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MinMax Mean-Field Team Approach for a Leader-Follower Network: A Saddle-Point Strategy

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Abstract— **This paper investigates a soft-constrained MinMax control problem of a leader-follower network. The network consists of one leader and an arbitrary number of followers that wish to reach consensus with minimum energy consumption in the presence of external disturbances. The leader and followers are coupled in the dynamics and cost function. Two non-classical information structures are considered: mean-field sharing and intermittent mean-field sharing, where the mean-field refers to the aggregate state of the followers. In mean-field sharing, every follower observes its local state, the state of the leader and the mean field while in the intermittent mean-field sharing, the mean-field is only observed at some (possibly no) time instants. A social welfare cost function is defined, and it is shown that a unique saddle-point strategy exists which minimizes the worst-case value of the cost function under mean-field sharing information structure. The solution is obtained by two scalable Riccati equations, which depend on a prescribed attenuation parameter, serving as a robustness factor. For the intermittent mean-field sharing information structure, an approximate saddle-point strategy is proposed, and its converges to the saddle-point is analyzed. Two numerical examples are provided to demonstrate the efficacy of the obtained results.**

I. Introduction

Recently, there has been a surge of interest in the application of networked control systems in various engineering problems such as sensor networks, swarm robotics and cooperative coordination of unmanned aerial vehicles, to name only a few. In this type of system, it is desired to achieve a global objective (such as consensus or flocking) using local control laws with limited information exchange. Many practical problems need to be taken into consideration in the design of a real-world networked-control system. For instance, multi-agent networks are often subject to external disturbances, which means that a practical control strategy needs to be robust to unwanted disturbances. However, in the theoretical analysis, such practical issues are usually neglected for simplicity.

Different robust control design techniques are introduced in the literature such as H_∞-control, risk-sensitive control and MinMax control, each of which has its own strengths and weaknesses. For example, in risk-sensitive asset management in the financial market, a risk factor is utilized in order to capture the randomness of the system; this suits applications with unknown disturbances. In MinMax control approach, on the other hand, the external disturbances are modeled as an adversarial player attempting to maximize the cost of the system. In general, there are two types of formulations for the MinMax control problem: (a) hard-constrained formulation, where an upper bound is set on the magnitude of the disturbance, and (b) soft-constrained formulation that penalizes the disturbance by a negative quadratic cost function [1]. It is demonstrated in [1] that the hard-constraint formulation is much more complex and less tractable than the soft-constraint formulation. Therefore, in this paper we focus on the soft-constraint formulation only.

There are two main challenges concerning a MinMax control setting in the leader-follower problem. The first one is the computational complexity that increases with the number of followers (curse of dimensionality). The second challenge is that it is not always feasible to assume that the states of all followers are available, specially when the number of followers is large. In such cases, a decentralized information structure is more desirable but it leads to a discrepancy in the followers' information. To address the above challenges, mean-field models are introduced in the literature to provide a tractable approximate solution to large-scale symmetric problems [2, 3]. In the control setting, mean-field-type game [4–6] studies an infinite-population model wherein the control optimization problem reduces to a one-body McKean-Vlasov optimization problem due to the simplification offered by the negligible effect in the infinite-population. In contrast, mean-field team [7] focuses on the finite-population model, where the effect of each individual player is not necessarily negligible. For linear quadratic mean-field teams, a transformation-based technique was first proposed in [8] and further extended in [7, 9, 10], that is similar in spirit to the completion-of-square approach in mean-field-type game [11, 12]. In general, mean-field-type game may be viewed as a special case of mean-field team, where the information structure is mean-field sharing and the number of players is infinite.

In the mean-field games literature, many researchers consider a leaderless network with a large number of homogeneous followers in the presence of an adversarial disturbance (player). For example, the authors in [13] study a MinMax mean-field-type game problem in the context of social networks. The proposed solution is an approximate robust mean-field equilibrium that is formulated in terms of two coupled forward-backward partial differential equations (i.e.,

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the Hamilton-Jacobi-Isaacs and Fokker-Planck-Kolmogorov equations). In [14], a Stackelberg-like mean-field game problem is considered, where an adversarial disturbance moves first and its worst-case value is computed irrespective of the actions of followers. Given the worst-case disturbance, an approximate robust Nash equilibrium is obtained for the followers using two coupled forward-backward stochastic differential equations. A similar Stackelberg-like mean-field problem is investigated in [15] with social cost function, where the variational analysis and person-by-person optimality principle are employed to derive an approximate robust mean-field-type strategy in terms of two coupled forward-backward stochastic differential equations. The authors in [16] integrate a leader player to the above setting in a sequential fashion, where the method is to first solve a MinMax control problem for the leader against its disturbance (irrespective of the actions of followers), and then solve a MinMax control problem for the followers that wish to track a convex combination of the mean-field and the state of the leader in the presence of disturbances. Unlike the above results in which the solution is presented in a semi-explicit manner, a leaderless MinMax mean-field-type game solution is computed explicitly in [12] in terms of two Riccati equations, where an existence condition for the solution in the case of scalar time-invariant coefficients is proposed.

In this paper, a leader-follower MinMax mean-field team problem with multivariate time-varying coefficients and an arbitrary number of homogeneous followers is investigated. In contrast to [13–16], we obtain the unique explicit saddle-point strategy under mean-field sharing information structure, and then give an approximate solution under intermittent mean-field sharing. It is to be noted that the solution concept of a saddle-point strategy (that simultaneously solves convex and concave optimization problems), is different from a Stackelberg strategy (that solves the two problems sequentially). Also, the Riccati equations obtained in [12, Equation 29] may be viewed as a special case of the continuous-time counterparts of the Riccati equations derived in this paper, where the coefficients associated with the leader are zero.

