# Combined Backstepping/Second-Order Sliding-Mode Boundary Stabilization of an Unstable Reaction–Diffusion Process

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***Abstract*—In this letter we deal with a class of open-loop unstable reaction-diffusion PDEs with boundary control and Robin-type boundary conditions. A second-order sliding mode algorithm is employed along with the backstepping method to asymptotically stabilize the controlled plant while providing at the same time the rejection of an external persistent boundary disturbance. A constructive Lyapunov analysis supports the presented synthesis, and simulation results are presented to validate the developed approach.**

***Index Terms*—Reaction-diffusion equation, boundary control, backstepping, second-order sliding modes.**

## I. Introduction

SLIDING mode control synthesis for processes governed by partial differential equations (PDEs) is well documented [11], [14] and it generally retains the distinguishing features of high accuracy and robustness that are possessed by its finite-dimensional counterpart. Particularly, the backstepping approach is a powerful strategy to deal with the boundary control of several classes of PDEs (see, e.g., [16]).

In recent years, the backstepping method and the sliding-mode control approach have been successfully combined in order to enhance the robustness features of the conventional backstepping design by admitting the presence of persistent boundary disturbances to be rejected via suitably chosen backstepping/sliding-mode boundary control inputs.

Examples of combined, synergic, use of backstepping and first-order sliding-mode techniques can be found in the recent works [1], [5], [6], [7], [9], [10], [17], [18]. In all the quoted references, the backstepping method is combined to first-order sliding-mode laws and, as a result, the designed boundary control law (or the observer output injection in [1]) is discontinuous.

More precisely, the class of PDEs dealt with in the present work encompasses boundary controlled reaction diffusion processes that possess three distinct instability sources: the reaction term, an unstable Robin's type boundary condition, and a matched persistent boundary disturbance located at the controlled boundary. In the disturbance-free case, the underlying stabilization problem was solved in [15] by means of the backstepping approach. In this letter we combine the backstepping approach and a second-order sliding mode algorithm to additionally reject a class of persistent matching disturbances by means of a continuous boundary control input. This robustness feature represents the main novelty behind [15].

This letter also extends the results of [13] by admitting the presence of a destabilizing reaction term and of a destabilizing boundary condition of the Robin's type. In [2], preliminary related results were announced. Namely, a backstepping and first-order sliding-mode based *discontinuous* boundary control scheme was proposed to ensure the system stabilization in the space $L^2(0,1)$ while rejecting a class of persistent boundary disturbances. In this letter we combine the backstepping approach with a continuous second-order sliding mode boundary control law, thereby obviating the chattering phenomenon, and we additionally prove stability in the higher-order Sobolev Space $H^2(0,1)$.

The rest of this letter is organized as follows. Section II states the problem under investigation and outlines the proposed solution. Section III illustrates the constructive Lyapunov-based synthesis of the boundary controller. Section IV deals with the simulation results and, finally, Section V collects some concluding remarks.

### A. Notation and Instrumental Lemmas

The notation used throughout is fairly standard. $H^l(0,1)$, with $l = 0, 1, 2, \ldots$, denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta)$ on $(0,1)$ with square

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integrable derivatives $z^{(i)}(\zeta)$ up to the order $l$ and the $H^l$-norm

$$\|z(\cdot)\|_l = \sqrt{\int_0^1 \Sigma_{i=0}^{l}[z^{(i)}(\zeta)]^2 d\zeta}. \tag{1}$$

Throughout this letter we shall also utilize the standard notation $H^0(0,1) = L^2(0,1)$. The following well-known results will be employed within this letter.

*Lemma 1 [13]:* Let $y(\xi) \in H^1(0,1)$. Then, the following inequalities hold:

$$y^2(i) \le y^2(1-i) + 2\|y(\cdot)\|_0\|y_\xi(\cdot)\|_0, \quad i=0,1 \tag{2}$$
$$y^2(i) \le 2y^2(1-i) + 2\|y_\xi(\cdot)\|_0^2, \quad i=0,1 \tag{3}$$

*Lemma 2:* Let $y_1(\xi), y_2(\xi) \in L^2(0,1)$. Then, the following inequality holds:

$$\|y_1(\cdot)+y_2(\cdot)\|_0^2 \le 2(\|y_1(\cdot)\|_0^2 + \|y_2(\cdot)\|_0^2) \tag{4}$$

*Lemma 3:* Let $a, b, \gamma \in \Re$ with $\gamma > 0$. Then, the following inequalities hold:

$$-\left(\frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2\right) \le ab \le \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2. \tag{5}$$

## II. Problem Formulation and Solution Outline

We consider the space- and time-varying scalar field $z(\xi,t)$ evolving in the space $H^2(0,1)$, with the spatial variable $\xi \in [0,1]$ and time variable $t \ge 0$. Let it be governed by the next Boundary Value Problem (BVP) equipped with controlled and perturbed Boundary Conditions (BCs)

$$z_t(\xi,t) = \theta z_{\xi\xi}(\xi,t) + \lambda z(\xi,t) \tag{6}$$
$$z_\xi(0,t) = -qz(0,t), \tag{7}$$
$$z_\xi(1,t) = v(t) + \psi(t) \tag{8}$$

where $v(t)$ is the manipulable boundary control input, $\theta, \lambda$ and $q$ are known system's parameters, and $\psi(t)$ is a nonvanishing disturbance. For large positive $\lambda$ and $q$, the BVP (6)-(8) possesses arbitrarily many unstable eigenvalues. The initial condition (IC) is $z(\xi,0) = z^0(\xi) \in H^2(0,1)$.

