Scalable Mesh Stability of Nonlinear Interconnected Systems

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***Abstract*—This paper deals with large-scale interconnected systems with general network topology, affected by external disturbances, and with the possibility to ensure the overall stability when some sufficient conditions are met by each agent with respect to its neighbors. We introduce the notion of scalable Mesh Stability (sMS), that requires the existence of trajectory bounds that do not depend on the number of subsystems. The immediate consequence is that perturbations originating in a point of the interconnected system do not amplify through it. A numerical example on interconnection of microgrids shows the interest and the effectiveness of the theoretical result.**

***Index Terms*—Mesh stability, large-scale systems, stability of nonlinear interconnected systems, scalability, ISS analysis**

I. Introduction

Large-scale interconnected systems have received considerable attention through the years [1], [2], [3], [4], [5] because of their relevance in several fields such as automotive [6], [7], power grids [8], [9] or opinion dynamics [10] among others. Target is to investigate stability analysis of large-scale systems in a decentralised/distributed framework. Recent results focus on scalable stability tests to be operated on single agents of the interconnected systems, without a central entity having global model knowledge. In [11], the authors introduce a scalable Input-to-State Stability test to be performed on a large-scale system composed by a number of agents described by the same dynamics, affected by disturbances, and with a symmetric graph modeling the agents' interconnection. In [12], a stability test is exploited for agents interconnected via locally Lipschitz conditions with different dynamics and no disturbances. In case of agents of first and second order, network stability is analyzed by means of port-Hamiltonian systems and properties of the Laplacian matrix describing the interconnections in [13]. Attention is also given to the possibility of limiting the sharing of information to guarantee the privacy of the agents belonging to a network [14]. The purpose of this paper is to introduce a distributed scalable stability condition for agents having different dynamics in a large-scale interconnected system, with no constraints on the topology of the interconnection, and with disturbances affecting each agent. A bound criterion is derived on the interconnection among the dynamics, with no prior knowledge of the underlying interconnection graph and exploitation of its structure properties.

The proposed approach is inspired by the notion of string stability, used to develop feedback controllers for a convoy of vehicles (platoon) [15]. Interconnected autonomous vehicles have the capability to avoid disturbance amplification backwards in the string via the concept of string stability, thus reducing stop-and-go waves propagation and traffic oscillations [16], [17]. An overview of the possibility to generalize the disturbance propagation avoidance via the concept of string stability to other classes of interconnected systems can be found in [18].

When modeling interconnected vehicles through nonlinear systems, Input-to-State Stability (ISS) [19], [20] is used to ensure string stability in the vehicular domain [15]. In this paper, we investigate the stability of interconnected autonomous systems affected by external disturbances by exploiting ISS concepts. Following the definition of scalable Input-to-State Stability (sISS) in [11], we introduce the scalable Mesh Stability (sMS) for general topological interconnection. We prove a local sufficient condition that, when it is satisfied by every subsystem in the interconnection, ensures global stability of the overall networked system, therefore obtaining a local test to be performed to assess global stability. We view the interaction of each agent i with its neighbors as a perturbation affecting the i-th dynamics, and this results in an enlargement of the stability region as in classical ISS theory. Then, in an inductive way, we derive trajectory bounds for the global system which are independent of the number of subsystems. We suppose that the underlying graph modeling the interactions among the systems is weakly connected. This assumption is without loss of generality since, whenever the graph is not weakly connected, it is possible to repeat our analysis for each weakly connected sub-component of the interconnection graph.

The formal results about string stability of vehicles in platoon formation described in [21] and [22] are extended here to large-scale systems with general interconnection topology and agents' dynamics affected by external disturbances. Similarly to [11], we obtain a stability condition that is scalable. However, the topological structure we consider is general, and the dynamics of each agent may be described by a different vector field. The proposed analysis does not exploit composite Lyapunov functions theory as in [12], where nonlinear systems with no disturbances are considered. As in [11] and [12] our approach relies on the small-gain theory for a network of

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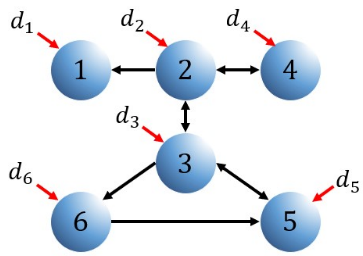

Fig. 1: Example of system described by (1): an heterogeneous large-scale system, where the interconnections can be represented by a weakly connected graph, subject to external disturbances.

interconnected agents. The controlled system obtained in [21] and [22], where the interconnection topology derives from the exploitation of macroscopic information in few aggregated variables, can be represented by the model presented in this paper, and its stability analysis can be derived as a special case of the one illustrated in the following sections.

Finally, we provide a numerical example representing a composition of interconnected electrical microgrids.