The remainder of the paper is organized as follows. The problem is defined and formulated in Section II. The main results are subsequently presented in Section III along with the required assumptions. In Section IV, two numerical examples are given to demonstrate the effectiveness of the results, and finally in Section V some concluding remarks are provided.

II. Problem Formulation

In this article, $\mathbb{R}$ and $\mathbb{N}$ denote, respectively, the sets of real and natural numbers. For any $k \in \mathbb{N}$, $\mathbb{N}_k$ denotes the finite set $\{1, \ldots, k\}$, and $x_{1:k}$ is short-hand notation for $\{x_1, \ldots, x_k\}$. $\mathbb{E}(\cdot)$ is the expectation of an event, $\operatorname{Cov}(\cdot)$ is the covariance matrix of a random vector, $\operatorname{Tr}(\cdot)$ is the trace of a matrix, and $\mathbf{I}$ and $\mathbf{0}$ are, respectively, the identity and zero matrices.

Consider a multi-agent system consisting of one leader and $n \in \mathbb{N}$ homogeneous followers. Let $x_t^0 \in \mathbb{R}^{\ell_x}$, $u_t^0 \in \mathbb{R}^{\ell_u}$, $d_t^0 \in \mathbb{R}^{\ell_x}$ and $w_t^0 \in \mathbb{R}^{\ell_x}$ denote, respectively, the state, action, disturbance and noise of the leader at time $t \in \mathbb{N}$, where $\ell_x, \ell_u \in \mathbb{N}$. Analogously, denote by $x_t^i \in \mathbb{R}^{\ell_x}$, $u_t^i \in \mathbb{R}^{\ell_u}$, $d_t^i \in \mathbb{R}^{\ell_x}$ and $w_t^i \in \mathbb{R}^{\ell_x}$, the state, action, disturbance and noise of follower $i \in \mathbb{N}_n$ at time $t \in \mathbb{N}$. In addition, define the aggregate state and aggregate action of followers as:

$$\bar{x}_t := \frac{1}{n} \sum_{i=1}^{n} x_t^i, \quad \bar{u}_t := \frac{1}{n} \sum_{i=1}^{n} u_t^i.$$

The dynamics of the leader at time $t \in \mathbb{N}$ is influenced by the aggregate state $\bar{x}_t$, the disturbance signal d_t^0 and noise w_t^0 as follows:

$$x_{t+1}^0 = A_t^0 x_t^0 + B_t^0 u_t^0 + S_t^0 \bar{x}_t + d_t^0 + w_t^0, \tag{1}$$

where A_t^0, B_t^0 and S_t^0 are matrices of appropriate dimensions. Furthermore, the dynamics of follower i at time t is affected by the state of the leader x_t^0, aggregate state $\bar{x}_t$, local disturbance d_t^i and local noise w_t^i as described below:

$$x_{t+1}^i = A_t x_t^i + B_t u_t^i + S_t \bar{x}_t + E_t x_t^0 + d_t^i + w_t^i, \quad i \in \mathbb{N}_n,\ t \in \mathbb{N}, \tag{2}$$

where A_t, B_t, S_t and E_t are matrices of appropriate dimensions. Let $T \in \mathbb{N}$ denote the control horizon, and assume that the primitive random variables $\{x_1^0, \{x_1^i\}_{i \in \mathbb{N}_n}, w_1^0, \{w_1^i\}_{i \in \mathbb{N}_n}, \ldots, w_T^0, \{w_T^i\}_{i \in \mathbb{N}_n}\}$ are mutually independent. In addition, it is assumed that the local noises of followers and the noise of the leader have zero mean and finite covariance matrices.

To be consistent with the terminology of the mean-field teams literature [7], the aggregate state of followers is called *mean-field* in this work. It is to be noted that the term mean-field has a slightly different meaning in mean-field games, where it refers to the aggregate state of an infinite population (as opposed to a finite population) of followers.

In this paper, we consider two non-classical information structures: mean-field sharing (MFS) and intermittent mean-field sharing (IMFS). In MFS information structure, the leader has access to its local state as well as the mean-field at any time t, i.e.,

$$u_t^0 = g_t^0(x_t^0, \bar{x}_t),$$

where $g_t^0 : \mathbb{R}^{2\ell_x} \to \mathbb{R}^{\ell_u}$. Furthermore, each follower $i \in \mathbb{N}_n$ has access to its local state as well as the state of the leader and the mean-field at time t, i.e.,

$$u_t^i = g_t^i(x_t^i, \bar{x}_t, x_t^0),$$

where $g_t^i : \mathbb{R}^{3\ell_x} \to \mathbb{R}^{\ell_u}$. In IMFS information structure, the mean-field is observed intermittently, i.e.,

$$u_t^0 = g_t^0(x_t^0, z_t), \quad u_t^i = g_t^i(x_t^i, z_t, x_t^0), \quad \forall i \in \mathbb{N}_n,$$

where $z_t := \bar{x}_t$ during the time when the mean-field is observed and $z_t :=$ `blank` during the time when the mean-field is not observed. In practice, IMFS information structure is useful when the number of followers is neither that small (so that the mean-field can be shared at each time instant) nor is very large (such that the strong law of large numbers can be applied to the mean-field). In such a case, it is feasible to obtain the mean-field intermittently such that at some time instants the information structure is observed while at some others it is not. When the number of followers is large, IMFS may be reduced to no-mean-field sharing, where the mean-field is not observed at all, i.e.,

$$u_t^0 = g_t^0(x_t^0), \quad u_t^i = g_t^i(x_t^i, x_t^0), \quad \forall i \in \mathbb{N}_n.$$

The set of all control laws $\mathbf{g} := \{g_{1:T}^0, g_{1:T}^1, \ldots, g_{1:T}^n\}$ is called the strategy of the network.

