Consensus of Heterogeneous Multi-agent Systems With Diffusive Couplings via Passivity Indices

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***Abstract*—This work is concerned with the problem of output consensus for two classes of heterogeneous nonlinear multi-agent systems which are interconnected via diffusive couplings over directed graphs. Specifically, for agents that are input feedforward passive (IFP), a condition in terms of passivity indices is proposed for asymptotic output consensus. Moreover, it is shown that the proposed condition can be exploited to design the coupling gain that ensures asymptotic consensus via a semidefinite program (SDP), and the existence of such a coupling gain can be guaranteed provided all the agents are IFP. For agents that are input feedforward output feedback passive (IF-OFP), a condition in terms of passivity indices for practical output consensus is provided, in which the relationship between the coupling gain and the consensus error bound is revealed.**

***Index Terms*—Agents-based systems, Cooperative control, LMIs**

I. Introduction

CONSENSUS, a fundamental issue in cooperative control of multi-agent systems, has received increasing research attention for decades due to its wide applications, e.g., frequency synchronization in power systems [1], formations of unmanned aerial vehicles [2] and coordination and control of distributed sensor networks [3]. Most of the pioneer works have discussed the consensus problems for systems with homogeneous agents, see, for instance, [2]–[4]. Recently, researchers have started to deal with the consensus of heterogeneous multi-agent systems. In fact, heterogeneity exists in most of the networked systems, e.g., a power system composed of individual generators with different dynamics due to different physical parameters. Among the remarkable works on output consensus of heterogeneous multi-agent systems, one should mention [5] and [6] where an internal model principle is proposed as a necessary and sufficient condition for output consensus of linear systems, and [7] where a general framework for robust output consensus is established.

In this work, we concentrate on heterogeneous nonlinear agents that can be characterized by passivity indices. It is well known that dissipativity (and its special case, passivity) is a useful tool for consensus analysis and control design. The output consensus for passive multi-agent systems over weight-balanced digraphs is studied in [8], which is further extended to general digraphs in [9], [10]. A passivity-based switching strategy is developed in [11] for general digraphs. The more general case wherein the agents can be described as input feedforward passive (IFP) systems, which encompasses the case of passive systems as a special case (see [12] for details), is considered in [13]–[15]. Particularly, it is shown in [13] that asymptotic consensus for IFP systems can be achieved via a simple diffusive coupling protocol provided that the couplings are sufficiently weak. In [14], [15], the non-trivial consensus and its synthesis for passivity-short IFP systems are addressed over general digraphs.

More recently, an emerging research aspect on heterogeneous multi-agent systems that has gained growing interests is the practical consensus. Generally, it is difficult to achieve complete asymptotic consensus in heterogeneous systems. Alternatively, the notion of the "practical consensus" is proposed in [16] to study the relationship between the coupling gain and the consensus error bound. Some related works of practical consensus are [17] where practical consensus of single integrator heterogeneous nonlinear time-varying systems over undirected graphs is studied, and [18] where asymptotic and practical consensus of QUAD nonlinear systems over weight-balanced digraphs are studied. To the best of our knowledge, the problem of practical consensus for heterogeneous multi-agent systems over general digraphs has not been addressed from the perspective of passivity indices yet.

Our contributions are as follows. First, a condition for asymptotic output consensus of nonlinear IFP systems is proposed. It is shown that asymptotic consensus can be achieved over general digraphs if agents can be characterized as IFP systems, which is an extension of [8]–[10] where all agents are required to be passive. Moreover, the proposed condition is exploited to design a suitable coupling gain via a semidefinite program (SDP). Second, for agents that can be characterized as input feedforward output feedback passive (IF-OFP) systems, a condition for practical output consensus is derived, which reveals the relationship between the coupling gain and the consensus error bound.

II. Preliminaries

A. Notation

Let $\mathbb{R}$ and $\mathbb{Z}$ be the set of real and integer numbers, respectively. The transpose of a matrix $A \in \mathbb{R}^{m\times n}$ is denoted by A^T. The notations $img(A)$ and $ker(A)$ denote the image and kernel of A, respectively. The Kronecker product is denoted as $\otimes$. $\|A\|$ denotes the 2-norm of A. Given a symmetric matrix $M \in \mathbb{R}^{m\times m}$, the notation $M > 0$ ($M \geq 0$) denotes that M is positive definite (positive semi-definite). Denote the eigenvalues of M in ascending order as $\lambda_1(M) \leq \lambda_2(M) \leq$

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$\ldots \leq \lambda_m(M)$. Denote I_m as the $m \times m$ identity matrix. $\mathbf{1}_m := (1, \ldots, 1)^T \in \mathbb{R}^m$ and $\mathbf{0}_m := (0, \ldots, 0)^T \in \mathbb{R}^m$. $col(v_1, \ldots, v_m) = (v_1^T, \ldots, v_m^T)^T$ denotes the column vector stacked with vectors $v_1, \ldots, v_m$. $diag\{\alpha_i\}$ is a diagonal matrix with its ith diagonal entry being α_i. The notation $\mathcal{C}^k$ is used to denote a $k \in \mathbb{Z}_{\geq 1}$ times continuously differentiable function.