The class of initial functions and admissible disturbances is specified by the following assumption.

*Assumption 1:* The initial function $z^0(\xi)$ is compatible to the perturbed BC's $z_\xi^0(0) = -qz^0(0)$, $z_\xi^0(1) = \psi(0)$, whereas the disturbance $\psi(t)$ is twice continuously differentiable, and there exists an *a priori* known constant $M > 0$ such that

$$|\dot{\psi}(t)| \le M, \quad \forall t \ge 0. \tag{9}$$

It should be stressed that the time derivative of the disturbance needs not to be calculated. Through the a-priori choice of parameter M in eq. (9), which reflects in the tuning parameters of the control law, one gives the proposed control algorithm the capability of rejecting the entire class of disturbances specified by (9). With the assumption above, the stability of the considered heat process is studied in an appropriate Sobolev space being specified to $H^2(0,1)$.

The proposed control strategy comprises two logical steps:

*Step 1:* Following [15], the backstepping transformation

$$x(\xi,t) = z(\xi,t) - \int_0^\xi k(\xi,y)z(y,t)dy \tag{10}$$

is employed to map system (6)-(8) into the *target dynamics*

$$x_t(\xi,t) = \theta x_{\xi\xi}(\xi,t) - cx(\xi,t) \tag{11}$$
$$x_\xi(0,t) = -qx(0,t), \tag{12}$$
$$x_\xi(1,t) = u(t) + \psi(t) \tag{13}$$
$$x(\xi,0) = x^0(\xi) = z^0(\xi) - \int_0^\xi k(\xi,y)z^0(y)dy \tag{14}$$

where $u(t)$ is a new manipulable boundary input, and $c$ is an arbitrarily chosen positive constant.

*Step 2:* A second-order sliding mode algorithm is employed to design a continuous boundary input $u(t)$ providing the asymptotic stabilization of the target dynamics (and, therefore, of the original dynamics (6)-(8) as well) in the space $L^2(0,1)$.

## III. Main Result

The proposed boundary control algorithm takes the form

$$v(t) = -\frac{\lambda^*}{2}z(1,t) + \int_0^1 k_\xi(1,y)z(y,t)dy + u(t) \tag{15}$$

where $\lambda^* = \frac{\lambda+c}{\theta}$ and

$$\begin{aligned} k_\xi(1,y) &= -\frac{\lambda^*}{2}z(1,t) - \lambda^* \int_0^1 \frac{I_1(\sqrt{\lambda^*(1-y^2)})}{\sqrt{\lambda^*(1-y^2)}}z(y,t)\,dy \\ &- \lambda^* \int_0^1 \frac{I_2(\sqrt{\lambda^*(1-y^2)})}{1-y^2}z(y,t)dy \\ &+ \frac{q\lambda^{*2}}{2\sqrt{\lambda^*+q^2}} \int_0^1 \left[\int_0^{1-y} e^{-\frac{q\tau}{2}} \sinh\left(\frac{\sqrt{\lambda^*+q^2}}{2}\tau\right)\right. \\ &\times \left.\frac{(2-\tau)I_0(\sqrt{\lambda^*(1+y)(1-y-\tau)})}{\sqrt{\lambda^*(1+y)(1-y-\tau)}}d\tau\right]z(y,t)\,dy \end{aligned} \tag{16}$$

and $u(t)$ is given by the following dynamical controller

$$\begin{aligned} \dot{u}(t) &= -\lambda_1 \mathrm{sgn}\, x(1,t) - \lambda_2 \mathrm{sgn}\, x_t(1,t) \\ &\quad - W_1 x(1,t) - W_2 x_t(1,t), \qquad u(0) = 0 \end{aligned} \tag{17}$$

The explicit representation of $x(1,t)$ and $x_t(1,t)$ in terms of the original systems coordinates is derived by (10) as

$$x(1,t) = z(1,t) - \int_0^1 k(1,y)z(y,t)dy \tag{18}$$
$$x_t(1,t) = z_t(1,t) - \int_0^1 k(1,y)z_t(y,t)dy \tag{19}$$

with

$$\begin{aligned} k(1,y) &= -\lambda \frac{I_1(\sqrt{\lambda^*(1-y^2)})}{\sqrt{\lambda^*(1-y^2)}} \\ &\quad + \frac{q\lambda^*}{\sqrt{\lambda^*+q^2}} \int_0^{1-y} e^{-q\tau/2} sinh\left(\frac{\sqrt{\lambda^*+q^2}}{2}\tau\right) \\ &\quad \times I_0\left(\sqrt{\lambda^*(1+y)(1-y-\tau)}\right)d\tau, \end{aligned} \tag{20}$$

whereas $\lambda_1$, $\lambda_2$, $W_1$ and $W_2$ are constant parameters subject to appropriate inequalities that shall be derived throughout this letter, and sign $\cdot$ stands for the multi-valued function

$$\mathrm{sgn}\, z \in \begin{cases} 1 & z > 0 \\ [-1,1] & z = 0 \\ -1 & z < 0. \end{cases} \tag{21}$$

*Remark 1:* Since the proposed dynamic boundary control input term $u(t)$ is governed by the ODE (17) with discontinuous right-hand side, the meaning of the solutions of the target dynamics (11)-(13) is then specified in the sense of *Filippov* [4]. Extension of the Filippov solution concept towards the infinite dimensional setting can be found, e.g., in [12].

