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Fair and Sparse Solutions in Network-Decentralized Flow Control

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***Abstract*—We proposed network-decentralized control strategies, in which each actuator can exclusively rely on local information, without knowing the network topology and the external input, ensuring that the flow asymptotically converges to the optimal one with respect to the p-norm. For $1 < p < \infty$, the flow converges to a unique constant optimal u_p^*. We show that the state converges to the optimal Lagrange multiplier of the optimization problem. Then, we consider networks where the flows are affected by unknown spontaneous dynamics and the buffers need to be driven exactly to a desired set-point. We propose a network-decentralized proportional-integral controller that achieves this goal along with asymptotic flow optimality; now it is the integral variable that converges to the optimal Lagrange multiplier. The extreme cases $p = 1$ and $p = \infty$ are of some interest since the former encourages sparsity of the solution while the latter promotes fairness. Unfortunately, for $p = 1$ or $p = \infty$ these strategies become discontinuous and lead to chattering of the flow, hence no optimality is achieved. We then show how to approximately achieve the goal as the limit for $p \to 1$ or $p \to \infty$.**

***Index Terms*—Decentralized control, p-norm minimization, flow networks, unknown dynamics, buffer level control.**

I. Introduction

FLOW networks [2] are relevant in many applications, including data transmission [16], [18], traffic and transportation networks [11], [13], [14], [15], [19], production-distribution systems [3], [5], irrigation [12], heating [20], [21], cyber-physical energy networks [1], and compartmental systems in general [7], [17], [22]. Large scale, geographical sparsity, and privacy issues often require decentralized control strategies. The concept of network-decentralized flow control was introduced by Iftar [15], Iftar and Davison [16], and later reconsidered in [5], [6], [7]. Given a dynamic network with buffers (associated with the nodes), controlled flows (associated with the arcs), and an external (uncontrolled) demand, a feedback control is called *network-decentralized* if each actuated arc decides its flow exclusively based on local information about the buffer levels at its extreme nodes.

We are concerned here with the asymptotic optimality of the resulting flow. A saturated network-decentralized control that asymptotically minimizes the 2-norm was proposed in [4], with extensions to more general classes of functionals in [8]. These results hold under the technical assumption that the functional is smooth and *strictly convex*. For flow networks with a single source and a single destination, [9] proposed a network-decentralized strategy that asymptotically drives the whole flow along the shortest path; this optimality mechanism can explain natural phenomena, e.g., lightning discharge [10].

Here, we consider the asymptotic optimization of

$$\lim_{t\to\infty} \|u(t)\|_p = \lim_{t\to\infty} \sqrt[p]{\sum_i |u_i(t)|^p}.$$

We first show that, for $1 < p < \infty$, the problem has a simple solution. Then, we investigate the limit cases of $p = \infty$ and $p = 1$. The former promotes *fair* solutions: in the ∞-optimal flow, the workload of any of the most exploited actuators cannot be reduced without imposing an even stronger effort to some other actuator. Conversely, the 1-norm encourages *sparse* solutions: the whole workload is assigned to some of the actuators, while the others are left inactive, although this is not a strict rule.

When considering ∞ and 1-norms, the lack of strict convexity renders the solution proposed in [8] not applicable. Indeed, the resulting controls would be discontinuous: although they may be stabilizing [5], they *introduce chattering*, and hence no asymptotic flow optimality can be ensured.

The contributions of this letter are summarized as follows.

- We propose a general network-decentralized control strategy that stabilizes the network and asymptotically minimizes the norm $\|u\|_p$, $1 < p < \infty$.
- The proposed control, for a given p with $1 < p < \infty$, is continuous. The state converges to the unique steady-state $\bar{x}_p$, corresponding to the Lagrange multiplier λ^* of the optimization problem.
- For $p \to \infty$ (respectively $p \to 1$), the ∞ (resp. 1) norm of the optimal solutions is arbitrarily close to the optimal ∞ (resp. 1) norm.

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- When an unknown, possibly unstable, dynamics affects the system flows, asymptotic optimality and buffer levels converging to 0 (i.e., to the desired set-point) can be achieved by adding an integrator to the proposed solution, if $p \geq 2$. We provide a counterexample explaining why the scheme does not work for $p < 2$.
- We propose a different solution for the case $1 < p < 2$ and show that the control ensures local stability.
- In the presence of the integrator, while the state converges to 0, the integral variable converges to the Lagrange multiplier λ^* of the optimization problem.

To focus on the main results, the proofs are in the Appendix.

II. Introduction and Motivation

We consider a class of systems of the form

$$\dot{x}(t) = Bu(t) - d, \tag{1}$$

where the equality holds component-wise; the state $x(t) \in \mathbb{R}^n$ is the vector of buffer levels, $u(t) \in \mathbb{R}^m$ is the vector of controlled flows, $B \in \mathbb{R}^{n\times m}$ is an assigned matrix and $d \in \mathbb{R}^n$ is an external *unknown* constant demand. We assume that $\bar{x} = 0$ is the reference (not the absolute) level: a negative state is to be interpreted as below this point. A negative flow is to be interpreted as directed in the opposite direction with respect to the assigned flow orientation.

