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Abstract—The *fundamental lemma* due to Willems et al. "A note on persistency of excitation," Syst. Control Lett., vol. 54, no. 4, pp. 325–329, 2005 plays an important role in system identification and data-driven control. One of the assumptions for the fundamental lemma is that the underlying linear time-invariant system is controllable. In this paper, the fundamental lemma is extended to address system identification for uncontrollable systems. Then, a data-driven algebraic test is derived to check whether the underlying system is controllable or not. An algorithm based on the singular value decomposition of a Hankel matrix constructed from the data is provided to implement the developed test. The algorithm has cubic computational cost. Examples are given to illustrate the theoretical results.

Index Terms—Behavioral approach, data-driven controllability, system identification.

I. INTRODUCTION

DATA-DRIVEN control is an approach to control problems where rather than using direct knowledge of the system's properties, one uses only the system's trajectories. Examples of questions that might be answered via data-driven approach include the determination of structural properties (*e.g.,* controllability, stability, and stabilizability) and the design of control laws to enforce desired properties (*e.g.,* state/output feedback design to enforce stability). This paper is mainly concerned with the former of these questions. In particular, we consider the problem of determining whether a discrete-time linear time-invariant (LTI) system is controllable.

Roughly speaking, a system is controllable if it is possible to "switch" between any two system trajectories. Many applications of control theory (for instance, system identification and feedback control) require the underlying system to be controllable. There are several equivalent criteria to verify the controllability of an LTI system that can be applied if one has a complete representation of the system under consideration. One such criterion due to Hautus [1] states that a state-space system with parameters A, B is controllable if and only if rank $\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$.

Recently, an algebraic data-driven test for controllability/stabilizability called a "data-driven Hautus test" [2, Theorem 8] was developed. While the result does not require a representation of the system, it is derived for state-space models and assumes knowledge of a state trajectory. In this paper, we formulate an analogous data-driven Hautus-type test for a general input/output system that assumes no knowledge of the state. We also provide an algorithm for data-driven verification of controllability. The algorithm is based on the singular value decomposition of the Hankel matrix built from the given data.

Theorem 1 of [3] (which we refer to throughout this paper as the *fundamental lemma*) states that if we are given an input/output trajectory of a controllable system and if the input trajectory is persistently exciting of sufficiently high order, then we can recover all trajectories of the system. In other words, we can uniquely recover the data generating system. This result has been utilized in developing identification algorithms [4, Chapter 8] and in data-driven simulation and control [5]. Recently, the fundamental lemma has received considerable attention due to growing interest in data-driven control problems [6], [7], [8], [9], [10], [11], [12], [13]. A weakness of the fundamental lemma, however, is that it requires an a priori assumption that the system under consideration is controllable.

Inspired by the results of [14], we prove an extension of the fundamental lemma that applies to uncontrollable systems (see Theorem 1). Our results show that it is possible to recover the controllable part of the data generating system and possibly, depending on the data, a (part of) the uncontrollable subsystem. Moreover, with a data-driven controllability test (see Theorem 3), it is possible to check whether the data are actually generated by a controllable system. With these results all assumptions of the fundamental lemma can be verified from the data with the only prior knowledge of the true system's order.

The rest of the paper is structured as follows: Section II defines the notation and recalls some mathematical preliminaries needed for the development of this work. In Section III, we consider the identification of uncontrollable systems. We prove that if the input is persistently exciting of sufficiently high order, the exact system must include the controllable part of the system. Then we state and prove our data-driven test for controllability in Section IV. An algorithm for the implementation of this data-driven controllability test is developed in Section V. The computational aspects of the developed algorithm is discussed in Section VI. Examples illustrating the developed results are presented in Section VII. Finally, conclusions are offered in the last section.

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II. NOTATION AND PRELIMINARIES

For any matrix $U \in \mathbb{R}^{p \times q}$, $U^\top$ denotes its transpose. The set of all generalized eigenvalues of a matrix pair (U, V) is denoted by $\Lambda(U, V)$. If U, V have the same number of

columns, $\text{col}(U, V) := \begin{bmatrix} U \\ V \end{bmatrix}$. For subspaces $\mathcal{R} \subseteq \mathbb{R}^p$ and $\mathcal{S} \subseteq \mathbb{R}^q$, we define the direct sum $\mathcal{R} \oplus \mathcal{S} \subseteq \mathbb{R}^{p+q}$ of these subspaces by

$$\mathcal{R} \oplus \mathcal{S} := \{\text{col}(u, v) : u \in \mathcal{R}, v \in \mathcal{S}\}$$

and the orthogonal complement $\mathcal{S}^\perp$ of $\mathcal{S}$ by

$$\mathcal{S}^\perp := \{x \in \mathbb{R}^q : x^T y = 0 \text{ for all } y \in \mathcal{S}\}.$$

I denotes the identity matrix, and 0 denotes the zero-matrix. Where it is helpful to do so, we may include subscripts to indicate the size of the matrix: $I_n \in \mathbb{R}^{n \times n}$, $0_n \in \mathbb{R}^n$, and $0_{m \times n} \in \mathbb{R}^{m \times n}$.

