Data-Driven Gain Scheduling Control of Linear Parameter-Varying Systems using Quadratic Matrix Inequalities

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Abstract

This paper synthesizes a gain-scheduled controller to stabilize all possible Linear Parameter-Varying (LPV) plants that are consistent with measured input/state data records. Inspired by prior work in data informativity and LTI stabilization, a set of Quadratic Matrix Inequalities is developed to represent the noise set, the class of consistent LPV plants, and the class of stabilizable plants. The bilinearity between unknown plants and 'for all' parameters is avoided by vertex enumeration of the parameter set. Effectiveness and computational tractability of this method is demonstrated on example systems.

1 Introduction

This paper performs Data Driven Control (DDC) of discrete-time Linear Parameter-Varying (LPV) systems using Quadratic Matrix Inequalities (QMIs). The problem setting involves parameter-affine LPV systems in which the parameter may vary arbitrarily within a polytope and the measured data admits a quadratic description in its noise. When the system has n states, m inputs, L parameters, and N_v vertices in the parameter polytope, we propose a non-conservative Linear Matrix Inequality (LMI) to find a quadratically stabilizing gain-scheduled controller for all consistent LPV plants involving N_v Positive Semidefinite (PSD) constraints of size $n(L+1)+m$ (continuous-time) or $n(L+2)+m$ (discrete-time) and a single Positive Definite (PD) constraint of size n.

LPVs systems are a class of linear systems whose plant dynamics depend on externally measured parameters. LPV systems have been employed to model and control nonlinear dynamics such as in vehicle control [1], missile control [2], and chemical processes [3]. Gain-scheduling control sets the input to be a function of the state and measured parameter [4]. Examples of quadratically stabilizing gain-scheduling through a common Lyapunov function include backsubstitution [5], interpolated vertex-controllers when the LPV dynamics are parameter-affine [2], and the use of a dynamic compensator when the plant dynamics are a Linear Fractional Transformation of the applied parameter [6]. The work in [7] applied different QMIs for robust control of a single given continuous-time LPV plant.

DDC is a methodology of formulating controllers for all possible plants that are consistent with measured input/output relations (data) [8]. Such algorithms avoid an expensive system-identification step to construct a generalized plant model. A survey of data-driven techniques is provided in [9]. One class of DDC methods applies Willem's Fundamental Lemma, which parameterizes all possible system responses by linear combinations of a single trajectory's Hankel matrices if a rank condition is satisfied (persistency of excitation) [10]. This Lemma can be used for stabilization/regulation [11] and Model Predictive Control [12, 13] with optional regularization to reduce sensitivity to noise.

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When the noise corrupting the recorded data admits a quadratic description, QMIs may be used in a non-conservative manner to describe the noise set and the set of consistent plants [14]. Their work forms a matrix S-Lemma [15], providing conditions under which the satisfaction of one QMI implies another QMI [16], in order to perform quadratic stabilization and robust control (H_2 and H_∞). The QMI-with-S-Lemma approach has also been used to stabilize nonlinear systems with state-dependent representations [17], to form a robust-control framework incorporating prior knowledge [18], to analyze and control continuous-time systems [19], to iteratively stabilize networked systems with block-structured controllers [20], and to impose LMI-region performance constraints on robust controllers [21].

DDC has been previously applied to LPV systems, as surveyed by [3]. Other instances of DDC for LPV include using Support Vector Machines [22], hierarchical control [23], and Willem's Fundamental Lemma [24]. The related problem of DDC of switched systems was studied in [25] using polynomial optimization. To the best of our knowledge, QMIs and the matrix S-Lemma have not been used for DDC of LPV systems.

The contributions of our work are

- A presentation of the Data-Driven LPV quadratic stabilization problem parameterized by QMIs
- An LMI to achieve quadratic stabilization via gain-scheduling vertex-QMIs with Kronecker structure in continuous-time and discrete-time
- An accounting of computational complexity which includes allowances for sparsity

This paper has the following structure: Section 2 reviews preliminaries such as notation, LPV stabilization, and the use of QMIs in forming stabilizing controllers. Section 3 applies this QMI method for LPV stabilization. Section 4 performs worst-case suboptimal H_2 control on LPV plants consistent with the noise structure. Section 5 demonstrates this stabilization approach on example systems. Section 6 concludes the paper.

