# Data-driven Meets Geometric Control: Zero Dynamics, Subspace Stabilization, and Malicious Attacks

Federico Celi and Fabio Pasqualetti

Abstract—Studying structural properties of linear dynamical systems through invariant subspaces is one of the key contributions of the geometric approach to system theory. In general, a model of the dynamics is required in order to compute the invariant subspaces of interest. In this paper we overcome this limitation by finding data-driven formulas for some of the foundational tools of geometric control. In particular, for an unknown linear system, we show how controlled and conditioned invariant subspaces can be found directly from experimental data. We use our formulas and approach to (i) find a feedback gain that confines the system state within a desired subspace, (ii) compute the invariant zeros of the unknown system, and (iii) design attacks that remain undetectable.

#### I. INTRODUCTION

The geometric approach is a collection of notions and algorithms for the analysis and control of dynamical systems. Differently from the classic methods in the frequency and state space domains [1], [2], the geometric approach offers an intuitive and coordinate-free analysis of the properties of dynamical systems in terms of appropriately defined subspaces, and synthesis algorithms based on subspace operations, such as sum, intersection, and orthogonal complementation. The geometric approach has been successfully used to solve a variety of complex control and estimation problems; we refer the interested reader to [3]–[5] for a detailed treatment of the main geometric control notions and their applications.

Similarly to the frequency and state-space approaches to control, the geometric approach assumes an accurate, in fact exact, representation of the system dynamics. To overcome this limitation and in response to an ever-increasing availability of sensors, historical data, and machine learning algorithms, the behavioral approach, and more generally a data-driven approach, has seen a rapid increase in popularity. Here, system analysis and control synthesis do not require a model of the dynamics and are instead obtained directly from experimental data reflecting the system dynamics [6].

While analysis, control and estimation problems can often be solved equivalently using different methods, the frequency, state-space, geometric, and data-driven approaches all offer complementary insights into the structure and properties of the system dynamics, and together contribute to forming a comprehensive theory of systems. In this paper we create the first connections between the geometric and datadriven approach to system analysis and control. In particular,

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we derive data-driven expressions of the fundamental sets used in the geometric approach to solve a variety of control and estimation problems, and show how these sets have an even more insightful and straightforward interpretation when analyzed in the higher-dimensional data space as compared to their geometric view in the lower-dimensional state space.

Related work. From the seminal work [7] that introduced the notions of controlled and conditioned invariants, the geometric approach to control has evolved over the last decades into a full theory and a set of algorithms for linear [3]–[5] and nonlinear [8] systems. Notable applications of the geometric approach are the disturbance decoupling [9] and fault detection [10] problems, the characterization of stealthy attacks in cyber-physical systems [11], and the secure state estimation problem [12]. In this paper we follow the notation and techniques of [3], which we briefly recall in Section II.

The data-driven approach to system analysis and control is receiving renewed and increased interest. While traditional indirect data-driven methods use data to identify a model of the system [13] and proceed to synthesize a controller in a second step, direct data-driven methods bypass (at least apparently [14]–[16]) the identification step and design control actions directly from data. In this framework, recent results tackle various problems for linear systems, including optimal [17], [18], robust [19], [20] and distributed [21]–[24] control, as well as unknown-input estimation [25]. We refer the reader to [26] for a recent survey on data-driven control.

Main contributions of this paper. The main contributions of this paper are as follows. First, for the linear, discrete, time-invariant systems described by the triple (A, B, C), we derive explicit, closed-form data-driven expressions of (i)  $\mathcal{V}^*$ , the largest  $(A, \operatorname{Im}(B))$ -controlled invariant subspace contained in  $\mathrm{Ker}(C)$ , (ii)  $\mathcal{S}^*$ , the smallest  $(A,\mathrm{Ker}(C))$ conditioned invariant subspace containing Im(B), (iii) the feedback gain F such that  $(A + BF)V^* \subseteq V^*$ , and (iv) the invariant zeros of (A, B, C). Since  $\mathcal{V}^*$  and  $\mathcal{S}^*$  are the basis of the geometric approach developed in [3], our data-driven formulas constitute the basis of a data-driven and modelfree theory of geometric control, and can be used to solve a variety of analysis, estimation, and control problems in a purely data-driven setting. Second, our results show that the fundamental invariant subspaces of the geometric approach, which are often computed recursively when operating in the state space, have a simple and direct interpretation in the higher-dimensional data space, where they can be computed by solving appropriately defined sets of linear equations. Third, we demonstrate the utility of our formulas to design undetectable data-driven attacks in a consensus system.