The paper is organized as follows: in Section II the class of considered interconnected systems is introduced as well as the definition of scalable Mesh Stability. In Section III, the main result about stability is presented, while Section IV describes a numerical example. In Section V, some conclusive remarks are provided.

Notation - $\mathbb{R}^+$ is the set of non-negative real numbers. For a vector $x \in \mathbb{R}^n$, $|x| = \sqrt{x^T x}$ is its Euclidean norm, $|x|_\infty = \max_{i=1\ldots n} |x_i|$ is its infinity norm and $|x(\cdot)|_\infty^{[t_0,t]} = \sup_{t_0 \le \tau \le t} |x(\tau)|$ is the $\mathcal{L}_\infty$ signal norm. Symbol $\lceil \cdot \rceil$ denotes the ceiling function, and $|| \cdot ||$ denotes the cardinality of the set in argument. We refer to [23] and [24] for the definition of Lyapunov functions, Input-to-State Stability (ISS), and class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions.

II. Preliminaries

Consider the large-scale system composed of N interconnected subsystems, with dynamics

$$\dot{x}_i = f_i\left(x_i, \{x_j\}_{j\in\mathcal{N}_i}, d_i\right), \quad i \in \mathcal{I}_N, \tag{1}$$

where $\mathcal{I}_N = \{1, ..., N\}$, $x_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$, is the state vector, $d_i \in \mathbb{R}^{m_i}$, $m_i \in \mathbb{N}$, is the disturbance, $\mathcal{N}_i \subseteq \mathcal{I}_N$ denotes the neighbors set, and $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_{i,1}} \cdots \times \mathbb{R}^{n_{i,||\mathcal{N}_i||}} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ is the vector field associated to the i-th subsystem (also called agent). Here, $n_{i,j}$ is the state dimension of the j-th neighbor of i. It is assumed that $f_i(0, 0, ..., 0, 0) = 0$, $\forall\, i \in \mathcal{I}_N$, and that the interconnected system (1) is forward complete, meaning that its solution exists for all initial states, and all disturbances d_i. Interconnection defined by (1) may be described by a directed graph $\mathcal{G} = (\mathcal{I}_N, E)$ (see Fig. 1)). The neighborhood of subsystem $i \in \mathcal{I}_N$ is defined as the set $\mathcal{N}_i = \{j \in \mathcal{I}_N \mid \exists\, (j, i) \in E\}$. For the problem not to be trivial, graph $\mathcal{G}$ is assumed to be weakly connected.

We recall that the subsystem $\dot{x}_i = f_i(x_i, \{x_j\}_{j\in\mathcal{N}_i}, d_i)$ is ISS if there exist $\beta_i \in \mathcal{KL}$, $\gamma_i \in \mathcal{K}_\infty$ and $\sigma_i \in \mathcal{K}_\infty$ such that

$$\begin{aligned} |x_i(t)| \le \beta_i\left(|x_i(t_0)|, t - t_0\right) + \gamma_i\left(\max_{j\in\mathcal{N}_i} |x_j(\cdot)|_\infty^{[t_0,t]}\right) \\ + \sigma_i\left(|d_i(\cdot)|_\infty^{[t_0,t]}\right), \ t \ge t_0 \ge 0. \end{aligned} \tag{2}$$

The following definition of scalable Mesh Stability (sMS) given for general topological interconnection, follows from the definition of scalable Input-to-State Stability (sISS) in [11]:

Definition 1. *The system (1) is said to be scalable Mesh Stable (sMS) if ISS property* (2) *of each subsystem i implies*

$$\max_{i\in\mathcal{I}_N} |x_i(t)| \le \beta\left(\max_{i\in\mathcal{I}_N} |x_i(t_0)|, t\right) + \sigma\left(\max_{i\in\mathcal{I}_N} |d_i(\cdot)|_\infty^{[t_0,t]}\right) \tag{3}$$

for some functions $\beta \in \mathcal{KL}$ and $\sigma \in \mathcal{K}_\infty$, for any initial condition $x_i(t_0) \in \mathbb{R}^n$ and any disturbance function $d_i(\cdot)$, $i \in \mathcal{I}_N$.

The definition above of sMS formalizes the desired property that the stability of a large-scale interconnected system, namely the trajectory bounds, does not dependent on the number of its subsystems. Ensuring the existence of such bounds guarantees that there is no amplification of perturbations through the network, due to disturbances acting on the agents or to the system being in a transient phase.

III. Main Results

In this section, we state sufficient conditions to be met locally in order to ensure stability of the overall interconnected system.

Theorem 1. *Assume that* (2) *holds for any $i \in \mathcal{I}_N$. If there exists $\tilde{\gamma} \in (0, 1)$ such that*

$$\gamma_i(s) \le \tilde{\gamma} s \tag{4}$$

for all $s \ge 0$ and $i \in \mathcal{I}_N$, then the interconnected system (1) is sMS.