The next standing assumption is required for stabilisability and ensures the existence of a solution u of $Bu = d$ for every possible d [4], [5].

Assumption 1: Matrix B has full row rank ($m \geq n$).

Definition 1: A state feedback control u is *network-decentralized* if each component u_k only depends on the buffer levels x_i corresponding to nonzero entries B_{ik} of the kth column of B, and is independent of d.

Our goal is to find a network-decentralized flow control strategy u that stabilizes the flow network and asymptotically yields the minimum $\|u\|_p$.

Remark 1: To minimize a weighted norm, $\sqrt[p]{\sum_i |u_i/\omega_i|^p}$, with ω_i assigned weights, we need to rescale the *actual* flow components u_i as $\hat{u}_i \doteq u_i/\omega_i$. The flow term in (1) is changed as $Bu = B\Omega \ \ (\Omega^{-1}u) = \hat{B}\hat{u}$, with $\Omega = \mathrm{diag}\{\omega_i\}$.

The network-decentralized minimization of the p-norm can lead to different outcomes depending on the value of p. Roughly speaking, small values of p tend to concentrate the flow along preferred channels with shortest path. Conversely, large values of p tend to spread the flow among the arcs.

Example 1: Consider the steady state equation $Bu - d = 0$ with $B = [4\ 3]$ and a generic d:

$$4u_1 + 3u_2 = d.$$

As shown in Fig. 1, the minimum p-norm flow is: for $p = 1$, $u^{(1)} = [d/4\ \ 0]^\top$ (only one actuator working); for $p = 2$, $u^{(2)} = (BB^\top)^{-1}B^\top d = [4d/25\ \ 3d/25]^\top$ (minimum "energy" $u_1^2 + u_2^2$); for $p = \infty$, $u^{(\infty)} = [d/7\ \ d/7]^\top$ (the actuators are working with equal intensity, $u_1 = u_2$).

In our model, each nonzero component of d can be either an outflow (when positive) or an inflow (when negative). In case d is not balanced, then the control u must have (possibly negative) flow components leaving or coming from the external environment. If B is an incidence matrix, each column B_k of B corresponds to a controlled flow arc and has a -1 in the departure node, 1 in the arrival node, 0 elsewhere; arcs from or to the external environment correspond to columns with a single nonzero entry equal to 1 or -1 (see Example VI). Assumption 1 requires that at least one column B_k of B has a single non-zero entry, associated with a controlled flow from or to the external environment.

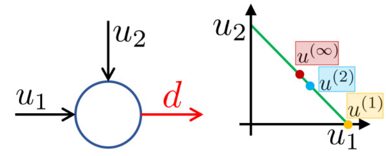

Fig. 1. The flow problem in Example 1 and the optimal controlled flows minimizing the p-norm for $p = 1$ (yellow), $p = 2$ (cyan), $p = \infty$ (red).

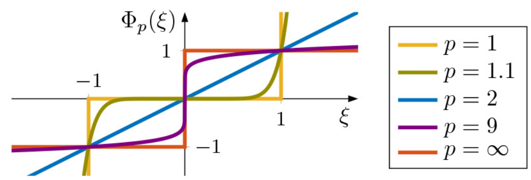

Fig. 2. Function $\Phi_p(\xi)$ for some values of p.

III. Preliminary: p-Norm Minimization

The following theorem is our starting point.

Theorem 1 (Strictly convex cost): Consider the cost

$$J(u) = \sum_{k=1}^{m} f_k(u_k),$$

where the functions $f_k : \mathbb{R} \to \mathbb{R}$ are continuously differentiable and strictly convex with strictly increasing derivatives, hence invertible. Consider the unique solution u^* to the problem

$$u^* = \arg\min_{Bu-d=0} J(u), \tag{2}$$

as well as the strictly increasing functions $g_k(u_k) = \frac{d}{du_k} f_k(u_k)$, $g(u) = [g_1(u_1), \ldots, g_m(u_m)]^\top$, and their inverse functions $\phi_k = g_k^{-1}$, $\phi(\xi) = [\phi_1(\xi_1), \ldots, \phi_m(\xi_m)]^\top$. Then, under Assumption 1, the network-decentralized control

$$u(t) = \phi(-B^\top x(t))$$

ensures convergence of the trajectories of system (1) to the unique steady state $\bar{x}$, whose components are equal to the Lagrange multipliers of the optimization problem (2), and $u(t) \to u^*$.

To consider the p-norm as a cost function, let us now define component-wise the control function

$$\Phi_p(\xi) = \mathrm{sign}(\xi)|\xi|^{\frac{1}{p-1}},$$

visualised in Fig. 2. The control law

$$u(t) = \Phi_p(-\gamma B^\top x(t)), \quad \gamma > 0, \tag{3}$$

is network-decentralized as required.