**Paper organization.** Section II contains our problem setup and some preliminary notions. Section III contains our datadriven formulas of the fundamental invariant subspaces of the geometric approach. Finally, Sections IV and V contain our illustrative examples and conclusion, respectively.

**Notation.** The set of real numbers is denoted with  $\mathbb{R}$ . The rank, range space, null space, transpose, and Moore-Penrose pseudoinverse of the matrix  $A \in \mathbb{R}^{n \times m}$  are denoted with  $\operatorname{rank}(A)$ ,  $\operatorname{Im}(A)$ ,  $\operatorname{Ker}(A)$ ,  $A^{\top}$ , and  $A^{\dagger}$  respectively. A basis of the subspace  $\mathcal{V}\subseteq\mathbb{R}^n$  is denoted with  $\mathrm{Basis}(\mathcal{V})$ . The Kronecker product between matrices A and B is denoted by  $A \otimes B$  and is defined as in [27].

#### II. PROBLEM SETUP AND PRELIMINARY NOTIONS

We consider the discrete-time linear time-invariant system

$$x(t+1) = Ax(t) + Bu(t)$$
 (1a)

$$y(t) = Cx(t) \tag{1b}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^p$  are the state, input and output vectors, respectively, and (A, B, C) are constant matrices of appropriate dimensions. For any horizon T > 1, the state and output trajectories of (1) can be written as

$$\underbrace{\begin{bmatrix} x_{(1)}^{(1)} \\ x_{(2)}^{(2)} \\ \vdots \\ x_{(T)} \end{bmatrix}}_{X_T} = \underbrace{\begin{bmatrix} A^2 \\ A^2 \\ \vdots \\ A^T \end{bmatrix}}_{O_T^T} x(0) + \underbrace{\begin{bmatrix} B & \cdots & 0 & 0 \\ AB & \cdots & 0 & 0 \\ \vdots \\ A^{T-1}B & \cdots & AB & B \end{bmatrix}}_{AB & \cdots & AB & B} \underbrace{\begin{bmatrix} u_{(0)} \\ u_{(1)} \\ \vdots \\ u_{(T-1)} \end{bmatrix}}_{U_T}, \quad (2)$$

and

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(T-1) \end{bmatrix}}_{Y_T} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix}}_{O_T^Y} x(0) + \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 0 \\ CB & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{T-2}B & \cdots & CB & 0 \end{bmatrix}}_{F_T^Y} U_T. (3)$$

Throughout the paper, we assume that the system matrices are not known and base our approach on a set of prerecorded trajectories obtained by arbitrarily probing the system (1).

### A. Data collection

The available data is collected from a set of N openloop control experiments with horizon T, and consist of the state and output trajectories obtained from (1) with initial condition  $x_0^i$  and control sequence  $U_T^i$ , for  $i \in \{1, \dots, N\}$ . In particular, the following data matrices are available:

$$X = \begin{bmatrix} X_T^1 & \cdots & X_T^N \end{bmatrix} \in \mathbb{R}^{nT \times N}, \tag{4a}$$

$$\begin{split} X &= \begin{bmatrix} X_T^1 & \cdots & X_T^N \end{bmatrix} \in \mathbb{R}^{nT \times N}, & \text{(4a)} \\ X_0 &= \begin{bmatrix} x_0^1 & \cdots & x_0^N \end{bmatrix} \in \mathbb{R}^{n \times N}, & \text{(4b)} \\ Y &= \begin{bmatrix} Y_T^1 & \cdots & Y_T^N \end{bmatrix} \in \mathbb{R}^{pT \times N}, & \text{(4c)} \end{split}$$

$$Y = \begin{bmatrix} Y_T^1 & \cdots & Y_T^N \end{bmatrix} \in \mathbb{R}^{pT \times N}, \tag{4c}$$

$$U = \begin{bmatrix} U_T^1 & \cdots & U_T^N \end{bmatrix} \in \mathbb{R}^{mT \times N}. \tag{4d}$$

From (2)-(3), we note the following relationships:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} O_T^X & F_T^X \\ O_T^Y & F_T^Y \end{bmatrix} \begin{bmatrix} X_0 \\ U \end{bmatrix}.$$
 (5)

We make the following assumption of persistently-exciting experimental inputs, which is generically satisfied by choosing the inputs and initial states independently and randomly.