We now recall some notions from behavioral system theory. For interested readers, we refer to [15]. A dynamical system is defined by the triplet $(\mathbb{T}, \mathbb{W}, \mathcal{B})$, where $\mathbb{T} \subseteq \mathbb{R}$ is the time axis, $\mathbb{W} \subseteq \mathbb{R}^q$ is the signal space, and $\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the behavior with $\mathbb{W}^{\mathbb{T}}$ the set of all functions (*i.e.,* trajectories or time series) $w : \mathbb{T} \to \mathbb{W}$. We consider discrete-time systems, so that $\mathbb{T} \subseteq \mathbb{N}$.

By $\mathcal{L}^q$, we denote the set of finite-order LTI systems with $\mathbb{W} \subseteq \mathbb{R}^q$. Each $\mathcal{B} \in \mathcal{L}^q$ admits a *kernel representation* $\mathcal{B} = \{w : R(\sigma)w = 0\}$, where $R \in \mathbb{R}^{g \times q}[\xi]$ is a polynomial matrix and σ is the *backward shift* operator defined as $(\sigma w)(t) = w(t+1)$. The number of inputs of $\mathcal{B} \in \mathcal{L}^q$ is denoted by $\mathbf{m}(\mathcal{B})$.

A system $\mathcal{B} \in \mathcal{L}^q$ over $\mathbb{T} = \mathbb{N}$ is *controllable* if and only if for every $w \in \mathcal{B}$, there exists a $\bar{w} \in \mathcal{B}$ and $t_1 < t_2 \in \mathbb{N}$, such that $\bar{w}(t) = w(t)$ for $t \le t_1$ and $\bar{w}(t) = 0$ for $t \ge t_2$. Equivalently, matrix $R(\lambda)$ has constant rank for all $\lambda \in \mathbb{C}$. Given a system $\mathcal{B}$, we define $\mathcal{B}_{\text{cont}}$ to be the *controllable part* of the system, i.e. the largest subset of $\mathcal{B}$ for which $\mathcal{B}_{\text{cont}}$ is itself an LTI system and $\mathcal{B}_{\text{cont}}$ is controllable.

Because we are primarily concerned with discrete-time systems, $[a, b]$ denotes an integer interval. For any trajectory $w : [1, T] \to \mathbb{R}^q$, we will use $w|_{[a,b]}$ to denote the restriction of w to the interval $[a, b] \subset [1, T]$. The restriction of the behavior $\mathcal{B}$ to the interval $[1, L]$ is defined as

$$\mathcal{B}|_L = \{w|_{[1,L]} : w \in \mathcal{B}\}.$$

Given a time series $w_d : [1, T] \to \mathbb{R}^q$, the associated Hankel matrix with $L \in \mathbb{N}$ block-rows is defined as

$$\mathcal{H}_L(w_d) = \begin{bmatrix} w_d(1) & w_d(2) & \cdots & w_d(T-L+1) \\ w_d(2) & w_d(3) & \cdots & w_d(T-L+2) \\ \vdots & \vdots & \ddots & \vdots \\ w_d(L) & w_d(L+1) & \cdots & w_d(T) \end{bmatrix}.$$

This time series is *persistently exciting of order L* if the Hankel matrix $\mathcal{H}_L(w_d)$ is full row rank.

For matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$, we define $\mathcal{B}(A, B, C, D)$ to be the associated input/state/output system *i.e.,* the set of trajectories $w = \text{col}(u, x, y)$ with $u : \mathbb{N} \to \mathbb{R}^m$, $x : \mathbb{N} \to \mathbb{R}^n$, and $y : \mathbb{N} \to \mathbb{R}^p$ that satisfy

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

for all $t \in \mathbb{N}$. Similarly, we define $\mathcal{B}(A, B)$ to be the associated input/state system, *i.e.,* the set of trajectories $w = \text{col}(u, x)$ for which u, x satisfy equation (1).

For a behavior $\mathcal{B}$ with input/output partition $\mathbb{R}^q = \mathbb{R}^m \oplus \mathbb{R}^p$, we say that $\mathcal{B}(A, B, C, D)$ is a *state-space representation* of $\mathcal{B}$ if

$$\mathcal{B} = \{(u, y) : (u, x, y) \in \mathcal{B}(A, B, C, D)$$
$$\text{for some } x : \mathbb{N} \to \mathbb{R}^n\}.$$

The *order* of $\mathcal{B}$, denoted by $\mathbf{n}(\mathcal{B})$, is defined to be the smallest n for which $\mathcal{B}$ has a state-space representation with state dimension n. The representation $\mathcal{B}(A, B, C, D)$ is *state minimal* if $n = \mathbf{n}(\mathcal{B})$. Note that a state minimal representation need not be state controllable, *i.e.,* the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

need not be full row rank. As shown in [16, p. 270], a state minimal representation is *observable* (considering the state variables as latent variables) and *state trim* (*i.e.,* for all $x_0 \in \mathbb{R}^n$ there exists $(u, x, y) \in \mathcal{B}(A, B, C, D)$ such that $x(0) = x_0$).

For $(u, x) \in \mathcal{B}(A, B)$, $\mathcal{H}_L(x, u)$ denotes the matrix

$$\mathcal{H}_L(x, u) = \text{col}(\mathcal{H}_1(x_{[1,T-L+1]}), \mathcal{H}_L(u_{[1,T]})),$$

$\mathcal{X} \subseteq \mathbb{R}^n$ denotes the subspace

$$\mathcal{X} = \text{span}\{x(1), x(2), \ldots, x(T-L+1)\},$$

and $\mathcal{R} \subseteq \mathbb{R}^n$ denotes the subspace of reachable states, *i.e.,* the column-space of the controllability matrix $\mathcal{C}$.