B. Passivity

Let us first give the definition of passivity for a nonlinear system described by

$$\Sigma : \begin{cases} \dot{x} = f(x, u) \\ y = h(x, u) \end{cases} \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$ and $y \in \mathcal{Y} \subset \mathbb{R}^m$ are the state, input and output, respectively, and $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are the state, input and output spaces, respectively.

Definition 1 (Dissipative/Passive System [19]). *System Σ with supply rate $\omega(t)$ is said to be dissipative if there exists a $\mathcal{C}^1$ nonnegative real function $V(x)$, called the storage function, such that for all $t \geq 0$,*

$$V(x(t)) - V(x(0)) \leq \int_0^t \omega(\tau)\, d\tau. \tag{2}$$

System Σ is called a passive system if the supply rate is $\omega(t) = u(t)^T y(t)$.

Throughout this work, we assume that any storage function V is radially unbounded, positive definite and $V(\mathbf{0}_n) = 0$.

Definition 2 (Excess/Shortage of Passivity [20]). *System Σ is said to be: Input Feedforward Passive (IFP) if it is dissipative with respect to the supply rate $\omega(u, y) = u^T y - \nu u^T u$ for some $\nu \in \mathbb{R}$, denoted as IFP(ν); Output Feedback Passive (OFP) if it is dissipative with respect to the supply rate $\omega(u, y) = u^T y - \rho y^T y$ for some $\rho \in \mathbb{R}$, denoted as OFP (ρ); Input Feedforward Output Feedback Passive (IF-OFP) if it is dissipative with respect to the supply rate $\omega(u, y) = u^T y - \nu u^T u - \rho y^T y$ for some $\nu \in \mathbb{R}$ and $\rho \in \mathbb{R}$, denoted as IF-OFP (ν, ρ).*

The signs of passivity indices ν and ρ denote an excess or shortage of passivity. Particularly, when $\nu > 0$ (respectively, $\rho > 0$), the system is said to be input strictly passive (ISP) (respectively, output strictly passive (OSP)).

C. Graph Theory

The information exchanging network is represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \ldots, N\}$ is the node set of all agents and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is the edge set. The edge $(i, j) \in \mathcal{E}$ denotes that agent i can obtain information from agent j. The graph $\mathcal{G}$ is said to be $undirected$ if $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$ and $directed$ otherwise. $\mathcal{G}$ is said to be $strongly\ connected$ if there exists a sequence of edges between any two agents. A sequence of time-varying graphs $\{\mathcal{G}(t)\}$ is said to be jointly strongly connected if there exists a $T > 0$ such that for any t_k, the union $\cup_{t \in [t_k, t_k+T]} \mathcal{G}(t)$ is strongly connected. The adjacency matrix is defined as $A = [a_{ij}]$, where $a_{ii} = 0$, $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. The in-degree and out-degree of the ith node are $d_{in}^i = \sum_{j=1}^N a_{ij}$ and $d_{out}^i = \sum_{j=1}^N a_{ji}$, respectively. The graph $\mathcal{G}$ is said to be $weight\text{-}balanced$ if $d_{in}^i = d_{out}^i,\ \forall i \in \mathcal{N}$. The in-degree matrix of $\mathcal{G}$ is defined as $W_{in} = diag\{d_{in}^i\}$. The Laplacian matrix of $\mathcal{G}$ is defined as $L = W_{in} - A$.

D. Preliminary Lemmas

Before stating our main results, we introduce some preliminary lemmas as follows.

Lemma 1 ([21]). *Let $\Xi = diag\{\xi_i\}$ where ξ is the left eigenvector of the Laplacian matrix L corresponding to the zero eigenvalue and satisfies that $\xi_i > 0$. Suppose the graph $\mathcal{G}$ is strongly connected, then $\Xi L + L^T \Xi \geq 0$.*

Lemma 2 ([22]). *Given a singular symmetric matrix $A \in \mathbb{R}^{N \times N}$ with eigenvalues $0 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_N$, and suppose $\mathbf{1}_N$ is the eigenvector corresponding to the zero eigenvalue, then*

$$\min_{x \neq \mathbf{0}_N,\ \mathbf{1}_N^T x = 0} x^T A x = \lambda_2 \|x\|^2, \quad \max_{x \neq \mathbf{0}_N} x^T A x = \lambda_N \|x\|^2.$$

Lemma 3 ([12]). *Let $V : \mathbb{R}^m \to \mathbb{R}$ be a continuous positive definite function that contains the origin. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|).$$

Moreover, if $V(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$.

