in response to the continuously changing feedstock supply and product market demand, the process operation strategy, including operating conditions and targets (e.g., setpoints of production rates and specifications) need to be frequently updated to improve the process economy. The new paradigm requires: (1) a flexible and reference-independent control design method, which does not need exact information on the reference trajectories ([1]); (2) an optimization-based online control approach that achieves optimal operational economy during the transitions ([2]); (3) a scalable distributed control framework for large-scale interacted networked systems.

One solution is economic model predictive control (EMPC), which integrates feedback strategy with dynamic plant economic optimization. Due to the use of the general cost functions (not necessarily positive-definite), the closed-loop system is not necessarily convergent [3]. In addition, the reference trajectories are often determined (and regularly changed) based on considerations other than the operational costs. In general, EMPCs without additional stability conditions do not converge to the reference trajectories. Several approaches have been proposed to address this issue. In [4], an additional terminal equality or terminal region constraint was added to the EMPC. Stability analysis of EMPC without terminal constraints are presented in [3], [5]. Alternatively, a Lyapunov-based stability constraint on the first step control action was proposed in [6]. The stability condition or constraint in most existing approaches are only valid at the equilibria/reference trajectories which are the prior knowledge in the stage of off-line control design. As such, they do not suit EMPCs for flexible manufacturing.

Another important issue is the scalability of EMPC for networked systems. In [7], a sequential distributed Lyapunov-based EMPC was proposed for nonlinear chemical process operation. In this approach, each controller optimizes a centralized economic cost to determine its own control action while inputs of other subsystems are either provided by the corresponding controllers or generated by a backup control law. The scalability of such an approach could be an issue when the system size increases. For many applications, e.g. geographically distributed energy storage systems, individual subsystems (which belong to different owners) optimize their own economic costs. In these situations, referred to as *self-interested networked systems*, controller coordination for network-wide stability can be achieved by using additional constraints for individual subsystems [8]. Similarly, a distributed EMPC (DEMPC) approach based on dissipation inequality constraints for subsystems was developed in [9]. These approaches are scalable as each agent exchanges trajectory information once and solves a small-scale EMPC problem in a distributed way. However, the predetermined stability constraints (e.g., dissipativity conditions) for individual subsystems in the above approaches can be very conservative.

In this work, a flexible and scalable DEMPC scheme based on control contraction metrics (CCM) [10] is proposed for self-interested large-scale networked systems. Unlike a Lyapunov function, a CCM defines a distance *between* trajectories, and hence provides a reference-independent stability constraint. Furthermore, the reference signals can be arbitrary admissible trajectories (including time varying setpoints), which is useful for flexible manufacturing. The use of separable CCMs [11], [12] ensures stability computations can be done with local node information.

Secondly, individual DEMPC controllers at each node op-

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timize a self-interest cost function, but are coordinated to share responsibility for the overall system stability. The coordination we propose is based on the alternating direction method of multipliers (ADMM) [13], which is motivated by the observation that DEMPC with a shared stability condition has the same general formulation as the well-studied Network Utility Maximization (NUM) problem. Unlike approaches based on stability constraints for individual subsystems ([8], [9]), the network contraction constraint is ensured by real-time negotiation between DEMPC controllers, which leads to less conservative stability conditions.

The proposed approach is scalable as it (1) allows individual DEMPC controllers to optimize their local cost functions subject to subsystem models in parallel, (2) requires only very limited information exchange among nodes as only the local stability constraints, not the input and state trajectories, are negotiated.

The remainder of this paper is organized as follows. In Section II, the problem of DEMPC for self-interested network systems is formulated and the preliminaries on CCM are introduced. The proposed CCM-based EMPC is presented in Section III and the distributed version with the ADMM framework is developed in Section IV. An illustrative example is presented in Section V.

II. Problem Formulation

Notations. Given a matrix $A \in \mathbb{R}^{n\times n}$, denote $\widehat{A} := A + A^T$. The notion $M \succeq 0$ stands for M being positive semidefinite. Let $\mathbb{S}^n_+$ denote the set of positive semidefinite $n \times n$ symmetric matrices. Given differentiable functions $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ the notation $\partial_f P$ stands for an $n \times n$ matrix with (i, j) element given by $\frac{\partial P_{ij}(x)}{\partial x} f(x)$.