Assumption 2.1: The experimental inputs and initial conditions are persistently exciting, that is,

$$\operatorname{rank} \begin{bmatrix} X_0 \\ U \end{bmatrix} = n + mT. \tag{6}$$

Let  $K_0 = \operatorname{Basis}(\operatorname{Ker}(X_0))$  and  $K_U = \operatorname{Basis}(\operatorname{Ker}(U))$ . From the Rank-nullity Theorem, Assumption 2.1 ensures that  $X_0K_U$  and  $UK_0$  are full-row rank, respectively. Assumption 2.1 is a standard assumption in data driven studies [19], [28].

Remark 1: (Alternative data-driven representations) Different data formats can be used to obtain a non-parametric data-driven representation of the system (1), including our representation (4) as well as Hankel and Page matrices [19], [28]. While Hankel and Page matrices are generated from a single controlled trajectory, the matrices in (4) use a collection of (possibly shorter) controlled trajectories. Different data collections can be more convenient for the solution of different problems, with, currently, Hankel and Page matrices being used mostly for feedback control problems [19] and multiple trajectories for robustness problems [17], [29].

#### B. Controlled and conditioned invariant subspaces

The notions of controlled and conditioned invariant subspaces are the basis of the geometric approach for the analysis and control of linear systems [7]. We now recall their definition and basic properties. We refer the interested reader to [3]–[5] for a detailed treatment of this subject.

Definition 1: ((A, B)-controlled invariant) Given a matrix  $A \in \mathbb{R}^{n \times n}$  and a subspace  $\mathcal{B} \subseteq \mathbb{R}^n$ , a subspace  $\mathcal{V} \subseteq \mathbb{R}^n$ is an  $(A, \mathcal{B})$ -controlled invariant subspace if

$$A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}.\tag{7}$$

When  $\mathcal{B} = \operatorname{Im}(B)$ , the notion of a controlled invariance refers to the possibility of confining the state trajectory of the system (1) within a subspace. Specifically, the subspace  $\mathcal V$ is an  $(A, \operatorname{Im}(B))$ -controlled invariant subspace if, for every initial state in V, there exists a control input such that the state belongs to  $\mathcal{V}$  at all times. Of particular interest is  $\mathcal{V}^*$ , the largest (A, Im(B))-controlled invariant subspace contained in Ker(C). The subspace  $\mathcal{V}^*$  contains all trajectories of (1) that generate an identically zero output. Hence, the subspace  $\mathcal{V}^*$  vanishes if and only if the system (1) features no invariant zeros, a notion that is at the basis of the analysis of stealthy attacks and unknown-input observers [11], among others.

Definition 2: ((A, C)-conditioned invariant) Given a matrix  $A \in \mathbb{R}^{n \times n}$  and a subspace  $\mathcal{C} \subseteq \mathbb{R}^n$ , a subspace  $\mathcal{S} \subseteq \mathbb{R}^n$ is an  $(A, \mathcal{C})$ -conditioned invariant subspace if

$$A(\mathcal{S} \cap \mathcal{C}) \subseteq \mathcal{S}. \tag{8}$$

When C = Ker(C), the notion of conditioned invariance arises in the context of state estimation. Specifically, the subspace S is an (A, Ker(C))-conditioned invariant subspace if it is possible to design an (asymptotic) observer

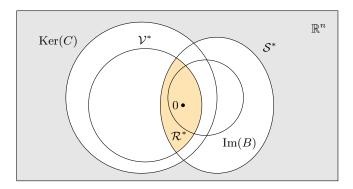


Fig. 1. This figure shows the inclusion relationships between the fundamental controlled and conditioned invariants defined in Section II. Notice how, by their definitions,  $\mathcal{V}^* \subseteq \operatorname{Ker}(C)$ ,  $\operatorname{Im}(B) \subseteq \mathcal{S}^*$  and  $\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{S}^*$ , highlighted in a different gradient.

that reconstructs the state  $x \setminus S$  by processing the initial condition, the input, and the measurements of the system (1). Of particular interest is  $S^*$ , the smallest  $(A, \operatorname{Ker}(C))$ -conditioned invariant subspace containing  $\operatorname{Im}(B)$ . In fact, the orthogonal complement of the subspace  $S^*$  is the largest subspace of the state space that can be estimated through a dynamic observer in the presence of an unknown input.

