On exact controllability of infinite-dimensional linear port-Hamiltonian systems*

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Abstract

Infinite-dimensional linear port-Hamiltonian systems on a one-dimensional spatial domain with full boundary control and without internal damping are studied. This class of systems includes models of beams and waves as well as the transport equation and networks of nonhomogeneous transmission lines. The main result shows that well-posed port-Hamiltonian systems, with state space $L^2((0,1);\mathbb{C}^n)$ and input space $\mathbb{C}^n$, are exactly controllable.

Keywords: Controllability, C_0-semigroups, port-Hamiltonian differential equations, boundary control systems.

Mathematics Subject Classification: 93C20, 93B05, 35L40, 93B52.

1 Introduction

In this article, we consider infinite-dimensional linear port-Hamiltonian systems on a one-dimensional spatial domain with boundary control of the form

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) =& \left(P_1\frac{\partial}{\partial \zeta}+P_0\right)(\mathcal{H}(\zeta)x(\zeta,t)),\\ x(\zeta,0) =& x_0(\zeta),\\ u(t) =& \widetilde{W}_B\begin{bmatrix}(\mathcal{H}x)(1,t)\\(\mathcal{H}x)(0,t)\end{bmatrix}, \end{aligned} \tag{1}$$

where $\zeta\in[0,1]$ and $t\geq 0$. Moreover, we assume that P_1 is an invertible $n\times n$ Hermitian matrix, P_0 is a $n\times n$ skew-adjoint matrix, $\widetilde{W}_B$ is a full row rank $n\times 2n$-matrix, and $\mathcal{H}(\zeta)$ is a positive $n\times n$ Hermitian matrix for a.e. $\zeta\in(0,1)$ satisfying $\mathcal{H},\mathcal{H}^{-1}\in L^\infty((0,1);\mathbb{C}^{n\times n})$. The matrix $P_1\mathcal{H}(\zeta)$ can be diagonalized as $P_1\mathcal{H}(\zeta)=S^{-1}(\zeta)\Delta(\zeta)S(\zeta)$, where $\Delta(\zeta)$ is a diagonal matrix and $S(\zeta)$ is an

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invertible matrix for a.e. $\zeta \in (0,1)$. We suppose the technical assumption that S^{-1}, S, $\Delta : [0,1] \to \mathbb{C}^{n\times n}$ are continuously differentiable.

Equation (1) describes a special class of port-Hamiltonian systems, which however is rich enough to cover in particular the wave equation, the transport equation and the Timoshenko beam equation, and also coupled beam and wave equations each with possibly damping on the boundary. For more information on this class of port-Hamiltonian systems we refer to the monograph [1] and the survey [2]. However, we note that here we always assume that there is no internal damping (the matrix P_0 is skew-adjoint) and that we have full boundary control ($\widetilde{W}_B$ is a full row rank $n \times 2n$-matrix).

Port-based network modeling of complex physical systems leads to port-Hamiltonian systems. For finite-dimensional systems there is by now a well-established theory [3, 4, 5]. The port-Hamiltonian approach has been extended to the infinite-dimensional situation by a geometric differential approach [6, 7, 8, 9] and by a functional analytic approach [10, 9, 1, 11, 12, 2]. Here we follow the functional analytic point of view. This approach has been successfully used to derive simple verifiable conditions for well-posedness [13, 10, 9, 1, 11, 14], stability [1, 15] and stabilization [16, 17, 15, 18] and robust regulation [19]. For example, the port-Hamiltonian system (1) is well-posed, if $v^*P_1v - w^*P_1w \le 0$ for every $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in \ker \widetilde{W}_B$.

Provided the port-Hamiltonian system (1) is well-posed, we aim to characterize *exact controllability*. Exact controllability is a desirable property of a controlled partial differential equation and has been extensively studied, see for example [20, 21, 22]. We call the port-Hamiltonian system exactly controllable, if every state of the system can be reached in finite time with a suitable control input. Triggiani [23] showed that exact controllability does not hold for many hyperbolic partial differential equations. However, in this paper we prove, that the port-Hamiltonian system (1) is exactly controllable whenever it is well-posed.