A. Problem statement

Let the set $\mathbf{d}$ be defined as $\{d_{1:T}^0, \{d_{1:T}^i\}_{i\in\mathbb{N}_n}\}$, and $\gamma > 0$ be a given attenuation parameter. Then the cost function of the system is defined as follows:

$$\begin{aligned} J_n^\gamma(\mathbf{g},\mathbf{d}) = \mathbb{E}\Big(\sum_{t=1}^T \Big[\frac{1}{n}\sum_{i=1}^n \big[(x_t^i)^\intercal Q_t x_t^i + (u_t^i)^\intercal R_t u_t^i - \gamma^2 (d_t^i)^\intercal d_t^i\big] \\ + (x_t^0)^\intercal Q_t^0 x_t^0 + (u_t^0)^\intercal R_t^0 u_t^0 - \gamma^2 (d_t^0)^\intercal d_t^0 \\ + (\bar{x}_t - x_t^0)^\intercal F_t (\bar{x}_t - x_t^0) + \bar{x}_t^\intercal P_t \bar{x}_t + \bar{u}_t^\intercal H_t \bar{u}_t\Big]\Big), \end{aligned} \tag{3}$$

where $Q_t, Q_t^0, R_t, R_t^0, F_t, P_t$ and H_t are symmetric matrices of appropriate dimensions. Note that the value of γ determines the relative importance of reaching consensus versus rejecting disturbance.

Problem 1 *Find the saddle-point strategy* $\mathbf{g}$ *under mean-field sharing information structure such that*

$$J_n^{\gamma,*} = \inf_{\mathbf{g}} \sup_{\mathbf{d}} J_n^\gamma(\mathbf{g},\mathbf{d}).$$

Problem 2 *Find an approximate saddle-point strategy* $\mathbf{g}_\varepsilon$ *under intermittent mean-field sharing information structure such that* $|\sup_{\mathbf{d}} J_n^\gamma(\mathbf{g}_\varepsilon, \mathbf{d}) - J_n^{\gamma,*}| \le \varepsilon(n)$, *where* $\varepsilon(n) \ge 0$ *and* $\lim_{n\to\infty} \varepsilon(n) = 0$.

Remark 1 Problems 1 and 2 can be extended to the case where each follower has an individual weight under the condition presented in [7, Assumption 3.6].

The main contributions of the paper are outlined below.

1) We present a unique saddle-point strategy in an explicit manner for a leader-follower network under mean-field sharing information structure, where the number of followers is not necessarily large, and hence the action of a single follower can not be neglected (note that the analysis of non-negligible players is more complex than the case with negligible players, in general).
2) We propose an approximate saddle-point strategy under intermittent mean-field sharing information structure, whose performance is different from that of mean-field sharing for any finite number of followers.

III. MAIN RESULTS

Define the following matrices at any time $t \in \mathbb{N}_T$:

$$\bar{A}_t := \begin{bmatrix} A_t^0 & S_t^0 \\ E_t & A_t + S_t \end{bmatrix}, \quad \bar{B}_t := \begin{bmatrix} B_t^0 & \mathbf{0}_{\ell_x\times\ell_u} \\ \mathbf{0}_{\ell_x\times\ell_u} & B_t \end{bmatrix},$$

$$\bar{Q}_t := \begin{bmatrix} Q_t^0 + F_t & -F_t \\ -F_t & Q_t + P_t + F_t \end{bmatrix}, \bar{R}_t := \begin{bmatrix} R_t^0 & \mathbf{0}_{\ell_u\times\ell_u} \\ \mathbf{0}_{\ell_u\times\ell_u} & H_t + R_t \end{bmatrix}.$$

Assumption 1 *At any time* $t \in \mathbb{N}_T$, *matrices* Q_t *and* $\bar{Q}_t$ *are positive semi-definite, and matrices* R_t *and* $\bar{R}_t$ *are positive definite.*

It will be shown later that Problem 1 under Assumption 1 can be cast as a strictly convex optimization problem with respect to the control actions of the leader and followers, and a strictly concave optimization problem with respect to the disturbances. Using the transformation technique introduced in [7], define the following variables: $\breve{x}_t^i := x_t^i - \bar{x}_t$, $\breve{u}_t^i := u_t^i - \bar{u}_t$, $\breve{d}_t^i := d_t^i - \bar{d}_t$ and $\breve{w}_t^i := w_t^i - \bar{w}_t$, where $\bar{d}_t := \frac{1}{n}\sum_{i=1}^n d_t^i$ and $\bar{w}_t := \frac{1}{n}\sum_{i=1}^n w_t^i$. It follows from (2) that:

$$\begin{aligned} \bar{x}_{t+1} &= (A_t + S_t)\bar{x}_t + B_t \bar{u}_t + E_t x_t^0 + \bar{d}_t + \bar{w}_t, \\ \breve{x}_{t+1}^i &= A_t \breve{x}_t^i + B_t \breve{u}_t^i + \breve{d}_t^i + \breve{w}_t^i. \end{aligned} \tag{4}$$

Note that $\frac{1}{n}\sum_{i=1}^n \breve{x}_t^i = \mathbf{0}_{\ell_x\times 1}$, $\frac{1}{n}\sum_{i=1}^n \breve{u}_t^i = \mathbf{0}_{\ell_u\times 1}$, $\frac{1}{n}\sum_{i=1}^n \breve{d}_t^i = \mathbf{0}_{\ell_x\times 1}$ and $\frac{1}{n}\sum_{i=1}^n \breve{w}_t^i = \mathbf{0}_{\ell_x\times 1}$.