III. IDENTIFICATION OF UNCONTROLLABLE SYSTEMS

The Hankel matrix $\mathcal{H}_L(w_d)$ provides a convenient way of encoding the information obtained from a trajectory w_d about a data generating system $\mathcal{B}$. In particular, we can deduce from the trajectory w_d that $\mathcal{B}|_L$ must contain all trajectories $w : [1, L] \to \mathbb{W}$ for which

$$\text{col}(w(1), \ldots, w(L)) = \mathcal{H}_L(w_d)g,$$

where $g \in \mathbb{R}^{q(T-L+1)}$. The set of such trajectories form the subspace $\mathcal{B}_{\text{MPUM}}|_L$, where $\mathcal{B}_{\text{MPUM}} \subseteq \mathcal{B}$ denotes the *most powerful unfalsified model* (MPUM), *i.e.,* the least complex model that fits the data exactly [4, Chapter 8].

A natural question that arises is that of when it is possible to guarantee that $\mathcal{B}_{\text{MPUM}}|_L = \mathcal{B}|_L$. In other words, under what conditions can we guarantee that $\mathcal{H}_L(w_d)$ completely specifies the behavior over $[1, L]$? The fundamental lemma provides conditions that guarantee $\mathcal{B}_{\text{MPUM}}|_L = \mathcal{B}|_L$; one of these conditions is that $\mathcal{B}$ is controllable. In the following theorem, we extend this result to show exactly how much of $\mathcal{B}|_L$ can be determined for an uncontrollable system $\mathcal{B}$.

Theorem 1: Consider a system $\mathcal{B} \in \mathcal{L}^q$. If $w_d = (u_d, y_d) \in \mathcal{B}|_T$ is such that u_d is persistently exciting of order $L + \mathbf{n}(\mathcal{B})$, then $\mathcal{B}_{\text{MPUM}}|_L \supseteq \mathcal{B}_{\text{cont}}|_L$.

In order to prove the theorem, we make use of the following two lemmas, which are extracted from the proof of Theorem 1 of [14].

Lemma 1: If $(u, x) \in \mathcal{B}(A, B)|_T$ and u is persistently exciting of order $L + n$, then $\mathcal{H}_L(x, u)$ has left kernel equal to $\mathcal{X}^\perp \oplus \{0_{mL}\}$. Equivalently, the column span of $\mathcal{H}_L(x, u)$ is $\mathcal{X} \oplus \mathbb{R}^{mL}$.

Lemma 2: If u is persistently exciting of order $L + n$, then $\mathcal{R} \subseteq \mathcal{X}$.

Proof: Lemma 1: It trivially holds that $\mathcal{X}^\perp \oplus \{0_{mL}\}$ is a subspace of the left kernel of $\mathcal{H}_L(x, u)$; we therefore only prove the reverse inclusion. To prove Lemma 1, it suffices to show that for any element $\mathrm{col}(\xi, \eta)$ of the left kernel with $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^{mL}$, we must have $\eta = 0$. This conclusion is indeed reached in the proof of Theorem 1 from [14].

Lemma 2: We equivalently show that $\mathcal{X}^\perp \subseteq \mathcal{R}^\perp$. Let $\xi \in \mathcal{X}^\perp$. Following the proof from [14], we conclude that $\xi^\top B = \xi^\top AB = \cdots = \xi^\top A^{n-1} B = 0$. In other words, ξ lies in the orthogonal complement to the column space of the controllability matrix $[B \quad AB \quad \cdots \quad A^{n-1}B]$. That is, $\xi \in \mathcal{R}^\perp$, as desired. ∎

Remark 1: As a consequence of the above lemma, we see that if u is persistently exciting of order $L + n$ and the input/state system defined by A and B is controllable, then $\mathcal{H}_L(x, u)$ has full row rank, which agrees with Theorem 1 of [14]. This also holds when u, x, y have complex entries since we could rewrite with the proof in this context using essentially the same steps.

Next, we prove Theorem 1.

Proof: Let $\mathcal{B}(A, B, C, D)$ be a state-space representation of $\mathcal{B}$, and let x be such that $\mathrm{col}(u_d, x, y_d) \in \mathcal{B}(A, B, C, D)|_T$. Define

$$\mathcal{T}_L = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{L-2}B & CA^{L-3}B & \cdots & D \end{bmatrix}, \quad \mathcal{O}_L = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{L-1} \end{bmatrix},$$

$$\mathbf{M}_L = \begin{bmatrix} 0 & I_{mL} \\ \mathcal{O}_L & \mathcal{T}_L \end{bmatrix}. \tag{3}$$

For $1 \le t \le T - L + 1$, we have

$$\mathbf{M}_L \, \mathrm{col}\,(x(t), u_d(t), \ldots, u_d(t + L - 1)) =$$
$$\mathrm{col}(u_d(t), \ldots, u_d(t + L - 1), y_d(t), \ldots, y_d(t + L - 1)). \tag{4}$$