Well-posedness of the underlying system, under the assumptions, imposed on the ICs and BCs, is actually verifiable in accordance with [3, Th. 3.3.3] by taking into account that the dynamic boundary control (17) is twice piece-wise continuously differentiable along the state trajectories of (11)-(13). Thus, in the remainder, it is assumed the following.

*Assumption 2:* The closed loop target system (11)-(13), (17) possesses a unique Filippov solution $x(\cdot,t) \in \mathrm{H}^2(0,1)$ and its time derivative $w(\cdot,t) = x_t(\cdot,t) \in \mathrm{L}^2(0,1)$ verifies the auxiliary boundary-value problem

$$w_t(\xi,t) = \theta w_{\xi\xi}(\xi,t) - cw(\xi,t) \tag{22}$$

$$w_\xi(0,t) = -qw(0,t), \quad w_\xi(1,t) = \dot{u}(t) + \dot{\psi}(t), \tag{23}$$

$$w(\xi,0) = w^0(\xi) = \theta x^0_{\xi\xi}(\xi) - cx^0(\xi) \tag{24}$$

Notice that (22)-(23) are formally obtained by differentiating (11)-(13), in the time variable $t$, whereas the IC (24) is straightforwardly derived from (14) and (11).

The next Theorem investigates the convergence features of the proposed boundary control design and summarizes the main result of this letter.

*Theorem 1:* Consider the boundary value problem (6)-(8), satisfying the Assumption 1, along with the boundary control strategy (15)-(20). Let the controller parameters be chosen such that

$$c > \theta q^2 \tag{25}$$

$$\lambda_1 - \lambda_2 > M, \quad \lambda_2 > M, \quad W_1 > \frac{q}{2}, \quad W_2 > q \tag{26}$$

Then, the global asymptotic convergence to zero of the augmented state $(z, z_t)$ in the space $H^2(0,1) \times L^2(0,1)$ is in force.

*Proof of Theorem 1:* The proof if splitted in three different steps.

1. *Backstepping transformation:* To map system (6)-(8) into the target dynamics (11)-(13) the backstepping transformation (10) is employed, and the associated kernel PDE was derived in [15, Sec. VIII] as follows:

$$k_{\xi\xi}(\xi,y) - k_{yy}(\xi,y) - \lambda^* k(\xi,y) = 0, \tag{27}$$

$$k_y(\xi,0) = -qk(\xi,0) \tag{28}$$

$$k(\xi,\xi) = -\frac{\lambda^*}{2}\xi \tag{29}$$

where $\lambda^* = \frac{\lambda+c}{\theta}$. These three conditions are compatible and form a well-posed PDE with the twice continuously differentiable solution [15, Sec. VIII]

$$\begin{aligned} k(\xi,y) = &-\lambda\xi \frac{I_1(\sqrt{\lambda^*(\xi^2-y^2)})}{\sqrt{\lambda^*(\xi^2-y^2)}} \\ &+ \frac{q\lambda^*}{\sqrt{\lambda^*+q^2}} \int_0^{\xi-y} e^{q\tau/2} sinh\left(\frac{\sqrt{\lambda^*+q^2}}{2}\tau\right) \\ &\times I_0\left(\sqrt{\lambda^*(\xi+y)(\xi-y-\tau)}\right) d\tau \end{aligned} \tag{30}$$

Spatial differentiation of (10) at $\xi = 0$ and $\xi = 1$ yields

$$x_\xi(0,t) = z_\xi(0,t) - k(0,0)z(x,t) \tag{31}$$

$$x_\xi(1,t) = z_\xi(1,t) - k(1,1)z(1,t) - \int_0^1 k_\xi(1,y)z(y,t)dy \tag{32}$$

Substituting into (31) the boundary condition (7), noticing that by (10) the equality $z(0,t) = x(0,t)$ is in force, and observing that, by (29), $k(0,0) = 0$, one derives that the boundary condition (12) is satisfied.

Substituting into (32) the boundary condition (8), and observing that, by (29), $k(1,1) = -\frac{\lambda^*}{2}$, one obtains

$$x_\xi(1,t) = v(t) + \psi(t) + \frac{\lambda^*}{2}z(1,t) - \int_0^1 k_\xi(1,y)z(y,t)dy \tag{33}$$

Therefore, choosing $v(t)$ as in (15)-(16), where (16) is obtained by spatial differentiation of (30) at $\xi = 1$, one derives that the boundary condition (13) is also satisfied. Thus, by means of the backstepping transformation (10) complemented by the boundary feedback (15)-(16) system (6)-(8) is transferred into the target dynamics (11)-(13).