III. Problem formulation

We consider a group of N heterogeneous agents of the general form

$$\begin{cases} \dot{x}_i = & f_i(x_i, u_i) \\ y_i = & h_i(x_i) \end{cases} \quad i = 1, \ldots, N \tag{3}$$

where $x_i \in \mathbb{R}^n$, $u_i, y_i \in \mathbb{R}^m$ are the state, input and output, respectively; f_i and h_i are general nonlinear functions. The dimension of the input and output of all agents are the same.

Definition 3. *The group of agents* (3) *is said to achieve asymptotic output consensus if* $\lim_{t \to +\infty} \|y_i(t) - y_j(t)\| = 0,\ \forall i, j \in \mathcal{N}$.

Define the average output as $\bar{y} := \frac{1}{N} \sum_{i=1}^N y_i$.

Definition 4. *The group of agents* (3) *is said to achieve practical output consensus if given $\epsilon > 0$, there exists a real number $T \geq 0$ (dependent on ϵ and $y_i(t_0)$ for all $i \in \mathcal{N}$) such that $\|y_i(t) - \bar{y}(t)\| \leq \epsilon,\ \forall t \geq t_0 + T, \forall i \in \mathcal{N}$.*

In this work we consider the scenario where the agents are diffusively coupled over a directed and strongly connected communication graph $\mathcal{G}$. To be specific, a consensus protocol based on relative output feedback is exploited, and the input $u_i,\ i \in \mathcal{N}$ are determined as

$$u_i = \sigma \sum_{j=1}^N a_{ij}(y_j - y_i), \quad i = 1, \ldots, N \tag{4}$$

where the coupling gain σ is a positive constant. It follows that a compact form of (4) is given by

$$u = -\sigma \left(L \otimes I_m \right) y \tag{5}$$

where L is the Laplacian matrix of $\mathcal{G}$, $u = col(u_1, \ldots, u_N)$ and $y = col(y_1, \ldots, y_N)$.

For a group of heterogeneous dissipative agents (3) that can be characterized by passivity indices, our goal is to investigate their consensus behaviours when they are interacting with each other by the diffusive coupling (4) over digraphs. Specifically, we aim to derive a condition for asymptotic output consensus and design a suitable coupling gain based on this condition. Moreover, for more general classes of systems that may not have the behaviour of asymptotic consensus, we aim to extend our condition to address practical output consensus and reveal the relationship between the coupling gain and the consensus error bound.

IV. Main Results

A. Asymptotic Consensus and the Coupling Gain

In this subsection, we first investigate asymptotic output consensus and then propose an optimization method to design a suitable coupling gain.

Assume that all agents in (3) can be represented as IFP systems. In particular, the ith agent can be characterized as a IFP(ν_i) system with the passivity index ν_i. Define the symmetric matrix

$$M = -\frac{\sigma}{2}(\Xi L + L^T \Xi) - \sigma^2 L^T \Xi \nu L \tag{6}$$

where $\nu = diag\{\nu_i\}$ and Ξ is defined in Lemma 1.

Theorem 1. *Consider the group of heterogeneous IFP agents* (3) *with diffusive couplings* (4)*. The interconnected system can achieve asymptotic output consensus if $M \leq 0$ and zero is a simple eigenvalue of M.*

Proof. Suppose $M \leq 0$ and zero is a simple eigenvalue of M. First, since each agent is IFP, there exists a storage function V_i for each agent i such that

$$\dot{V}_i \leq y_i^T u_i - \nu_i u_i^T u_i, \ \forall i \in \mathcal{N}. \tag{7}$$

Select the candidate Lyapunov function as $V = \sum_{i=1}^N \xi_i V_i$, where ξ_i is the ith element of the left eigenvector of the Laplacian matrix L corresponding to the zero eigenvalue and satisfies $\xi_i > 0$. Hence, V is positive definite. The derivative of V gives

$$\begin{aligned}
\dot{V} \leq & \sum_{i=1}^N \xi_i y_i^T u_i - \xi_i \nu_i u_i^T u_i \\
= & \sum_{i=1}^N y_i^T (\xi_i \otimes I_m) u_i - u_i^T (\xi_i \nu_i \otimes I_m) u_i \\
= & -\sigma y^T (\Xi \otimes I_m)(L \otimes I_m) y \\
& - \sigma^2 y^T (L^T \otimes I_m)(\Xi \nu \otimes I_m)(L \otimes I_m) y \\
= & y^T \left\{ \left[-\frac{\sigma}{2}(\Xi L + L^T \Xi) - \sigma^2 L^T \Xi \nu L \right] \otimes I_m \right\} y \\
= & y^T (M \otimes I_m) y.
\end{aligned}$$

By properties of Kronecker product, one has $M \leq 0 \Rightarrow M \otimes I_m \leq 0$. Therefore, $\dot{V} \leq 0$.