Consequently,

$$\mathrm{col}(\mathcal{H}_L(u_d), \mathcal{H}_L(y_d)) = \mathbf{M}_L \mathcal{H}_L(x, u_d).$$

The column-space of $\mathrm{col}(\mathcal{H}_L(u_d), \mathcal{H}_L(y_d))$ consists of vectors of the form $\mathrm{vec}(w|_{[1, L]})$, where we define

$$\mathrm{vec}(w|_{[1,L]}) = \mathrm{col}(u(1), \ldots, u(L), y(1), \ldots, y(L)). \tag{5}$$

Thus, this column-space is equal to

$$\{\mathrm{vec}(w_L) : w_L \in \mathcal{B}_{\mathrm{MPUM}}|_L\}.$$

On the other hand, by Lemma 1, the column-space of $\mathcal{H}_L(x, u_d)$ is equal to $\mathcal{X} \oplus \mathbb{R}^{mL}$, which means that the column-space of $\mathrm{col}(\mathcal{H}_L(u_d), \mathcal{H}_L(y_d))$ is equal to $\{\mathbf{M}_L v : v \in \mathcal{X} \oplus \mathbb{R}^{mL}\}$.

The controllable sub-behavior $\mathcal{B}_{\mathrm{cont}} \subseteq \mathcal{B}$ consists of all trajectories $w \in \mathcal{B}$, for which the corresponding state-trajectory satisfies $x(t) \in \mathcal{R}$ for all t. Since $x(t)$ could be any element of $\mathcal{R}$ and by equation (4), $w_L \in \mathcal{B}|_L$ corresponds to a controllable trajectory if and only if $\mathrm{vec}(w_L) \in \{\mathbf{M}_L v : v \in \mathcal{R} \oplus \mathbb{R}^{mL}\}$.

With these observations and by Lemma 2,

$$\{\mathrm{vec}(w_L) : w_L \in \mathcal{B}_{\mathrm{cont}}|_L\} = \{\mathbf{M}_L v : v \in \mathcal{R} \oplus \mathbb{R}^{mL}\}$$
$$\subseteq \{\mathbf{M}_L v : v \in \mathcal{X} \oplus \mathbb{R}^{mL}\}$$
$$= \{\mathrm{vec}(w_L) : w_L \in \mathcal{B}_{\mathrm{MPUM}}|_L\}.$$

Therefore, $\mathcal{B}_{\mathrm{MPUM}}|_L \supseteq \mathcal{B}_{\mathrm{cont}}|_L$, as desired. ∎

IV. DATA-DRIVEN CONTROLLABILITY TESTS

We first summarize results from [2] that we use. For any trajectory $f : [1, T] \to \mathbb{R}^k$, we will use the notation $f^+ := f|_{[2,T]}$ and $f^- := f|_{[1,T-1]}$. Suppose that $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $(u_d, x_d) \in \mathcal{B}(A, B)|_T$. Define $\Sigma_n(u_d^-, x_d)$ to be the set of all pairs (A, B) whose associated system contains trajectory (u_d^-, x_d).

We say that the data (u_d^-, x_d) are *informative for controllability/stabilizability* if every $(A, B) \in \Sigma_n(u_d^-, x_d)$ is such that the associated input-state system is controllable/stabilizable.

Theorem 2 ([2]): The data (u_d^-, x_d) are informative for controllability if and only if for all $\lambda \in \mathbb{C}$,

$$\mathrm{rank}\,\big(\mathcal{H}_1(x_d^+) - \lambda \mathcal{H}_1(x_d^-)\big) = n. \tag{6}$$

Likewise, the data (u_d^-, x_d) are informative for stabilizability if and only if equation (6) holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \ge 1$.

We now consider the extension of this condition to arbitrary LTI systems. Let $\mathcal{B} \in \mathcal{L}^q$ be such a system. Suppose we have access to a trajectory $w_d \in \mathcal{B}|_T$.

It is known that from a given trajectory $w_d = (u_d, y_d)$, we can construct an associated state trajectory by concatenating the trajectory with its shifts. That is, it is possible to build a state-space representation of w_d in which $(u_d, x_c, y_d) \in \mathcal{B}(A, B, C, D)$ is such that $x_c(t) = \mathrm{vec}(w_d|_{[t, t+L-1]})$, with $L = \mathbf{n}(\mathcal{B})$ and vec as defined in (5). See, for example, [17, Section 2.3.3] and [18]; more recently, the technique has been exploited in [8, Section VI]. Generally, the constructed state-space representation may not be state minimal.

In view of this construction, a natural approach to extending Theorem 2 is to replace the state-data with this concatenated input/output trajectory. That is, one might consider using $x_c(t)$ and then considering the rank of $\mathcal{H}_1(x_c^+) - \lambda \mathcal{H}_1(x_c^-)$. However, because of the general non-minimality of the state-space trajectory in $\mathbb{R}^{qL}$, the matrix $\mathcal{H}_1(x_c^+) - \lambda \mathcal{H}_1(x_c^-)$ can never achieve full row rank, and hence Theorem 2 cannot be applied directly.

Nevertheless, we can indeed determine whether a system is controllable by considering the rank of $\mathcal{H}_1(x_c^+) - \lambda \mathcal{H}_1(x_c^-)$ (which has the same rank as $\mathcal{H}_L(w_d^+) - \lambda \mathcal{H}_L(w_d^-)$ since the two matrices have the same rows arranged in different orders) by the following result.