The subspaces  $\mathcal{V}^*$  and  $\mathcal{S}^*$  can be conveniently computed using simple recursive algorithms [3]. Further, these subspaces can be used to characterize important properties of the system (1). For instance, the system (1) is right invertible if and only if  $\mathcal{V}^* \cup \mathcal{S}^* = \mathbb{R}^n$ , and left invertible if and only if the subspace  $\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{S}^*$  is empty [3]. It should be noticed that  $\mathcal{R}^*$  coincides with the largest subspace that can be reached from the origin with trajectories that belong to  $\mathcal{V}^*$  at all times (hence, generating an identically zero output).

The definition of the subspaces  $\mathcal{V}^*$ ,  $\mathcal{S}^*$  and  $\mathcal{R}^*$ , as well as the algorithms to compute them, assume the exact knowledge of the system matrices. Instead, in the remainder of the paper we derive purely data-driven expressions of these subspaces, which also offers an alternative interpretation of them. Similarly to how  $\mathcal{V}^*$ ,  $\mathcal{S}^*$  and  $\mathcal{R}^*$  are used in the geometric approach, our data-driven formulas can also be used to solve a variety of estimation and control problems.

#### III. DATA-DRIVEN GEOMETRIC CONTROL

We begin with finding a data-driven expression of the subspace  $\mathcal{V}^*$  for the system (1), the largest  $(A, \operatorname{Im}(B))$ -controlled invariant subspace contained in  $\operatorname{Ker}(C)$ .

Theorem 3.1: (Data driven formula for  $V^*$ ) Let (4) be the data generated by the system (1) with  $T \ge n$ . Then,

$$\mathcal{V}^* = \begin{bmatrix} X_0 K_U & 0 \end{bmatrix} \operatorname{Ker} \begin{bmatrix} Y K_U & Y K_0 \end{bmatrix}. \tag{9}$$

To prove Theorem 3.1, recall that  $\mathcal{V}^*$  is the set of initial states for which there exists a control input such that the resulting state trajectory generates an identically zero output. Since the system is linear, under our assumption of persistently exciting experimental inputs, any system trajectory can

be expressed as an appropriate linear combination of the experimental trajectories. We next formalize this intuition.

Lemma 3.2: (Data-driven trajectories of (1)) Let (4) be the data generated by the system (1) with  $T \geq n$ . Let  $\bar{X}_T$  and  $\bar{Y}_T$  be the state and output trajectories of (1) generated with some initial condition and control input. Then,

$$\begin{bmatrix} \bar{X}_T \\ \bar{Y}_T \end{bmatrix} = \begin{bmatrix} XK_U & XK_0 \\ YK_U & YK_0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \tag{10}$$

for some vectors  $\alpha$  and  $\beta$ .

*Proof:* Let  $\bar{x}_0$  and  $\bar{U}_T$  be the initial condition and input to (1). Since the matrices  $X_0K_U$  and  $UK_0$  are full-row rank (see Assumption 2.1), there exists  $\alpha$  and  $\beta$  such that

$$\bar{x}_0 = X_0 K_U \alpha$$
 and  $\bar{U}_T = U K_0 \beta$ . (11)

From (2) we have

$$\bar{X}_T = O_T^X \bar{x}_0 + F_T^X \bar{U}_T = O_T^X X_0 K_U \alpha + F_T^X U K_0 \beta$$
$$= X K_U \alpha + X K_0 \beta,$$

where the last equality follows from (5). Similarly from (3),

$$\bar{Y}_T = O_T^Y \bar{x}_0 + F_T^Y \bar{U}_T = O_T^Y X_0 K_U \alpha + F_T^Y U K_0 \beta$$
$$= Y K_U \alpha + Y K_0 \beta,$$

which concludes the proof.

Lemma 3.2 shows how any state and output trajectory of (1) can be written as a linear combination of the available data. In particular, state and output trajectories are obtained in (10) as the sum of the free and forced responses, which are reconstructed from data of arbitrary control experiments. In fact,  $XK_U\alpha$  is the state trajectory of (1) with initial condition  $X_0K_u\alpha$  and zero input (free response), while  $XK_0\beta$  is the state trajectory of (1) with zero initial condition and input  $UK_0\beta$  (forced response). We remark that Assumption 2.1 of persistently exciting inputs is necessary to obtain this result.

The following instrumental Lemma shows that it is sufficient to consider trajectories of any finite length  $T \geq n$  to compute  $\mathcal{V}^*$ , and is instrumental to the proof of Theorem 3.1.