2. *Stability of the target dynamics:* Differentiating with respect to time system (11)-(13), and substituting (17) in the corresponding BC, one obtains

$$x_{tt}(\xi,t) = \theta x_{\xi\xi t}(\xi,t) - cx_t(\xi,t), \tag{34}$$

$$x_{\xi t}(0,t) = -qx_t(0,t), \tag{35}$$

$$\begin{aligned} x_{\xi t}(1,t) &= \dot{u}(t) + \dot{\psi}(t) = -\lambda_1 \mathrm{sgn}\, x(1,t) \\ &-\lambda_2 \mathrm{sgn}\, x_t(1,t) - W_1 x(1,t) - W_2 x_t(1,t) + \dot{\psi}(t) \end{aligned} \tag{36}$$

whose augmented state vector $(x, x_t)$ evolves in the Hilbert space $H^2(0,1) \times L^2(0,1)$. Introduce the next Lyapunov function

$$V_1(t) = \lambda_1\theta|x(1,t)| + \frac{1}{2}\theta W_1 x^2(1,t) + \frac{1}{2}\|x_t(\cdot,t)\|_0^2. \tag{37}$$

In light of (34), the corresponding time derivative takes the form:

$$\begin{aligned} \dot{V}_1(t) &= \lambda_1\theta x_t(1,t)\mathrm{sgn}(x(1,t)) + \theta W_1 x(1,t)x_t(1,t) \\ &+ \int_0^1 x_t(\nu,t)x_{tt}(\nu,t)d\nu = \lambda_1\theta x_t(1,t)\mathrm{sgn}(x(1,t)) \\ &+ \theta W_1 x(1,t)x_t(1,t) + \theta\int_0^1 x_t(\nu,t)x_{\xi\xi t}(\xi,t)d\nu \\ &- c\int_0^1 x_t^2(\nu,t)d\nu \end{aligned} \tag{38}$$

Integration by parts along with the BCs (35) and (36) yield

$$\begin{aligned} \theta\int_0^1 & x_t(\nu,t)x_{\xi\xi t}(\nu,t)d\nu = \theta x_t(1,t)x_{\xi t}(1,t) \\ &- \theta x_t(0,t)x_{\xi t}(0,t) - \theta\|x_{t\xi}(\cdot,t)\|_0^2 = -\lambda_1\theta x_t(1,t)\mathrm{sgn}x(1,t) \\ &- \lambda_2\theta|x_t(1,t)| - W_1\theta x_t(1,t)x(1,t) - W_2\theta x_t^2(1,t) \\ &+ \theta x_t(1,t)\dot{\psi}(t) + \theta q x_t^2(0.t) - \theta\|x_{t\xi}(\cdot,t)\|_0^2 \end{aligned} \tag{39}$$

Now substituting (39) into (38), and rearranging, one straightforwardly derives the next

$$\begin{aligned} \dot{V}_1(t) = &-\theta\lambda_2|x_t(1,t)| - \theta W_2 x_t^2(1,t) + \theta x_t(1,t)\dot{\psi}(t) \\ &+ q\theta x_t^2(0,t) - \theta\|x_{t\xi}(\cdot,t)\|_0^2 - c\|x_t(\cdot,t)\|_0^2 \end{aligned} \tag{40}$$

By (2), specified with $y(\cdot) = x_t(\cdot)$ and $i = 0$, and by then applying inequality (5), one derives that

$$q\theta x_t^2(0,t) \le q\theta\Big(x_t^2(1,t) + 2\|x_t(\cdot,t)\|_0\|x_{t\xi}(\cdot,t)\|_0\Big) \le q\theta\Big(x_t^2(1,t) + \gamma_1\|x_t(\cdot,t)\|_0^2 + \frac{1}{\gamma_1}\|x_{t\xi}(\cdot,t)\|_0^2\Big) \quad (41)$$

where $\gamma_1$ is an arbitrary positive constant. Substituting (41) and (9) into (40), and making simple manipulations, one derives the next estimation

$$\dot V_1(t) \le -\theta(\lambda_2 - M)|x_t(1,t)| - \theta(W_2 - q)x_t^2(1,t) - \theta\Big[1 - \frac{q}{\gamma_1}\Big]\|x_{t\xi}(\cdot,t)\|_0^2 - \big[c - q\theta\gamma_1\big]\|x_t(\cdot,t)\|_0^2 \quad (42)$$