Proposition 1 (p-norm minimization): Let Assumption 1 be satisfied. For any real p, with $1 < p < \infty$, consider the vector u_p^* as the unique solution to the problem

$$u_p^* = \arg\min_{Bu-d=0} \|u\|_p. \tag{4}$$

For any $\gamma > 0$, control (3) ensures convergence of the state of (1) to the equilibrium $\bar{x} = \lambda^*$, the Lagrange multiplier of the optimization problem (4), unique solution of

$$B\phi(-B^\top\lambda^*) - d = 0. \tag{5}$$

The control at steady state $u_p^* = \Phi_p(-\gamma B^\top\bar{x})$ minimizes $\|u\|_p$ under the constraint $Bu - d = 0$.

The proposition considers values of p with $1 < p < \infty$. The limit for $p \to \infty$ of control (3) is no longer continuous: $\Phi_\infty(\xi) = \mathrm{sign}(\xi)$. For $p \to 1$, (3) is not even a proper function: $\Phi_1(\xi) = 0$ for $|\xi| \le 1$ and $\Phi_1(\xi) = \mathrm{sign}(\xi)\infty$ elsewhere (see Fig. 2). To face this discontinuity we use the continuous control (3) for p either large or close to 1.

The following property concerns the p-optimal u_p^*, in the case of a flow graph in which B is an incidence matrix.

Proposition 2 (No-waste at steady-state): Let B be an incidence matrix under Assumption 1. Then, the total controlled net inflow (i.e., sum of the controlled inflows minus sum of the controlled outflows) matches the total uncontrolled net outflow $\sum_k d_k$. Moreover, assume that $d_k \ge 0\ \forall k$ (resp. $d_k \le 0\ \forall k$). Then, the optimal p-norm controlled flow u_p^*, $1 < p < \infty$, has no outflow (resp. no inflow) components associated with arcs to/from the external environment.

The proposition means that resources injected to meet a positive demand are not subsequently ejected and wasted.

IV. Sub-Optimality

Here we study the limits of $\|u_p^*\|_p$ for $p \to \infty$ and $p \to 1$.

Theorem 2 (∞-norm): As $p \to \infty$, the p-norm optimal costs converge from above to the ∞-norm optimal cost:

$$\|u_p^*\|_p \to \|u_\infty^*\|_\infty.$$

To assess sub-optimality, we compare $\|u_p^*\|_\infty$ and its limit $\|u_\infty^*\|_\infty$ by considering the bound (see the proof of Th. 2)

$$\|u_p^*\|_\infty \le \|u_p^*\|_p \le \|u_\infty^*\|_p \le \sqrt[p]{m}\|u_\infty^*\|_\infty.$$

Theorem 3 (1-norm): As $p \to 1$ from above, the p-norm optimal costs converge from below to the 1-norm optimal cost:

$$\|u_p^*\|_p \to \|u_1^*\|_1.$$

Again we may compare $\|u_p^*\|_1$ with its limit $\|u_1^*\|_1$:

$$\|u_p^*\|_1 \le m^{(1-\frac{1}{p})}\|u_p^*\|_p \le m^{(\frac{p-1}{p})}\|u_1^*\|_1.$$

Since the 1-norm (resp. the ∞-norm) is not strictly convex, the optimal solution u_1^* (resp. u_∞^*) may be non-unique. If it is unique, however, the optimal flow u_p^* converges to the unique optimal solution when $p \to 1$ (resp. $p \to \infty$).

V. Networks With Unknown Dynamics and Buffer Level Control

We consider the generalised model

$$\dot{x}(t) = A(x) + Bu(t) - d. \tag{6}$$

Assumption 2: The nonlinear term $A(\cdot)$ is unknown. We assume $A(0) = 0$ and $\|A(z) - A(x)\|_2 \le L\|z - x\|_2$.

The assumption $A(0) = 0$ does not compromise generality, because a nonzero term $A(0)$ could always be embedded in d, by redefining $A(x) := A(x) - A(0)$ and $\hat{d} := d - A(0)$. It is also reasonable to assume that $A(x)$ is Lipschitz in physical systems, since realistic dynamics of interest have a finite rate of variation in practice.

Due to the presence of $A(x)$, the previous control law (3) does no longer ensure optimality unless $x = 0$. Hence, our goal is now to find a network-decentralized flow control strategy u that stabilizes the flow network, asymptotically yields the minimum $\|u\|_p$, and simultaneously guarantees that $x(t) \to 0$, *the reference set-point*, as $t \to \infty$.

We need the next technical lemma to arrive to a domain of attraction measured by a parameter ρ.

Lemma 1: The following identity holds

$$\left[\Phi_p(-\gamma B^\top(z + \bar{\xi})) - \Phi_p(-\gamma B^\top\bar{\xi})\right] = -B\Delta(z, \bar{\xi})B^\top z,$$

where Δ is a positive diagonal matrix, for any $z, \gamma > 0, \bar{\xi}$, $p > 1$, and B satisfying Assumption 1. Moreover, assume $p \ge 2$ and $\rho > 0$ be given such that $\|\bar{\xi}\| \le \rho/2$. Then, for all $\delta > 0$, there exists $\gamma > 0$ such that $\Delta \succeq \delta I$, for all z such that $\|z\| \le \rho/2$.

The next is the first result of the section.