Theorem 3: Let $w_d \in \mathcal{B}|_T$ and $\mathcal{B} \in \mathcal{L}^q$. Suppose that we have window-length $L \ge \mathbf{n}(\mathcal{B})$, and that $w_d = (u_d, y_d)$ is such that the input u_d is persistently exciting of order $\mathbf{n}(\mathcal{B}) + L + 1$. Then, $\mathcal{B}$ is controllable if and only if for all $\lambda \in \mathbb{C}$,

$$\mathrm{rank}\,\big(\mathcal{H}_L(w_d^+) - \lambda \mathcal{H}_L(w_d^-)\big) = \mathbf{n}(\mathcal{B}) + \mathbf{m}(\mathcal{B})L. \tag{7}$$

Likewise, $\mathcal{B}$ is stabilizable if and only if equation (7) holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \ge 1$.

Proof: We only prove the statement regarding controllability since the proof of the statement regarding stabilizability is essentially the same. Let $\mathcal{B}(A, B, C, D)$ be a state minimal state-space representation of $\mathcal{B}$. Then the trajectory $w_d(t) = \text{col}(u_d(t), y_d(t))$ of $\mathcal{B}$ is such that $(u_d, x, y_d) \in \mathcal{B}(A, B, C, D)|_T$ for some $x : [1, T] \to \mathbb{R}^n$ with $n = \mathbf{n}(\mathcal{B})$. Let $m := \mathbf{m}(\mathcal{B})$. We note that by the minimality of the representation, the resulting state-space representation is observable. Let $\mathbf{M}_L$ denote the block-matrix associated with this system defined in equation (3).

Suppose that $\mathcal{B}$ is not controllable. It follows that the pair (A, B) describes an uncontrollable input/state system. By Theorem 2, there exists a $\lambda \in \mathbb{C}$ such that $\text{rank}(\mathcal{H}_1(x^+) - \lambda \mathcal{H}_1(x^-)) < n$. For this λ, it follows that

$$\text{rank} \begin{bmatrix} \mathcal{H}_1(x^+) - \lambda \mathcal{H}_1(x^-) \\ \mathcal{H}_L(u_d^+) - \lambda \mathcal{H}_L(u_d^-)) \end{bmatrix} < mL + n, \qquad (8)$$

so that

$$\text{rank}(\mathcal{H}_L(w_d^+) - \lambda \mathcal{H}_L(w_d^-))$$
$$= \text{rank} \left(\mathbf{M}_L \begin{bmatrix} \mathcal{H}_1(x^+) - \lambda \mathcal{H}_1(x^-) \\ \mathcal{H}_L(u_d^-) - \lambda \mathcal{H}_L(u_d^-) \end{bmatrix} \right)$$
$$\leq \text{rank} \begin{bmatrix} \mathcal{H}_1(x^+) - \lambda \mathcal{H}_1(x^-) \\ \mathcal{H}_L(u_d^+) - \lambda \mathcal{H}_L(u_d^-) \end{bmatrix} < mL + n.$$

Conversely, suppose that $\mathcal{B}$ is controllable. Let $\lambda \in \mathbb{C}$, and let $\tilde{u}, \tilde{x}$ be new input and state variables defined by $\tilde{u} = u_d^+ - \lambda u_d^-$ and $\tilde{x} = x^+ - \lambda x^-$. Let $\bar{\mathcal{B}}(A, B)$ denote the input/state system described by A, B whose inputs and states are allowed to have complex entries. Because $\mathcal{B}(A, B)$ is controllable, $\bar{\mathcal{B}}(A, B)$ is controllable as well. By the LTI property, $(\tilde{u}, \tilde{x})$ must also be a valid trajectory of $\bar{\mathcal{B}}(A, B)$.

Because u is persistently exciting of order $L + n + 1$, $\tilde{u}$ must be persistently exciting of order $L + n$. Indeed,

$$\begin{bmatrix} -\lambda I & I & & \\ & \ddots & \ddots & \\ & & -\lambda I & I \\ I & & & 0 \end{bmatrix} \mathcal{H}_{L+n+1}(u_d) = \begin{bmatrix} \mathcal{H}_{L+n}(\tilde{u}) \\ \mathcal{H}_1(u_d|_{[1, T-L-n]}) \end{bmatrix}. \quad (9)$$

Because $\mathcal{H}_{L+n}(\tilde{u})$ is obtained by selecting rows from the above full row rank matrix, $\mathcal{H}_{L+n}(\tilde{u})$ is full row rank.