Lemma 3.3: (Computing  $V^*$  from trajectories of finite length) For the system (1), any initial state  $x_0$ , and any finite horizon  $T \ge n$ , the following statements are equivalent:

- (i)  $x_0 \in \mathcal{V}^*$ ;
- (ii) there exists an input sequence  $u(0),\ldots,u(T-1)$  such that y(t)=0 for all  $t\in\{0,\ldots,T-1\}$ . Proof:
- (i)  $\Rightarrow$  (ii) Follows from the definition of  $\mathcal{V}^*$ .
- (ii)  $\Rightarrow$  (i) Notice that y(T-1) = Cx(T-1) = 0. Thus,  $x(T-1) \in \text{Ker}(C) = \mathcal{V}_0$ . Similarly, x(T-2) satisfies

$$x(T-1) = Ax(T-2) + Bu(T-2)$$
, and  $y(T-2) = Cx(T-2) = 0$ .

This implies that

$$x(T-2) \in A^{-1}(x(T-1) - Bu(T-2))$$
  
$$\subseteq A^{-1}(\mathcal{V}_0 + \operatorname{Im}(B)) \cap \operatorname{Ker}(C) = \mathcal{V}_1$$

Iterating this procedure yields

$$x(T-1) \in \mathcal{V}_0 = \text{Ker}(C), \text{ and}$$
 (12a)

$$x(T-i) \in \mathcal{V}_i = A^{-1}(V_{i-1} + \text{Im}(B)) \cap \text{Ker}(C).$$
 (12b)

Since  $V_i$  converges to  $V^*$  is at most n steps [7], we have that  $x(T-\tau) \in V^*$  for all  $\tau \geq n$ , which concludes the proof.

We now prove Theorem 3.1 using Lemma 3.2 and 3.3.

*Proof of Theorem 3.1:* From Lemma 3.3 we seek all initial conditions  $x_0$  for which the output can be maintained at zero for  $T \ge n$  steps. From (10), the vectors  $\alpha$  and  $\beta$  that identify state trajectories with identically zero output must satisfy

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \operatorname{Ker} \begin{bmatrix} YK_U & YK_0 \end{bmatrix}. \tag{13}$$

The initial condition corresponding to such trajectories is  $x_0 = X_0 K_U \alpha$  (see (11)). Thus, the set  $\mathcal{V}^*$  can be written as

$$\mathcal{V}^* = \begin{bmatrix} X_0 K_U & 0 \end{bmatrix} \operatorname{Ker} \begin{bmatrix} Y K_U & Y K_0 \end{bmatrix}.$$

We next find a data-driven expression for  $S^*$ , the smallest (A, Ker(C)) conditioned invariant containing Im(B).

Theorem 3.4: (Data driven formula for  $S^*$ ) Let (4) be the data generated by the system (1) with  $T \ge n$ . Then,

$$S^* = HXK_0 \operatorname{Ker}(YK_0), \tag{14}$$

where 
$$H = \underbrace{\begin{bmatrix} 0_n & \cdots & 0_n & I_n \end{bmatrix}}_{T \text{ matrices}}$$
.

To prove Theorem 3.4, we first show that, similarly to the case of  $\mathcal{V}^*$ , the subspace  $\mathcal{S}^*$  can be computed from a collection of trajectories of finite length T > n.

Lemma 3.5: (Computing  $S^*$  from trajectories of finite length) For the system (1) and any finite horizon  $T \ge n$ , the following statements are equivalent:

- (i)  $x(T) \in \mathcal{S}^*$ ;
- (ii) there exists an input sequence  $u(0), \ldots, u(T-1)$  such that y(t) = 0 for all  $t \in \{0, \ldots, T-1\}$  and x(0) = 0. *Proof:*
- (i)  $\Rightarrow$  (ii) Follows from the definition of  $\mathcal{S}^*$ . For example, with u(t)=0, for  $t\in\{0,\ldots,T-2\}$  and  $u(T-1)\neq 0$ . Then  $x(T)=Ax(T-1)+Bu(T-1)=Bu(t-1)\in \mathrm{Im}(B)\subset \mathcal{S}^*$ .
- (ii)  $\Rightarrow$  (i) Because x(1) = Bu(0) and y(1) = Cx(1) = 0, we have  $x(1) \in \text{Im}(B) \cap \text{Ker}(C) = \mathcal{S}_1 \cap \text{Ker}(C)$ . Similarly,

$$x(2) \in A(\mathcal{S}_1 \cap \operatorname{Ker}(C)) + \operatorname{Im}(B) = \mathcal{S}_2,$$

and  $x(2) \in \text{Ker}(C)$  since y(2) = Cx(2) = 0. Recursively:

$$x(1) \in \mathcal{S}_1 = \operatorname{Im}(B), \text{ and}$$
 (15a)

$$x(i) \in \mathcal{S}_i = A(\mathcal{S}_{i-1} \cap \text{Ker}(C)) + \text{Im}(B).$$
 (15b)