Theorem 4 (Dynamic network-decentralized control): For $p \ge 2$, under Assumptions 1 and 2, consider the proportional-integral control

$$u = \Phi_p\Big(-\gamma B^\top(x + \xi)\Big), \tag{7}$$

$$\dot{\xi} = \alpha x, \quad \xi(0) = 0, \tag{8}$$

with $\alpha > 0$ arbitrarily given. Consider the initial domain

$$x(0) \in \mathcal{X}_0 = \left\{x : \|x\|^2 \le \rho_0^2 = \frac{\rho^2}{8} - \frac{3}{2}\|\bar{\xi}\|^2\right\},$$

with given $\rho^2 > 12\|\bar{\xi}\|^2$, where $\bar{\xi} = \lambda^*$, the Lagrange multiplier of the optimization problem (4) (we remind that $A(0) = 0$ at $\bar{x} = 0$), is the unique vector that solves

$$B\Phi_p\Big(-\gamma B^\top\bar{\xi}\Big) - d = 0.$$

Then, there exist $\gamma > 0$ such that $x(t) \to 0$, $u(t) \to u_p^*$ and $\xi(t) \to \bar{\xi}$.

Remark 2: Instead of assuming $A(x)$ is Lipschitz everywhere, we can assume $A(x)$ is smooth on a compact set $\mathcal{C}$ (hence Lipschitz in $\mathcal{C}$) including $\|x\| \le \rho$, our domain of attraction (as defined in the proof of Theorem 4).

Assuming $p \ge 2$ is crucial to apply Lemma 1. Indeed, the following example shows a case in which control (7) cannot be effective when $1 < p < 2$.

Example 2: Consider $\dot{x} = ax + u$, with $a > 0$. Apply control (7)-(8) and let $\kappa = 1/(p-1)$. We get

$$\dot{x} = ax - \gamma(x + \xi)|x + \xi|^{\kappa-1},$$

$$\dot{\xi} = \alpha x,$$

with equilibrium $\bar{x} = 0$ and $\bar{\xi} = 0$. When $1 < p < 2$, we have $\kappa > 1$. The linearised system in $(0, 0)$ has matrix $\begin{bmatrix} a & 0 \\ \alpha & 0 \end{bmatrix}$, hence the equilibrium is unstable for any $a > 0$.

We face the problem of $p \le 2$ by changing the control strategy: we stabilise the system by means of a linear term; then, we insert the integral variable in the nonlinear function

$$u = -\gamma B^\top x + \Phi_p\Big(-\gamma B^\top\xi\Big), \tag{9}$$

$$\dot{\xi} = \alpha x, \quad \xi(0) = 0. \tag{10}$$

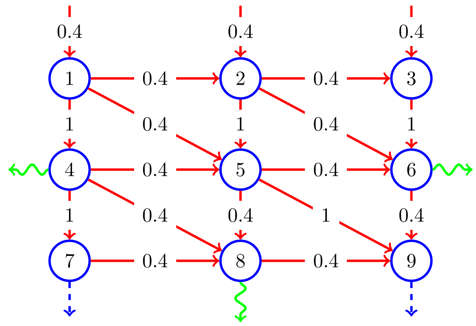

Fig. 3. Fluid network: controlled arcs (red arrows), with weight $1/\omega_k$ for each controlled arc k; losses (green arrows); demands (blue arrows).

The following assumption basically rules out both singularities and under-exploitation of the network.

Assumption 3: The optimal flow u_p^* corresponding to d has at least n nonzero components. The submatrix of B formed by the corresponding columns has rank n.

Theorem 5: Let $p \leq 2$ and Assumptions 1, 2 and 3 be satisfied with $A(x)$ smooth. The closed-loop system admits the unique steady state $x = 0$ and $\xi = \bar{\xi}$ and u is the optimal u_p^*. The steady state is locally stable for $\gamma > 0$ large enough (which exists because $BB^\top$ is positive definite), such that

$$[\bar{A} - \gamma BB^\top]^\top + [\bar{A} - \gamma BB^\top] = -Q \prec 0,$$

where $\bar{A}$ is the Jacobian of $A(x)$ evaluated at 0.

Remark 3: Proposition 2 holds as well if the external uncontrolled demand also takes into account the effect of the dynamics: $\hat{d} = d - A(0)$.

VI. Example: System of Interconnected Tanks

Consider the fluid network in Fig. 3. There are $n = 9$ tanks, whose levels are $h \in \mathbb{R}^9$, and $m = 19$ controlled flows. The graph incidence 9×19 matrix is

$$\tilde{B} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Given the weights, $\omega = [1, 2.5, 2.5, 1, 2.5, 2.5, 1, 2.5, 2.5, 1, 2.5, 2.5, 2.5, 1, 2.5, 2.5, 2.5, 2.5, 2.5]$, we take $B = \tilde{B}\Omega$ (Remark 1). Let $d = [0, 0, 0, 0, 0, 0, 0.7, 0, 0.3]^\top$. The state is $x(t) = h(t) - \bar{h}$ with set-point $\bar{h} = [17.69, 20.37, 22.70, 16.59, 22.42, 17.93, 19.54, 20.68, 15.66]^\top$.