Denote the set $\mathcal{S} = \{ y \mid y_i = y_j, \forall\, i, j \}$. Clearly, $y \in \mathcal{S}$ is equivalent to $y = \mathbf{1}_N \otimes \bar{y}$. Zero is the simple eigenvalue of M and it can be observed that $M \mathbf{1}_N = \mathbf{0}_N$. Then, $y^T (M \otimes I_m) y = 0$, if and only if $y \in \mathcal{S}$, and $y^T (M \otimes I_m) y < 0 \; \forall y \notin \mathcal{S}$. Since $V \geq 0$ and $\dot{V} \leq 0$, there exists a constant $c \geq 0$ such that $\lim_{t \to +\infty} V = c$. When $V = c$, $\dot{V} = 0$, and $\dot{V} = 0$ only if $y \in \mathcal{S}$. Therefore, $\lim_{t \to +\infty} \| y_i(t) - y_j(t) \| = 0$, $\forall i \in \mathcal{N}$, and asymptotic output consensus can be achieved. □

The next step is to design the coupling gain σ. In the following result, it is shown that the condition in Theorem 1 can be satisfied if σ takes any value within an interval $(0, \sigma_e)$ where σ_e depends on the graph topology and the IFP indices $\nu_i, \ i \in \mathcal{N}$. With a linear transformation technique introduced in [23], the condition of Theorem 1 can be transformed into a linear matrix inequality (LMI) condition.

Let $\bar{M} := \frac{1}{\sigma} M = -\frac{1}{2}(\Xi L + L^T \Xi) - \sigma L^T \Xi \nu L$. Since $\sigma > 0$, the condition of Theorem 1 is equivalent to $\bar{M} \leq 0$ and zero is a simple eigenvalue of $\bar{M}$. Let us define a matrix $R \in \mathbb{R}^{N \times (N-1)}$ such that $img(R) = ker(\mathbf{1}_N^T)$ and it follows that $\tilde{M} = R^T \bar{M} R$ has the same eigenvalues with $\bar{M}$ except for zero. The design of σ is converted to solving a SDP problem.

Corollary 1. *The group of IFP agents* (3) *with the diffusive coupling* (4) *can achieve consensus if the coupling gain $\sigma \in (0, \sigma_e)$, where*

$$\begin{aligned}
\sigma_e = \sup_{\sigma \in \mathbb{R}_+} \quad & \sigma \\
\textit{subject to} \quad & \tilde{M} < 0.
\end{aligned} \tag{8}$$

Remark 1. *It should be noted that since the matrix $\Xi L + L^T \Xi \geq 0$ according to Lemma 1, the LMI constraint in* (8) *is always strictly feasible. In other words, there must exist $\sigma > 0$ such that $\tilde{M} < 0$ regardless of the sign or value of $\nu_i, i \in \mathcal{N}$. Moreover, the condition in Theorem 1 does not impose any constraint on the sign of passivity indices $\nu_i, \ i \in \mathcal{N}$, which implies that the agents can be non-passive, and all IFP systems are output-consensusable. However, in order to compensate for the shortage of passivity of agents, the coupling gain should be chosen within the interval $(0, \sigma_e)$ instead of any positive value. Therefore, Theorem 1 is more general than results obtained in [8]–[10] where all agents are required to be passive. Moreover, when all agents are passive, i.e., $\nu_i \leq 0, \ \forall i \in \mathcal{N}$, it follows that the LMI condition $\tilde{M} < 0$ in* (8) *is satisfied automatically and $\sigma_e \to \infty$, which recovers the results in [8]–[10].*

Remark 2. *The conservatism of the condition in Theorem 1 stems from the choice of Lyapunov function. Moreover, it can be observed that if the condition in Theorem 1 is satisfied with some σ and $\nu_i, i \in \mathcal{N}$, it is also satisfied with the same σ and with $\hat{\nu}_i, \ i \in \mathcal{N}$ where $\hat{\nu}_i \geq \nu_i, \ i \in \mathcal{N}$. For a nonlinear system, it is generally difficult to derive the exact IFP index, and only its lower bound can be obtained by specifying the storage function, which narrows the feasible range of σ. The conservatism is illustrated in Example 1 by checking the tightness of the bound σ_e.*

B. Extensions of Asymptotic Consensus among IFP agents

An extension of Theorem 1 is to consider the case where agents interact with each other using different coupling gains, i.e.,

$$u_i = \sigma_i \sum_{j=1}^{N} a_{ij}(y_j - y_i), \quad i = 1, \ldots, N \tag{9}$$

where σ_i, $\forall i \in \mathcal{N}$ denote different coupling gains for different agents.

Corollary 2. *The group of IFP agents* (3) *with the diffusive coupling* (9) *can achieve asymptotic output consensus if the symmetric matrix* $Q = -\frac{1}{2}(\Xi L + L^T \Xi) - L^T diag\{\sigma_i\} \Xi \nu L \leq 0$ *and zero is its simple eigenvalue.*

Its proof follows from a similar argument of the proof of Theorem 1 by selecting $V = \sum_{i=1}^{N} \xi_i \sigma_i^{-1} V_i$.