For given $i, j \in \mathbb{N}$ and $i < j$, the notation $\mathbb{N}_{[i,j]}$ stands for the set $\{k \in \mathbb{N}\colon i \le k \le j\}$. A *graph* $\mathcal{G} = (V_\mathcal{G}, E_\mathcal{G})$ is a pair of *node set* $V_\mathcal{G} := \mathbb{N}_{[1,N]}$ and *edge set* $E_\mathcal{G} \subset V_\mathcal{G} \times V_\mathcal{G}$. A node $i \in V_\mathcal{G}$ is said to be *adjacent* to node $j \in V_\mathcal{G}$ if $(i, j) \in E_\mathcal{G}$. The set of nodes that are adjacent to j is defined as $\mathcal{N}(j) := \{i \in V_\mathcal{G} : i \neq j, (i, j) \in E_\mathcal{G}\}$. Given two nodes $i, j \in V_\mathcal{G}$, an ordered sequence of edges $\{(v_k, v_{k+1}) \in \mathcal{E}\}_{k=0}^{L}$ with $v_0 = i$ and $v_L = j$ is said to be a *path from node i to node j*. A graph is said to be *strongly connected* if, for every two nodes $i, j \in V_\mathcal{G}$, there exists a path connecting them.

Consider the network described by a strongly connected graph $\mathcal{G}$ with N nodes. The system evolving on each node $i \in V_\mathcal{G}$ is described by

$$\dot{x}_i = f_i(x_i, \breve{x}_i) + B_i(x_i)u_i \tag{1}$$

where state x_i and input u_i are constrained by bounded closed sets $\mathbb{X}_i \subset \mathbb{R}^{n_i}$ and $\mathbb{U}_i \subset \mathbb{R}^{m_i}$, respectively. Consider the subsystems indexed by $j_1, \dots, j_k \in \mathcal{N}(i)$ that are adjacent to i, the vector $\breve{x}_i := \mathrm{col}(x_{j_1}, x_{j_2}, \dots, x_{j_k}) \in \mathbb{R}^{\breve{n}_i}$ is the *interconnecting input*, where $\breve{n}_i := n_{j_1} + n_{j_2} \cdots + n_{j_k}$. Each subsystem (1) is given by the smooth vector fields f_i and B_i. By employing a more compact notation, the networked dynamics can be represented by

$$\dot{x} = f(x) + B(x)u \tag{2}$$

where $x = \mathrm{col}(x_1, x_2 \dots, x_N) \in \mathbb{X} \subset \mathbb{R}^n$ with $n = n_1 + \cdots + n_N$ and $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_N$, $u = \mathrm{col}(u_1, u_2, \dots, u_N) \in \mathbb{U} \subset \mathbb{R}^m$ with $m = m_1 + m_2 + \cdots + m_N$ and $\mathbb{U} = \mathbb{U}_1 \times \mathbb{U}_2 \times \cdots \times \mathbb{U}_N$, $f = \mathrm{col}(f_1, f_2, \dots, f_N)$ and $B = \mathrm{diag}(B_1, B_2, \dots, B_N)$. A function $u(\cdot)$ is said to be an *input signal* or *control* for (2) if it is locally essential bounded. Denote $X(\cdot, x_0, u)$ as the trajectory of (solution to) (2) under the input signal u with initial condition x_0.

The control objective is to design a distributed EMPC law μ such that for any initial state x_0, closed-loop trajectories $X(\cdot, x_0, \mu)$ exponentially converge to the admissible reference $x^*(\cdot)$. Meanwhile, economic criteria for individual subsystems are optimized during the transitions from initial states to the neighborhood of the reference trajectory.

Problem 1. Consider the networked system described by (2) with self-interested stage economic costs $\ell_i(x_i, u_i)$ for each subsystem $i \in \mathbb{N}_{[1,N]}$. For any initial state x_0 and admissible reference trajectory $x^*(\cdot)$, design a distributed EMPC law $\mu : \mathbb{X} \to \mathbb{U}$, such that

- P_1. The closed-loop trajectory $X(\cdot, x_0, \mu)$ exponentially converges to the neighborhood of $x^*(\cdot)$.
- P_2. Each subsystem i optimizes its own stage cost function $\ell_i(x_i, u_i)$ based on its own model (1) and state trajectory information of its neighbors $\mathcal{N}(i)$.

Since the reference trajectory $x^*(\cdot)$ could be any feasible solution, we need to develop a reference-independent stability analysis and control design method. Contraction theory [10] can be used to solve P_1 of Problem 1. Some mathematical background on Riemannian geometry is presented here before the introduction of contraction theory.