Define the following variables backward in time:

$$\begin{cases} \breve{M}_t := Q_t + A_t \breve{M}_{t+1} \breve{\Delta}_t^{-1} A_t^\intercal, \\ \bar{M}_t := \bar{Q}_t + \bar{A}_t \bar{M}_{t+1} \bar{\Delta}_t^{-1} \bar{A}_t^\intercal, \\ \breve{\Delta}_t := \mathbf{I}_{\ell_x\times\ell_x} + B_t R_t^{-1} B_t^\intercal \breve{M}_{t+1} - \gamma^{-2} \breve{M}_{t+1}, \\ \bar{\Delta}_t := \mathbf{I}_{2\ell_x\times 2\ell_x} + \bar{B}_t \bar{R}_t^{-1} \bar{B}_t^\intercal \bar{M}_{t+1} - \gamma^{-2} \bar{M}_{t+1}, \\ \breve{c}_t^i := \breve{c}_{t+1}^i + \mathrm{Tr}(\breve{M}_{t+1} \operatorname{Cov} \breve{w}_t^i), \\ \bar{c}_t := \bar{c}_{t+1} + \mathrm{Tr}(\bar{M}_{t+1} \operatorname{Cov}([w_t^0, \bar{w}_t])), \end{cases} \tag{5}$$

where $\breve{M}_{T+1} := \mathbf{0}_{\ell_x\times\ell_x}$, $\bar{M}_{T+1} := \mathbf{0}_{2\ell_x\times 2\ell_x}$, $\bar{c}_{T+1} := 0$ and $\breve{c}_{T+1}^i := 0$, $i \in \mathbb{N}_n$.

Theorem 1 *Let Assumption 1 hold. Then:*

1) *Given any* $\gamma > 0$, *Problem 1 admits a unique feedback saddle-point strategy if matrices* $\gamma^2 \mathbf{I}_{\ell_x\times\ell_x} - \breve{M}_{t+1}$ *and* $\gamma^2 \mathbf{I}_{2\ell_x\times 2\ell_x} - \bar{M}_{t+1}$ *are positive definite for every* $t \in \mathbb{N}_{T-1}$, *where* $\breve{M}_{t+1}$ *and* $\bar{M}_{t+1}$ *are given by* (5).
2) *The saddle-point strategy of the leader is described by:*
$$u_t^0 = \bar{L}_t^{1,1} x_t^0 + \bar{L}_t^{1,2} \bar{x}_t, \tag{6}$$
and for every follower $i \in \mathbb{N}_n$:
$$u_t^i = \breve{L}_t x_t^i + \bar{L}_t^{2,1} x_t^0 + (\bar{L}_t^{2,2} - \breve{L}_t)\bar{x}_t, \tag{7}$$
where $\begin{bmatrix} \bar{L}_t^{1,1} & \bar{L}_t^{1,2} \\ \bar{L}_t^{2,1} & \bar{L}_t^{2,2} \end{bmatrix} := -\bar{B}_t \bar{M}_{t+1} \bar{\Delta}_t^{-1} \bar{A}_t$ *and* $\breve{L}_t := -B_t \breve{M}_{t+1} \breve{\Delta}_t^{-1} A_t$.
3) *The performance under the saddle-point strategy is given by:* $J_n^{\gamma,*} = \frac{1}{n}\sum_{i=1}^n [\mathrm{Tr}(\breve{M}_1 \operatorname{Cov}(\breve{x}_1^i)) + \breve{c}_1^i] + \mathrm{Tr}(\bar{M}_1 \operatorname{Cov}([x_1^0, \bar{x}_1])) + \bar{c}_1$.

PROOF The proof is presented in Appendix I.

We now impose the following two assumptions on the model.

Assumption 2 *The initial states and local noises of the followers are i.i.d. random variables and independent of those of the leader.*

Assumption 3 *All matrices in the dynamic equations* (1) *and* (2) *and cost function* (3) *as well as the covariance matrices are independent of the number of followers.*

Let $\hat{m}_1$ be the expected value of the initial states of the followers, and $\hat{m}_t$ denote an estimate of the mean-field $\bar{x}_t$ at time t such that if $z_{t+1} = \texttt{blank}$ at time $t+1$:

$$\hat{m}_{t+1} := (A_t + S_t + B_t \bar{L}_t^{2,2})\hat{m}_t + (B_t \bar{L}_t^{2,1} + E_t)x_t^0 + \bar{d}_t,$$

and if $z_{t+1} = \bar{x}_{t+1}$, $\hat{m}_{t+1} := \bar{x}_{t+1}$. Under Assumptions 2 and 3, it can be shown that $\hat{m}_{t+1}$ almost surely converges to $\bar{x}_{t+1}$ at every time instant, as $n \to \infty$, due the strong law of large numbers, on noting that the dynamics of the mean-field under the saddle-point strategy is described by:

$$\bar{x}_{t+1} = (A_t + S_t + B_t \bar{L}_t^{2,2})\bar{x}_t + (B_t \bar{L}_t^{2,1} + E_t)x_t^0 + \bar{d}_t + \bar{w}_t.$$