2 Reminder on port-Hamiltonian systems

We define

$$\mathfrak{A}x := \left(P_1 \frac{d}{d\zeta} + P_0\right)(\mathcal{H}x), \qquad x \in \mathcal{D}(\mathfrak{A}), \tag{2}$$

on $X := L^2((0,1);\mathbb{C}^n)$ with the domain

$$\mathcal{D}(\mathfrak{A}) := \left\{x \in X \mid \mathcal{H}x \in H^1((0,1);\mathbb{C}^n)\right\} \tag{3}$$

and $\mathfrak{B} : \mathcal{D}(\mathfrak{A}) \to \mathbb{C}^n$ by

$$\mathfrak{B}x = \widetilde{W}_B \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix}. \tag{4}$$

Here $H^1((0,1);\mathbb{C}^n)$ denotes the first order Sobolev space. We call $\mathfrak{A}$ the *(maximal) port-Hamiltonian operator* and equip the state space $X = L^2((0,1);\mathbb{C}^n)$

with the energy norm $\sqrt{\langle \cdot, \mathcal{H}\cdot\rangle}$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $L^2((0,1);\mathbb{C}^n)$. We note that the energy norm is equivalent to the standard norm on $L^2((0,1);\mathbb{C}^n)$.

Then the partial differential equation (1) can be written as a *boundary control system*

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\ u(t) &= \mathfrak{B}x(t). \end{aligned}$$

The first important question is whether the port-Hamiltonian system (1) is *well-posed* in the sense that for every initial condition $x_0 \in X$ and every $u \in L^2_{\text{loc}}([0,\infty);\mathbb{C}^n)$ equation (1) has a unique mild solution.

In [10, 9, 1] it is shown that the port-Hamiltonian system (1) is well-posed if and only if the operator $A : \mathcal{D}(A) \subset X \to X$, defined by

$$Ax := \left(P_1 \frac{d}{d\zeta} + P_0\right)(\mathcal{H}x), \qquad x \in \mathcal{D}(A), \tag{5}$$

with the domain

$$\mathcal{D}(A) := \left\{ x \in \mathcal{D}(\mathfrak{A}) \mid \widetilde{W}_B \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix} = 0 \right\} \tag{6}$$

generates a strongly continuous semigroup on X. We recall, that A generates a contraction semigroup on X if and only if A is dissipative on X, c.f. [13, 1, 15]. Further, matrix conditions to guarantee generation of a contraction semigroup have been obtained in [13, 1, 15] and matrix conditions for the generation of strongly continuous semigroups can be found in [11].

For the proof of the main theorem feedback techniques are needed and therefore we investigate port-Hamiltonian systems with boundary control and observations. These are systems of the form

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)(\mathcal{H}(\zeta)x(\zeta,t)), \\ x(\zeta,0) &= x_0(\zeta), \\ u(t) &= \widetilde{W}_B \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix} \\ y(t) &= \widetilde{W}_C \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix}, \end{aligned} \tag{7}$$

where we restrict ourselves in this article to case where P_1, P_0, $\mathcal{H}$ and $\widetilde{W}_B$ satisfy the condition described in Section 1 and $\widetilde{W}_C$ is a full row rank $k \times 2n$ matrix, $k \in \{0, \cdots, n\}$, such that the matrix $\begin{bmatrix} \widetilde{W}_B \\ \widetilde{W}_C \end{bmatrix}$ has full row rank. We call system (7) a *(boundary control and observation) port-Hamiltonian system.* The case $k = 0$ refers to the case of a system without observation, that is, every definition or statement of the port-Hamiltonian system (7) also applies to the port-Hamiltonian system (1).

We define $\mathfrak{C} : \mathcal{D}(\mathfrak{A}) \to \mathbb{C}^k$ by

$$\mathfrak{C}x = \widetilde{W}_C \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix}. \tag{8}$$

Then we can write the port-Hamiltonian system (7) in the following form

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\ u(t) &= \mathfrak{B}x(t), \\ y(t) &= \mathfrak{C}x(t). \end{aligned} \tag{9}$$

If the operator A, defined by (5)-(6), generates a strongly continuous semigroup on the state space X, then (9) defines a *boundary control and observation system*, see [1, Theorem 11.3.2 and Theorem 11.3.5].