If the inequalities $\lambda_2 > M, \quad W_2 > q, \quad \gamma_1 > q, \quad c > q^2$ are in force then the negative semidefiniteness condition $\dot V_1(t) \le 0$ holds. Notice that the positive parameter $\gamma_1$ is arbitrary, as it does not enter in the plant closed-loop dynamics but just in the convergence proof, and the obtained inequalities on $\lambda_2, W_2$ and $c$ coincide with those appearing in (25)-(26). Condition $\dot V_1(t) \le 0$ allows one to conclude that for all constants $R \ge V_1(0) \ge 0$ the domain $V_1(t) \le R$ will be invariant. As a trivial consequence, the next inequalities

$$|x(1,t)| \le \frac{R}{\lambda_1\theta}, \qquad \|x_t(\cdot,t)\|_0^2 \le 2R \quad (43)$$

hold. Now define

$$V_2(t) = V_1(t) + \frac{1}{2}\kappa_1\theta W_2 x^2(1,t) + \frac{1}{2}\kappa_2\int_0^1 x^2(\nu,t)d\nu + \kappa_1 x(1,t)\int_0^1 x_t(\nu,t)d\nu \quad (44)$$

where $\kappa_1$ and $\kappa_2$ are positive constants to subsequently be specified. Since the last term in the right hand side of (44) is sign-indefinite, it must be proven that $V_2(t)$ is a positive definite function. Applying the triangle inequality, and taking into account (43), it yields

$$x(1,t)\int_0^1 x_t(\nu,t)\,d\nu \ge -\frac{1}{2}\Big[x^2(1,t) + \|x_t(\cdot,t)\|_0^2\Big] \ge -\frac{1}{2}\Big[\frac{R}{\lambda_1\theta}|x(1,t)| + \|x_t(\cdot,t)\|_0^2\Big]. \quad (45)$$

By (45) and (44), function $V_2(t)$ can be estimated as

$$\begin{aligned} V_2(t) &\ge \lambda_1\theta|x(1,t)| + \frac{1}{2}\theta(W_1 + \kappa_1 W_2)x^2(1,t) + \frac{1}{2}\|x_t(\cdot,t)\|_0^2 \\ &\quad - \frac{\kappa_1}{2}\Big[\frac{R}{\lambda_1\theta}|x(1,t)| + \|x_t(\cdot,t)\|_0^2\Big] + \frac{1}{2}\kappa_2\|x^2(\cdot,t)\|_0^2 \\ &= \Big(\lambda_1\theta - \frac{\kappa_1 R}{2\lambda_1\theta}\Big)|x(1,t)| + \frac{1}{2}\theta(W_1 + \kappa_1 W_2)x^2(1,t) \\ &\quad + \frac{1}{2}(1-\kappa_1)\|x_t(\cdot,t)\|_0^2 + \frac{1}{2}\kappa_2\|x^2(\cdot,t)\|_0^2. \end{aligned} \quad (46)$$

Let us specify $\kappa_1 > 0$, $\kappa_2 > 0$ such that

$$\kappa_1 < \min\left\{\frac{2\lambda_1^2\theta^2}{R}, 1\right\}. \quad (47)$$

Then, it follows from (44) and (46)-(47) that the augmented functional (44) is estimated according to

$$\rho_1\Big(|x(1,t)| + x^2(1,t) + \|x(\cdot,t)\|_0^2 + \|x_t(\cdot,t)\|_0^2\Big) \le V_2(t) \le \rho_2\Big(|x(1,t)| + x^2(1,t) + \|x(\cdot,t)\|_0^2 + \|x_t(\cdot,t)\|_0^2\Big) \quad (48)$$

for some positive constants $\rho_1$ and $\rho_2$, with $\rho_1 < \rho_2$, and it is thus positive definite and radially unbounded.

Let us now evaluate the time derivative of $V_2(t)$ along the closed-loop system trajectories. One derives

$$\dot V_2(t) = \dot V_1(t) + \kappa_1\theta W_2 x(1,t)x_t(1,t) + \kappa_2\int_0^1 x(\nu,t) \times x_t(\nu,t)d\nu + \kappa_1 x_t(1,t)\int_0^1 x_t(\nu,t)d\nu + \kappa_1 x(1,t)\int_0^1 x_{tt}(\nu,t)d\nu \quad (49)$$

Substituting (11), (12) and (34)-(36) into the corresponding terms of (49), and performing lengthy but straightforward manipulations, yield

$$\begin{aligned} \dot V_2(t) &\le -\theta(\lambda_2 - M)|x_t(1,t)| - \theta(W_2 - q)x_t^2(1,t) \\ &\quad - \theta\Big[1 - \frac{q}{\gamma_1}\Big]\|x_{t\xi}(\cdot,t)\|_0^2 - \big[c - q\theta\gamma_1\big]\|x_t(\cdot,t)\|_0^2 \\ &\quad + \kappa_1 x_t(1,t)\int_0^1 x_t(\nu,t)d\nu - c\kappa_1 x(1,t)\int_0^1 x_t(\nu,t)d\nu \\ &\quad - \kappa_1\theta(\lambda_1 - \lambda_2 - M)|x(1,t)| - \kappa_1\theta W_1 x^2(1,t) \\ &\quad + \kappa_1\theta q x(1,t)x_t(0,t) + \kappa_2\theta x(1,t)x_\xi(1,t) + \kappa_2\theta q x^2(0,t) \\ &\quad - \kappa_2\theta\|x_\xi(\cdot,t)\|_0^2 - \kappa_2 c\|x(\cdot,t)\|_0^2 \end{aligned} \quad (50)$$

The sign-undefined terms in the right hand side of (50) are going to be estimated.