We now replace the mean-field $\bar{x}_t$ in the saddle-point strategy of Theorem 1 with the estimate $\hat{m}_t$ (that is measurable under IMFS information structure)[1] to construct the following approximate saddle-point strategy:

$$\begin{cases} v_t^0 = \bar{L}_t^{1,1} x_t^0 + \bar{L}_t^{1,2} \hat{m}_t, \\ v_t^i = \breve{L}_t x_t^i + \bar{L}_t^{2,1} x_t^0 + (\bar{L}_t^{2,2} - \breve{L}_t)\hat{m}_t, \quad i \in \mathbb{N}_n. \end{cases} \tag{8}$$

Since the dynamic equations (1) and (2), cost function (3) and the saddle-point strategies (6), (7) and (8) are bounded and continuous in $\bar{x}_t$, it results that (8) is an approximate saddle-point strategy. The reader is referred to [10] for a detailed proof in the context of optimal control, which is similar, to a great extent, to the convergence proof of MinMax control problem considered in this subsection, but note that the Riccati equations here are different and the relative errors defined in [10] are of intermittent nature. However, these differences do not add much complexity to the convergence proof because the Riccati equations (5) do not depend on the number of followers according to Assumption 3. Hence, the rate of convergence with respect to the number of followers is $1/n$, similar to [10, Theorem 2]. This leads to the following theorem.

Theorem 2 *Let Assumptions 1–3 hold. The strategy* (8) *is an approximate saddle-point strategy for Problem 2.*

IV. Numerical Examples

In this section, two numerical examples are provided to illustrate the efficacy of the obtained results.

Example 1. Consider a multi-agent network consisting of one leader and 100 identical followers whose dynamics are described by equations (1) and (2), respectively, with the following numerical parameters: $A_t^0 = 0.85$, $B_t^0 = 0.15$, $A_t = 0.85$, $B_t = 0.85$, $S_t^0 = 0.03$, $S_t = 0.1$, $E_t = 0.01$, $w_t^i \sim \mathcal{N}(0, 0.3)$, $\forall i \in \mathbb{N}_n$, $T = 20$.

Let the initial state of the leader be $x_1^0 = 30$ and the initial states of the followers be chosen randomly (with uniform distribution) in the interval $[0, 20]$. The followers are exposed to an external disturbance given by: $d_t^i = 0.6 sin(t)$, $t \in \mathbb{N}_{20}$, $i \in \mathbb{N}_{100}$. The objective of the leader and followers is to minimize the cost function (3) under the worst-case

[1]Note that the worse-case value of $\bar{d}_t$ is given by (15) in the Appendix.

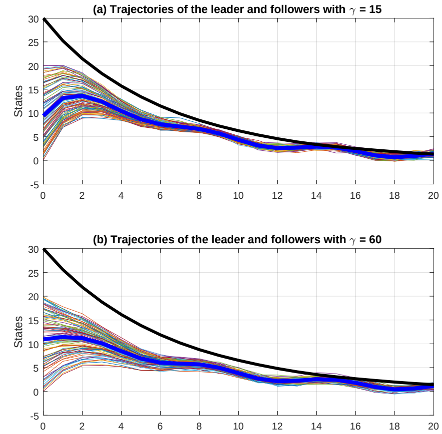

Fig. 1. Sample trajectories of the leader and followers in Example 1, where thin colored curves are the states of the followers, thick blue curve is the mean-field, and thick black curve is the state of the leader.

disturbance, where at any time $t \in \mathbb{N}_{20}$: $R_t = 70$, $Q_t = 8$, $F_t = 11$, $P_t = 0.4$, $R_t^0 = 50$, $Q_t^0 = 0.5$, $H_t = 0.1$.

Sample trajectories of the leader and followers are depicted in Figure 1. It is shown that as the attenuation parameter γ increases, the fluctuations of the mean-field decrease which means better disturbance rejection.

Example 2. Consider 100 followers with identical dynamics that wish to track a reference signal, which may be viewed as a virtual leader with constant state $x_t^0 = 10$, $\forall t \in \mathbb{N}_{30}$. The initial states of the followers are randomly chosen in the interval $[0, 8]$ with a uniform distribution. The dynamics of the leader and followers are expressed by the following parameters: $A_t^0 = A_t = 1$, $B_t^0 = 0$, $w_t^i \sim \mathcal{N}(0, 0.3)$, $\forall i \in \mathbb{N}_{100}$, $S_t^0 = 0$, $S_t = 0.04$, $B_t = 1$, $T = 30$, $E_t = 0.001$. The followers are subject to local external disturbances given below: $d_t^i = 0.4 sin(t)$, $i \in \mathbb{N}_{100}$, $t \in \mathbb{N}_{30}$. The weight matrices in the cost function are given by: $R_t = 0.11$, $Q_t = 0.01$, $F_t = 0.07$, $P_t = 0.001$, $R_t^0 = 10^{-4}$, $Q_t^0 = 10^{-4}$, $H_t = 1$.

In Figure 2, three sample trajectories of the states of followers are displayed for three different values of the attenuation parameter. It is observed that as the attenuation parameter increases, the disturbance is rejected more strongly at the cost of prolonging the consensus process.