Definition 2.1. *Let $\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \subset X \to X$, $\mathfrak{B} : \mathcal{D}(\mathfrak{A}) \to \mathbb{C}^n$ and $\mathfrak{C} : \mathcal{D}(\mathfrak{A}) \to \mathbb{C}^k$ be linear operators. Then $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a* boundary control and observation system *if the following hold:*

1. *The operator $A : \mathcal{D}(A) \subset X \to X$ with $\mathcal{D}(A) = \mathcal{D}(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and $Ax = \mathfrak{A}x$ for $x \in \mathcal{D}(A)$ is the infinitesimal generator of a strongly continuous semigroup on X.*
2. *There exists a right inverse $\widetilde{B} \in \mathcal{L}(\mathbb{C}^n, X)$ of $\mathfrak{B}$ in the sense that for all $u \in \mathbb{C}^n$ we have $\widetilde{B}u \in \mathcal{D}(\mathfrak{A})$, $\mathfrak{B}\widetilde{B}u = u$ and $\mathfrak{A}\widetilde{B} : \mathbb{C}^n \to X$ is bounded.*
3. *The operator $\mathfrak{C}$ is bounded from $\mathcal{D}(A)$ to $\mathbb{C}^k$, where $\mathcal{D}(A)$ is equipped with the graph norm of A.*

We recall, that if A, defined by (5)-(6), generates a strongly continuous semigroup on the state space X, then the port-Hamiltonian system (7) is a boundary control and observation system.

We note that for $x_0 \in \mathcal{D}(\mathfrak{A})$ and $u \in C^2([0,\tau]; \mathbb{C}^n)$, $\tau > 0$, satisfying $\mathfrak{B}x_0 = u(0)$, a boundary control and observation system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ possesses a *unique classical solution* [1, Lemma 13.1.5].

For technical reasons we formulate the boundary conditions equivalently via the boundary flow and the boundary effort. As the matrix $\left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right]$ is invertible, we can write the port-Hamiltonian system (7) equivalently as

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)(\mathcal{H}(\zeta)x(\zeta,t)), \\ x(\zeta, 0) &= x_0(\zeta), \\ u(t) &= W_B \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}, \\ y(t) &= W_C \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}, \end{aligned} \tag{10}$$

where

$$\begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix}$$

and

$$\widetilde{W}_B = W_B \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right], \widetilde{W}_C = W_C \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right]. \tag{11}$$

Here $f_{\delta,\mathcal{H}x}$ is called the *boundary flow* and $e_{\delta,\mathcal{H}x}$ the *boundary effort*. The port-Hamiltonian system (7) is uniquely described by the tuple $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ given by (2), (3), (4) and (8).

Well-posedness is a fundamental property of boundary control and observation systems.

Definition 2.2. *We call a boundary control and observation system* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ well-posed *if there exist a* $\tau > 0$ *and* $m_\tau \geq 0$ *such that for all* $x_0 \in \mathcal{D}(\mathfrak{A})$ *and* $u \in C^2([0,\tau]; \mathbb{C}^n)$ *with* $u(0) = \mathfrak{B}x_0$ *the classical solution* x, y *satisfy*

$$\begin{aligned} \|x(\tau)\|_X^2 &+ \int_0^\tau \|y(t)\|^2 dt \\ &\leq m_\tau \left(\|x_0\|_X^2 + \int_0^\tau \|u(t)\|^2 dt \right). \end{aligned}$$

There exists a rich literature on well-posed systems, see e.g. Staffans [24] and Tuscnak and Weiss [25]. In general, it is not easy to show that a boundary control and observation system is well-posed. However, for the port-Hamiltonian system (7) well-posedness is already satisfied if A generates a strongly continuous semigroup.

Theorem 2.3. *[1, Theorem 13.2.2] The port-Hamiltonian system* (7) *is well-posed if and only if the operator* A *defined by* (5)-(6) *generates a strongly continuous semigroup on* X.

There is a special class of port-Hamiltonian systems for which well-posedness follows immediately.

Definition 2.4. *A port-Hamiltonian systems* (7) *is called* impedance passive*, if*

$$\operatorname{Re} \langle \mathfrak{A}x, x \rangle \leq \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x \rangle \tag{12}$$

for every $x \in \mathcal{D}(\mathfrak{A})$*. If we have equality in* (12)*, then the port-Hamiltonian system is called* impedance energy preserving.

The fact that a port-Hamiltonian system is impedance energy preserving can be characterized by a easy checkable matrix condition.