By the Cauchy-Schwartz inequality and (43), one obtains

$$\Big|\kappa_1 x_t(1,t)\int_0^1 x_t(\nu,t)d\nu\Big| \le \kappa_1|x_t(1,t)|\int_0^1 |x_t(\nu,t)|d\nu \le \kappa_1|x_t(1,t)|\|x_t(\cdot,t)\|_0 \le \kappa_1\sqrt{2R}|x_t(1,t)| \quad (51)$$

Applying the extended triangle inequality (3) along with the Cauchy-Schwartz inequality yields

$$\Big|c\kappa_1 x(1,t)\int_0^1 x_t(\nu,t)d\nu\Big| \le \frac{1}{2}c\kappa_1\Big[\gamma_2 x^2(1,t) + \frac{1}{\gamma_2} \times \Big(\int_0^1 x_t(\nu,t)d\nu\Big)^2\Big] \le \frac{1}{2}c\kappa_1\gamma_2 x^2(1,t) + \frac{1}{2}c\frac{\kappa_1}{\gamma_2}\|x_t(\cdot,t)\|_0^2, \quad (52)$$

$$|\kappa_1\theta q x(1,t)x_t(0,t)| \le \frac{1}{2}\kappa_1\theta q\Big(x^2(1,t) + x_t^2(0,t)\Big). \quad (53)$$

Rewriting (41) using a different arbitrary constant $\gamma_3$ in place of $\gamma_1$, and substituting the resulting relation into (53), one gets

$$|\kappa_1\theta q x(1,t)x_t(0,t)| \le \frac{1}{2}\kappa_1\theta q x^2(1,t) + \frac{1}{2}\kappa_1\theta q x_t^2(1,t) + \frac{\gamma_3}{2}\kappa_1\theta q\|x_t(\cdot,t)\|_0^2 + \frac{1}{2\gamma_3}\kappa_1\theta q\|x_{t\xi}(\cdot,t)\|_0^2 \quad (54)$$

By applying (3), one also derives that

$$|\kappa_2\theta x(1,t)x_\xi(1,t)| + \kappa_2\theta q x^2(0,t) \leq \frac{1}{2}\gamma_4\theta\kappa_2 x^2(1,t) + \frac{\kappa_2}{2\gamma_4}\theta x_\xi^2(1,t) + \kappa_2\theta q x^2(0,t) \tag{55}$$

where $\gamma_4$ is an arbitrary positive constant. By property (3), specialized with $y(\cdot) = x_\xi(\cdot)$, relation (11), which yields $x_{\xi\xi}(\xi,t) = \frac{1}{\theta}x_t(\xi,t) + \frac{c}{\theta}x(\xi,t)$, relation (12), and inequality (4), one gets the upper-estimation $\theta x_\xi^2(1,t) \leq 2\theta q^2 x^2(0,t) + \frac{4}{\theta}\|x_t(\cdot,t)\|_0^2 + \frac{4c^2}{\theta}\|x(\cdot,t)\|_0^2$. Substituting this inequality into (55) it yields

$$|\kappa_2\theta x(1,t)x_\xi(1,t)| + \kappa_2\theta q x^2(0,t) \leq \frac{\gamma_4}{2}\theta\kappa_2 x^2(1,t) + \frac{2\kappa_2}{\theta\gamma_4}\|x_t(\cdot,t)\|_0^2 + \frac{2\kappa_2 c^2}{\theta\gamma_4}\|x(\cdot,t)\|_0^2 + \kappa_2\theta q\left(1+\frac{q}{\gamma_4}\right)x^2(0,t) \tag{56}$$

By exploiting (2), specified with $y(\cdot) = x(\cdot)$ and $i = 0$, and relation (3) signal $x^2(0,t)$ can be estimated as $x^2(0,t) \leq x^2(1,t) + +2\|x(\cdot,t)\|_0\|x_\xi(\cdot,t)\|_0 \leq x^2(1,t) + \gamma_5\|x(\cdot,t)\|_0^2 + \frac{1}{\gamma_5}\|x_\xi(\cdot,t)\|_0^2$, where $\gamma_5$ is an arbitrary positive constant. Substituting this estimation into (56) it finally yields

$$|\kappa_2\theta x(1,t)x_\xi(1,t)| + \kappa_2\theta q x^2(0,t) \leq \theta\kappa_2\left[\frac{\gamma_4}{2} + q\left(1+\frac{q}{\gamma_4}\right)\right] \times x^2(1,t) + \kappa_2\left[\frac{2c^2}{\theta\gamma_4} + \theta q\gamma_5\left(1+\frac{q}{\gamma_4}\right)\right]\|x(\cdot,t)\|_0^2 + \frac{2\kappa_2}{\theta\gamma_4}\|x_t(\cdot,t)\|_0^2 + \frac{\kappa_2\theta q}{\gamma_5}\left(1+\frac{q}{\gamma_4}\right)\|x_\xi(\cdot,t)\|_0^2, \tag{57}$$