Throughout the paper, the state and input manifolds $\mathcal{X}, \mathcal{U}$ are assumed to be Euclidean manifolds $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For two points $x_0, x_1 \in \mathcal{X}$, there exists a smooth curve $\gamma : [0, 1] \to \mathcal{X}$ connecting x_0 and x_1, i.e., $\gamma(0) = x_0$ and $\gamma(1) = x_1$. $T_x\mathcal{X}$ denotes the tangent space at $x \in \mathcal{X}$ and $T\mathcal{X} := \bigcup_{x\in\mathcal{X}}\{x\} \times T_x\mathcal{X}$ denotes the tangent bundle. The Riemannian metric is a uniformly bounded matrix function $M : \mathcal{X} \to \mathbb{S}^n_+$ (i.e., there exist constants $0 < \alpha_1 < \alpha_2$ such that $\alpha_2 I_n \succeq M(x) \succeq \alpha_1 I_n$). The inner product of $v, w \in T_x\mathcal{X}$ induced by Riemannian metric can be defined as $\langle v, w\rangle_x := v^T M(x) w$. The geodesic connecting two points $x_0, x_1 \in \mathcal{X}$ with respect to Riemann metric $M(x)$ can be defined as follows:

$$\begin{aligned} &\Gamma(\cdot) := \arg\min_{\gamma} \int_0^1 \langle \gamma_s, \gamma_s\rangle_{\gamma(s)}\, ds \\ &\text{s.t. } \gamma(\cdot) \in \mathcal{C}^1,\ \gamma(0) = x_0,\ \gamma(1) = x_1 \end{aligned} \tag{3}$$

where $\gamma(\cdot) \in \mathcal{C}^1$ means $\gamma_s(\cdot) := \partial\gamma/\partial s(\cdot)$ is a piecewise continuous vector function. Note that if the Riemann metric is a constant matrix, then the geodesic $\Gamma(\cdot)$ is the straight line connecting x_0 to x_1. For state-dependent $M(x)$, the geodesic $\Gamma(\cdot)$ is in general a curve, and methods for computing it were proposed in [14]. We can define the Riemannian distance

$d(x_0, x_1)$ and energy $\varepsilon(x_0, x_1)$ as follows:

$$d(x_0, x_1) := \int_0^1 \sqrt{\langle \Gamma_s, \Gamma_s \rangle_{\Gamma(s)}}\, ds \tag{4}$$

$$\varepsilon(x_0, x_1) := \int_0^1 \langle \Gamma_s, \Gamma_s \rangle_{\Gamma(s)}\, ds. \tag{5}$$

Now we can define a framework to deal with the exponential convergence of any pair of solutions.

Definition 1. System (2) is said to be *metric exponentially stabilizable with rate* λ if, for every solution $X(\cdot, x^*, u^*)$ to (2), there exist a Riemannian distance function $d(\cdot, \cdot)$ and a feedback control law μ such that, for every initial condition $x \in \mathcal{X}$, the corresponding solution $X(\cdot, x, \mu)$ to (2) satisfies the inequality

$$d(X(t, x^*, u^*), X(t, x, \mu)) \leq e^{-\lambda t} d(x^*, x),\ \forall t > 0. \tag{6}$$

Consider the unconstrained dynamical system of (2) and assume that the state and input involve on manifolds $\mathcal{X}$ and $\mathcal{U}$, respectively. Then the contraction property between any pair of solutions is analyzed by investigating the stability of its associated *differential dynamics* [15], [16]:

$$\dot{\delta}_x = A(x, u)\delta_x + B(x)\delta_u, \tag{7}$$

where $(\delta_x, \delta_u) \in T(\mathcal{X} \times \mathcal{U})$, $A(x, u) = \frac{\partial f(x)}{\partial x} + \sum_{j=1}^m u_j \frac{\partial b_j(x)}{\partial x}$ and $b_j(x)$ is the jth column of $B(x)$. It can be understood as a linear time-varying system (continuous linearization) along any given trajectory $X(\cdot, x, u)$ of (2). The CCM, which is similar to control Lyapunov functions, can be defined as follows.

Definition 2 ([10])**.** A Riemann metric $M(x)$ is said to be a *contraction metric* for system (2) and (7) if there exists a constant $\lambda > 0$ such that the inequality

$$\frac{d}{dt} V(x, \delta_x) \leq -2\lambda V(x, \delta_x) \tag{8}$$

where $V(x, \delta_x) := \delta_x^T M(x) \delta_x$ is denoted as the differential Lyapunov function, holds for every trajectory to system (2) and (7). Furthermore, if following two conditions

$$\partial_{b_j} M(x) + \widehat{M(x) \tfrac{\partial b_j(x)}{\partial x}} = 0,\ j = 1, \ldots, m \tag{9a}$$

$$\delta_x^T \left(\partial_f M(x) + \widehat{M(x) \tfrac{\partial f(x)}{\partial x}} \right) \delta_x < -2\delta_x^T M(x) \delta_x \tag{9b}$$

hold for all $\delta_x \neq 0$ and $\delta_x^T M(x) B(x) = 0$, then $M(x)$ is said to be a *control contraction metric*.