Theorem 2.5. *[13, Theorem 4.4] The port-Hamiltonian systems* (7) *is impedance energy preserving if and only if it holds*

$$\begin{bmatrix} W_B\Sigma W_B^* & W_B\Sigma W_C^* \\ W_C\Sigma W_B^* & W_C\Sigma W_C^* \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \tag{13}$$

where $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$.

Remark 2.6. *Every impedance energy preserving port-Hamiltonian system* (7) *is well-posed;* $W_B\Sigma W_B^* = 0$ *even implies that* A *generates a unitary strongly continuous group, c.f. [11, Theorem 1.1].*

In order to formulate the *mild solution* of a well-posed port-Hamiltonian system (7) we need to introduce some notation. Let X_{-1} be the completion of X with respect to the norm $\|x\|_{X_{-1}} = \|(\beta I - A)^{-1}x\|_X$ for some β in the resolvent set $\rho(A)$ of A, this implies,

$$X \subset X_{-1}$$

and X is continuously embedded and dense in X_{-1}. Furthermore, let $(T(t))_{t\geq 0}$ be the strongly continuous semigroup generated by A. The semigroup $(T(t))_{t\geq 0}$ extends uniquely to a strongly continuous semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} whose generator A_{-1}, with domain equal to X, is an extension of A, see e.g. [26]. Moreover, we can identify X_{-1} with the dual space of $\mathcal{D}(A^*)$ with respect to the pivot space X, see [22], that is $X_{-1} = \mathcal{D}(A^*)'$. If the port-Hamiltonian system (7) is well-posed, then the unique mild solution is given by

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)(\mathfrak{A}\widetilde{B} - A_{-1}\widetilde{B})u(s)\,ds.$$

Here the operator $\widetilde{B} : \mathbb{C}^n \to L^2((0,1);\mathbb{C}^n)$ can be defined as follows

$$(\widetilde{B}u)(\zeta) := (\mathcal{H}(\zeta))^{-1}\left(S_1\zeta + S_2(1-\zeta)\right)u,$$

where S_1 and S_2 are $n \times n$-matrices given by

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} := \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}^{-1} \widetilde{W}_B^*(\widetilde{W}_B\widetilde{W}_B^*)^{-1}.$$

For a well-posed port-Hamiltonian system (7) the *transfer function* is given by [1, Theorem 12.1.3]

$$G(s) = \mathfrak{C}(sI - A)^{-1}(\mathfrak{A}\widetilde{B} - s\widetilde{B}) + \mathfrak{C}\widetilde{B}, \qquad s \in \rho(A),$$

where $\rho(A)$ denotes the resolvent set of A. The transfer function is bounded on some right half plane and equals the Laplace transform of the mapping $u(\cdot) \mapsto y(\cdot)$ if $x_0 = 0$.

Definition 2.7. *[1, Definition 13.1.11] A well-posed port-Hamiltonian system* (7) *with transfer function G is called* regular *if* $\lim_{s\in\mathbb{R},s\to\infty} G(s)$ *exists. In this case the* feedthrough operator *D is defined as*

$$D := \lim_{s\in\mathbb{R},s\to\infty} G(s).$$

Lemma 2.8. *[1, Lemma 13.2.22] Under the standing assumptions every well-posed port-Hamiltonian system* (7) *is regular.*

So far, we have only considered *open-loop system*, that is, the input $u(t)$ is independent of the output $y(t)$, see Figure 1. Systems, where input and output are connected via a feedback law

$$u(t) = Fy(t) + v(t), \tag{14}$$

are called *closed-loop systems*, see Figure 2. Here F denotes the so called *feedback operator* and $v(t)$ the new input.

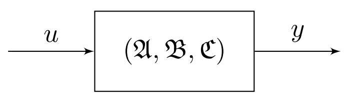

Figure 1: open-loop system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$

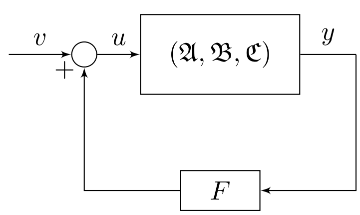

Figure 2: closed-loop system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with feedback F

Definition 2.9. *([1, Theorem 13.2.2] and [27, Proposition 4.9])* (7) *and we denote by D the corresponding feedthrough. A $n \times n$-matrix F is called an* admissible feedback operator *for a regular port-Hamiltonian system* (7) *with feedthrough operator D, if $I - DF$ is invertible.*