Remark 3. *If* $\exists \nu_i < 0$*, the approximation of the condition in this corollary in terms of eigenvalues gives* $\sigma_i < \frac{\lambda_2(\Xi L + L^T \Xi)}{-2 \min_i\{\nu_i\} \lambda_N(L^T \Xi L)}$*, showing that local gains can be designed independent of other agents' indices provided the minimum index is known [14]. It also reveals what kinds of graph can tolerate more non-passive systems and ensure larger coupling gains. However, it adopts approximation in terms of eigenvalues and thus certainly reduces the feasible range.*

Another extension is to consider asymptotic consensus over time-varying graphs, where $\mathcal{G}(t)$ at each time t is weight-balanced. Denote $L(t)$ as the graph Laplacian and assume it is not zero at any time. The input u can be written as

$$u = -\sigma \left(L(t) \otimes I_m\right) y. \tag{10}$$

Corollary 3. *Suppose* $\{\mathcal{G}(t)\}$ *is a sequence of jointly strongly connected weight-balanced digraphs with* $L(t) \neq \mathbf{0}$*, then the group of IFP agents* (3) *with the diffusive coupling* (10) *can achieve asymptotic output consensus if* $\exists \nu_i < 0$ *and the coupling gain* σ *satisfies* $0 < \sigma < \frac{\lambda_+\left(L(t) + L^T(t)\right)}{-2 \min_i\{\nu_i\} \lambda_N(L^T(t) L(t))}$, $\forall t > 0$ *where* $\lambda_+(\cdot)$ *denotes the nonzero smallest eigenvalue.*

Its proof lies in the fact that $ker\left(L(t)\right) = ker\left(L^T(t)\right)$ and the existence of a coordinate transformation for $L(t) + L^T(t)$ and $L^T(t)L(t)$. The rest of the argument is similar to [8].

C. Practical Consensus and the Coupling Gain

Theorem 1 is developed based on the assumption that all agents are IFP systems. When a wider class of agents, the IF-OFP agents, is considered, the results proposed in Theorem 1 is no longer applicable. Alternatively, we will investigate practical output consensus and reveal the relationship between the consensus error bound and the coupling gain hereafter.

Consider the scenario where all agents in (3) can be represented as IF-OFP systems. In particular, the ith agent can be characterized as a IF-OFP (ν_i, ρ_i) system and there exist some $\rho_i < 0$. In fact, if $\rho_i \geq 0$, $\forall i \in \mathcal{N}$, the term $u_i^T y_i - \nu_i u_i^T u_i - \rho_i y_i^T y_i$ is upper bounded by $u_i^T y_i - \nu_i u_i^T u_i$ due to $-\rho_i y_i^T y_i \leq 0$. Then, the inequalities (7) are satisfied and Theorem 1 still holds, so the asymptotic consensus can still be achieved.

Assumption 1. *For each individual agent, there exist constants* $\underline{C}_i, \overline{C}_i > 0$*, such that* $\underline{C}_i \|x_i\| \leq \|y_i\| \leq \overline{C}_i \|x_i\|$.

This assumption requires that each $h_i(x_i)$ is upper bounded and lower bounded by some linear functions.

Assumption 2. *The average output* $\bar{y}$ *of agents* (3) *with the diffusive couplings* (4) *is uniformly bounded, i.e., there exists* $p > 0$*, such that* $\|\bar{y}\| \leq p$.

This assumption is not restrictive. In fact, some of the agents are allowed to be unstable so long as the instability is compensated by other agents. Similar assumptions can be found in [17], [18].

Theorem 2. *Under Assumption 1 and 2, the group of IF-OFP agents* (3) *with the diffusive coupling* (4) *can achieve practical output consensus if* $M \leq 0$ *where* M *is defined in* (6)*, and*

$$\lambda_{N-1}(M) < \min_i \{\xi_i \rho_i\} \tag{11}$$

where $\lambda_{N-1}(M)$ *denotes the second largest eigenvalue of* M*. The error bound* ϵ *defined in Definition 4 is given by*

$$\epsilon = \alpha_1^{-1} \left(\alpha_2 \left(\frac{(\sqrt{b^2 - ac} + a + b) p \sqrt{N}}{a} \right) \right) \tag{12}$$

where $a = -\lambda_{N-1}(M) + \min_i\{\xi_i \rho_i\}$*,* $b = \max_i\{|\xi_i \rho_i|\}$*,* $c = \min_i\{\xi_i \rho_i\} < 0$ *and* α_1*,* α_2 *are some class* $\mathcal{K}$ *functions dependent on storage functions* V_i*,* $i \in \mathcal{N}$*.*