Since  $S_i$  converges to  $S^*$  in at most n steps [7], we have that  $x(\tau) \in S^*$  for all  $\tau \geq n$ , which concludes the proof.

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4: From (10), when x(0) = 0, any state trajectory of length T that generates an identically zero output of length T can be parametrized with  $\alpha = 0$  and  $\beta \in \text{Ker}(YK_0)$ . Using Lemma 3.5, the set  $\mathcal{S}^*$  can be equivalently

written as the final states reached by such trajectories, that is,  $S^* = HXK_0 \operatorname{Ker}(YK_0)$ , which concludes the proof.

Remark 2: (Obtaining  $\mathcal{R}^*$  from  $\mathcal{V}^*$  and  $\mathcal{S}^*$ ) The combined knowledge of  $\mathcal{V}^*$  and  $\mathcal{S}^*$  allows us to find  $\mathcal{R}^*$  as [3]

$$\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{S}^*. \tag{16}$$

Alternatively, one can also find explicit data-driven expression for  $\mathcal{R}^*$  using trajectories of finite-length. For example, one can show that when the condition  $u(0) \neq 0$  is imposed in Lemma 3.5, then  $x(T) \in \mathcal{R}^* \subseteq \mathcal{S}^*$ . We omit the proof of this result, and use (16) to directly compute  $\mathcal{R}^*$ .

The state of a system can be confined within a subspace  $\mathcal V$  through a state-feedback controller if and only if  $\mathcal V$  is a controlled invariant subspace. We continue this section with the data-driven design of such state-feedback controller, that is, the data-driven design of a matrix F such that

$$(A+BF)\mathcal{V}\subseteq\mathcal{V}.\tag{17}$$

For a trajectory  $X_T$  and input  $U_T$ , let

$$X_{0,T} = [x(1) \quad x(2) \quad \cdots \quad x(T-1)],$$
 (18a)

$$X_{1,T} = [x(2) \ x(3) \ \cdots \ x(T)], \text{ and}$$
 (18b)

$$U_{0,T} = [u(0) \quad u(1) \quad \cdots \quad u(T-1)].$$
 (18c)

Theorem 3.6: (Data-driven feedback for invariant subspace) Let  $X_T$  be the trajectory of (1) with input  $U_T$  and some initial condition. Let  $\mathcal{V} = \operatorname{Im}(V)$  be an  $(A, \operatorname{Im}(B))$ -controlled invariant subspace, and let

$$F = U_{0,T}(X_{0,T}^{\dagger} + K\gamma), \tag{19}$$

with  $K = Ker(X_{0:T})$  and

$$\gamma = -((I - VV^{\dagger})X_{1,T}K)^{\dagger}(I - VV^{\dagger})X_{1,T}X_{0,T}^{\dagger}VV^{\dagger}.$$

If  $[U_{0,T}^{\top} X_{0,T}^{\top}]^{\top}$  is full row rank,<sup>2</sup> then  $(A+BF)\mathcal{V}\subseteq\mathcal{V}$ . *Proof:* From [19, Theorem 2], for any state-feedback gain F, the closed loop matrix can be written as

$$A + BF = X_{1,T}G$$

where the matrix G satisfies  $X_{0,T}G = I$  and  $U_{0,T}G = F$ . Further, F renders the subspace  $\mathcal{V}$  invariant if and only if

$$(A + BF)\mathcal{V} = X_{1,T}G\mathcal{V} \subseteq \mathcal{V},$$

or, equivalently,

$$(I - VV^{\dagger})X_{1,T}GV = 0,$$

where  $V=\operatorname{Basis}(\mathcal{V})$  and  $(I-VV^\dagger)$  is a projector onto  $\mathcal{V}^\perp$ . From  $X_{0,T}G=I$  we obtain  $G=X_{0,T}^\dagger+K\gamma$ , where  $\gamma$  is any matrix of

$$(I - VV^{\dagger})X_{1,T}(X_{0,T}^{\dagger} + K\gamma)V = 0.$$

Solving for  $\gamma$  (a solution  $\gamma$  exists because  $\mathcal V$  is an (A,B)-controlled invariant subspace and  $[U_{0,T}^\top X_{0,T}^\top]^\top$  is full-row rank) and using  $U_{0,T}G=F$  concludes the proof.