2 Preliminaries

DDC Data Driven Control

LMI Linear Matrix Inequality

LPV Linear Parameter-Varying

LPVA LPV A-affine

PD Positive Definite

PSD Positive Semidefinite

SDP Semidefinite Program

QMI Quadratic Matrix Inequality

2.1 Notation

The double dots in $1..L$ represent the sequence of natural numbers between 1 and L. The n-dimensional real vector space is $\mathbb{R}^n$. The nonnegative real orthant is $\mathbb{R}^n_{\geq 0}$ and the cone of positive vectors is $\mathbb{R}^n_{>0}$. The set of real-valued $m \times n$ matrices is $\mathbb{R}^{m \times n}$. The transpose of a matrix M is M^T. The kernel (nullspace) of a matrix M is $\ker(M)$. The set of symmetric matrices of size n is $\mathbb{S}^n$, and its subsets of PSD and PD matrices are $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$. The vertical concatenation of matrices A and B of compatible dimensions is $[A; B]$ and their horizontal concatenation is $[A, B]$. The symmetrization operator applied to $M \in \mathbb{R}^{n \times n}$ is $\mathbf{sym}(M) = (M + M^T)/2$. The pseudoinverse of a matrix M is $M^\dagger$.

The matrices I_n, $\mathbf{0}_{m \times n}$, $\mathbf{1}_{m \times n}$ are respectively the identity, zeros, and ones matrices of appropriate dimensions. The dimension subscripts will be dropped when the matrix sizes are unambiguous. The $*$ marking will be used in block matrices to refer to the canonical transpose of oppositely-indexed elements.

The Kronecker product of matrices P and Q is $P \otimes Q$. The Hadamard (elementwise) product of matrices is $P \odot Q$. The symbol $\otimes_{\text{col}}$ will denote the column-wise Khatri-Rao product for matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times n}$ [26]

$$A \otimes_{\text{col}} B = (\mathbf{1}_{p\times 1} \otimes A) \odot (B \otimes \mathbf{1}_{m\times 1}). \tag{1}$$

The convex hull of a set of points $P = \{p_j\}_{j=1}^N$ is $\text{conv}(P)$. The notation δx will mean the derivative $\dot{x}$ in continuous-time or the next state x_+ in discrete-time.

2.2 LPV Stabilization

LPV dynamics with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and measurable parameter $\theta \in \Theta \subset \mathbb{R}^L$ are

$$\delta x = A(\theta)x + B(\theta)u. \tag{2}$$

The LPV A-affine (LPVA) structure [27] has B constant and A θ-affine for some set of matrices $\forall \ell : A_\ell \in \mathbb{R}^{n\times n}$ if

$$\delta x = \left(\textstyle\sum_{\ell=1}^L A_\ell \theta_\ell\right) x + Bu. \tag{3}$$

This preliminary subsection will deliver exposition on the case where $(\{A_\ell\}, B)$ are known and fixed while θ is unknown and measured on-line. The main body of the paper will focus on the setting where the plant $(\{A_\ell\}, B)$ is unknown but consistent with observed data.

Remark 1. *LPVA structure may be rendered affine in the parameter by adjoining a new constant $\theta_0 = 1$ to θ.*

Let $\Omega = \{\omega_v\}_{v=1}^{N_v}$ be a finite set of N_v points in $\mathbb{R}^L$. In this paper, the parameter set Θ will be chosen to be the compact convex polytope $\Theta = \text{conv}(\Omega)$. We will refer to Ω as the vertices of Θ (or as vertices more generally).