There are unknown losses from 3 tanks, n. 4, 6 and 8, which are modelled *for numerical purposes* by function $A(h) = b - \sqrt{b^2 + Hh}$. We take $(H_4, b_4) = (0.001, 0.002)$, $(H_6, b_6) = (0.002, 0.003)$ and $(H_8, b_8) = (0.001, 0.003)$. This information *is not used in the control synthesis*. The system (6) is $\dot{x} = A(x + \bar{h}) + Bu - d$, where the nonzero components of A are those related to tanks 4, 6 and 8.

We apply control (7)-(8) for $p \geq 2$ and control (9)-(10) for $p < 2$, with $\alpha = 0.05$. We consider three consecutive time intervals of length 600. In these intervals we take: first, $p = 2$ and $\gamma = 0.03$; second, $p = 9$ and $\gamma = 10^{-6}$; third, $p = 1.1$ and $\gamma = 0.06$. The initial conditions are $h_0 = [15.51, 18.41, 19.01, 18.80, 17.34, 18.36, 19.63, 18.12, 19.77]^\top$.

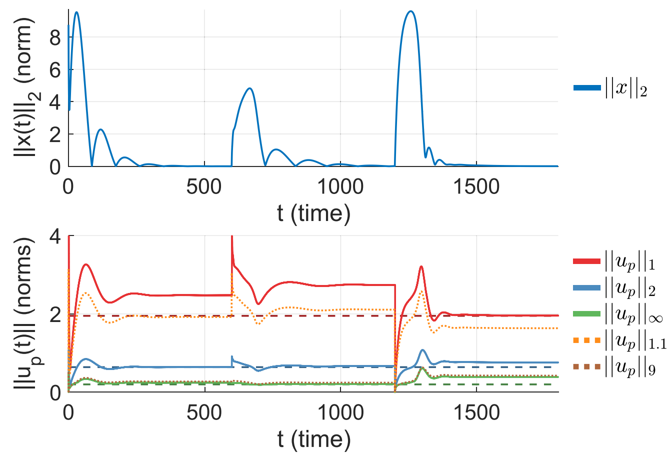

Fig. 4. *Top:* norm of the state $\|x(t)\|_2$. *Bottom:* solid lines represent the norms $\|u_p\|_1$ (red), $\|u_p\|_2$ (blue) and $\|u_p\|_\infty$ (green), which respectively get close to the optimal $\|u_1^*\|_1$, $\|u_2^*\|_2$ and $\|u_\infty^*\|_\infty$ (dashed lines) in the third ($p = 1.1$), first ($p = 2$), and second ($p = 9$) intervals. Animations are available at: https://users.dimi.uniud.it/~franco.blanchini/oneinf.html

In Fig. 4 we report the norm of the state $x(t)$ (top) and relevant norms of the inputs $u(t)$ (bottom). As expected, the state $x(t)$ converges to zero in all cases (i.e., $h(t) \rightarrow \bar{h}$). Moreover, the steady-state control u_p has 1, 2 and ∞-norms that get close to the optimal values for $p = 1.1$, $p = 2$ and $p = 9$. The steady-state controls are reported in Table I. The steady-state total *actual* controlled inflow (which is given by $\omega_{17}u_{p,17}^* + \omega_{18}u_{p,18}^* + \omega_{19}u_{p,19}^* \approx 1.454$ for any p) matches the total uncontrolled outflow (given by $\sum_k \hat{d}_k = \sum_k [d - A(\bar{h})]_k = 1.454$) including both the demand and the losses modelled by the nonlinear dynamics.

VII. Conclusion

We proposed a robust network-decentralized proportional integral controller for flow systems ensuring exact convergence to the desired steady-state set-point and asymptotic flow optimality. The control works in the presence of unknown Lipschitz dynamics and external demand flows. In view of its structure, the scheme works in the presence of failures as long as the rank assumptions remain satisfied.

Appendix

Proof of Theorem 1: The proof can be inferred from [8]. Here we give a different proof that points out the uniqueness of the steady state. Consider the Lagrange multiplier vector λ and the Lagrangian

$$L(u, \lambda) = J(u) + \lambda^\top (Bu - d).$$

The optimality condition with respect to u requires

$$\nabla J(u) + \lambda^\top B = [g(u)]^\top + \lambda^\top B = 0.$$

Considering the inverse, the optimal flow is

$$u^* = \phi(-B^\top \lambda^*), \tag{11}$$

and λ^* is the unique vector that satisfies (5). To prove that $x(t) \rightarrow \lambda^*$, take the Lyapunov function $V(x) = \frac{1}{2}\|x - \lambda^*\|^2$.