<sup>&</sup>lt;sup>1</sup>We use  $I_n \in \mathbb{R}^{n \times n}$  and  $0_n \in \mathbb{R}^{n \times n}$  to denote the identity and zero matrices of appropriate dimensions, respectively.

<sup>&</sup>lt;sup>2</sup>This condition requires the trajectory to be sufficiently informative and is related to the notion of persistency of excitation [19], [28], [30].

Theorem 3.6 details the computation of a feedback matrix that renders a subspace invariant, from sufficiently informative state and input trajectories. It should be noticed that Theorem 3.6 does not guarantee the internal, nor external, stability of the subspace, which imposes additional constraints on  $\gamma$ . This is left as a topic of future investigation.

To conclude this section we present a strategy to identify the invariant zeros of (1) from data. We make the assumption that (1) is *non-degenerate*, i.e.,  $\mathcal{R}^*$  is empty. Degenerate systems are intrinsically vulnerable to, e.g., undetectable malicious attacks with unstable state trajectories. On the other hand, for non-degenerate systems, the existence of unstable invisible trajectories depends on the modulo of its invariant zeros. In fact, the knowledge of the number and magnitude of the invariant zeros of a non-degenerate system is essential when studying problems such as noninteracting control [7] and malicious attack detection [11], motivating our interest in their identification.

Theorem 3.7: (Data-driven invariant zeros) Let X and  $\mathcal{V}^*$  be as in (4a) and (9), respectively, with  $T \geq n$ . Let  $V = \operatorname{Im}(\mathcal{V}^*)$  and assume that  $\mathcal{R}^* = \emptyset$ . Then,  $z \in \mathbb{C}$  is an invariant zero of (1) if and only if the matrix

$$\begin{bmatrix} XX^{\dagger}(I\otimes V) & -\left([z\ z^2\ \cdots\ z^T]\otimes I\right)^{\top} \end{bmatrix} \quad (21)$$

has a nontrivial kernel.

*Proof:* When  $\mathcal{V}^* \neq \emptyset$  and  $\mathcal{R}^* = \emptyset$ , there exists a trajectory  $x(t) = z^t x(0)$ , with  $x(t) \in \mathcal{V}^*$  for all  $t \geq 0$  and z an invariant zero of (1) [3]. We write such trajectory as

$$X_T^V = \begin{bmatrix} zI\\z^2I\\\vdots\\z^TI \end{bmatrix} v = ([z\ z^2\ \cdots\ z^T] \otimes I)^\top v. \tag{22}$$

With Assumption 2.1, any trajectory belongs to the image of the data matrix X. Then, when the trajectory  $X_T^V$  above exists, there also exists a vector  $\bar{w} \in X^\dagger(I \otimes V)$  such that  $X\bar{w} = X_T^V$ . The condition on  $\bar{w}$  imposes that the trajectory is compatible with (1) while evolving inside  $\mathcal{V}^*$ . Both vectors  $v \neq 0$  and  $\bar{w} = X^\dagger(I \otimes V)w \neq 0$  exist if and only if

$$XX^{\dagger}(I \otimes V)w = ([z \ z^2 \ \cdots \ z^T] \otimes I)^{\top}v$$
 (23)

i.e., the kernel of  $[XX^{\dagger}(I \otimes V) - ([z \ z^2 \ \cdots \ z^T] \otimes I)^{\top}]$  is non-empty, concluding the proof.

The invariant zeros of the system (1) can be equivalently characterized using data collected as in (18).

Lemma 3.8: (Data-driven invariant zeros) Let  $\mathcal{V}^*$  be as in (9) and assume that  $\mathcal{R}^* = \emptyset$ . Let  $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ , with  $T_1 = \mathcal{V}^*$ , and  $T_2$  chosen such that T is nonsingular. Finally, let  $G = X_{0,T}^{\dagger} + K\gamma$ , with  $\gamma$  defined as in (20). Then, the invariant zeros of (1) are the eigenvalues of  $A_{11}$ , where

$$T^{-1}(X_{1,T}G)T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$
 (24)

*Proof:* This result derives from the facts that (i) the closed loop system with the state feedback u = Fx satisfies

$$A + BF = X_1 TG, \tag{25}$$

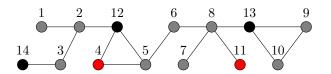


Fig. 2. An example of consensus network from [31]. Agents are numbered from 1 through 14, where nodes  $\{12, 13, 14\}$  (in black) are the leaders and nodes  $\{4, 11\}$  (in red) are the network monitors.

