Errata for "Comments on Truncation Errors for Polynomial Chaos Expansions"

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***Abstract*—We fix errata encountered in letter T. Mühlpfordt *et al.* "Comments on truncation errors for polynomial chaos expansions," *IEEE Control Systems Letters*, 2.1, Jan. 2018, pp. 169–174.**

I. Proof of Theorem 1

The content of Theorem 1 remains untouched. It is only its preceding assumption and the proof itself that we rectify.

First, we make Assumption 1 more precise.

Assumption 1 (Exact PCE *Input):* For a given n_ξ-variate orthogonal polynomial basis $\{\phi_j\}_{j=0}^{\ell_z}$, the PCE of the real-valued random variable $\mathsf{z} \in \mathrm{L}^2(\Omega, \mu; \mathbb{R})$ has the known and finite minimum degree $d_z \in \mathbb{N}_0$, and $\ell_z + 1 \geq (n_\xi + d_z)!/(n_\xi! d_z!)$ PCE coefficients, see (6). Unless stated otherwise we assume $\ell_z + 1 = (n_\xi + d_z)!/(n_\xi! d_z!)$.

With this we correct the proof of Theorem 1 as follows.

Proof: The PCE for z_i is given by

$$\mathsf{z}_i = \sum_{j=0}^{\ell_z} z_{i,j}\phi_j,$$

where $\ell_z + 1 = (n_\xi + d_z)!/(n_\xi! d_z!)$. Without loss of generality we assume that

$$f(\mathsf{z}_1, \ldots, \mathsf{z}_{n_z}) = \mathsf{z}_1^{\alpha_1} \cdots \mathsf{z}_{n_z}^{\alpha_{n_z}} + \cdots,$$

where $\alpha = [\alpha_1 \cdots \alpha_{n_z}]^\top \in \mathbb{N}_0^{n_z}$ is a multi-index with $|\alpha| = d_f$. We insert the PCE for every z_i

$$\begin{aligned} f&\left(\sum_{j=0}^{\ell_z} z_{1,j}\phi_j, \ \ldots, \ \sum_{j=0}^{\ell_z} z_{n_z,j}\phi_j\right) \\ &= \left(\sum_{j=0}^{\ell_z} z_{1,j}\phi_j\right)^{\alpha_1} \cdots \left(\sum_{j=0}^{\ell_z} z_{n_z,j}\phi_j\right)^{\alpha_{n_z}} + \cdots \\ &= \gamma \xi_1^{\beta_1} \cdots \xi_{n_\xi}^{\beta_{n_\xi}} + \cdots, \end{aligned}$$

where γ is some constant, and the multi-index $\beta = [\beta_1 \cdots \beta_{n_\xi}]^\top \in \mathbb{N}_0^{n_\xi}$ satisfies $d_z|\alpha| = d_z d_f$. Hence, the highest-degree polynomial term of y has degree $d_z d_f$, yielding a total number of $\ell + 1$ basis elements given by (6) with $d \leftarrow d_z d_f$, thus enlarging the basis by $\ell - \ell_z$ elements. The orthogonal projection of y onto Z yields $\mathrm{P}_n \mathsf{y} = \sum_{j=0}^{n} y_j \phi_j$. Consequently, the truncation error e_n becomes $\mathsf{e}_n = \sum_{j=n+1}^{\ell} y_j \phi_j$, which is zero in case of $n \geq \ell$. For $n < \ell$, apply Parseval's identity to obtain $\|\mathsf{e}_n\|$ (equation (9) in the original reference [1]), see [2]. ■

II. Proof of Theorem 2

The first part of Theorem 2 was misleading. We revise Theorem 2 as follows.

Theorem 2 (Uncertainty Quantification for Problem 1): For all realizations of z, let the active constraints in Problem 1 satisfy the linear inequality constraint qualification (LICQ) at the optimal solution y. If the set of active constraints $\mathcal{A} = \{a_1, \ldots, a_{n_{\mathrm{act}}}\} \subseteq \{1, \ldots, n_{\mathrm{con}}\}$ is the same for all realizations of $\mathsf{z} = [\mathsf{h}^\top \ \mathsf{b}^\top]^\top$, then the element-wise truncation error becomes

$$\|\mathsf{y}_i - \mathrm{P}_n \mathsf{y}_i\| = \begin{cases} \sqrt{\sum_{j=n+1}^{d} \left(w_i^{h\top} h_j + w_i^{b\top} M_\mathcal{A} b_j\right)^2 \|\phi_j\|^2}, & \text{if } n < d, \\ 0, & \text{if } n \geq d, \end{cases}$$

where $w_i^{h\top}$, $w_i^{b\top}$ are the ith rows with $i = 1, \ldots, n_\chi$ of the matrices W^h, W^b that satisfy

$$\begin{bmatrix} W^h & W^b \\ V^h & V^b \end{bmatrix} = -\begin{bmatrix} H & A^\top M_\mathcal{A}^\top \\ M_\mathcal{A} A & 0 \end{bmatrix}^{-1}.$$

The active constraint selection matrix $M_\mathcal{A} \in \mathbb{N}_0^{n_{\mathrm{act}} \times n_{\mathrm{con}}}$ is constructed from the active set $\mathcal{A}$ and has elements $(M_\mathcal{A})_{i a_i} = 1$ for $i = 1, \ldots, n_{\mathrm{act}}$, zero elsewhere.

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Proof: Because the set of active constraints is supposed to be $\mathcal{A}$ for all realizations, the Karush-Kuhn-Tucker conditions hold in terms of a function of random variables

$$\begin{bmatrix} \mathsf{y} \\ \lambda^\star \end{bmatrix} = \begin{bmatrix} W^h & W^b \\ V^h & V^b \end{bmatrix} \begin{bmatrix} \mathsf{h} \\ M_{\mathcal{A}}\mathsf{b} \end{bmatrix},$$

where (2) and $\mathsf{z} = [\mathsf{z}_1^\top \ \mathsf{z}_2^\top]^\top = [\mathsf{h}^\top \ \mathsf{b}^\top]^\top$ are used. Invertibility follows from LICQ. Consequently for all $i = 1, \ldots, n_\chi$,

$$\mathsf{y}_i = w_i^{h\top}\mathsf{h} + w_i^{b\top}M_{\mathcal{A}}\mathsf{b} = \sum_{j=0}^{d}(w_i^{h\top}h_j + w_i^{b\top}M_{\mathcal{A}}b_j)\phi_j,$$

and the result follows from Theorem 1 with $d_f = 1$. ■

With the new version of Theorem 2, the related remark reads.

Remark 2 (Extension to Changes in the Active Set): Note that even if the active set changes, Theorem 2 still holds locally on a polytope. Recall that multiparametric QPs admit locally affine solutions defined on polytopes [3]. Furthermore, the error description from can be turned into an upper bound by considering the *worst case* active set, which maximizes $\|\mathsf{y}_i - \mathrm{P}_n\mathsf{y}_i\|$. Due to space limitations, we leave the details to future work.

References

[1] T. Mühlpfordt, R. Findeisen, V. Hagenmeyer, and T. Faulwasser, "Comments on truncation errors for polynomial chaos expansions," *IEEE Control Syst. Lett.*, vol. 2, no. 1, pp. 169–174, Jan. 2018.

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