By Remark 1, the matrix $\mathcal{H}_L(\tilde{u}, \tilde{x})$ has full row rank $mL + n$. So,

$$\mathcal{H}_L(\tilde{x}, \tilde{u}) = \begin{bmatrix} \mathcal{H}_1(x^+ - \lambda x^-) \\ \mathcal{H}_L(u_d^+ - \lambda u_d^-) \end{bmatrix} = \begin{bmatrix} \mathcal{H}_1(x^+) - \lambda \mathcal{H}_1(x^-) \\ \mathcal{H}_L(u_d^+) - \lambda \mathcal{H}_L(u_d^-) \end{bmatrix}$$

has full row rank as well. Finally, having $L \geq n$ ensures that $\mathbf{M}_L$ has full column rank. It follows that

$$\text{rank}(\mathcal{H}_L(w_d^+) - \lambda \mathcal{H}_L(w_d^-))$$
$$= \text{rank} \left(\mathbf{M}_L \begin{bmatrix} \mathcal{H}_1(x^+) - \lambda \mathcal{H}_1(x^-) \\ \mathcal{H}_L(u_d^+) - \lambda \mathcal{H}_L(u_d^-) \end{bmatrix} \right)$$
$$= \text{rank}(\mathbf{M}_L) = mL + n,$$

as desired. ∎

Remark 2 (Connection with the fundamental lemma): The prerequisites for the fundamental lemma to hold are: (i) $\mathcal{B}$ is controllable, and (ii) the input is persistently exciting of order $\mathbf{n}(\mathcal{B}) + L$. Condition (ii) can be verified from the data, but it was believed that (i) can only be assumed since it is not verifiable from the data. However, by using Theorem 3, condition (i) can also be checked from the given data only provided that the input is persistently exciting of sufficiently high order. Thus, we can verify both conditions required for the fundamental lemma directly from the given data.

Remark 3 (Continuous-time systems): Theorems 1 and 3 apply to continuous-time systems. For a given sampling time $\delta T > 0$, let

$$u_d = (u_d(1), u_d(1 + \delta T), \ldots, u_d(1 + (T-1)\delta T)),$$
$$y_d = (y_d(1), y_d(1 + \delta T), \ldots, y_d(1 + (T-1)\delta T)).$$

be a sampled trajectory and the input u_d is persistently exciting of sufficiently high order. Then the zero order hold input signal corresponding to u_d and the sampled output y_d suffice to obtain the full behavior of the sampled-data system for generic choices of δT.

Remark 4 (When an upper bound to $\mathbf{n}(\mathcal{B})$ is known):

As is allowed for by Theorem 1 and exemplified in Examples 1 and 2 (see Section VII), it is possible (even with persistently exciting input of sufficiently high order) for the MPUM of an uncontrollable system $\mathcal{B}$ to be controllable with $\mathbf{n}(\mathcal{B}_{\text{MPUM}}) < \mathbf{n}(\mathcal{B})$. Thus, in the absence of additional information, it is generally impossible to verify the controllability of a system without exact knowledge of the order.

However, if we have access to an upper bound to $\mathbf{n}(\mathcal{B})$, *i.e.*, we know a positive integer n_u such that $\mathbf{n}(\mathcal{B}) \leq n_u$, then it is possible to verify the uncontrollability of a system. Since Theorem 3 guarantees that $\text{rank}(\mathcal{H}_L(w_d^+) - \lambda \mathcal{H}_L(w_d^-))$ will be constant for all $\lambda \in \mathbb{C}$ if $\mathcal{B}$ is controllable, we can conclude that $\mathcal{B}$ is uncontrollable if $\text{rank}(\mathcal{H}_L(w_d^+) - \lambda \mathcal{H}_L(w_d^-))$ fails to be constant (provided that u_d is persistently exciting of order $n_u + L + 1$).

V. THE ALGORITHM

In this section, we develop an algorithm to check condition (7). It is based on the singular value decomposition (SVD) of the matrix $\mathcal{H}_L(w_d^-)$.

Algorithm 1: Data-driven controllability test
Input: Observed time series $w_d \in \mathcal{B}|_T$, input cardinality $\mathbf{m}(\mathcal{B})$, order $\mathbf{n}(\mathcal{B})$, and window-length $L \geq \mathbf{n}(\mathcal{B})$.
Output: $\mathcal{B}$ is controllable/uncontrollable.

1: Perform the SVD: $U^\top \mathcal{H}_L(w_d^-) V = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$, and let $r = \text{rank} \, \mathcal{H}_L(w_d^-)$.

2: Partition the matrix $\mathcal{H}_L(w_d^+)$ as $U^\top \mathcal{H}_L(w_d^+) V = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$, where $H_{11} \in \mathbb{R}^{r \times r}$.

3: Compute the generalized eigenvalues of the matrix pair (H_{11}, S).

4: Compute the rank of $H_\lambda = \begin{bmatrix} H_{11} - \lambda S & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ for all $\lambda \in \Lambda(H_{11}, S)$.

5: If $\text{rank} \, H_\lambda = \mathbf{n}(\mathcal{B}) + \mathbf{m}(\mathcal{B})L$ for all $\lambda \in \Lambda(H_{11}, S)$, then $\mathcal{B}$ is controllable. Otherwise, $\mathcal{B}$ is uncontrollable.

Note that to test for stabilizability, we only need to consider the generalized eigenvalues λ of (H_{11}, S) with absolute value greater than or equal to 1.

VI. COMPUTATIONAL ASPECTS

In this section, we discuss the computational complexity and numerical stability of the algorithm from Section V. For the sake of simplicity, denote by M and N the number of rows and columns in $\mathcal{H}_L(w_d^-)$. Note that $M = qL$, and $N = T - L + 1$. The computational cost of each step of Algorithm 1 is as follows.