Proof. Suppose that $M \leq 0$, and (11) holds. Since each agent is IF-OFP, there exists a storage function V_i such that

$$\dot{V}_i \leq y_i^T u_i - \nu_i u_i^T u_i - \rho_i y_i^T y_i, \ \forall i \in \mathcal{N}.$$

Following similar lines of proof of Theorem 1, we select the candidate Lyapunov function as $V = \sum_{i=1}^{N} \xi_i V_i$. Since $\xi_i > 0$, V is positive definite. The derivative of V gives

$$\begin{aligned}
\dot{V} \leq & \sum_{i=1}^{N} \xi_i y_i^T u_i - \xi_i \nu_i u_i^T u_i - \xi_i \rho_i y_i^T y_i \\
= & \sum_{i=1}^{N} y_i^T (\xi_i I_m) u_i - u_i^T (\xi_i \nu_i I_m) u_i - y_i^T (\xi_i \rho_i I_m) y_i \\
= & -\sigma y^T (\Xi \otimes I_m)(L \otimes I_m) y - y^T (\Xi \rho \otimes I_m) y \\
& - \sigma^2 y^T (L^T \otimes I_m)(\Xi \nu \otimes I_m)(L \otimes I_m) y.
\end{aligned}$$

Denote $K = I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T$ and $\varepsilon = y - \mathbf{1}_N \otimes \bar{y}$. Then, $(\mathbf{1}_N \otimes \gamma)^T \varepsilon = (\mathbf{1}_N \otimes \gamma)^T (K \otimes I_m) y = (\mathbf{1}_N^T K \otimes \gamma^T) y = 0$ where γ is an arbitrary vector $\mathbb{R}^n$. Since $M \mathbf{1}_N = 0$, it can be obtained that

$$\begin{aligned}
\dot{V} \leq & y^T (M \otimes I_m) y - y^T (\Xi \rho \otimes I_m) y \\
= & \varepsilon^T (M \otimes I_m) \varepsilon - (\varepsilon + \mathbf{1}_N \otimes \bar{y})^T (\Xi \rho \otimes I_m)(\varepsilon + \mathbf{1}_N \otimes \bar{y}) \\
= & \varepsilon^T (M \otimes I_m - \Xi \rho \otimes I_m) \varepsilon - 2 \left(\mathbf{1}_N \otimes \bar{y}\right)^T (\Xi \rho \otimes I_m) \varepsilon \\
& - \left(\mathbf{1}_N \otimes \bar{y}\right)^T (\Xi \rho \otimes I_m)(\mathbf{1}_N \otimes \bar{y}) \\
\leq & \left[\lambda_{N-1}(M) - \min_i\{\xi_i \rho_i\}\right] \|\varepsilon\|^2 \\
& + 2 \|\mathbf{1}_N \otimes \bar{y}\| \|\Xi \rho \otimes I_m\| \|\varepsilon\| - \min_i\{\xi_i \rho_i\} \|\mathbf{1}_N \otimes \bar{y}\|^2 \\
\leq & -a \|\varepsilon\|^2 + 2bp \|\varepsilon\| \sqrt{N} - cp^2 N \\
\leq & 0, \quad \forall \|\varepsilon\| \geq \frac{\left(\sqrt{b^2 - ac} + b\right) p \sqrt{N}}{a}
\end{aligned}$$

where the second inequality follows from Lemma 2; the third inequality follows from $\|\Xi\rho \otimes I_m\| = \max_i \{|\xi_i\rho_i|\}$ and $\|\mathbf{1}_N \otimes \bar{y}\| \leq \sqrt{N}p$ based on Assumption 2. Moreover, by the reverse triangle inequality, one has $\|\varepsilon\| \geq \|y\| - \|\mathbf{1}_N \otimes \bar{y}\| \geq \|y\| - \sqrt{N}p$, which follows that $\dot{V} \leq 0$ whenever $\|y\| \geq \frac{(\sqrt{b^2-ac}+b)p\sqrt{N}}{a} + \sqrt{N}p$.

Denote $x = col(x_1, \ldots, x_N)$, it follows that $\|x\|^2 = \sum_{i=1}^{N} \|x_i\|^2$. By the inequality of arithmetic and geometric means, $\|x\| \leq \sum_{i=1}^{N} \|x_i\| \leq \sqrt{N}\|x\|$, and similarly, $\|y\| \leq \sum_{i=1}^{N} \|y_i\| \leq \sqrt{N}\|y\|$. Combining these inequalities and under Assumption 1, one obtains

$$\frac{1}{\sqrt{N}} \min_{i\in\mathcal{N}}\{\frac{1}{\overline{C}_i}\} \|y\| \leq \|x\| \leq \sqrt{N} \max_{i\in\mathcal{N}}\{\frac{1}{\underline{C}_i}\} \|y\| . \tag{13}$$