Proof. The proof is divided into two main steps, similarly to [25] and [26]: first, a common bound for all trajectories $x_i(t)$ valid for all $t \ge t_0$ is derived, and then an estimation of (3) for the interconnected system is proven to exist.

Let us introduce $\hat{\beta}(r, s) = \max_{i\in\mathcal{I}_N} \beta_i(r, s)$ and $\hat{\sigma} = \max_i \sigma_i(r)$, $r, s \ge 0$. It is readily seen that $\hat{\beta} \in \mathcal{KL}$, $\hat{\sigma} \in \mathcal{K}_\infty$. Also, let $|x(\bar{t})|_\infty = \max_{i\in\mathcal{I}_N} |x_i(\bar{t})|$ and $d_\infty^{[t_1,t_2]} = \max_{i\in\mathcal{I}_N} |d_i(\cdot)|_\infty^{[t_1,t_2]}$. Since (4) holds, and $\hat{\sigma} \in \mathcal{K}_\infty$ implies $\hat{\sigma}\left(|d_i(\cdot)|_\infty^{[t_0,t]}\right) \le \hat{\sigma}\left(d_\infty^{[t_0,t]}\right)$, then

$$\begin{aligned} |x_i(\cdot)|_\infty^{[t_0,t]} \le \hat{\beta}\left(|x(t_0)|_\infty, 0\right) + \tilde{\gamma} \max_{j\in\mathcal{N}_i} |x_j(\cdot)|_\infty^{[t_0,t]} \\ + \hat{\sigma}\left(d_\infty^{[t_0,t]}\right). \end{aligned} \tag{5}$$

By applying the $\max$ operator on both sides of inequality (5), the following holds:

$$\max_{i\in\mathcal{I}_N}|x_i(\cdot)|_\infty^{[t_0,t]} \leq \hat{\beta}\left(|x(t_0)|_\infty, 0\right) + \tilde{\gamma}\max_{i\in\mathcal{I}_N}\left\{\max_{j\in\mathcal{N}_i}|x_j(\cdot)|_\infty^{[t_0,t]}\right\} + \hat{\sigma}\left(d_\infty^{[t_0,t]}\right) \leq \hat{\beta}\left(|x(t_0)|_\infty, 0\right) + \tilde{\gamma}\max_{i\in\mathcal{I}_N}|x_i(\cdot)|_\infty^{[t_0,t]} + \hat{\sigma}\left(d_\infty^{[t_0,t]}\right). \tag{6}$$

Grouping the term $\max_{i\in\mathcal{I}_N}|x_i(\cdot)|_\infty^{[t_0,t]}$ on the left side, we obtain

$$(1-\tilde{\gamma})\max_{i\in\mathcal{I}_N}|x_i(\cdot)|_\infty^{[t_0,t]} \leq \hat{\beta}(|x(t_0)|_\infty, 0) + \hat{\sigma}\left(d_\infty^{[t_0,t]}\right). \tag{7}$$

Then,

$$\max_{i\in\mathcal{I}_N}|x_i(\cdot)|_\infty^{[t_0,t]} \leq \frac{1}{1-\tilde{\gamma}}\left(\hat{\beta}(|x(t_0)|_\infty, 0) + \hat{\sigma}\left(d_\infty^{[t_0,t]}\right)\right), \tag{8}$$

where the last inequality holds because of the assumption $\tilde{\gamma} \in (0,1)$. Inequality (8) holds $\forall\ t \geq t_0$, and it does not depend on the i-th agent dynamics and on the system dimension. Thereby, it provides a common bound for all trajectories of the interconnected system. However, no information about system evolution over time is provided. For this reason, we look for an estimation of (3) for all the agents. As before, the result is obtained by an inductive method. Let us define $\hat{\beta}_0(r) = \frac{1}{1-\tilde{\gamma}}\hat{\beta}(r,0)$ and $\bar{\sigma}(s) = \frac{1}{1-\tilde{\gamma}}\hat{\sigma}(s)$, $r,s \geq 0$. Moreover, let be $\tau \in [t_0, t]$. Then, from (8) for each $i \in \mathcal{I}_N$ the following inequality holds:

$$|x_i(\tau)| \leq \hat{\beta}_0(|x(t_0)|_\infty) + \bar{\sigma}\left(d_\infty^{[t_0,t]}\right). \tag{9}$$

For simplicity, $\mu(r,s) = \hat{\beta}_0(r) + \bar{\sigma}(s)$ is introduced. By (2) with $t_0 = \tau$, and definition of $\tilde{\gamma}$, for each $i \in \mathcal{I}_N$ we get:

$$|x_i(t)| \leq \hat{\beta}(|x_i(\tau)|, t-\tau) + \tilde{\gamma}\max_{j\in\mathcal{N}_i}|x_j(\cdot)|_\infty^{[\tau,t]} + \hat{\sigma}\left(|d_i(\cdot)|_\infty^{[\tau,t]}\right). \tag{10}$$

Since $\hat{\sigma} \in \mathcal{K}_\infty$, then $\hat{\sigma}\left(|d_i(\cdot)|_\infty^{[\tau,t]}\right) \leq \hat{\sigma}\left(d_\infty^{[t_0,t]}\right)$.