Proposition 2.10. *[1, Theorem 13.1.12] Let* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *be a well-posed port-Hamiltonian system* (7)*. Assume that F is an admissible feedback operator.*

Then the closed-loop system $(\mathfrak{A}, (\mathfrak{B} - F\mathfrak{C}), \mathfrak{C})$*, i.e.,*

$$\begin{aligned}\frac{\partial x}{\partial t}(\zeta,t) =& \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)(\mathcal{H}(\zeta)x(\zeta,t)),\\ x(\zeta,0) =& x_0(\zeta),\\ v(t) =& (\mathfrak{B} - F\mathfrak{C})x(t),\\ y(t) =& \mathfrak{C}x(t)\end{aligned} \tag{15}$$

with input v *and output* y *is a well-posed port-Hamiltonian system.*

Definition 2.11. *The well-posed port-Hamiltonian system* (7) *is* exactly controllable*, if there exists a time* $\tau > 0$ *such that for all* $x_1 \in X$ *there exists a control function* $u \in L^2((0,\tau);\mathbb{C}^n)$ *such that the corresponding mild solution satisfies* $x(0) = 0$ *and* $x(\tau) = x_1$*.*

Proposition 2.12. *[27, c.f. Remark 6.9] Let* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *be a well-posed port-Hamiltonian system* (7)*. Assume that* F *is an admissible feedback operator. Then the closed-loop system* $(\mathfrak{A}, (\mathfrak{B} - F\mathfrak{C}), \mathfrak{C})$ *is exactly controllable if and only if the open-loop system* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is exactly controllable.*

3 Exact controllability for port-Hamiltonian systems

This section is devoted to the main result of this paper, that is, we show that every well-posed port-Hamiltonian system (1) is exactly controllable.

Exact controllability for impedance energy preserving port-Hamiltonian system has been studied in [2].

Proposition 3.1. *[2, Corollary 10.7] An impedance energy preserving port-Hamiltonian system* (7) *is exactly controllable.*

For completeness we include the proof of Proposition 3.1.

Proof. As the port-Hamiltonian system (7) is impedance energy preserving the corresponding operator A generates a unitary strongly continuous group. Thus, $-A$ generates a bounded strongly continuous semigroup and exact controllability is equivalent to optimizability, [28, Corollary 2.2]. The system is called *optimizable* if for all $x_0 \in X$ there exists a control function $u \in L^2((0,\infty);\mathbb{C}^n)$ such that the corresponding mild solution x satisfies $x \in L^2((0,\infty);X)$. Thus it is sufficient to show that the port-Hamiltonian system (7) is optimizable. Let $x_0 \in X$ be arbitrarily. In [19, Lemma 7] it is shown that for every $k > 0$ the choice $u(t) = -ky(t)$ leads to a mild solution in $L^2((0,\infty);X)$. This shows optimizability of system (7) and concludes the proof. □

Now we can formulate our main result.

Theorem 3.2. *Every well-posed port-Hamiltonian system* (1) *is exactly controllable.*

For the proof of our main result we need the following lemmas.

Lemma 3.3. *Let* $[\, W_1 \; W_0 \,] \in \mathbb{C}^{n\times 2n}$ *have full row rank with* $W_1, W_0 \in \mathbb{C}^{n\times n}$. *Then, there exist invertible matrices* $\tilde{R}_1, \tilde{R}_0 \in \mathbb{C}^{n\times n}$ *such that* $[\, W_1 \; W_0 \,]\begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_0 \end{bmatrix} = I$.

Proof. Let $[\, W_1 \; W_0 \,]$ have full row rank with $\operatorname{rank} W_1 = n-k$, $k \in \{0,\dots,n\}$, and $\operatorname{rank} W_0 = n-\ell$ with $\ell \in \{0,\dots,n\}$. Clearly $n-k+n-\ell \geq n$, or equivalently, $k+\ell \leq n$.

By W_1^{n-k} we denote the first $n-k$ rows of W_1 and W_1^k denotes the last k rows. Similarly, by $W_0^{n-\ell}$ we denote the last $n-\ell$ rows of W_0 and by W_0^ℓ the first ℓ rows. That is

$$W_1 = \begin{bmatrix} W_1^{n-k} \\ W_1^k \end{bmatrix} \qquad \text{and} \qquad W_0 = \begin{bmatrix} W_0^\ell \\ W_0^{n-\ell} \end{bmatrix}.$$

Without loss of generality, using row reduction and the fact that $\operatorname{rank}[\, W_1 \; W_0 \,] = n$, we may assume that $W_1^k = 0$ and that W_1^{n-k} and $W_0^{n-\ell}$ have full row rank.