III. CCM BASED EMPC

Condition (9) can be transformed into following state-dependent LMI [10]

$$\partial_f W - \widehat{\tfrac{\partial f}{\partial x} W} + \gamma(x) B B^T - 2\lambda W \succeq 0 \tag{10}$$

where $W(x) = M(x)^{-1}$ and $\gamma(x) > 0$. If system (2) can be modeled as, transformed into or approximated by polynomial systems, then (10) can be solved by the sum-of-squares (SOS) programming [17]. We can then obtain a differential feedback law for system (7)

$$\delta_u = -K(x)\delta_x = -\frac{1}{2}\gamma(x) B^T M(x) \delta_x \tag{11}$$

and a reference-independent feedback law for system (2)

$$u = u^* - \int_0^1 K(\Gamma(s)) \Gamma_s\, ds \tag{12}$$

where $\Gamma(s)$ is a geodesic connecting $x^*(\cdot)$ and $x(\cdot)$ with respect to contraction metric $M(x)$, $u^*(\cdot)$ is the corresponding nominal input for $x^*(\cdot)$. By integrating (8) along the geodesic $\Gamma(x^*, x)$, we have following contraction inequality [10]:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\varepsilon(x^*, x) &= \langle \Gamma_s(1), \dot{x} \rangle_x - \langle \Gamma_s(0), \dot{x}^* \rangle_{x^*} \\ &\leq -\lambda \varepsilon(x^*, x) \end{aligned} \tag{13}$$

which can be rewritten as follows:

$$\begin{aligned} g(x, u, x^*, \lambda) := &\langle \Gamma_s(1), f(x) + B(x)u \rangle_x - \\ &\langle \Gamma_s(0), \dot{x}^* \rangle_{x^*} + \lambda \varepsilon(x^*, x) \leq 0. \end{aligned} \tag{14}$$

It is a linear inequality constraint on u since $\Gamma(x^*, x)$ and $\varepsilon(x^*, x)$ can be calculated at the beginning of each sampling period. It can also be understood as a implicit incremental Lyapunov constraint since inequality (14) implies that the length of the minimal path (geodesic) between x and x^* decreases exponentially. The contraction constraint (14) may not be feasible because the synthesis problem (10) does not take the state/input constraints into account. A possible way to resolve this issue is to search for a uniform feasible convergence rate λ_m for a given CCM $M(x)$ as follows:

$$\begin{aligned} &g(x, u, x^*, \lambda_m) \leq 0,\ 0 < \lambda_m < \lambda, \\ &x, x^* \in \mathbb{X},\ u \in \mathbb{U},\ \|\dot{x}^*\| \leq v_m \end{aligned} \tag{15}$$

where v_m is the maximum velocity of the reference trajectory. The above feasibility problem can be solved off-line by implementing a line search on λ_m and a sampling method for constraints verification. For non-uniform convergence rates, it can be obtained by constructing a family of subsets covering $\mathbb{X}$ and searching for an individual λ_m on each subset. Here we assume that there exists a feasible λ_m for (15).

A CCM-based EMPC, which guarantees closed-loop stability to any admissible reference trajectories, can be formulated as follows:

$$\min_{\tilde{u} \in S(\tau)} \sum_{j=0}^{H-1} \ell(\tilde{x}(j), \tilde{u}(j))$$

$$\text{s.t.} \quad \tilde{x}(j+1) = f^\tau(\tilde{x}(j), \tilde{u}(j)),\ \tilde{x}(0) = x(t_k) \tag{16a}$$

$$\tilde{u}(j) \in \mathbb{U},\ \tilde{x}(j+1) \in \mathbb{X},\ 0 \leq j \leq H-1 \tag{16b}$$

$$g(\tilde{x}(0), \tilde{u}(0), x^*, \lambda_m) \leq 0 \tag{16c}$$

where $\ell(\tilde{x}, \tilde{u}) = \sum_{i=1}^N \ell_i(\tilde{x}_i, \tilde{u}_i)$ is the stage economic cost function, H is the prediction horizon, and $S(\tau)$ is the family of piece-wise constant functions with sampling period τ. Eq. (16a) is the sampled-data representation of (2). The state

measurements $x(t_k)$ at time step $t_k = k\tau$, $k = 0, 1, \dots$ are assumed to be available. The optimal state and input trajectories to (16) are denoted by $x^{\rm opt}(\mathbb{N}_{[1,H]})$ and $u^{\rm opt}(\mathbb{N}_{[0,H-1]})$. The EMPC is implemented in a receding horizon fashion, that is, solve (16) and apply the control action $\mu(x(t_k)) := u^{\rm opt}(0)$ to system (2) for $t \in [t_k, t_{k+1})$.