Given $\bar{\omega} \in (0,1)$, for any $t > t_0 > 0$ define $M = \lceil(t-t_0)\rceil$, and $\tau_k = t_0 + \bar{\omega}^{M-k+1}(t-t_0)$. By construction, sequence $\{\tau_k\}_{k=0}^M$ is increasing and $\tau_k \in (t_0, t)$. As a consequence, from (10) with $\tau = \tau_M$ we get:

$$|x_i(t)| \leq \hat{\beta}\left(|x_i(\tau_M)|, t-\tau_M\right) + \tilde{\gamma}\max_{j\in\mathcal{N}_i}|x_j(\cdot)|_\infty^{[\tau_M,t]} + \hat{\sigma}\left(d_\infty^{[t_0,t]}\right), \tag{11}$$

where $t - \tau_M = (1-\bar{\omega})(t-t_0)$. At this point, a bound for $\max_{j\in\mathcal{N}_i}|x_j(\cdot)|_\infty^{[\tau_M,t]}$ needs to be derived. For this purpose, let us consider the trajectories at time instant τ_M with initial condition in $\tau = \tau_{M-1}$. Then, from (10), for each $j \in \mathcal{I}_N$ we get:

$$|x_j(\tau_M)| \leq \hat{\beta}\left(|x_j(\tau_{M-1})|, (1-\bar{\omega})\bar{\omega}(t-t_0)\right) + \tilde{\gamma}\max_{v\in\mathcal{N}_j}|x_v(\cdot)|_\infty^{[\tau_{M-1},\tau_M]} + \hat{\sigma}\left(d_\infty^{[t_0,t]}\right), \tag{12}$$

where $(1-\bar{\omega})\bar{\omega}(t-t_0) = \tau_M - \tau_{M-1}$. By recalling that $\max_{v\in\mathcal{N}_j}|x_v(\cdot)|_\infty^{[\tau_{M-1},\tau_M]} \leq \max_{v\in\mathcal{I}_N}|x_v(\cdot)|_\infty^{[\tau_{M-1},t]}$, we obtain:

$$|x_j(\cdot)|_\infty^{[\tau_M,t]} \leq \hat{\beta}\left(|x_j(\tau_{M-1})|, (1-\bar{\omega})\bar{\omega}(t-t_0)\right) + \tilde{\gamma}\max_{v\in\mathcal{I}_N}|x_v(\cdot)|_\infty^{[\tau_{M-1},t]} + \hat{\sigma}\left(d_\infty^{[t_0,t]}\right). \tag{13}$$

From (9), (11) and (13), the following holds

$$|x_i(t)| \leq \sum_{k=0}^{1}\tilde{\gamma}^k\hat{\beta}\left(\mu\left(|x(t_0)|_\infty, d_\infty^{[t_0,t]}\right), (1-\bar{\omega})\bar{\omega}^k(t-t_0)\right) + \sum_{k=0}^{1}\tilde{\gamma}^k\hat{\sigma}\left(d_\infty^{[t_0,t]}\right) + \tilde{\gamma}^2\max_{j\in\mathcal{I}_N}|x_j(\cdot)|_\infty^{[\tau_{M-1},t]}. \tag{14}$$

By repeating the previous steps for the entire time partition, we obtain

$$|x_i(t)| \leq \sum_{k=0}^{M}\tilde{\gamma}^k\hat{\beta}\left(\mu\left(|x(t_0)|_\infty, d_\infty^{[t_0,t]}\right), (1-\bar{\omega})\bar{\omega}^k(t-t_0)\right) + \sum_{k=0}^{M}\tilde{\gamma}^k\hat{\sigma}\left(d_\infty^{[t_0,t]}\right) + \tilde{\gamma}^{M+1}\max_{j\in\mathcal{I}_N}|x_j(\cdot)|_\infty^{[\tau_0,t]}. \tag{15}$$

As in [25], a $\mathcal{KL}$ function ϕ is defined as

$$\phi(r,s) = \sup_{\omega\in(0,1]}\omega^q\hat{\beta}(r,\omega s), \tag{16}$$

for some $q > 0$. Then, multiplying and dividing $\hat{\beta}$ in (15) for $\left((1-\bar{\omega})\bar{\omega}^k\right)^q$,

$$|x_i(t)| \leq \sum_{k=0}^{M}\frac{(\bar{\omega}^{-q}\tilde{\gamma})^k}{(1-\bar{\omega})^q}\phi\left(\mu\left(|x(t_0)|_\infty, d_\infty^{[t_0,t]}\right), t-t_0\right) + \sum_{k=0}^{M}\tilde{\gamma}^k\hat{\sigma}\left(d_\infty^{[t_0,t]}\right) + \tilde{\gamma}^{M+1}\mu\left(|x(t_0)|_\infty, d_\infty^{[t_0,t]}\right). \tag{17}$$