TABLE I
THE STEADY-STATE SOLUTION u_p^* AND THE OPTIMAL u_q^*

p	q	u_p^* *(steady-state control)*	u_q^* *(computed via linear/quadratic programming)*	$\|u_p^*\|_p$	$\|u_p^*\|_q$	$\|u_q^*\|_q$
2	2	$[0.132, -0.033, -0.081, 0.118, -0.109, -0.037, 0.120, 0.155, 0.172, 0.102, -0.233, -0.054, 0.222, 0.047, 0.218, 0.063, 0.243, 0.210, 0.129]^\top$	$[0.132, -0.033, -0.081, 0.118, -0.109, -0.037, 0.120, 0.155, 0.172, 0.102, -0.233, -0.054, 0.222, 0.047, 0.218, 0.063, 0.243, 0.210, 0.129]^\top$	0.641	0.641	0.641
9	∞	$[0.170, -0.063, -0.126, 0.184, -0.131, -0.052, 0.170, 0.131, 0.131, 0.169, -0.206, -0.058, 0.189, 0.117, 0.189, 0.074, 0.194, 0.194, 0.193]^\top$	$[0.178, -0.065, -0.126, 0.200, -0.119, -0.037, 0.173, 0.141, 0.150, 0.173, -0.200, -0.046, 0.189, 0.064, 0.183, 0.060, 0.196, 0.191, 0.195]^\top$	0.244	0.206	0.200
1.1	1	$[0.015, 0.000, -0.001, 0.000, -0.045, 0.000, 0.001, 0.120, 0.337, 0.000, -0.280, 0.000, 0.381, 0.000, 0.194, -0.000, 0.387, 0.193, 0.001]^\top$	$[0.000, 0.0000, 0.000, -0.000, -0.051, 0.000, 0.000, 0.120, 0.336, 0.000, -0.280, 0.000, 0.387, 0.000, 0.195, 0.000, 0.387, 0.195, 0.000]^\top$	1.634	1.955	1.950

Being ϕ_k strictly increasing, $(z_1^\top - z_2^\top)(\phi(-z_1) - \phi(-z_2)) < 0$ for $z_1 \neq z_2$. Therefore,

$$\dot V(x) = (x - \lambda^*)^\top \Big[B\phi(-B^\top x) - d\Big]$$
$$= (x^\top B - \lambda^{*\top} B)\Big[\phi(-B^\top x) - \phi(-B^\top \lambda^*)\Big] < 0,$$

for $x \neq \lambda^*$. Hence, x converges to the multiplier.

Proof of Proposition 1: Minimising $\|u\|_p$ is equivalent to minimising

$$\frac{1}{p\gamma}\|u\|_p^p = \sum_{k=1}^m \frac{1}{p\gamma}|u_k|^p,$$

which is a cost of the same type as that considered in Theorem 1, with $f_k(u_k) = \frac{1}{p\gamma}|u_k|^p$. The derivative of $f_k(u_k)$ is $g_k(u_k) = \text{sign}(u_k)\ |u_k|^{p-1}/\gamma$, whose inverse function is

$$u_k(\xi_k) = \phi_k(\xi_k) = \text{sign}(\xi_k)\ \sqrt[p-1]{\gamma|\xi_k|} = \Phi_p(\gamma\xi_k).$$

Then, if we evaluate $\phi(\xi)$ at $\xi = -B^\top x$, we get (3). The statement follows from Theorem 1.

Proof of Theorem 2: We first show that $\|u_p^*\|_p$ is decreasing as p increases. In fact, for $p_2 > p_1$,

$$\|u_{p_2}^*\|_{p_2} \leq \|u_{p_1}^*\|_{p_2} \leq \|u_{p_1}^*\|_{p_1}.$$

The first inequality is true because, by definition, $u_{p_2}^*$ is the minimizer of $\|\cdot\|_{p_2}$. The second follows from the property that, for any vector u of size m and $1 \leq k \leq h$,

$$\|u\|_h \leq \|u\|_k \leq m^{(\frac{1}{k} - \frac{1}{h})}\|u\|_h. \tag{12}$$

The decreasing sequence $\|u_p^*\|_p$ has a limit μ

$$\mu \doteq \lim_{p\to\infty} \|u_p^*\|_p = \inf_{p\geq 1} \|u_p^*\|_p.$$

We show that $\mu = \|u_\infty^*\|_\infty$. Since, for the same reasons invoked above, $\|u_\infty^*\|_\infty \leq \|u_p^*\|_\infty \leq \|u_p^*\|_p$, we have $\|u_\infty^*\|_\infty \leq \mu$, because $\|u_p^*\|_p$ converges to μ.

To show that $\|u_\infty^*\|_\infty \geq \mu$, consider that, in view of (12), for $h = \infty$ and $k = p$, $\|u\|_\infty \leq \|u\|_p \leq \sqrt[p]{m}\|u\|_\infty$, for all vectors. Hence, for all p,

$$\|u_\infty^*\|_\infty \geq \|u_\infty^*\|_p/\sqrt[p]{m} \geq \|u_p^*\|_p/\sqrt[p]{m}.$$

When $p \to \infty$, $\|u_p^*\|_p/\sqrt[p]{m} \to \mu$, because $\sqrt[p]{m} \to 1$, and hence $\|u_\infty^*\|_\infty \geq \mu$. Therefore, it must be $\mu = \|u_\infty^*\|_\infty$.

Proof of Theorem 3: It is almost identical to that of Theorem 2, and it is hence omitted.