We choose right inverses $R_1^{n-k} \in \mathbb{C}^{n\times(n-k)}$ for W_1^{n-k} and $R_0^{n-\ell} \in \mathbb{C}^{n\times(n-\ell)}$ for $W_0^{n-\ell}$. Thus,

$$W_1^{n-k}R_1^{n-k} = I \qquad \text{and} \qquad W_0^{n-\ell}R_0^{n-\ell} = I.$$

Clearly, the columns of R_1^{n-k} and $R_0^{n-\ell}$ are linearly independent and are not elements of the kernel of W_1 and W_0, respectively.

Let $R_1^k \in \mathbb{C}^{n\times k}$ consisting of columns spanning the kernel of W_1, and let $R_0^\ell \in \mathbb{C}^{n\times \ell}$ consisting of columns spanning the kernel of W_0. We define $R_1 = \begin{bmatrix} R_1^{n-k} & R_1^k \end{bmatrix} \in \mathbb{C}^{n\times n}$ and $R_0 = \begin{bmatrix} R_0^\ell & R_0^{n-\ell} \end{bmatrix} \in \mathbb{C}^{n\times n}$. Thus, R_1 and R_0 are invertible and it yields

$$\begin{aligned} &W_1R_1 + W_0R_0 \\ &= \begin{bmatrix} I_{n-k} & 0_{(n-k)\times k} \\ 0_{k\times(n-k)} & 0_{k\times k} \end{bmatrix} + \begin{bmatrix} 0_{\ell\times\ell} & W_0^l R_0^{n-\ell} \\ 0_{(n-\ell)\times\ell} & I_{n-\ell} \end{bmatrix}. \end{aligned}$$

Thus, $W_1R_1 + W_0R_0 := M$ is invertible as an upper triangular matrix and we define $\tilde{R}_1 := R_1M^{-1}$ and $\tilde{R}_0 := R_0M^{-1}$ to obtain the assertion of the lemma. □

Lemma 3.4. *Let* $\alpha \neq 0$ *and* $(\mathfrak{A}, \mathfrak{B})$ *be a well-posed port-Hamiltonian system. Then the port-Hamiltonian system* $(\mathfrak{A}, \alpha\mathfrak{B})$ *is well-posed as well. Moreover, the system* $(\mathfrak{A}, \mathfrak{B})$ *is exactly controllable if and only if the system* $(\mathfrak{A}, \alpha\mathfrak{B})$ *is exactly controllable.*

Proof. Well-posed of the scaled system follows immediately. The controllability of the two systems is equivalent, since we can scale the input function u of one system by α or $\frac{1}{\alpha}$ to get an input for the other system without changing the mild solution. □

Proof of Theorem 3.2: We start with an arbitrary port-Hamiltonian system (1) described by the tuple $(\mathfrak{A}, \mathfrak{B})$.

By Lemma 3.4, this system is exactly controllable if and only if for some $\alpha > 0$ the system $(\mathfrak{A}, \alpha\mathfrak{B})$ is exactly controllable. We aim to prove that there exists an $\alpha > 0$ such that the system $(\mathfrak{A}, \alpha\mathfrak{B})$ is exactly controllable.

Thus, we aim to write the system $(\mathfrak{A}, \alpha\mathfrak{B})$ as a closed-loop system of an exactly controllable system $(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$. To construct $(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ we find an impedance energy preserving system $(\mathfrak{A}, \mathfrak{B}_o, \tilde{\mathfrak{C}})$ which is exactly controllable by Proposition 3.1.

By (4) and (11), the operator $\mathfrak{B}$ is described by a full row rank $n \times 2n$-matrix

$$W_B = \begin{bmatrix} W_1 & W_0 \end{bmatrix}.$$

Using Lemma 3.3 there exists a matrix $R = \left[\begin{smallmatrix} R_1 \\ R_0 \end{smallmatrix}\right] \in \mathbb{C}^{2n \times n}$ such that

$$W_B R = I$$

and $R_1, R_0 \in \mathbb{C}^{n \times n}$ are invertible. If $W_0 = 0$, without loss of generality we may assume that $R_0 = I$ and $R_1 = W_1^{-1}$.