The discrete-time model in (16a) is only used for MPC optimization and the contraction constraint (16c), which is derived from the continuous time model, needs to be satisfied only by the actual control action $\mu(x(t_k))$. The convergence property of $x(t)$, $t \in [t_k, t_{k+1})$ needs to be addressed. Denote the left hand side of (13) as $L(x^*, x, u)(t)$. For $t \in [t_k, t_{k+1})$, the constant control action $\mu_k = \mu(x(t_k))$ satisfying

$$\frac{1}{2}\frac{d}{dt}\varepsilon(x^*, x)(t_k) \le -\lambda_m \varepsilon(x^*, x)(t_k) \tag{17}$$

is applied to system (2). Since function $f(\cdot), B(\cdot)$ are continuous and $\mathbb{X}, \mathbb{U}$ are bounded closed sets, then one can find, for all $t \in [t_k, t_{k+1})$, a positive constant K, such that

$$\|L(x^*, x, \mu_k)(t) - L(x^*, x, \mu_k)(t_k)\| \le K\tau. \tag{18}$$

Due to the stability margin provided by convergence rate λ_m, we have following metric exponentially stability property.

Theorem 1. *Assume that system* (2) *is controlled by EMPC* (16) *with CCM $M(x)$ and contraction rate λ_m. For any given initial state x_0 and continuous admissible reference trajectory $x^*(t)$, the closed-loop trajectory $x(t)$ will metric exponentially converge to the tube*

$$B_{\rho_m}(x^*(t)) := \{x : d(x^*(t), x) \le \rho_m\} \tag{19}$$

where $\rho_m = \sqrt{K\tau/\lambda_m}$. More specifically, the convergence rate at $x \in \mathbb{X}\backslash B_\rho(x^*(t))$ (with $\rho > \rho_m$) is no less than $c\lambda_m$, where $c = 1 - (\rho_m/\rho)^2$.

Proof. Using inequalities (13), (17) and (18), for any $x \in \mathbb{X}\backslash B_\rho(x^*(t))$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\varepsilon(x^*, x)(t)\\
&\quad = L(x^*, x, \mu_k)(t_k) + [L(x^*, x, \mu_k)(t) - L(x^*, x, \mu_k)(t_k)]\\
&\quad \le -\lambda_m \varepsilon(x^*, x)(t_k) + K\tau\\
&\quad = -c\lambda_m \varepsilon(x^*, x)(t_k) + K\tau - (1-c)\lambda_m \varepsilon(x^*, x)(t_k)\\
&\quad \le -c\lambda_m \varepsilon(x^*, x)(t_k) + K\tau - (1-c)\lambda_m \rho^2\\
&\quad = -c\lambda_m \varepsilon(x^*, x)(t_k) < 0,
\end{aligned}$$

which implies $\varepsilon(x^*, x)(t) < \varepsilon(x^*, x)(t_k)$. Then we can conclude that

$$\frac{1}{2}\frac{d}{dt}\varepsilon(x^*, x)(t) \le -c\lambda_m \varepsilon(x^*, x)(t). \tag{20}$$

□

Remark 1. The admissible reference trajectory $x^*(t)$ in EMPC problem (16) only represents user-specified control tasks, which may not be optimal with respect to the economic cost function. For example, a robotic arm needs to track different trajectories while minimizing the energy cost. Clearly, the economical optimal trajectory for the robotic arm is to remain at a certain position. These applications require that the stability property of EMPC is independent of the cost function, which cannot be achieved by EMPC schemes without stability constraints [3], [5], [18]. The proposed CCM-based EMPC optimizes the system economy while controlling the state to be within a certain exponential shrinking tube. The convergence rate λ_m becomes a tuning parameter for the trade-off between the economic cost and the stability requirement. Larger λ_m implies fast convergence to $x^*(t)$ and better robustness with respect to disturbances but the economy during the transition period may be sacrificed as the feasible region shrinks.

IV. Distributed EMPC with Separable CCM

As the scale of networked systems increases, the implementation of centralized control becomes very difficult and often infeasible. In [1], [11], distributed state feedback control based on separable CCM and differential dissipativity were developed respectively.

Definition 3 (Sum-separable CCM [11])**.** A CCM $M(x)$ for system (2) is said to be *sum-separable* if, for each subsystem i, there exists a contraction metric $M_i(x_i)$ satisfying $M(x) = \mathrm{diag}(M_1(x_1), M_2(x_2), \dots, M_N(x_N))$.