If parameter $\bar{\omega}$ is chosen such that $\tilde{\gamma} < \bar{\omega}^q < 1$,

$$|x_i(t)| \leq \frac{(1-\bar{\omega})^{-q}}{\bar{\omega}^q - \tilde{\gamma}}\phi\left(\mu\left(|x(t_0)|_\infty, d_\infty^{[t_0,t]}\right), t-t_0\right) + \bar{\sigma}\left(d_\infty^{[t_0,t]}\right) + \tilde{\gamma}^{M+1}\mu\left(|x(t_0)|_\infty, d_\infty^{[t_0,t]}\right). \tag{18}$$

From the definition of M and $\tilde{\gamma}$, $\tilde{\gamma}^{M+1}r = \tilde{\gamma}^{\lceil s\rceil+1}r < \tilde{\gamma}^s r$, where $\tilde{\gamma}^s r$ is $\mathcal{KL}$, $r,s \geq 0$. As a consequence, function $\bar{\beta}(r,s) = \frac{(1-\bar{\omega})^{-q}}{\bar{\omega}^q-\tilde{\gamma}}\phi(r,s) + \tilde{\gamma}^s r$ is $\mathcal{KL}$, and the following holds

$$|x_i(t)| \leq \bar{\beta}\left(\mu\left(|x(t_0)|_\infty, d_\infty^{[t_0,t]}\right), t-t_0\right) + \bar{\sigma}\left(d_\infty^{[t_0,t]}\right). \tag{19}$$

Since $\mu(r,s) = \hat{\beta}_0(r) + \bar{\sigma}(s)$, the presence of $\bar{\sigma}$ in the first argument of function $\bar{\beta}$ in (19) needs to be addressed. Each trajectory $x_i(t)$ satisfies both (8) and (19); therefore, we introduce the following function:

$$k(r,\bar{\sigma}(s), t-t_0) = \min\left\{\bar{\beta}(\hat{\beta}_0(r) + \bar{\sigma}(s), t-t_0), \hat{\beta}_0(r)\right\}. \tag{20}$$

As shown in equations (98), (99) and (100) in [26], for any $\alpha \in \mathcal{K}_\infty$

$$k(r,\bar{\sigma}(s), t-t_0) \leq \bar{\beta}(\hat{\beta}_0(r) + \alpha^{-1}(r), t-t_0) + \hat{\beta}_0 \circ \alpha(\bar{\sigma}(s)). \tag{21}$$

Then, by defining the class $\mathcal{KL}$ function $\beta(r,s) = \bar{\beta}(\hat{\beta}_0(r) + \alpha^{-1}(r), s)$, and the class $\mathcal{K}_\infty$ function $\sigma(r) = \bar{\sigma}(r) + \hat{\beta}_0 \circ \alpha(\bar{\sigma}(r))$, for each $i \in \mathcal{I}_N$ and for each $t \geq t_0$ we obtain

$$|x_i(t)| \leq \beta\left(\max_{j\in\mathcal{I}_N} |x_j(t_0)|, t - t_0\right) + \sigma\left(\max_{j\in\mathcal{I}_N} |d_j(\cdot)|_\infty^{[t_0,t]}\right). \tag{22}$$

Then, the sMS of (1) is proven. □

Theorem 1 ensures stability for the overall large-scale system described by a general interconnection graph if the sufficient condition in (4) is satisfied by each subsystem. As a consequence, stability of the overall system can be assessed through the analysis of local dynamics satisfying ISS-like trajectory bounds (2). Hence, condition (4) can be used as a local test to determine the sMS of the whole system, as illustrated in Section IV. Moreover, boundedness with respect to external disturbances is ensured.

Remark 1. *Since Theorem 1 holds for a general class of interconnected systems, the obtained result can be applied also to the analysis of vehicular platoons, composed by cascaded interconnected systems [21], [22].*

IV. Numerical Example

In this section a numerical example is presented to validate the theoretical results of Theorem 1. As power systems are naturally modeled as large-scale and highly nonlinear structured dynamical systems [27], [28], the case study we propose deals with stability analysis of a complex interconnection of microgrids, i.e. clusters of local power sources and loads [29].