V. Conclusions

In this paper, a robust control strategy was proposed for a class of leader-follower networks. Two decentralized information structures were studied. For mean-field sharing structure, it was shown that a unique saddle-point strategy exists under some mild assumptions. For intermittent mean-field sharing, the proposed strategy was shown to converge to the saddle-point strategy as the number of followers tends to infinity. The main feature of the obtained results is the fact that the solutions are identified by two Riccati equations that are not in the form of forward-backward equations, and their dimensions do not increase with the number of followers. In addition, it was numerically verified that the disturbance

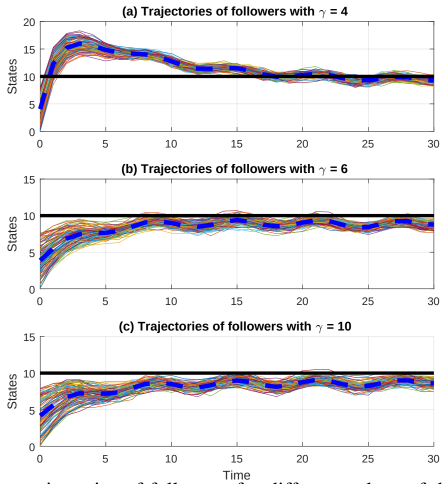

Fig. 2. The trajectories of followers for different values of the attenuation parameter γ in Example 2.

rejection property of the solution outweighs the consensus-reaching behaviour when the attenuation parameter is large.

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Appendix I
Proof of Theorem 1

Rewrite the cost function defined in (3) in terms of the new variables as:

$$J_n^{\gamma}(\mathbf{g},\mathbf{d})=\mathbb{E}\Big(\sum_{t=1}^{T}\Big[\frac{1}{n}\sum_{i=1}^{n}(\breve{x}_t^i)^\intercal Q_t\breve{x}_t^i+(\breve{u}_t^i)^\intercal R_t\breve{u}_t^i-\gamma^2(\breve{d}_t^i)^\intercal\breve{d}_t^i\Big]$$
$$+\begin{bmatrix} x_t^0\\ \bar{x}_t\end{bmatrix}^\intercal\bar{Q}_t\begin{bmatrix} x_t^0\\ \bar{x}_t\end{bmatrix}+\begin{bmatrix} u_t^0\\ \bar{u}_t\end{bmatrix}^\intercal\bar{R}_t\begin{bmatrix} u_t^0\\ \bar{u}_t\end{bmatrix}-\gamma^2\begin{bmatrix} d_t^0\\ \bar{d}_t\end{bmatrix}^\intercal\begin{bmatrix} d_t^0\\ \bar{d}_t\end{bmatrix}\Big).$$

At any time t, define the following augmented vectors: $\mathbf{x}_t := [\breve{x}_t^{1\intercal},\ldots,\breve{x}_t^{n\intercal},x_t^{0\intercal},\bar{x}_t^\intercal]^\intercal$, $\mathbf{u}_t := [\breve{u}_t^{1\intercal},\ldots,\breve{u}_t^{n\intercal},u_t^{0\intercal},\bar{u}_t^\intercal]^\intercal$, $\mathbf{d}_t := [\breve{d}_t^{1\intercal},\ldots,\breve{d}_t^{n\intercal},d_t^{0\intercal},\bar{d}_t^\intercal]^\intercal$. Suppose for now that $\mathbf{x}_t$ is known, and solve the corresponding Isaacs' equation according to [17, Theorem 3.2], i.e., the cost-to-go function at the terminal time T is: $V_T(\mathbf{x}_T)=\frac{1}{n}\sum_{i=1}^{n}(\breve{x}_T^i)^\intercal\breve{M}_T\breve{x}_T^i+\begin{bmatrix} x_T^0\\ \bar{x}_T\end{bmatrix}^\intercal\bar{M}_T\begin{bmatrix} x_T^0\\ \bar{x}_T\end{bmatrix}+\frac{1}{n}\sum_{i=1}^{n}\breve{c}_T^i+\bar{c}_T$. Suppose that the cost-to-go function takes the following form at time $t+1$:

$$V_{t+1}(\mathbf{x}_{t+1})=\frac{1}{n}\sum_{i=1}^{n}(\breve{x}_{t+1}^i)^\intercal\breve{M}_{t+1}\breve{x}_{t+1}^i$$
$$+\begin{bmatrix} x_{t+1}^0\\ \bar{x}_{t+1}\end{bmatrix}^\intercal\bar{M}_{t+1}\begin{bmatrix} x_{t+1}^0\\ \bar{x}_{t+1}\end{bmatrix}+\frac{1}{n}\sum_{i=1}^{n}\breve{c}_{t+1}^i+\bar{c}_{t+1}. \quad (9)$$

It is now desired to show that (9) holds at time t as well. It follows from Isaacs' equation that:

$$
\begin{aligned}
V_t(\mathbf{x}_t) = &\sup_{\mathbf{d}_t} \inf_{\mathbf{u}_t} \Big(\frac{1}{n} \sum_{i=1}^{n} \big[(\breve{x}_t^i)^\intercal Q_t \breve{x}_t^i + (\breve{u}_t^i)^\intercal R_t \breve{u}_t^i - \gamma^2 (\breve{d}_t^i)^\intercal \breve{d}_t^i \big] \\
&+ \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix}^\intercal \bar{Q}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} + \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix}^\intercal \bar{R}_t \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix} - \gamma^2 \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix}^\intercal \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix} \\
&+ \mathbb{E}\big[V_{t+1}(\mathbf{x}_{t+1}) \mid \mathbf{x}_t, \mathbf{u}_t, \mathbf{d}_t \big] \Big). \qquad (10)
\end{aligned}
$$