Since $\xi_i > 0$, it is obvious that $\|x\| \to \infty \Rightarrow V = \sum_{i=1}^{N} \xi_i V_i \to \infty$. Then, by Lemma 3 there exist class $\mathcal{K}$ functions $\overline{\alpha}$ and $\underline{\alpha}$, such that $\underline{\alpha}(\|x\|) \leq V \leq \overline{\alpha}(\|x\|)$. By properties of class $\mathcal{K}$ functions and (13),

$$\underline{\alpha}\left(\frac{1}{\sqrt{N}} \min_{i\in\mathcal{N}}\{\frac{1}{\overline{C}_i}\} \|y\|\right) \leq V \leq \overline{\alpha}\left(\sqrt{N} \max_{i\in\mathcal{N}}\{\frac{1}{\underline{C}_i}\} \|y\|\right).$$

Define $\alpha_1(\|y\|) = \underline{\alpha}\left(\frac{1}{\sqrt{N}} \min_{i\in\mathcal{N}}\{\frac{1}{\overline{C}_i}\} \|y\|\right)$, $\alpha_2(\|y\|) = \overline{\alpha}\left(\sqrt{N} \max_{i\in\mathcal{N}}\{\frac{1}{\underline{C}_i}\} \|y\|\right)$. Since $\sqrt{N}$, $\min_{i\in\mathcal{N}}\{\frac{1}{\overline{C}_i}\}$ and $\max_{i\in\mathcal{N}}\{\frac{1}{\underline{C}_i}\}$ are all positive constants, it follows that α_1, α_2 are also class $\mathcal{K}$ functions and $\alpha_1(\|y\|) \leq V \leq \alpha_2(\|y\|)$.

Finally, according to Theorem 4.18 in [12], there exists a T (dependent on $y(t_0)$ and $\frac{(\sqrt{b^2-ac}+a+b)p\sqrt{N}}{a}$), such that $\|y - \mathbf{1}_N \otimes \bar{y}\| = \|(K \otimes I_m)\, y\| \leq \|y\| \leq \epsilon$, $\forall t > T$, where $\epsilon = \alpha_1^{-1}\left(\alpha_2\left(\frac{(\sqrt{b^2-ac}+a+b)p\sqrt{N}}{a}\right)\right)$. Since $\|y - \mathbf{1}_N \otimes \bar{y}\|^2 = \sum_{i=1}^{N} \|y_i - \bar{y}\|^2$, it follows that $\|y_i - \bar{y}\| \leq \epsilon$, $\forall i \in \mathcal{N}$, $\forall t > T$. □

Note that the eigenvalues of M are dependent upon the indices $\nu_i,\ i \in \mathcal{N}$. This implies that the IF-OFP indices ρ_i, ν_i are constrained by the inequality (11). Intuitively, with fixed indices $\nu_i,\ i \in \mathcal{N}$, the indices $\rho_i,\ i \in \mathcal{N}$ should not be too small in order to reach practical consensus. Moreover, one can observe that the value of a increases as the coupling gain σ increases since $M \leq 0$. Therefore, it can be inferred from (12) that the error bound ϵ becomes smaller as the coupling gain σ increases while satisfying the condition in (11).

V. Numerical Examples

Example 1 (Asymptotic Consensus):

Consider three agents with the following dynamics

$$\begin{cases} \dot{x}_{11} = -2x_{11} + u_1 \\ \dot{x}_{12} = u_1 \\ y_1 = x_{12} - x_{11} \end{cases},$$

$$\begin{cases} \dot{x}_{i1} = -i(x_{i1} + x_{i2})^3 - i(x_{i1} + x_{i2}) + iu_i \\ \dot{x}_{i2} = -i(x_{i1} + x_{i2})^3 - i(x_{i1} + x_{i2}) + (i-1)u_i \\ y_i = -2x_{i2}, \quad i = 2, 3 \end{cases}$$

which are interconnected via a weight-unbalanced digraph shown in Fig. 1. The corresponding Laplacian matrix is obtained as $L = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$ with $\xi = \begin{pmatrix} 0.25 & 0.25 & 0.50 \end{pmatrix}^T$.

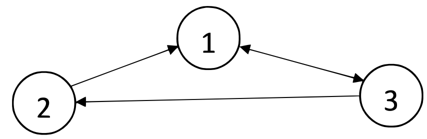


Fig. 1: Digraph of three interconnected agents.

Let us first verify that these three systems are IFP(ν_i). It can be obtained by exploiting Corollary 1 in [24] that $\nu_1 = -0.50$. The indices of the nonlinear agents are estimated as $\nu_2 = -0.75$ and $\nu_3 = -0.83$ respectively, by using storage function $V_i = \frac{1}{i(i-1)+4}\|x_{i1} + x_{i2}\|^2 + \frac{1}{2}\|x_{i1} - x_{i2}\|^2$, $i = 2, 3$.