We consider a network representing an aggregate of microgrids composed by two different kinds of nodes, with an asymmetric interconnection topology. We consider a two-levels large-scale network (see Fig. 2). The first level is composed by groups of interacting agents (microgrids), representing a local area (the blue nodes in Fig. 2) [30]. The second level is composed by the agents denoted by ρ_i (the red nodes in Fig. 2), $i \in \mathbb{N}$, each one interacting with its own local area and with the other ρ_j agents, $j \neq i$. Those second level nodes interconnect the other local areas, and represent supervisory controllers targeting power balance with respect to the power disturbances both from a local and a global point of view. For both agents' levels, an asymmetric topology is considered. We denote with $\mathcal{N}_{\rho_i} \subseteq \{i_j \,|\, j = 1, 2, ..., n_i\} \cup \{\rho_p \,|\, p = 1, 2, ..., r\}$ the neighborhood set of the i-th node ρ_i, with $i = 1, ..., r$. Symbol $n_i \in \mathbb{N}$ represents the number of first-level node associated to the i-th local area, while $r \in \mathbb{N}$ is the number of second-level nodes in the system. We define the neighborhood of each first-level node as $\mathcal{N}_{i_j} \subseteq \{i_k \,|\, k = 1, 2, ..., n_i\} \cup \{\rho_i\}$.

Remark 2. *Although in the proposed example we suppose that all the interconnection links are bidirectional, the theoretical framework allows more general interconnections topologies.*

We define the state of the first-level nodes as $x_{i_j} \in \mathbb{R}$, and the state of the second-level ones as $x_{\rho_i} \in \mathbb{R}$, where $i = 1, ..., r$ and $j = 1, ..., n_i$. The states $x_{i_j} \in \mathbb{R}$ represent power mismatch in the microgrid, and are affected by

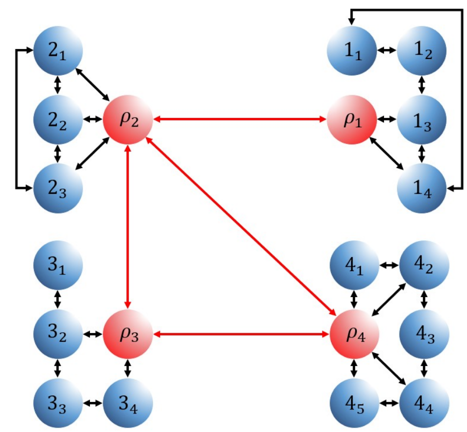

Fig. 2: The interconnected systems considered as case study: the blue nodes represent microgrids whose dynamics is described in terms of power exchanges in a local area, while red nodes describes supervisory controllers injecting/absorbing power with the target to ensure power balance with respect to both the local area and its neighborhood.

the power unbalance d_{i_j}. Consequently, the time derivative of x_{i_j} describes the power mismatch variation. The states $x_{\rho_i} \in \mathbb{R}$ describe supervisory controllers injecting/absorbing power with the target to ensure power balance with respect to both the local area and its neighborhood, and are affected by errors d_{ρ_i}. The associated dynamics are reported in the following:

$$\dot{x}_{i_j} = -3x_{i_j} + d_{i_j} + \sum_{k\in K} a_k^{i_j} x_k + a_{\rho_i}^{i_j} x_{\rho_i} \tag{23a}$$

$$\dot{x}_{\rho_i} = -3x_{\rho_i} + d_{\rho_i} + \left(\prod_{l\in L} b_l^{\rho_i} |x_l|\right)^{\frac{1}{||L||}} + \left(\prod_{h\in H} c_h^{\rho_i} |x_h|\right)^{\frac{1}{||H||}} \tag{23b}$$

where $K = \mathcal{N}_{i_j}/\{\rho_i\}$, $L = \mathcal{N}_{\rho_i} \cap \{i_j \mid j = 1, 2, ..., n_i\}$, $H = \mathcal{N}_{\rho_i} \cap \{\rho_p \,|\, p = 1, 2, ..., r\}$. We consider the coefficients $a_k^{i_j}, a_{\rho_i}^{i_j} \in \mathbb{R}$, $b_l^{\rho_i}, c_h^{\rho_i} \in \mathbb{R}^+$, and $d_{i_j}, d_{\rho_i} \in [\underline{d}, \bar{d}] \subset \mathbb{R}$, $\forall\, i_j, \rho_i$.

We now prove that the large-scale system under consideration satisfies the local condition (4) that is required for sMS. We choose the positive definite function $V_{i_j} = x_{i_j}^2/2$, the time derivative of which is as follows:

$$\begin{aligned}\dot{V}_{i_j} &= -3x_{i_j}^2 + x_{i_j} d_{i_j} + x_{i_j} \cdot \sum_{k\in K} a_k^{i_j} x_k + a_{\rho_i}^{i_j} x_{i_j} x_{\rho_i} \\ &\leq -3x_{i_j}^2 + |x_{i_j}| \cdot \left| d_{i_j} + \sum_{k\in\mathcal{N}_{i_j}} a_k^{i_j} x_k \right| \\ &\leq -\frac{5}{2} x_{i_j}^2 + \frac{1}{2}\left| |d_{i_j}| + \sum_{k\in\mathcal{N}_{i_j}} |a_k^{i_j}| \max_{k\in\mathcal{N}_{i_j}} |x_k| \right|^2, \end{aligned} \tag{24}$$