A vertex-controller $K_v \in \mathbb{R}^{m\times n}$ is defined at each vertex ω_v in Ω, yielding the state-feedback law $u = K_v x$. Given a parameter $\theta \in \mathbb{R}^L$, a gain-scheduled controller $u = K(\theta)x$ may be found by first solving for a feasible $c \in \mathbb{R}^{N_v}$ using Linear Programming

$$\text{find } c \in \mathbb{R}_+^{N_w} \qquad \textstyle\sum_{v=1}^{N_v} c_v = 1 \qquad \textstyle\sum_{v=1}^{N_v} c_v \omega_v = \theta, \tag{4a}$$

and then returning the control policy,

$$K(\theta) = \textstyle\sum_{v=1}^{N_v} c_v K_v \qquad u = K(\theta)x. \tag{4b}$$

Any feasible point c of (4a) will serve: uniqueness of $K(\theta)$ is not required. Application of the gain-scheduled $u = K(\theta)x$ to the LPVA system (3) leads to the decomposed dynamics

$$\begin{aligned} \delta x &= A(\theta)x + BK(\theta)x && (5a)\\ &= \textstyle\sum_{\ell=1}^L \theta_\ell A_\ell x + \sum_{v=1}^{N_v} c_v B K_v x && (5b)\\ &= \left[\textstyle\sum_{v=1}^{N_v} c_v \left(\sum_{\ell=1}^L \omega_{\ell v} A_\ell\right) + c_v B K_v\right] x. && (5c) \end{aligned}$$

The open-loop system A_v for each vertex ω_v (multiplied in (5) by c_v) may be defined as

$$A_v = \textstyle\sum_{\ell=1}^L \omega_{\ell v} A_\ell. \tag{6}$$

Lemma 2.1. *If C is a convex cone with elements z and Θ= conv(Ω) , then the following statements are equivalent:*

$$\begin{aligned} &\textstyle\sum_{\ell=1}^L \theta_\ell z_\ell \in C && \forall \theta \in \Theta && (7a)\\ &\textstyle\sum_{\ell=1}^L \omega_{\ell v} z_\ell \in C && \forall v = 1..N_v && (7b) \end{aligned}$$

Proof. Statement (7a) implies (7b) because each vertex ω_v is an element of Θ. Every point $\theta \in \Theta$ may be represented by a possibly non-unique convex combination of vertices with coordinates $\theta_\ell = \sum_{v=1}^{N_v} c_v \omega_{\ell v}$ given that $\Theta = \text{conv}(\Omega)$ ((4a) and Section 2.1.4 of [28]). Eq. (7b) implies (7a), because $\sum_{\ell=1}^{L} \theta_\ell z_\ell$ may be expressed as the convex combination of C-elements $\sum_{\ell=1}^{L} \sum_{v=1}^{N_v} (c_v \omega_{\ell v}) z_\ell$. □

Definition 2.1. *The controller* $u = K(\theta)x$ *from Eq.* (4) *quadratically stabilizes the LPVA system* (3) *if there exists a* θ*-independent* $Y \in \mathbb{S}^n_{++}$ *(for continuous-time) or a* $P \in \mathbb{S}^n_{+}+$ *(for discrete-time)*

$$-2\ \boldsymbol{sym}(Y(A(\theta) + BK(\theta))) \in \mathbb{S}^n_{++} \qquad \forall \theta \in \Theta \tag{8a}$$

$$\begin{bmatrix} P & (A(\theta) + BK(\theta))P \\ * & P \end{bmatrix} \in \mathbb{S}^{2n}_{++} \qquad \forall \theta \in \Theta \tag{8b}$$

Lemma 2.2. *Equations* (8a) *and* (8b) *are equivalent to the following respective conditions,*

$$-2\ \boldsymbol{sym}(Y(A_v + BK_v)) \in \mathbb{S}^n_{++} \qquad \forall v = 1..N_v \tag{9a}$$

$$\begin{bmatrix} P & (A_v + BK_v)P \\ * & P \end{bmatrix} \in \mathbb{S}^{2n}_{++} \qquad \forall v = 1..N_v \tag{9b}$$

Proof. Equivalence of the respective pairs [(8a), (9a)] and [(8b), (9b)] holds by Lemma 2.1 with regard to the cones $\mathbb{S}^n_{++}$ and $\mathbb{S}^{2n}_{++}$ [2]. □

Pre- and post-multiplying (9a) by Y^{-1} yields

$$-2\ \mathbf{sym}((A_v + BK_v)Y^{-1}) \in \mathbb{S}^n_{++} \qquad \forall v = 1..N_v. \tag{10}$$

Problems (9a) and (9b) are convex after substituting $S_v = K_v Y^{-1}$ (using (10)) and $S_v = K_v P$ respectively [29].