We now consider the port-Hamiltonian system $(\mathfrak{A}, \mathfrak{B}_o, \widetilde{\mathfrak{C}})$, where

$$\mathfrak{B}_o x = \begin{bmatrix} R_1^{-1} & 0 \end{bmatrix} \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}$$

and

$$\widetilde{\mathfrak{C}} x = \begin{bmatrix} 0 & R_1^* \end{bmatrix} \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}.$$

Obviously, the port-Hamiltonian system $(\mathfrak{A}, \mathfrak{B}_o, \widetilde{\mathfrak{C}})$ is impedance energy preserving. Then it follows from Proposition 3.1 that $(\mathfrak{A}, \mathfrak{B}_o, \widetilde{\mathfrak{C}})$ is exactly controllable.

If $W_0 = 0$, then $(\mathfrak{A}, \mathfrak{B}) = (\mathfrak{A}, \mathfrak{B}_o)$ and thus the statement is proved with $\alpha = 1$.

We now assume that $W_0 \neq 0$. In this case we consider the port-Hamiltonian system $(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$, where

$$\mathfrak{C}_o x = \begin{bmatrix} \alpha R_1^{-1} & \alpha R_0^{-1} \end{bmatrix} \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}.$$

The constant $\alpha > 0$ will be chosen later. The matrix $\left[\begin{smallmatrix} R_1^{-1} & 0 \\ \alpha R_1^{-1} & \alpha R_0^{-1} \end{smallmatrix}\right]$ is invertible and the port-Hamiltonian system $(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ is still exactly controllable, since changing the output does not influence controllability.

The port-Hamiltonian system $(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ is regular, see Theorem 2.3 and Lemma 2.8. By D we denote the feedthrough operator of $(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ and we choose

$$\alpha = \begin{cases} 2 \|D\| \|W_0 R_0\|, & D \neq 0 \\ 1, & D = 0 \end{cases}.$$

Then $\alpha > 0$ and the matrix

$$F = \frac{1}{\alpha} W_0 R_0$$

is an admissible feedback operator for $(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ as $\|DF\| < 1$ (which implies invertibility of $I - DF$).

We now consider the closed-loop system as shown in Figure 3 and obtain

$$\begin{aligned}
\dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\
u_\alpha(t) &= \alpha(u_o(t) - F y_o(t)) \\
&= \alpha(\mathfrak{B}_o - F\mathfrak{C}_o)x(t) \\
&= \left(\alpha \begin{bmatrix} R_1^{-1} & 0 \end{bmatrix} - W_0 R_0 \begin{bmatrix} \alpha R_1^{-1} & \alpha R_0^{-1} \end{bmatrix}\right) \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix} \\
&= \alpha W_B \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}.
\end{aligned}$$

Thus, the closed-loop system equals the port-Hamiltonian system $(\mathfrak{A}, \alpha\mathfrak{B})$. As the open-loop system $(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ is exactly controllable, by Theorem 2.12 the port-Hamiltonian system $(\mathfrak{A}, \alpha\mathfrak{B})$ is exactly controllable.

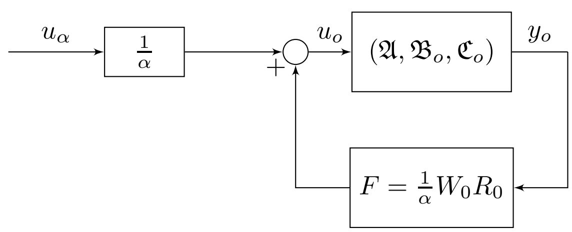

Figure 3: $(\mathfrak{A}, \alpha\mathfrak{B})$ as a closed-loop system

Thus, every well-posed port-Hamiltonian system is exactly controllable. □

4 Example of an exactly controllable port-Hamiltonian system

An (undamped) vibrating string can be modeled by

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left(T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right), \tag{16}$$