where we used the Young's inequality $|a||b| \leq (a^2 + b^2)/2$. Through straightforward calculations, we obtain the following state inequality for all $t \geq t_0$:

$$\begin{aligned}|x_{i_j}(t)| \leq \beta_{i_j}(|x_{i_j}(t_0)|, t) + \sigma_{i_j}\left(|d_{i_j}(\cdot)|_\infty^{[t_0,t]}\right) \\ + \tilde{\gamma}_{i_j} \max_{k\in\mathcal{N}_{i_j}} |x_k(\cdot)|_\infty^{[t_0,t]}, \end{aligned} \tag{25}$$

with

$$\beta_{i_j}(s,r) = e^{-\frac{5}{2}r}s, \;\; \sigma_{i_j}(s) = \sqrt{\frac{1}{5}}s,$$

and

$$\tilde{\gamma}_{i_j} = \sqrt{\frac{1}{5}}\sum_{k\in\mathcal{N}_{i_j}}|a_k^{i_j}|.$$

We proceed to prove similar inequalities for the system (23b). Let us consider again a quadratic function $V_{\rho_i} = x_{\rho_i}^2/2$ and its derivative

$$\begin{aligned}\dot{V}_{\rho_i} =& -3x_{\rho_i}^2 + x_{\rho_i}\left(d_{\rho_i} + \Psi_b + \Psi_c\right) \\ & \leq -3x_{\rho_i}^2 + |x_{\rho_i}|\left||d_{\rho_i}| + \Psi \max_{k\in\mathcal{N}_{\rho_i}}|x_k|\right| \\ & \leq -\frac{5}{2}x_{\rho_i}^2 + \frac{1}{2}\left||d_{\rho_i}| + \Psi \max_{k\in\mathcal{N}_{\rho_i}}|x_k|\right|^2, \end{aligned} \quad (26)$$

with $\Psi_b = \left(\prod_{l\in L} b_l^{\rho_i}|x_l|\right)^{\frac{1}{||L||}}$, $\Psi_c = \left(\prod_{h\in H} c_h^{\rho_i}|x_h|\right)^{\frac{1}{||H||}}$ and $\Psi = \left(\prod_{l\in L} b_l^{\rho_i}\right)^{\frac{1}{||L||}} + \left(\prod_{h\in H} c_h^{\rho_i}\right)^{\frac{1}{||H||}}$. As before, we obtain the following state trajectory inequality:

$$\begin{aligned}|x_{\rho_i}(t)| \leq \beta_{\rho_i}(|x_{\rho_i}(0)|,t) + \sigma_{\rho_i}\left(|d_{\rho_i}(\cdot)|_\infty^{[0,t]}\right) \\ + \tilde{\gamma}_{\rho_i}\max_{k\in\mathcal{N}_{\rho_i}}|x_k(\cdot)|_\infty^{[0,t]}, \end{aligned} \quad (27)$$

with

$$\beta_{\rho_i}(s,r) = e^{-\frac{5}{2}r}s, \;\; \sigma_{\rho_i}(s) = \sqrt{\frac{1}{5}}s,$$

and

$$\tilde{\gamma}_{\rho_i} = \sqrt{\frac{1}{5}}\Psi.$$

Inequality (27) is similar to (25), except for the interconnection term $\tilde{\gamma}_{i_j} \neq \tilde{\gamma}_{\rho_i}$. Thus, following Theorem 1, if coefficients $a_k^{i_j}, a_{\rho_i}^{i_j}, b_l^{\rho_i}, c_h^{\rho_i}$ are such that $\tilde{\gamma} = \max\left\{\tilde{\gamma}_{i_j}, \tilde{\gamma}_{\rho_i}\right\} \in (0,1)$, the stability of the overall system is ensured.

Simulations with respect to the system depicted in Fig. 2 are reported in Fig. 3. The simulation time is 20 s and it is split into two main phases: for $t \in [0,8]$ s the system is not affected by any disturbance and each agent has a non-zero initial condition, randomly generated between $[-4,4]$. In the second phase ($t \in [8,20]$), the set of nodes $\{1_1, 2_2, 4_3, 4_5, \rho_1, \rho_3, \rho_4\}$ is affected by sinusoidal disturbances $d_p(t) = A_p\sin(\omega_p t + \phi_p)$, whose parameters are chosen in a random way: $A_p \in [-3,3]$, $\omega \in [1,4]$ and $\phi_p \in [0,\pi]$. Since the coefficients $a_k^{i_j}, a_{\rho_i}^{i_j}, b_l^{\rho_i}, c_h^{\rho_i}$ are such that the maximum interconnection parameter is $\tilde{\gamma} \approx 0.98$ (see Table I), the results of Theorem 1 apply.