From (4), (9) and (10), one arrives at:

$$
\begin{aligned}
V_t(\mathbf{x}_t) = &\sup_{\mathbf{d}_t} \inf_{\mathbf{u}_t} \Big(\frac{1}{n} \sum_{i=1}^{n} \big[(\breve{x}_t^i)^\intercal Q_t \breve{x}_t^i + (\breve{u}_t^i)^\intercal R_t \breve{u}_t^i - \gamma^2 (\breve{d}_t^i)^\intercal \breve{d}_t^i \big] \\
&+ \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix}^\intercal \bar{Q}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} + \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix}^\intercal \bar{R}_t \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix} - \gamma^2 \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix}^\intercal \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix} \\
&+ \mathbb{E}\Big[\frac{1}{n} \sum_{i=1}^{n} \big[(A_t \breve{x}_t^i + B_t \breve{u}_t^i + \breve{d}_t^i + \breve{w}_t^i)^\intercal \breve{M}_{t+1} \\
&\quad \times (A_t \breve{x}_t^i + B_t \breve{u}_t^i + \breve{d}_t^i + \breve{w}_t^i) \big] + \frac{1}{n} \sum_{i=1}^{n} \breve{c}_{t+1}^i + \bar{c}_{t+1} \\
&+ \Big(\bar{A}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} + \bar{B}_t \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix} + \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix} + \begin{bmatrix} w_t^0 \\ \bar{w}_t \end{bmatrix} \Big)^\intercal \bar{M}_{t+1} \\
&\quad \times \Big(\bar{A}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} + \bar{B}_t \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix} + \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix} + \begin{bmatrix} w_t^0 \\ \bar{w}_t \end{bmatrix} \Big) \Big] \Big).
\end{aligned}
$$

This yields:

$$
\begin{aligned}
V_t(\mathbf{x}_t) = &\sup_{\mathbf{d}_t} \inf_{\mathbf{u}_t} \Big(\frac{1}{n} \sum_{i=1}^{n} \big[(\breve{x}_t^i)^\intercal Q_t \breve{x}_t^i + (\breve{u}_t^i)^\intercal R_t \breve{u}_t^i - \gamma^2 (\breve{d}_t^i)^\intercal \breve{d}_t^i \big] \\
&+ \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix}^\intercal \bar{Q}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} + \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix}^\intercal \bar{R}_t \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix} - \gamma^2 \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix}^\intercal \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix} \\
&+ \frac{1}{n} \sum_{i=1}^{n} [(A_t \breve{x}_t^i + B_t \breve{u}_t^i)^\intercal \breve{M}_{t+1} (A_t \breve{x}_t^i + B_t \breve{u}_t^i) + (\breve{d}_t^i)^\intercal \breve{M}_{t+1} \breve{d}_t^i \\
&+ 2(\breve{d}_t^i)^\intercal \breve{M}_{t+1} (A_t \breve{x}_t^i + B_t \breve{u}_t^i) + 2 \breve{w}_t^i \breve{M}_{t+1} (A_t \breve{x}_t^i + B_t \breve{u}_t^i + \breve{d}_t^i) \\
&+ 2(\breve{d}_t^i)^\intercal \breve{M}_{t+1} \breve{w}_t^i + \mathrm{Tr}(\breve{M}_{t+1} \operatorname{Cov}(\breve{w}_t^i))] + \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix}^\intercal \bar{M}_{t+1} \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix} \\
&+ \Big(\bar{A}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} + \bar{B}_t \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix} \Big)^\intercal \bar{M}_{t+1} \Big(\bar{A}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} + \bar{B}_t \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix} \Big) \\
&+ 2 \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix}^\intercal \bar{M}_{t+1} \Big(\bar{A}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} + \bar{B}_t \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix} \Big) + \bar{c}_{t+1} \\
&+ 2 \begin{bmatrix} \breve{w}_t^0 \\ \bar{w}_t \end{bmatrix}^\intercal \bar{M}_{t+1} \Big(\bar{A}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} + \bar{B}_t \begin{bmatrix} u_t^0 \\ \bar{u}_t \end{bmatrix} \Big) + \frac{1}{n} \sum_{i=1}^{n} \breve{c}_{t+1}^i \\
&+ 2 \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix}^\intercal \bar{M}_{t+1} \begin{bmatrix} \breve{w}_t^0 \\ \bar{w}_t \end{bmatrix} + \mathrm{Tr}(\bar{M}_{t+1} \operatorname{Cov}([w_t^0, \bar{w}_t])) \Big). \qquad (11)
\end{aligned}
$$