1: The SVD of $\mathcal{H}_L(w_d^-)$ has computational cost of $O\left(N(M^2 + N^2)\right)$ flops.[1]
2: The matrix multiplication $U^\top \mathcal{H}_L(w_d^-) =: \mathcal{G}$ requires $O(2M^2N - MN)$ flops, and the multiplication $\mathcal{G}V$ requires $O(2MN^2 - MN)$ flops. Hence, this step requires $O\left(2(M^2N + MN^2 - MN)\right)$ flops in total.
3: Solving the generalized eigenvalue problem has computational cost of $O(30M^3)$ flops [19].
4: Since computation of rank done by computing the SVD, this step has a computational cost of $O\left(N(M^2 + N^2)\right)$ flops.
5: This step has been executed at no cost.

Summarizing, the computational cost of Algorithm 1 is:

$$O\big(N(M^2 + N^2) + 2(M^2N + MN^2 - MN)$$
$$+ 30M^3 + N(M^2 + N^2)\big),$$

and hence the algorithm is cubic in both M and N, and hence cubic in q, L, and T. In other words, the algorithm is cubic in $\max\{q, L, T\}$. Usually, T is higher than q and L, and hence the algorithm is cubic in T. This fact is confirmed empirically in the result shown in Fig. 1.
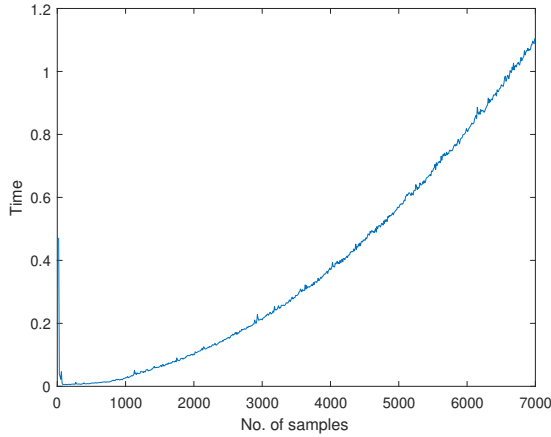

Fig. 1. Illustration of the result that the computational complexity of Algorithm 1 is cubic in the number of samples of a given trajectory. We applied the algorithm to a single-input single-output (SISO) system of order 4, where the output data are generated with random initial conditions and random input.

All computations performed in the algorithm (*viz.*, SVD, rank determination, and generalized eigenvalue computation) are numerically stable. For this reason, we believe that Algorithm 1 is numerically stable, and our experiments suggests the same.

[1] A flop is a floating-point operation, *i.e.,* a scalar addition or multiplication.

VII. ILLUSTRATIVE EXAMPLES

Example 1 (Overlap of $\mathcal{B}_{cont}, \mathcal{B}_{\mathrm{MPUM}}, \mathcal{B}$):

Suppose that an uncontrollable system $\mathcal{B}$ has an observable state-space representation $\mathcal{B}_{SS} = \mathcal{B}(A, B, C, D)$ presented in its Kalman decomposition, so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

That is, A_{11} is an $r \times r$ matrix (for some $r < n$), the pair (A_{11}, B_1) describes a controllable input/state system, and the set of reachable states of $\mathcal{B}_{SS}$ is $\mathcal{R} = \mathbb{R}^r \oplus \{0_{n-r}\}$. Let $L \geq n$ and $(u, x, y) \in \mathcal{B}_{SS}|_T$, with u persistently exciting of order $L + n$; note that $(u, y) \in \mathcal{B}|_T$. For $1 \leq t \leq T$, write $x(t) = \mathrm{col}(x_1(t), x_2(t))$, with $x_1(t) \in \mathbb{R}^r$. The observability of $\mathcal{B}_{SS}$ and the choice $L \geq n$ ensures that $\mathbf{M}_L$ has full column rank, which means that $\mathcal{R} \subset \mathcal{X}$ implies that $\{\mathbf{M}_L v : v \in \mathcal{R} \oplus \mathbb{R}^{mL}\} = \{\mathbf{M}_L v : v \in \mathcal{X} \oplus \mathbb{R}^{mL}\}$. That is, $\mathcal{B}_{cont} = \mathcal{B}_{\mathrm{MPUM}}$ if and only if $\mathcal{R} = \mathcal{X}$. Note moreover that the column space of $\mathbf{M}_L$ is equal to the (vectorized trajectories of the) true behavior $\mathcal{B}|_L$, so that $\mathcal{B}_{\mathrm{MPUM}} = \mathcal{B}$ if and only if $\mathcal{X} = \mathbb{R}^n$.

Lemma 2 ensures that $\mathcal{X} = \mathrm{span}\{x(1), x(2), \ldots, x(T - L + 1)\}$ contains $\mathcal{R}$. It follows that for any vector $v \in \mathcal{X}$, the projection $v^\perp$ of v onto $\mathcal{R}^\perp$ is also in $\mathcal{X}$. In other words, $\mathcal{X}$ contains the vectors $\mathrm{col}(0_r, x_2(t))$ for $1 \leq t \leq T - L + 1$. Note that $x_2(t) = A_{22}^{t-1} x_2(1)$, and the persistently exciting condition on u ensures that $T - L + 1 \geq m(L + n) \geq n$. Thus, the span of $\{x_2(t) : 1 \leq t \leq T - L + 1\}$ is equal to the A_{22}-invariant subspace generated by $x_2(1)$.