Substituting (51)-(57) into (50) gives

$$\begin{aligned}\dot V_2(t) \leq &-\theta\left(\lambda_2 - M - \kappa_1\frac{\sqrt{2R}}{\theta}\right)|x_t(1,t)| \\ &- \theta\left[W_2 - q\left(1+\frac{1}{2}\kappa_1\right)\right]x_t^2(1,t) \\ &- \theta\left[1 - \frac{q}{\gamma_1} - \kappa_1\frac{q}{2\gamma_3}\right]\|x_{t\xi}(\cdot,t)\|_0^2 \\ &- \left[c\left(1-\frac{\kappa_1}{2\gamma_2}\right) - q\theta\left(\gamma_1 + \kappa_1\frac{\gamma_3}{2}\right) - \frac{2\kappa_2}{\theta\gamma_4}\right]\|x_t(\cdot,t)\|_0^2 \\ &- \kappa_1\theta(\lambda_1 - \lambda_2 - M)|x(1,t)| - \theta\left[\kappa_1\left(W_1 - \frac{1}{2}q - \frac{c}{2\theta}\gamma_2\right)\right. \\ &\left.- \kappa_2\left(\frac{\gamma_4}{2} + q + \frac{q^2}{\gamma_4}\right)\right]x^2(1,t) \\ &- \kappa_2\left(c - \frac{\gamma_5}{\gamma_4}\theta q^2 - \gamma_5\theta q - \frac{2c^2}{\theta\gamma_4}\right)\|x(\cdot,t)\|_0^2 \\ &- \kappa_2\theta\left(1 - \frac{q^2}{\gamma_4\gamma_5} - \frac{q}{\gamma_5}\right)\|x_\xi(\cdot,t)\|_0^2. \end{aligned} \tag{58}$$

In order to have a nonpositive right hand side of (58), the next system of inequalities should be satisfied

$$\lambda_2 - M - \frac{\kappa_1\sqrt{2R}}{\theta} > 0, \qquad W_2 - q - \frac{\kappa_1 q}{2} > 0, \tag{59}$$

$$1 - \frac{q}{\gamma_1} - \frac{\kappa_1 q}{2\gamma_3} > 0, \qquad \lambda_1 - \lambda_2 - M > 0, \tag{60}$$

$$c\left(1-\frac{\kappa_1}{2\gamma_2}\right) - q\theta\left(\gamma_1 + \kappa_1\frac{\gamma_3}{2}\right) - \frac{2\kappa_2}{\theta\gamma_4} > 0, \tag{61}$$

$$\kappa_1\left(W_1 - \frac{1}{2}q - \frac{c}{2\theta}\gamma_2\right) - \kappa_2\left(\frac{\gamma_4}{2} + q + \frac{q^2}{\gamma_4}\right) > 0, \tag{62}$$

$$c - \frac{\gamma_5}{\gamma_4}\theta q^2 - \gamma_5\theta q - \frac{2c^2}{\theta\gamma_4} > 0, \tag{63}$$

$$1 - \frac{q^2}{\gamma_4\gamma_5} - \frac{q}{\gamma_5} > 0. \tag{64}$$

Inequalities (59)-(60) follow from (25)-(26), and by imposing the following inequality

$$\kappa_1 < \min\left\{\frac{\theta(\lambda_2 - M)}{\sqrt{2R}}, \frac{2(W_2 - q)}{q}, \frac{2\gamma_3(1-\frac{q}{\gamma_1})}{q}\right\} \tag{65}$$

on the arbitrary positive parameter $\kappa_1$ in addition to (47). Relation (61) is rewritten as

$$c - q\theta\gamma_1 - \frac{\kappa_1}{2}\left(\frac{c}{\gamma_2} + q\theta\gamma_3\right) - \kappa_2\frac{2}{\theta\gamma_4} > 0 \tag{66}$$

which follows from (25)-(26) and by imposing the following inequalities

$$\kappa_1 < \frac{2(c - \gamma_1\theta q)}{\gamma_3\theta q + \frac{c}{\gamma_2}} \tag{67}$$

$$\kappa_2 < \frac{\theta\gamma_4\left[c\left(1-\frac{\kappa_1}{2\gamma_2}\right) - q\theta\left(\gamma_1 + \kappa_1\frac{\gamma_3}{2}\right)\right]}{2} \tag{68}$$

in addition to (47) and (65). Relation (62) is satisfied due to (26) and by imposing the following inequalities

$$\gamma_2 < \frac{2\theta(W_1 - \frac{q}{2})}{c}, \quad \kappa_2 < \frac{\kappa_1\theta\left(W_1 - \frac{1}{2}q - \frac{c}{2\theta}\gamma_2\right)}{\left(\frac{\bar\gamma_4}{2} + q + \frac{q^2}{\bar\gamma_4}\right)} \tag{69}$$

on the $\gamma_2$ and $\kappa_2$ parameters in addition to (68). Relation (64) is equivalent to

$$\gamma_5 > \frac{q(q+\gamma_4)}{\gamma_4} \tag{70}$$

Substituting (70) in (63), one obtains

$$c > \theta q^2\left(\frac{q+\gamma_4}{\gamma_4}\right)^2 + \frac{2c^2}{\theta\gamma_4}, \tag{71}$$

and it is clear that $\forall c > \theta q^2 \quad \exists\ \bar\gamma_4$ such that relation (71) is in force $\forall\gamma_4 \geq \bar\gamma_4$. Thus, under the tuning conditions (25)-(26), and the additional inequalities (65), (67)-(70) along with $\gamma_4 \geq \bar\gamma_4$, all terms in the right hand-side of (58) are nonnegative, and it can be readily found $\rho_3 > 0$ such that

$$\dot V_2(t) \leq -\rho_3\left(|x(1,t)| + x^2(1,t) + \|x(\cdot,t)\|_0^2 + \|x_t(\cdot,t)\|_0^2\right) \tag{72}$$