Given any disturbance vector $\mathbf{d}_t$, we now compute the gradient vector with respect to $\mathbf{u}_t$ and set each component to zero in order to obtain the optimal actions of the leader and followers, i.e., one has:

$$
\begin{aligned}
&2(\breve{u}_t^i)^\intercal R_t + 2(\breve{u}_t^i)^\intercal B_t^\intercal \breve{M}_{t+1} B_t + 2(\breve{x}_t^i)^\intercal A_t^\intercal \breve{M}_{t+1} B_t \\
&\qquad + 2(\breve{d}_t^i)^\intercal \breve{M}_{t+1} B_t = \mathbf{0}_{1 \times \ell_u}, \quad \forall i \in \mathbb{N}_n,
\end{aligned}
$$

$$
\begin{aligned}
&2[u_t^{0\intercal}, \bar{u}_t^\intercal] \bar{R}_t + 2[u_t^{0\intercal}, \bar{u}_t^\intercal] \bar{B}_t^\intercal \bar{M}_{t+1} \bar{B}_t \\
&+ 2 \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix}^\intercal \bar{A}_t^\intercal \bar{M}_{t+1} \bar{B}_t + 2 \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix}^\intercal \bar{M}_{t+1} \bar{B}_t = \mathbf{0}_{1 \times 2\ell_u},
\end{aligned}
$$

which leads to:

$$
\begin{aligned}
\breve{u}_t^{i*} = &-(R_t + B_t^\intercal \breve{M}_{t+1} B_t)^{-1} (B_t^\intercal \breve{M}_{t+1} A_t) \breve{x}_t^i \\
&- (R_t + B_t^\intercal \breve{M}_{t+1} B_t)^{-1} (B_t^\intercal \breve{M}_{t+1}) \breve{d}_t^i, \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
\begin{bmatrix} u_t^{0,*} \\ \bar{u}_t^* \end{bmatrix} = &-(\bar{R}_t + \bar{B}_t^\intercal \bar{M}_{t+1} \bar{B}_t)^{-1} (\bar{B}_t^\intercal \bar{M}_{t+1} \bar{A}_t) \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} \\
&- (\bar{R}_t + \bar{B}_t^\intercal \bar{M}_{t+1} \bar{B}_t)^{-1} (\bar{B}_t^\intercal \bar{M}_{t+1}) \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix}. \qquad (13)
\end{aligned}
$$

It is observed that the Hessian matrix is diagonal, with matrices $R_t + B_t^\intercal \breve{M}_{t+1} B_t$ and $\bar{R}_t + \bar{B}_t^\intercal \bar{M}_{t+1} \bar{B}_t$ as its diagonal terms that are positive definite according to Assumption 1. Hence, the cost function is strictly convex in the newly defined control actions.

By incorporating the optimal strategies (12) and (13) into (11), calculating the gradient with respect to $\mathbf{d}_t$ and setting each component to zero, one arrives at the following $n+1$ equations:

$$
\begin{aligned}
&- \breve{M}_{t+1} B_t (R_t + B_t^\intercal \breve{M}_{t+1} B_t)^{-1} B_t^\intercal \breve{M}_{t+1} \breve{d}_t^i \\
&\quad - \breve{M}_{t+1} B_t (R_t + B_t^\intercal \breve{M}_{t+1} B_t)^{-1} B_t^\intercal \breve{M}_{t+1} A_t \breve{x}_t^i \\
&- \gamma^2 \breve{d}_t^i + \breve{M}_{t+1} \breve{d}_t^i + \breve{M}_{t+1} A_t \breve{x}_t^i = \mathbf{0}_{\ell_x \times 1}, \quad \forall i \in \mathbb{N}_n,
\end{aligned}
$$

and

$$
\begin{aligned}
&- \bar{M}_{t+1} \bar{B}_t (\bar{R}_t + \bar{B}_t^\intercal \bar{M}_{t+1} \bar{B}_t)^{-1} \bar{B}_t^\intercal \bar{M}_{t+1} \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix} \\
&\quad - \bar{M}_{t+1} \bar{B}_t (\bar{R}_t + \bar{B}_t^\intercal \bar{M}_{t+1} \bar{B}_t)^{-1} \bar{B}_t^\intercal \bar{M}_{t+1} \bar{A}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} \\
&- \gamma^2 \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix} + \bar{M}_{t+1} \begin{bmatrix} d_t^0 \\ \bar{d}_t \end{bmatrix} + \bar{M}_{t+1} \bar{A}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix} = \mathbf{0}_{2\ell_x \times 1}.
\end{aligned}
$$

After some manipulations, the worst-case disturbances can be obtained as:

$$
\breve{d}_t^{i*} = \gamma^{-2} \breve{M}_{t+1} \breve{\Delta}_t^{-1} A_t \breve{x}_t^i, \qquad (14)
$$

and

$$
\begin{bmatrix} d_t^{0,*} \\ \bar{d}_t^* \end{bmatrix} = \gamma^{-2} \bar{M}_{t+1} \bar{\Delta}_t^{-1} \bar{A}_t \begin{bmatrix} x_t^0 \\ \bar{x}_t \end{bmatrix}, \qquad (15)
$$

where $\breve{\Delta}_t$ and $\bar{\Delta}_t$ are given by (5). In addition, we find the Hessian matrix which is diagonal with matrices $-\gamma^2 \mathbf{I}_{\ell_x \times \ell_x} + \breve{M}_{t+1}$ and $-\gamma^2 \mathbf{I}_{2\ell_x \times 2\ell_x} + \bar{M}_{t+1}$ as its diagonal terms. Therefore, if these matrices are negative definite, it is concluded that the cost function is strictly concave with respect to disturbances. The recursion (5) is finally obtained by incorporating the worst-case disturbances (14) and (15) into the optimal strategies (12) and (13) and comparing the expressions (9) and (10) at times $t+1$ and t, respectively, which completes the proof.