With the block diagonal structure, we can have $V(x, \delta_x) = \sum_{i=1}^N \delta_{x_i}^T M_i(x_i)\delta_{x_i}$ and $\varepsilon(x, x^*) = \sum_{i=1}^N \varepsilon(x_i, x_i^*)$. The geodesic computation (3) can be decentralized [11]. Moreover, the contraction constraint for the networked system can be rewritten in the following sum-separable form

$$g(x, u, x^*, \lambda_m) = \sum_{i=1}^N g_i(x_i, u_i, x_i^*, \lambda_m) \le 0 \tag{21}$$

where g_i is the contraction representation corresponding to the i-th subsystem in (1). The study on the existence of separable CCM for monotone nonlinear system is presented in [12].

The DEMPC for self-interested systems can be represented as follows:

$$\begin{aligned}
J_i(\phi_i) := \min &\sum_{j=0}^{H-1} \ell_i(\tilde{x}_i(j), \tilde{u}_i(j))\\
\text{s.t. } &\tilde{x}_i(j+1) = f_i^\tau(\tilde{x}_i(j), \breve{x}_i(j), \tilde{u}_i(j)) && \text{(22a)}\\
&\tilde{x}_i(0) = x_i(t_k),\ \tilde{u}_i(j) \in \mathbb{U}_i,\ \tilde{x}_i(j+1) \in \mathbb{X}_i,\\
&\quad 0 \le j \le H-1 && \text{(22b)}\\
&g(\tilde{x}_i(0), \tilde{u}_i(0), x_i^*, \lambda_m) \le \phi_i && \text{(22c)}
\end{aligned}$$

where (22a) is the discrete-time model of the i-th subsystem with sampling period τ, $\mathbb{X}$ and $\mathbb{U}$ are state and input constraint sets respectively, and (22c) is the contraction constraint for the i-th subsystem. The external inputs are defined by

$$\breve{x}_i(\mathbb{N}_{[0,H-1]}) := \left[\breve{x}_i(t_k) \quad \breve{x}_i^{\rm opt}\left(\mathbb{N}_{[2,H]}\right)\right] \tag{23}$$

where $\breve{x}_i(t_k)$ and $\breve{x}_i^{\rm opt}$ are current state measurements of neighborhood subsystems and their predicted state trajectories at previous step, respectively. At the beginning of each time step, only the current statement $x_i(t_k)$ and predicted optimal state trajectory $x_i^{\rm opt}$ of the i-th subsystems are transmitted to its neighborhoods, which satisfies the item of P_2 in Problem 1. The self-interest of the systems is reflected by an individual

local cost function each agent is trying to minimize [8]. The coordination between individual DEMPC controllers is needed to ensure the network stability constraints:

$$\sum_{i=1}^{N} g(\tilde{x}_i(0), \tilde{u}_i(0), x_i^*, \lambda_m) \leq \sum_{i=1}^{N} \phi_i \leq 0. \tag{24}$$

The coordination problem can be reformulated as follows:

$$\min_{\phi} \sum_{i=1}^{N} J_i(\phi_i), \quad \text{s.t. } \mathbf{1}^T \phi \leq 0 \tag{25}$$

where $\mathbf{1} = [1, 1, \ldots, 1]^T$ and $\phi = \text{col}(\phi_1, \ldots, \phi_N)$. It is clear from the construction (22) that each $J_i(\phi_i)$ is a monotonically decreasing function of ϕ_i. We note that in the special case of linear dynamics and linear or quadratic costs l_i, the $J_i(\phi_i)$ are also convex.

It is interesting to note that the problem (25) has the same general formulation as the well-studied network utility maximization problem (e.g. [19] and references therein), which allocates communication channels to different nodes subject to limited link capacities and time-varying transmission rates. In this case, ϕ_i and $J_i(\phi_i)$ can be understood as the transmission rate and (negative) utility function at node i, respectively. The constraint $\mathbf{1}^T \phi \leq 0$ corresponds to the link capacity.

In this work, we apply an ADMM framework similar to [19], [20] to solve the problem in (25). Unlike the use of ADMM in DMPC proposed in [20], in our approach the controllers only negotiate the stability constraints (ϕ_i), rather than the state and input trajectories of each subsystem.

In general, the ADMM algorithm solves the following problem [13]:

$$\min_{\theta, \phi} \; f(\phi) + g(\psi) \quad \text{s.t} \quad A\phi + B\psi = c, \tag{26}$$

by iteratively updating the primal variables ϕ, ψ and dual variable ν as follows:

$$\begin{aligned} \phi^{p+1} &= \arg\min_{\phi} L_\beta(\phi, \psi^p, \nu^p) \\ \psi^{p+1} &= \arg\min_{\psi} L_\beta(\phi^{p+1}, \psi, \nu^p) \\ \nu^{p+1} &= \nu^k + (A\phi^{p+1} + B\psi^{p+1} - c) \end{aligned} \tag{27}$$

where p is the iteration step, ν is a scaled dual variable and the augmented Lagrangian is

$$L_\beta = f(\phi) + g(\psi) + \nu^T (A\phi + B\psi - c) + \frac{\beta}{2} \|A\phi + B\psi - c\|^2$$

with a constant $\beta > 0$. We can rewrite (25) in an ADMM amenable form:

$$\min_{\phi, \psi} \sum_{i=1}^{N} J_i(\phi_i), \quad \text{s.t.} \quad \phi - \psi = 0, \; \mathbf{1}^T \psi \leq 0. \tag{28}$$