Relations (72) and (48), considered together, imply that $V_2(t)$ asymptotically escapes to zero. Particularly, by (48), it implies that the norms $\|x(\cdot,t)\|_0^2$ and $\|x_t(\cdot,t)\|_0^2$ asymptotically vanish. Then, it follows from (11) that $\|x_{\xi\xi}(\cdot,t)\|_0^2$ asymptotically vanishes as well. This, in turns, guarantees the asymptotic stability of (11)-(13), (17), (25)-(26) whose augmented state vector $(x, x_t)$ is evolving in the space $H^2(0,1) \times L^2(0,1)$.

3. *Stability of the original system:* To prove stability of the original system (6)-(8), (15)-(16) it must be proven that the

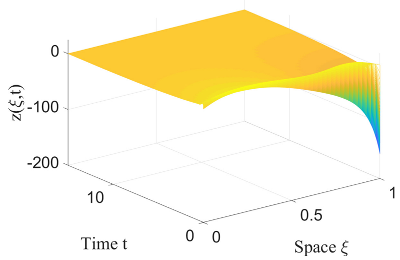


Fig. 1. Solution $z(\xi, t)$ spatiotemporal profile.

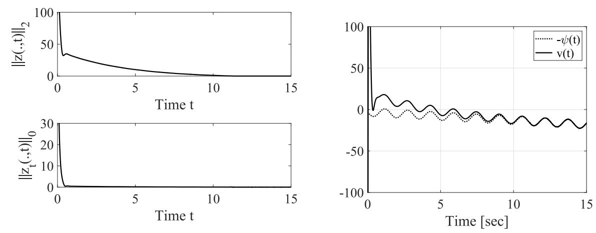


Fig. 2. Left plot: time evolutions of $\|z(\cdot, t)\|_2$ and $\|z_t(\cdot, t)\|_0$. Right plot: boundary control $v(t)$ and the disturbance $-\psi(t)$.

transformation (10) is invertible. Associated computations can be found in [2, Proof of Th. 1]. Equivalence of norms of $z(\xi, t)$ and $x(\xi, t)$ in $H^2(0, 1)$ is thus proven and the same norm equivalence straightforwardly holds as well for the norms of $z_t(\xi, t)$ and $x_t(\xi, t)$ in $L^2(0, 1)$, thereby ensuring that $\|z(\cdot, t)\|_2$ and $\|z_t(\cdot, t)\|_0$ asymptotically decay, too. This completes the proof of Theorem 1. ■

## IV. Simulations

Consider the BVP (6)-(8), with parameters $\theta = 0.5$, $\lambda = 10$ and $q = 2$, and the disturbance $\psi(t)$ chosen as $\psi(t) = 3 + t + 5\mathtt{sin}(4t)$. The magnitude of the disturbance time derivative $\dot{\psi}(t)$ is upper-estimated as $M = 25$, according to (9). The state initial condition has been set to $z(\xi, 0) = 20 sin(\pi\xi) + 10 sin(3\pi\xi)$. The finite-difference method is used by discretizing the spatial solution domain $\xi \in [0, 1]$ into a finite number of $N = 40$ uniformly spaced solution nodes $\xi_i = ih$, $h = 1/(N + 1)$, $i = 1, 2, \ldots, 40$. The resulting 40-th order discretized system is solved by fixed-step Euler method with step $T_s = 10^{-4}$.

The proposed controller (15)-(20) has been then implemented with the parameters $c = 5$, $\lambda_1 = 150$, $\lambda_2 = 50$, $W_1 = 2$, $W_2 = 4$, which are selected in accordance with (25)-(26). Figure 1 depicts the spatiotemporal profile of the closed-loop solution $z(\xi, t)$, which, as expected, is asymptotically steered to the zero value. Figure 2-left shows the time evolutions of the norms $\|z(\cdot, t)\|_2$ and $\|z_t(\cdot, t)\|_0$, which both eventually tend to zero according to the theoretical analysis of Theorem 1, whereas Figure 2-right depicts the time evolution of the boundary control input $v(t)$, showing that it is continuous and, in addition, that it eventually approaches the sign-reversed disturbance $-\psi(t)$ as $t \to \infty$.

## V. Conclusion

A class of reaction-diffusion processes has been stabilized by employing a synergic combination between the backstepping and sliding mode control methodologies. The natural continuation of this letter, to be addressed in next publications, will be the design of a state observer capable of reconstructing the state $z(\xi, t)$ and its time derivative using boundary sensing, thereby enabling, once coupled to the presented state-feedback controller, the output-feedback implementation of the considered technique.

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