2.3 Quadratic Matrix Inequalities

This section reviews QMIs and the matrix S-Lemma approach proposed by [14, 16].

Definition 2.2. *Given a matrix* $M \in \mathbb{S}^n$*, a QMI is the quadratic statement in* $X \in \mathbb{R}^{n \times k}$ *that* $X^T M X \in \mathbb{S}^k_{+}$*.*

QMIs can also be strict with $X^T M X \in \mathbb{S}^k_{++}$. The works in [14, 16] present conditions under which one QMI implies another QMI, with specific attention on the scenario where X can be partitioned as $X = [I; Z^T]$ for some Z. In this case, the variable Z is referred to as satisfying a QMI constraint.

Definition 2.3. *Let* $\Phi \in \mathbb{S}^{n+k}$ *be a partitioned matrix,*

$$\Phi_{11} \in \mathbb{S}^n, \ -\Phi_{22} \in \mathbb{S}^k_{+}. \tag{11a}$$

A matrix $Z \in \mathbb{R}^{n \times k}$ *satisfies the Quadratic Boundedness Property with respect to* Φ *(*$Z \in QBP(\Phi)$*) if*

$$\begin{bmatrix} I_n \\ Z^T \end{bmatrix}^T \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} \begin{bmatrix} I_n \\ Z^T \end{bmatrix} \in \mathbb{S}^k_{+}. \tag{11b}$$

Lemma 2.3 (Theorem 3.2b of [16])**.** *Assuming that* Φ *satisfies* (11a)*, let* $\Phi \mid \Phi_{22}$ *be the Generalized Schur complement* $\Phi_{11} - \Phi_{12}\Phi_{22}^{\dagger}\Phi_{12}$*,* $\|\cdot\|_F$ *be the Frobenius norm, and* $\lambda_{\max}$ *(*$\lambda_{\min}$*) be the maximum (minimum) matrix eigenvalue. Then for all matrices* $Z \in QBP(\Phi)$ *:*

$$\|Z + \Phi_{22}^{-1}\Phi_{12}\|_F^2 < k\lambda_{\max}(\Phi \mid \Phi_{22})/\lambda_{\min}(-\Phi_{22}).$$

Z *is therefore bounded if* $-\Phi_{22} \in \mathbb{S}^k_{++}$*.*

Definition 2.4. *The Strict Quadratic Boundedness Property ($Z \in SQBP(\Phi)$) holds if the matrix in* (11b) *is in* $\mathbb{S}^k_{++}$.

Structures of Φ are listed in Section 2 of [16]. Particular instances include energy bounds $\Phi_{11} - ZZ^T \in \mathbb{S}^n_+$ (with $\Phi_{12} = \mathbf{0}$, $\Phi_{22} = -I_k$) and individual sample L_2 bounds (adding some conservatism) $\forall k' = 1..k$: $\|z_{k'}\|_2 \leq \epsilon$, (with $\Phi_{11} = \epsilon^2 k I_n, \Phi_{12} = \mathbf{0}, \Phi_{22} = -I_k$).

Theorem 2.4 (Strict Matrix S-Lemma, [Cor. 4.13 of [16]])**.** *Let $M, N \in \mathbb{S}^{m+k}$ be matrices satisfying* (11a) *with the same partitioning scheme and let $Z \in \mathbb{R}^{n \times k}$. The following conditions are equivalent under the assumptions that $ker N_{22} \subseteq ker N_{12}$, $N \mid N_{22} \in \mathbb{S}^n_+$, and $-M_{22} \in \mathbb{S}^k_{++}$:*

$$Z \in SQBP(M), \quad \forall Z \in QBP(N) \tag{12a}$$

$$\exists \alpha \geq 0, \beta > 0 : \tag{12b}$$

$$M - \alpha N - \begin{bmatrix} \beta I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{k \times k} \end{bmatrix} \in \mathbb{S}^{m+k}_+.$$

3 LPV Stabilization with QMIs

3.1 Problem Description

A sampling process records a set of T observations from an unknown LPVA system (3) under a bounded noise process $w(\cdot)$ (discrepancy) for $t = 0..T$

$$\delta x(t) = \left(\textstyle\sum_{\ell=1}^L A_\ell \theta_\ell \right) x(t) + Bu(t) + w(t). \tag{13}$$