$t \geq 0$, $\zeta \in (0,1)$, where $\zeta \in [0,1]$ is the spatial variable, $w(\zeta, t)$ is the vertical position of the string at place ζ and time t, $T(\zeta) > 0$ is the Young's modulus of the string, and $\rho(\zeta) > 0$ is the mass density, which may vary along the string. We assume that T and ρ are positive and continuously differentiable functions on $[0,1]$. By choosing the state variables $x_1 = \rho \frac{\partial w}{\partial t}$ (momentum) and $x_2 = \frac{\partial w}{\partial \zeta}$ (strain), the partial differential equation can equivalently be written as

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} \right) \\ &= P_1 \frac{\partial}{\partial \zeta} \left(\mathcal{H}(\zeta) \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} \right), \end{aligned} \tag{17}$$

where

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{H}(\zeta) = \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}.$$

The boundary control for (17) is given by

$$\begin{bmatrix} \widetilde{W_1} & \widetilde{W_0} \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix} = u(t),$$

where $\begin{bmatrix} \widetilde{W_1} & \widetilde{W_0} \end{bmatrix}$ is a 2×4-matrix with rank 2, or equivalently, the partial differential equation is equipped with the boundary control

$$\begin{bmatrix} \widetilde{W_1} & \widetilde{W_0} \end{bmatrix} \begin{bmatrix} \rho \frac{\partial w}{\partial t}(1,t) \\ \frac{\partial w}{\partial \zeta}(1,t) \\ \rho \frac{\partial w}{\partial t}(0,t) \\ \frac{\partial w}{\partial \zeta}(0,t) \end{bmatrix} = u(t). \tag{18}$$

Defining $\gamma = \sqrt{T(\zeta)/\rho(\zeta)}$, the matrix function $P_1\mathcal{H}$ can be factorized as

$$P_1 \mathcal{H} = \underbrace{\begin{bmatrix} \gamma & -\gamma \\ \rho^{-1} & \rho^{-1} \end{bmatrix}}_{S^{-1}} \underbrace{\begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} (2\gamma)^{-1} & \rho/2 \\ (2\gamma)^{-1} & \rho/2 \end{bmatrix}}_{S}.$$

In [11] it is shown that the port-Hamiltonian system (16), (18) is well-posed if and only if

$$\widetilde{W_1} \begin{bmatrix} \gamma(1) \\ T(1) \end{bmatrix} \oplus \widetilde{W_0} \begin{bmatrix} -\gamma(0) \\ T(0) \end{bmatrix} = \mathbb{C}^2,$$

or equivalently if the vectors $\widetilde{W_1} \left[\begin{smallmatrix} \gamma(1) \\ T(1) \end{smallmatrix} \right]$ and $\widetilde{W_0} \left[\begin{smallmatrix} -\gamma(0) \\ T(0) \end{smallmatrix} \right]$ are linearly independent.

By Theorem 3.2 the port-Hamiltonian system (16), (18) is exactly controllable if the vectors $\widetilde{W_1} \left[\begin{smallmatrix} \gamma(1) \\ T(1) \end{smallmatrix} \right]$ and $\widetilde{W_0} \left[\begin{smallmatrix} -\gamma(0) \\ T(0) \end{smallmatrix} \right]$ are linearly independent. Here we consider $\widetilde{W_1} := I$ and $\widetilde{W_0} := \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right]$. Then the port-Hamiltonian system (16), (18) is exactly controllable if the vectors $\left[\begin{smallmatrix} \gamma(1) \\ T(1) \end{smallmatrix} \right]$ and $\left[\begin{smallmatrix} \gamma(0) \\ T(0) \end{smallmatrix} \right]$ are linearly independent.

5 Conclusions

In this paper we have studied the notion of exact controllability for a class of linear port-Hamiltonian system on a one dimensional spacial domain with full boundary control and no internal damping. We showed that for this class well-posedness implies exact controllability. Further, we applied the obtained results to the wave equation.

By duality a well-posed port-Hamiltonian system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with state space $L^2((0,\infty);\mathbb{C}^n)$ and output space $\mathbb{C}^n$ is exactly observable. An interesting problem for future research is the characterization of exact controllability for port-Hamiltonian systems with internal damping, i.e, port-Hamiltonian systems where P_0 is not necessarily skew-adjoint. We note, that the condition that $\widetilde{W}_B$ has full rank cannot be neglected, as in general without full boundary control a port-Hamiltonian system is not exact controllable. Another open question is the characterization of exact controllability for port-Hamiltonian systems of higher order, see [10]. However, for these systems even the characterization of well-posedness is an open problem.

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