Next, we solve the SDP in (8) and obtain that $\sigma_e = 0.5438$. Hence, for any $\sigma \in (0, 0.5438)$, asymptotic output consensus can be achieved. For example, the outputs when $\sigma = 0.50$ is shown in the middle trajectories of Fig. 2. For asymptotic consensus among passivity-short IFP agents, the coupling gain cannot be arbitrarily large. As the coupling gain grows larger, asymptotic consensus is no longer guaranteed. We check the tightness of the bound and find that the outputs obviously diverge when $\sigma \geq 1.10$, which can be observed in the bottom trajectories of Fig. 2. When individual agents take different coupling gains, it is possible to choose some σ_i larger than the threshold σ_e obtained in Corollary 1. For instance, given $\sigma_1 = 0.10$, $\sigma_2 = 1.10$ and $\sigma_3 = 0.30$, the condition in Corollary 2 is satisfied, so asymptotic consensus can be achieved, which is shown by the upper trajectories in Fig. 2.

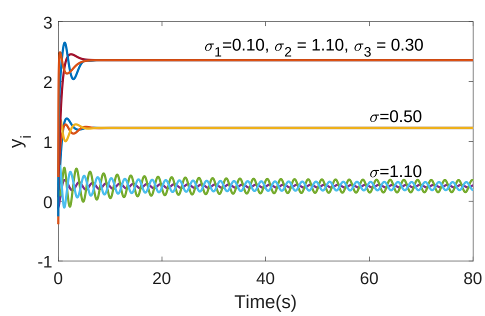


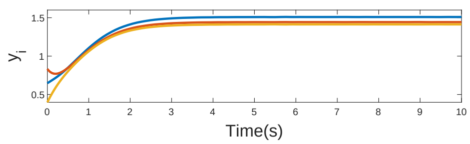
Fig. 2: Outputs with different choices of coupling gains.

Example 2 (Practical Consensus):

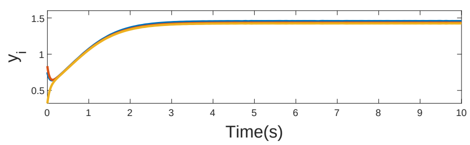
Consider three agents interconnected through Fig. 1 with the following dynamics

$$\begin{cases} \dot{x}_i = -ix_i + (i+1)\sin x_i + u_i \\ y_i = x_i, \quad i = 1, 2, 3. \end{cases}$$

It can be shown by $V_i = \frac{1}{2}x_i^2$ that each agent without input is exponentially bounded in $\|x_i\| \leq \frac{i+1}{i-\delta}$, where $0 < \delta < 1$. Since exponentially bounded systems cannot be destabilized by diffusive couplings [18], [25], the average output $\bar{y}$ is bounded. All agents are IF-OFP systems whose indices are estimated by the storage function $V_i = \frac{1}{2}x_i^2$ as $\rho_i = -1$ and $\nu_i = 0$, $i = 1, 2, 3$. Then, $M \leq 0$ for any $\sigma > 0$. When $\sigma \in (1.33, +\infty)$, the inequality (11) holds and thus practical consensus is guaranteed. The output trajectories when $\sigma = 3,\ 10$ are shown respectively in Fig. 3. The relationship between σ and $\sum_{i=1}^{3} \|y_i(t) - \bar{y}(t)\|$ is shown in Fig. 4. It can be observed that the consensus error becomes smaller as the coupling gain σ increases.



(a) Outputs when $\sigma = 3$.



(b) Outputs when $\sigma = 10$.

Fig. 3: Outputs with different coupling gains.

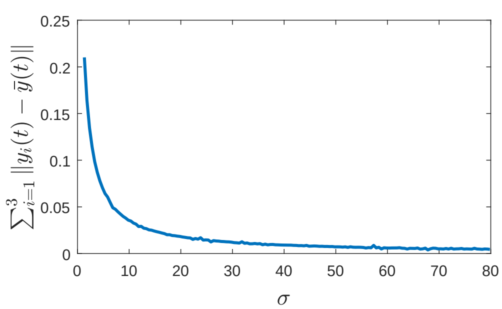


Fig. 4: The relationship between σ and $\sum_{i=1}^{3} \|y_i(t) - \bar{y}(t)\|$.

VI. Conclusions

This work has addressed the problem of output consensus for two classes of heterogeneous nonlinear multi-agent systems interconnected via diffusive couplings over directed graphs. Sufficient conditions in terms of passivity indices have been proposed for asymptotic consensus of nonlinear IFP agents and practical consensus of nonlinear IFP-OFP agents. It has been shown that the interconnected system can achieve asymptotic consensus by choosing a proper coupling gain if all the agents are IFP. For agents that can be characterized as IF-OFP systems, it has been shown that if the average output is uniformly bounded, the interconnected system can achieve practical consensus, i.e., a small enough consensus error bound can be guaranteed given a sufficiently large coupling gain.

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