Proof of Lemma 1: Given any increasing function φ of a real variable y, defined on $|y| \leq a$, which admits a (possibly unbounded) derivative, we can write, for any $y_1, y_2 \in \mathbb{R}$,

$$\varphi(y_1) - \varphi(y_2) = D(y_1, y_2)(y_1 - y_2), \tag{13}$$

with $D(y_1, y_2) \doteq [\varphi(y_1) - \varphi(y_2)]/[y_1 - y_2]$ (assuming $D(y_1, y_1) \doteq \varphi'(y_1)$). Note that $D(y_1, y_2) \geq \min_{|y|\leq a}\ \varphi'(y)$. If we consider function $\varphi(y) = \text{sign}(y)\ \sqrt[p-1]{|y|}$ with $p \geq 2$, the minimum of the derivative φ' is at the extrema and

$$D(y_1, y_2) \geq a^{-\frac{p-2}{p-1}}/(p-1).$$

Let B_k be the kth column of B, $y_1 = -\gamma B_k^\top(z + \bar\xi)$ and $y_2 = -\gamma B_k^\top \bar\xi$. Assume $\|z\| \leq \rho/2$ and $\|\bar\xi\| \leq \rho/2$. Then,

$$|y_1| = |\gamma B_k^\top(z + \bar\xi)| \leq \gamma\|B_k\|\|\bar\xi + z\|$$
$$\leq \gamma\|B_k\|(\|\bar\xi\| + \|z\|) \leq \gamma\|B\|\rho \doteq a,$$
$$|y_2| = |\gamma B_k^\top \bar\xi| \leq \gamma\|B_k\|\|\bar\xi\| \leq \gamma\|B\|\|\bar\xi\| \leq a$$

(note that $\|B_k\| \leq \|B\|$). For each k, let D_k be defined as in (13). Given δ, take γ such that

$$D_k\gamma \geq \gamma\frac{1}{p-1}(\gamma\|B\|\rho)^{-\frac{p-2}{p-1}} = \frac{(\|B\|\rho)^{-\frac{p-2}{p-1}}}{p-1}\ \gamma^{\frac{1}{p-1}} \geq \delta,$$

for all k. Denoting as Φ_{pk} the kth component of Φ_p

$$B\Big[\Phi_p(-\gamma B^\top(z + \bar\xi)) - \Phi_p(-\gamma B^\top \bar\xi)\Big]$$
$$= \sum_k\ B_k\Big[\Phi_{pk}(-\gamma B_k^\top(z + \bar\xi)) - \Phi_{pk}(-\gamma B_k^\top \bar\xi)\Big]$$
$$= \sum_k\ B_k D_k\Big[-\gamma B_k^\top z\Big] = \sum_k\ -B_k\delta_k B_k^\top z = -B\Delta B^\top z,$$

with Δ diagonal matrix with entries $\delta_k = D_k\gamma$ not smaller than δ, $\Delta \succeq \delta I$.

Proof of Proposition 2: The fact that the net controlled inflow compensates the demand d is trivial. For the next step consider equation (5) and write it as

$$B\phi(-B^\top\lambda^*) = -BD(\lambda^*)B^\top\lambda^* = d \geq 0,$$

where $D(\lambda^*)$ is a positive diagonal matrix, computed as in the proof of Lemma 1. We have that $-BD(\lambda^*)B^\top$ is a Metzler matrix which is negative definite, and hence Hurwitz. Its inverse is thus non-positive and $\lambda^* = [-BD(\lambda^*)B^\top]^{-1}d \leq 0$ component-wise. From (11), since ϕ_k has the same sign as its argument, we have that all the flows u_k corresponding to columns of B that have a single nonzero component, B_{ik}, are inflows coming from the external environment (if $B_{ik} > 0$, $u_k \geq 0$, while if $B_{ik} < 0$, $u_k \leq 0$).

Proof of Theorem 4: First note that $\|\bar\xi\| \leq \rho/2$, a condition we need to apply Lemma 1. Define $z \doteq x + \omega$, and $\omega \doteq \xi - \bar\xi$ and write the system as

$$\dot x = A(x) + B\Phi_p\Big(-\gamma B^\top(z + \bar\xi)\Big) - B\Phi_p\Big(-\gamma B^\top\bar\xi\Big),$$
$$\dot\xi = \alpha x.$$

Considering that $\dot{\omega} = \dot{\xi}$ and exploiting Lemma 1, we have

$$\begin{aligned}\dot{z} &= \dot{x} + \dot{\omega} = A(z-\omega) - B\Delta B^\top z + \alpha(z-\omega),\\ \dot{\omega} &= \alpha(z-\omega).\end{aligned}$$