This data is collected into matrices $(\mathbf{X}_-, \mathbf{U}, \mathbf{\Theta})$

$$\begin{array}{ccccccc} \mathbf{X}_- & := & [x(0) & x(1) & \dots & x(T-1)] \\ \mathbf{U} & := & [u(0) & u(1) & \dots & u(T-1)] \\ \mathbf{\Theta} & := & [\theta(0) & \theta(1) & \dots & \theta(T-1)]. \end{array} \tag{14}$$

The derivative observations $\dot{\mathbf{X}}$ (continuous-time) and one-step-ahead records $\mathbf{X}_+$ (discrete-time) are

$$\begin{array}{ccccccc} \dot{\mathbf{X}} & := & [\dot{x}(0) & \dot{x}(1) & \dots & \dot{x}(T-1)] \\ \mathbf{X}_+ & := & [x(1) & x(2) & \dots & x(T)]. \end{array} \tag{15}$$

The symbol $\mathbf{X}_\delta$ will refer to $\dot{\mathbf{X}}$ or $\mathbf{X}_+$ as appropriate. The data $\mathcal{D}$ will denote the tuple $(\mathbf{X}_-, \mathbf{U}, \mathbf{\Theta}, \mathbf{X}_\delta)$.

Let $\mathbf{\Theta}_\ell \in \mathbb{R}^{1 \times T}$ be the row of $\mathbf{\Theta}$ associated with parameter θ_ℓ. The discrepancy $\mathbf{W}$ collected from (13) (mathematically equivalent to process noise for discrete-time) associated with the observations in $\mathcal{D}$ for a given LPVA $(A(\theta), B)$ is

$$\mathbf{W} = \mathbf{X}_\delta - \left(\textstyle\sum_{\ell=1}^L \mathbf{\Theta}_\ell \otimes_{\text{col}} A_\ell \right) \mathbf{X}_- - B\mathbf{U}. \tag{16}$$

The following assumptions will be imposed,

- A1 n, m, L, T are all finite and known.
- A2 The set Θ is a known compact non-empty polytope with vertices Ω.
- A3 The ground truth system has LPVA structure (3).
- A4 There exists a known $\Phi \in \mathbb{S}^{n+T}$ satisfying (11a) such that $\mathbf{W} \in \text{QBP}(\Phi)$ for the ground-truth system.

The consistency set of plants $(A(\theta), B)$ compatible with $\mathcal{D}$ given Φ is

$$\Sigma_{\mathcal{D}}(\Phi) = \{(\{A_\ell\}_{\ell=1}^L, B) \mid \mathbf{W} \text{ from (16)} \in \text{QBP}(\Phi)\}.$$

Remark 2. *Data matrices arising from multiple trajectories may be horizontally concatenated if the noise structure in* Φ *is compatible with the arrangement (Example 2 of [30]).*

Our goal is to solve the following problem,

Problem 3.1. *Find a gain-scheduled (Eq. (4)) control policy* $u = K(\theta)x$ *such that* $x_+ = (A(\theta) + BK(\theta))x$ *is quadratically stable for all* $(\{A_\ell\}, B) \in \Sigma_\mathcal{D}$.

Remark 3. *Problem* (3.1) *will be solved by enforcing that* (9) *holds for all* $(\{A_\ell\}, B) \in \Sigma_\mathcal{D}$ *(Lemma 2.2).*

3.2 Data Consistency QMI

The set $\Sigma_\mathcal{D}(\Phi)$ may be represented as a QMI.

Using the convention that $\{A_\ell\} = [A_1, A_2, \ldots, A_L]$ and $\{A_\ell^T\} = [A_1^T; A_2^T; \ldots; A_L^T]$, the discrepancy matrix $\mathbf{W}$ from (16) may be represented as

$$\begin{bmatrix} I_n \\ \mathbf{W}^T \end{bmatrix} = \begin{bmatrix} I_n & \mathbf{X}_\delta \\ \mathbf{0}_{n\times Ln} & -\mathbf{\Theta} \otimes_{\text{col}} \mathbf{X}_- \\ \mathbf{0}_{n\times m} & -\mathbf{U} \end{bmatrix}^T \begin{bmatrix} I_n \\ \{A_\ell^T\} \\ B^T \end{bmatrix}. \tag{17}$$