TABLE I: Interconnection values $\tilde{\gamma}$

	$\min\tilde{\gamma}$	$\max\tilde{\gamma}$
Group #1	0.1342	0.9839
Group #2	0.1968	0.6708
Group #3	0.0358	0.5241
Group #4	0.4651	0.8721
ρ_i	0.6872	0.9214

Fig. 3a shows the trajectories of the overall system, while Fig. 3b refers to the sub-group identified by the node ρ_1 and

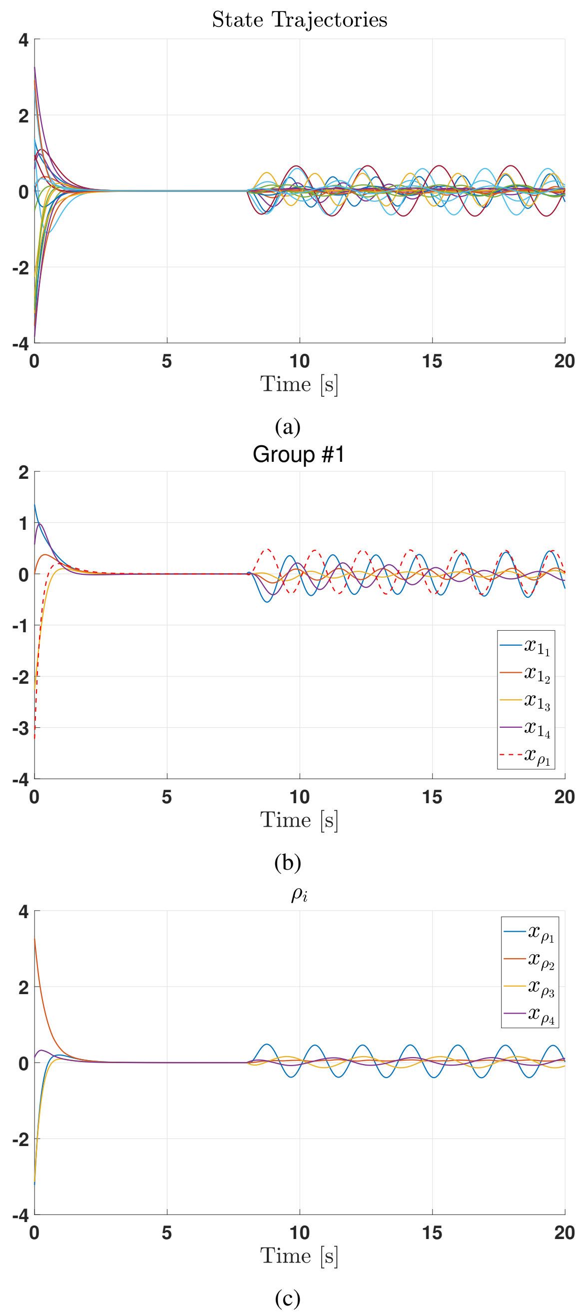

Fig. 3: Numerical simulations corresponding to the example in Fig. 2. In (a) are shown the overall system trajectories, in (b) are reported the trajectories of the subsystem corresponding to $i = 1$ and (c) are the trajectories of the ρ_i nodes.

Fig. 3c reports the trajectories of the ρ_i nodes. In the first phase, the simulation shows an asymptotic stable behavior for the system. In particular, the perturbations due to non-zero initial conditions vanish. From the interconnected-system point of view, this means that such perturbations do not amplify through the network leading to instability. Similarly, in the second phase where the nodes are affected by disturbances $d_p(t)$, the perturbations due to them are shown not to intensify along the network. As it is possible to see in Fig. 3b, despite the disturbances act only on nodes 1_1 and ρ_1, the other trajectories exhibit as well a sinusoidal behaviour. However, such perturbations are attenuated and not amplified. A similar behavior is shown in Fig. 3c, where the variables x_{ρ_i} are depicted.

V. Conclusions

In this paper we introduce the scalable Mesh Stability notion for interconnected systems with general topology. We analyze sufficient conditions for the stability of generic large-scale interconnected systems, with a non-predefined structure and subject to external disturbances. The derived sufficient conditions lead to a novel test exploiting the Input-to-State Stability concepts, that, if verified locally by each agent, ensures the overall system's stability. Moreover, the stability result is scalable with respect to the number of agents composing the system; indeed, the state trajectories are bounded in norm by functions that do not depend on the number of agents. The proposed theoretical framework is used to analyze the stability of a network of microgrids, and simulation results are illustrated.

Future work will focus on the analysis of non-autonomous large-scale systems, showing different and complex dynamics, where the sufficient conditions proposed in this paper have to be ensured by means of a controller.

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