Therefore, we have $\mathcal{B}_{cont} = \mathcal{B}_{\mathrm{MPUM}}$ if and only if $\mathcal{R} = \mathcal{X}$, which occurs if and only if $x_2(1) = 0$; otherwise, we have a strict containment $\mathcal{B}_{cont} \subset \mathcal{B}_{\mathrm{MPUM}}$. Similarly, we have $\mathcal{B}_{\mathrm{MPUM}} = \mathcal{B}$ if and only if $x_2(1)$ is a cyclic vector for A_{22}; otherwise, we have a strict containment $\mathcal{B}_{\mathrm{MPUM}} \subset \mathcal{B}$.

If the input u fails to be persistently exciting of sufficiently high order, then it is possible that we have neither $\mathcal{B}_{\mathrm{MPUM}}|_L \supseteq \mathcal{B}_{cont}|_L$ nor $\mathcal{B}_{\mathrm{MPUM}}|_L \subseteq \mathcal{B}_{cont}|_L$. For example: consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad C = I, \quad D = 0,$$

and the trajectory with initial state $x(1) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$ and input $u(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ for $t = 1, 2, \ldots, T$. Note that u fails to be persistently exciting of any order. We can see that

$$\mathcal{X} = \mathrm{span}\{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\top\}, \quad \mathcal{R} = \mathrm{span}\{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top\}.$$

Note that neither of the spaces $\mathcal{R}, \mathcal{X}$ contain the other and, as we argue in the first paragraph above, the same applies to $\mathcal{B}_{cont}$ and $\mathcal{B}_{\mathrm{MPUM}}$.

Example 2 (RLC circuit):

Consider the RLC circuit shown in Fig. 2. The physical model parameters are C, R_C, L, and R_L. By selecting all of these parameters equal to 1, we obtain an uncontrollable system.

After elimination of the latent variables, the behavioral equation for the manifest variables i, v is given by the linear constant coefficients differential equation (see [20])

$$\frac{d^2}{dt^2} i + 2 \frac{d}{dt} i + i = \frac{d^2}{dt^2} v + 2 \frac{d}{dt} v + v.$$

This, in turn, gives us a kernel representation of the manifest behavior with parameter $R(s) = \begin{bmatrix} P(s) & -Q(s) \end{bmatrix}$, with $P(s) = Q(s) = s^2 + 2s + 1$.
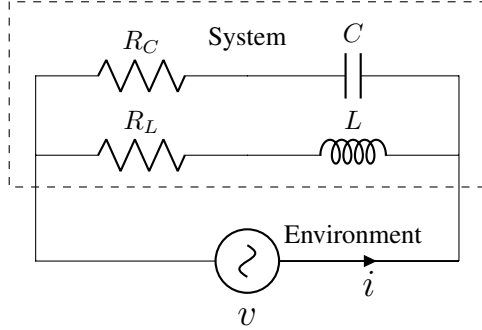

Fig. 2. RLC circuit example from [20]. The behavior of the manifest variables $w = (i, v)$ is uncontrollable for specific values of the physical model parameters C, R_C, L, and R_L.

If we consider the system with v as an input and i as an output, then the corresponding transfer function is

$$H(s) = \frac{s^2 + 2s + 1}{s^2 + 2s + 1},$$

i.e., from a classical input/output point of view the system is a static unit gain. However, the behavior is second order and uncontrollable. The system has an observable canonical state space representation $\mathcal{B}(A, B, C, D)$ with

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \text{ and } D = 1.$$

In view of Remark 3, we discretize the continuous-time model and simulate it with a zero mean random input. First, we produce a trajectory $w_d = (v_d, i_d) \in \mathcal{B}|_T$ under zero initial conditions, where $\mathcal{B}$ is the discretized true system. Physically, the zero initial conditions mean that the energy stored in the capacitor and the inductor is initially zero. We obtain the corresponding MPUM, which turns out to be the static model $\{ \begin{bmatrix} i \\ v \end{bmatrix} \mid i = v \}$. Indeed, as proved in Theorem 1, the uncontrollable sub-behavior may not be identifiable. In this case, the MPUM coincides with the controllable subsystem of $\mathcal{B}$. Even though we can not identify $\mathcal{B}$ from the data w_d, we can detect from w_d and the prior knowledge that the order is 2 that $\mathcal{B}$ is uncontrollable using Algorithm 1.

Next, we simulate a trajectory $w_d \in \mathcal{B}|_T$ under nonzero initial conditions. In this case, the MPUM coincides with $\mathcal{B}$. Again, this is consistent with Theorem 1, and Algorithm 1 correctly detects the uncontrollability property of the system directly from w_d.

VIII. CONCLUDING REMARKS

We have investigated the assumption of controllability in the fundamental lemma. In this regard, we have shown that for a given time series w_d such that the input is persistently exciting of sufficiently high order, it is always possible to recover the controllable part of the system. We have developed a data-driven test to check controllability of the underlying LTI system: for a given time series w_d with input that is

persistently exciting of sufficiently high order, we can check directly whether it is generated by a controllable system. If it is generated by a controllable system, we can employ the fundamental lemma to recover the full data generating system; otherwise, we can apply our result to recover the controllable part of the system.

For the implementation of this controllability test, we have given a numerical algorithm, which is based on the singular value decomposition of the Hankel-structure matrix constructed from an observed trajectory and the generalized eigenvalue computation.

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