Then the iterative primal-dual subproblems become:

$$\phi_i^{p+1} = \arg\min_{\phi_i} J_i(\phi_i) + \nu_i^p \phi_i + \frac{\beta}{2} (\phi_i - \psi_i^p)^2 \tag{29a}$$

$$\psi^{p+1} = \arg\min_{\mathbf{1}^T \psi \leq 0} \left(-\nu^p \psi + \frac{\beta}{2} \left\| \phi^{p+1} - \psi \right\|^2 \right) \tag{29b}$$

$$\nu_i^{p+1} = \nu^k + \beta(\phi_i^{k+1} - \psi_i^{p+1}). \tag{29c}$$

The ϕ_i-update in (29a) can be rewritten as follows:

$$\begin{aligned} &\min_{\tilde{u}_i, \phi_i} \; v_i^p \phi_i + \beta(\phi_i - \psi_i^p)^2 + \sum_{j=0}^{H-1} \ell_i(\tilde{x}_i(j), \tilde{u}_i(j)) \\ &\text{s.t. (22a)} - \text{(22c)}. \end{aligned} \tag{30}$$

This is a decentralized optimization problem which optimizes a local cost function subject to a local model of the i-th node.

Due to the network contraction constraint (24), the ψ-update (29b) needs to be performed in a centralized fashion. However, in many large-scale system, strong interactions are often within certain clusters of subsystems while the clusters are sparsely connected with other clusters. Thus we can divide the entire network into clusters and impose individual constraint on each cluster. Here we define a *cluster* $\mathcal{C}$ to be a strongly connected subgraph of $\mathcal{G}$. A family set of clusters $\{\mathcal{C}_s\}_{1 \leq s \leq M}$ is said to be a *partition* of $\mathcal{G}$ if $\bigcup_{s=1}^{M} V_{\mathcal{C}_s} = V_{\mathcal{G}}$ and $\mathcal{C}_r \cap \mathcal{C}_s = \emptyset$, $1 \leq r < s \leq M$ are satisfied. Let $\mathcal{C}_s = \{i_{k_1}, \ldots, i_{k_s}\}$, $1 \leq s \leq M$ be a cluster partition of the network $\mathcal{G}$. We can replace the network constraint with

$$\mathbf{1}^T \psi_s \leq 0, \; 1 \leq s \leq M \tag{31}$$

where $\psi_s = \text{col}(\psi_{i_{k_1}}, \ldots, \psi_{i_{k_s}})$. Note that the above constraint is a sufficient condition for network stability. The ψ-update (29b) can be rewritten as follows:

$$\psi_s^{p+1} = \arg\min_{\mathbf{1}^T \psi_s \leq 0} \left(-\nu_s^p \psi_s + \frac{\beta}{2} \left\| \phi_s^{p+1} - \psi_s \right\|^2 \right) \tag{32}$$

where $\phi_s = \text{col}(\phi_{i_{k_1}}, \ldots, \phi_{i_{k_s}})$. From [19], the analytical solution to the above problem can be expressed as follows:

$$\psi_s^{p+1} = \phi_s^{p+1} - (\theta_s^{p+1} \mathbf{1} - \nu_s^p)/\beta \tag{33}$$

where

$$\theta_s^{p+1} = \frac{1}{|V_{\mathcal{C}_s}|} \max \left\{ 0, \mathbf{1}^T (\beta \phi_s^{p+1} + \nu_s^p) \right\} \tag{34}$$

with $|V_{\mathcal{C}_s}|$ denoting the number of nodes in the cluster $\mathcal{C}_s$.