(ii) the subspace  $\mathcal{V}^*$  is invariant for the closed-loop matrix A+BF, and (iii) the invariant zeros of (1) are the eigenvalues of the closed-loop matrix A+BF contained in  $\mathcal{V}^*$ .

#### IV. MALICIOUS ATTACKS: AN ILLUSTRATIVE EXAMPLE

To illustrate a possible use of the theory developed in this paper, consider the leader-follower consensus network in Fig. 2. The dynamics of the followers are given by the matrices

The network is equipped with two monitoring nodes, specifically, nodes 4 and 11. The state of the monitoring nodes is used to detect any anomalous behavior of the network from its nominal dynamics (see also [32]). We let an attacker take control of the leader nodes, and seek for an attack strategy that remains undetectable from the monitoring nodes, and leverages only historical data of the network dynamics. In particular, the attacker strategy is designed as follows: (i) compute  $\mathcal{V}^*$  and  $\mathcal{S}^*$  using Theorems 3.1 and 3.4, respectively, and find  $\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{S}^*$ ; (ii) for  $R = \operatorname{Basis}(\mathcal{R}^*)$ , and X, U and  $K_0$  defined as in (4a), (4d) and Assumption 2.1, compute  $^3$  P as

$$\begin{bmatrix} XK_0 & I \otimes R \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = 0; \tag{26}$$

and (iii) choose the attack input  $A_T$  as  $A_T \in \text{Im}(UK_0P)$ . Then, for any initial state x(0) and nominal control input  $U_T$ , the output of (2)-(3) with input  $U_T$  is indistinguishable from the output with input  $U_T + A_T$ . As can be seen in Fig. 3

<sup>&</sup>lt;sup>3</sup>Similarly to the proof of Theorem 3.7, it can be shown that, for nontrivial v and w, any trajectory satisfying  $XK_0v=(I\otimes R)w$  (i) starts at the origin, (ii) evolves in  $\mathcal{R}^*$ , and (iii) is compatible with the data (4) of (1).

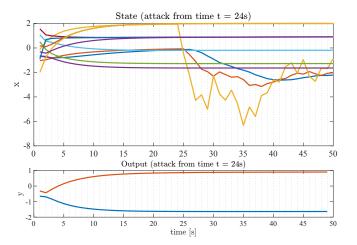


Fig. 3. In this figure we show an attack on the network of Fig. 2. The systems initial condition is chosen randomly and the leaders impose  $u = [-2\ 2\ 4]^{\top}$ . The attacker waits for the system to reach its equilibrium and then, at time t = 24s, injects an attack  $A_T$  as proposed in Sec IV. We notice how the system state its perturbed from the equilibrium, while the output of the system remains unaffected by the attack, rendering the attack action effectively invisible at the output.

from time t=24s, the attacker strategy perturbs the state of the network but does not affect the monitoring nodes, thus remaining undetectable. In fact, it can be shown that any input  $A_T \in \operatorname{Im}(UK_0P)$  moves the state trajectory within the controlled invariant  $\mathcal{R}^* \subseteq \operatorname{Ker}(C)$ , thus affecting the state of the system but not its output.

#### V. CONCLUSION

In this paper we show how experimental data can be used to learn key invariant subspaces of a linear system. In particular, we derive data-driven expressions for  $\mathcal{V}^*$ , the largest  $(A, \operatorname{Im}(B))$ -controlled invariant contained in  $\operatorname{Ker}(C)$ , and  $S^*$ , the smallest (A, Ker(C))-conditioned invariant containing Im(B). Being able to identify these subspaces from data suggests that much of the results and intuitions of the geometric approach to control can be conveniently reworked in a data-driven framework. To support this point, we leverage the identified invariant subspaces to design a data-driven feedback controller to force the state inside a desired controlled invariant subspace, and to compute the invariant zeros of the system. Finally, as an example of the theoretical results, we design a data-driven undetectable attack. Applications and extensions of the proposed results are numerous, and are left as the subject of future investigation.

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