Consider the Lyapunov function $V = [\|z\|^2 + \|\omega\|^2]/2$. Matrix $BB^\top$ is positive definite in view of Assumption 1. Denoting by $\sigma > 0$ the smallest eigenvalue of $BB^\top \succ 0$ and exploiting the Lipschitz assumption on $A(\cdot)$, Lemma 1 under the assumption that

$$\|z\| \le \rho/2, \tag{14}$$

and $\|z-\omega\| \le \|z\| + \|\omega\|$, we get

$$\begin{aligned}\dot{V} &= z^\top A(z-\omega) - z^\top B\Delta B^\top z + \alpha z^\top z - \alpha z^\top \omega + \\ &\quad + \alpha\omega^\top z - \alpha\omega^\top\omega\\ &\le \|z\|\|A(z-\omega)\| - z^\top B\Delta B^\top z + \alpha\|z\|^2 - \alpha\|\omega\|^2\\ &\le L\|z\|\|z-\omega\| - \delta z^\top BB^\top z + \alpha\|z\|^2 - \alpha\|\omega\|^2\\ &\le L\|z\|\|z-\omega\| - \delta\sigma\|z\|^2 + \alpha\|z\|^2 - \alpha\|\omega\|^2\\ &\le -\delta\sigma\|z\|^2 + L\|z\|^2 + \alpha\|z\|^2 + L\|z\|\|\omega\| - \alpha\|\omega\|^2\\ &= \begin{bmatrix}\|z\| & \|\omega\|\end{bmatrix}\begin{bmatrix}-\delta\sigma + L + \alpha & L/2\\ L/2 & -\alpha\end{bmatrix}\begin{bmatrix}\|z\|\\ \|\omega\|\end{bmatrix} < 0,\end{aligned}$$

for $(z,\omega) \ne 0$, as long as $\alpha(\delta\sigma - L - \alpha) - L^2/4 > 0$.

According to Lemma 1 and its proof, this condition can be ensured under (14) by taking a large γ to ensure that δ is large enough, precisely $\delta > (L + \alpha + L^2/(4\alpha))/\sigma$.

So, we prove that (14) is satisfied for all t, if the initial value of x is $x(0) \in \mathcal{X}_0$ (and the condition $\|\bar{\xi}\| \le \rho/2$ is true for $\mathcal{X}_0 \ne \emptyset$). We have

$$\begin{aligned}V(0) &= \|z(0)\|^2/2 + \|\omega(0)\|^2/2 = \|x(0) - \bar{\xi}\|^2/2\\ &\quad + \|-\bar{\xi}\|^2/2 \le \|x(0)\|^2 + 3\|\bar{\xi}\|^2/2 \le \rho^2/8,\end{aligned}$$

since $\|x(0) - \bar{\xi}\|^2 \le \|x(0)\|^2 + \|\bar{\xi}\|^2 + 2\|x(0)\|\|\bar{\xi}\| \le 2(\|x(0)\|^2 + \|\bar{\xi}\|^2)$. Note that this implies $\|z(0)\|^2/2 \le \rho^2/8$, too. So, initially, V is not greater than $\rho^2/8$ and hence (14) is satisfied: this means that V is initially decreasing. As long as (14) holds, V decreases; consequently, we have $V(t) = \|z(t)\|^2/2 + \|\omega(t)\|^2/2 \le \rho^2/8$, meaning that (14) will always be satisfied for all $t > 0$.

Observe that, by the Lyapunov theorem, $\dot{V} < 0$ implies $z(t), \omega(t) \to 0$, so $\xi(t) \to \bar{\xi}$ and $x(t) = z(t) - \omega(t) \to 0$.

Since $B\Phi_p(-\gamma B^\top \bar{\xi}) = d$ and $\bar{\xi}$ is the Lagrange multiplier, the limit $u_p^* = \Phi_p(-\gamma B^\top \bar{\xi})$ is the optimal flow.

Proof of Theorem 5: The linearised system has the form

$$\begin{aligned}\dot{x} &= [\bar{A} - \gamma BB^\top]x - \gamma B\Delta B^\top \omega,\\ \dot{\xi} &= \alpha x,\end{aligned}$$

where $\omega = \xi - \bar{\xi}$. Δ is the diagonal matrix including the derivative of Φ_p. By Assumption 3, Δ has at least n nonzero entries corresponding to columns of B having rank n. Write $B\Delta B^\top = \tilde{B}\tilde{\Delta}\tilde{B}^\top$, where $\tilde{B}$ and $\tilde{\Delta}$ are restrictions achieved by eliminating the zero elements of Δ and the corresponding columns of B. Assumptions 1 and 3 imply that $\tilde{B}$ has rank n, hence $B\Delta B^\top = \tilde{B}\tilde{\Delta}\tilde{B}^\top$ is positive definite.

Take the (local) Lyapunov function $V = x^\top x + (\gamma/\alpha)\omega^\top B\Delta B^\top \omega$. Then,

$$\begin{aligned}\dot{V} &= 2x^\top[\bar{A} - \gamma BB^\top]x - 2\gamma x^\top B\Delta B^\top \omega + 2\gamma\omega^\top B\Delta B^\top x\\ &= x^\top[\bar{A} + \bar{A}^\top - 2\gamma BB^\top]x = -x^\top Qx \le 0,\end{aligned}$$

and $\dot{V} < 0$ for $x \ne 0$, where $-Q = \bar{A} + \bar{A}^\top - 2\gamma BB^\top$. According to the LaSalle invariance principle, the state converges to the set where $x = 0$. There is no trajectory of the system included in the set $\{x = 0\}$, other than the steady-state trajectory given by $x \equiv 0$ and $\omega \equiv 0$, because on such set $\dot{x} = -B\Delta B^\top \omega \ne 0$ unless $\omega \equiv 0$. By the Krasowskii theorem, both x and ω converge to 0.

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