In order to reduce the iteration steps of ADMM, the following stopping criteria can be adopted:

$$\mathbf{1}^T (\phi_s^{p+1} - \psi_s^{p+1}) \leq \max \left(\lambda_e \mathbf{1}^T \varepsilon(x_s, x_s^*), |V_{\mathcal{C}_s}| \epsilon \right) \tag{35}$$

where $\varepsilon(x_s, x_s^*) = \text{col}\left(\varepsilon(x_{i_{k_1}}, x_{k_1}^*), \ldots, \varepsilon(x_{i_{k_s}}, x_{k_s}^*) \right)$ and $\lambda_e, \epsilon > 0$ are chosen to be sufficiently small. The basic idea is to terminate the ADMM algorithm once a certain converge rate is achieved. When the current state is far away from the reference point, i.e., $\varepsilon(x, x^*)$ is very large, the converge rate for closed-loop system becomes $\lambda_m - \lambda_e$ since

$$\begin{aligned} &g(x, u, x^*, \lambda_m - \lambda_e) = g(x, u, x^*, \lambda_m) - \lambda_e \varepsilon(x, x^*) \\ &\leq \sum_{i=1}^{N} \phi_i^{p+1} - \lambda_e \varepsilon(x, x^*) \leq \sum_{i=1}^{N} (\phi_i^{p+1} - \psi_i^{p+1}) - \lambda_e \varepsilon(x, x^*) \\ &= \sum_{s=1}^{M} \mathbf{1}^T (\phi_s^{p+1} - \psi_s^{p+1} - \lambda_e \varepsilon(x_s, x_s^*)) \leq 0. \end{aligned}$$

In the proposed ADMM scheme, ϕ-update in (29a) can be formulated as relative small-scale optimization problems. Only small amount of information on ϕ, ψ, ν is exchanged among

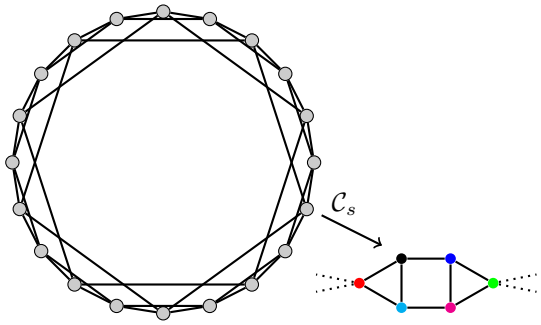

Figure 1. Network topology of the illustrative example

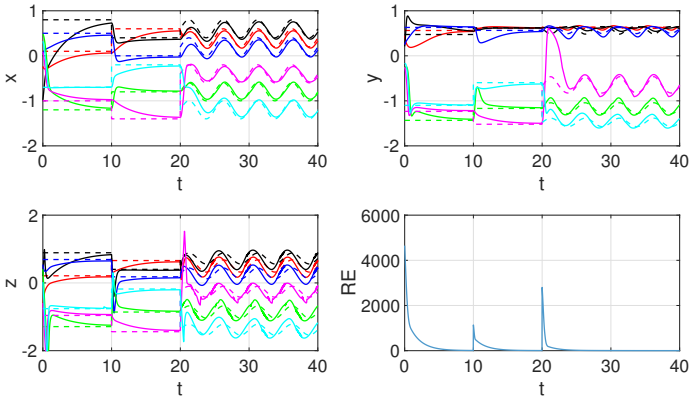

Figure 2. Closed-loop trajectories of Cluster 1 (dashed – references, solid – state trajectories) and Riemannian energy (RE) of the entire network.

the members of each cluster during iterations. Moreover, the iteration steps can be reduced by using a state-dependent tolerance level, as shown in (35). The above features make the proposed approach scalable.

V. Illustrative Example

We consider a network consisting of 20 clusters, each of which contains six color-coded nodes, as depicted in Figure 1. The clusters are connected through red and green nodes. The dynamics of node $i \in \mathbb{N}_{[1,120]}$ is represented by

$$\begin{cases} \dot{x}_i = -x_i + z_i - 0.1(x_i - \sum_{j\in\mathcal{N}(i)} x_j) \\ \dot{y}_i = x_i^2 - y_i^3 - 2x_i z_i + z_i \\ \dot{z}_i = -y_i + u_i \end{cases} \tag{36}$$

where $x_i, y_i, z_i \in [-2, 2]$ and $u_i \in [-4, 4]$. The stage economic cost function for node i is given by $\ell_i(q_i, u_i) = 2x_i - y_i$, where $q_i := [x_i, y_i, z_i]^T$. The main parameters are listed as follows: $\lambda_m = 0.3$, $\beta = 1$, $\lambda_e = \epsilon = 0.01$, $T_s = 0.01$, $H = 10$. Note that in total, this system has 360 states and unstable nonlinear dynamics. The simulation results are shown in Figure 2 (where the state trajectories of the nodes are plotted in the colors of the nodes), from which it can be seen that the proposed approach can drive the networked system to admissible reference trajectories.

VI. Conclusion

In this work, we develop a flexible DEMPC approach for networked systems. Individual subsystems are controlled to optimize their own economic costs while being coordinated to reach arbitrary admissible network-wide reference trajectories with a shared responsibility for overall network stability. The latter is ensured by using sum-separable control contraction metrics and an ADMM framework for online iterative negotiations among the distributed controllers.

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