Defining the matrix $\Psi \in \mathbb{S}^{n+(Ln+m)}$ as

$$\Psi = \begin{bmatrix} I_n & \mathbf{X}_\delta \\ \mathbf{0}_{n\times Ln} & -\mathbf{\Theta} \otimes_{\text{col}} \mathbf{X}_- \\ \mathbf{0}_{n\times m} & -\mathbf{U} \end{bmatrix} \Phi \begin{bmatrix} I_n & \mathbf{X}_\delta \\ \mathbf{0}_{n\times Ln} & -\mathbf{\Theta} \otimes_{\text{col}} \mathbf{X}_- \\ \mathbf{0}_{n\times m} & -\mathbf{U} \end{bmatrix}^T, \tag{18}$$

it holds that the following two descriptions are identical:

$$(\{A_\ell\}, B) \in \Sigma_\mathcal{D}(\Phi) \qquad \leftrightarrow \qquad [\{A_\ell\}, B] \in \text{QBP}(\Psi). \tag{19}$$

3.3 Stabilization QMI

This section will form a QMI for stabilization of the subsystem $A_v \in \mathbb{R}^{n\times n}$ at vertex v from (6) by a controller $K_v \in \mathbb{R}^{m\times n}$. The continuous-time LMI criterion in (10) is equivalent to the following QMI

$$[\{A_\ell\}, B] \in \text{SQBP}\begin{pmatrix} \mathbf{0} & * & * \\ -\omega_v \otimes_{\text{col}} Y^{-1} & \mathbf{0} & * \\ -K_v Y^{-1} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{20}$$

as obtained by pre- and post-multiplying (9a) by the invertible $Y^{-1} \in \mathbb{S}^n_{++}$. The discrete-time LMI criterion in (9b) is equivalent to the following QMI by collecting terms

$$[\{A_\ell\}, B] \in \text{SQBP}\begin{pmatrix} P & * & * \\ \mathbf{0} & -(\omega_v \omega_v^T) \otimes P & * \\ \mathbf{0} & -(\omega_v^T) \otimes (K_v P) & -K_v P K_v^T \end{pmatrix}. \tag{21}$$

Theorem 3.2 (Continuous-Time)**.** *Under assumptions A1-A5, QMI* (20) *holds for all* $(\{A_\ell\}, B) \in \Sigma_\mathcal{D}(\Phi)$ *if and only if* $\exists \alpha_v \geq 0, \beta_v > 0$ *such that*

$$\begin{bmatrix} -\beta_v I_n & * & * \\ -\omega_v \otimes_{col} Y^{-1} & \mathbf{0} & * \\ -K_v Y^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix} - \alpha_v \Psi \in \mathbb{S}_+^{(L+1)n+m}. \tag{22}$$

Proof. This will follow a similar proof strategy as Sections IV of [14] and V.I of [16]. The (α_v, β_v) structure follows from Theorem 2.4. It remains to affirm the assumptions under which this theorem is valid. Given that $Y \in \mathbb{S}^n_{++}$ and $-\Phi_{22} \in \mathbb{S}^T_+$ (A4), the lower-right corner of the matrix in (20) and Ψ may each be expressed as

$$-\mathbf{0}_{(L+1)n+m} \in \mathbb{S}^{Ln+m}_+ \tag{23a}$$

$$-\begin{bmatrix} \mathbf{\Theta} \otimes_{\text{col}} \mathbf{X}_- \\ \mathbf{U} \end{bmatrix} \Phi_{22} \begin{bmatrix} \mathbf{\Theta} \otimes_{\text{col}} \mathbf{X}_- \\ \mathbf{U} \end{bmatrix}^T \in \mathbb{S}^{Ln+m}_+. \tag{23b}$$

The final condition is that $\ker\Psi_{22} \subseteq \ker\Psi_{12}$ with

$$\ker\Psi_{22} = \ker \begin{bmatrix} \mathbf{\Theta} \otimes_{\text{col}} \mathbf{X}_- \\ \mathbf{U} \end{bmatrix} \tag{24a}$$

$$\ker\Psi_{12} = \ker(\Phi_{12} + \Phi_{22}\mathbf{X}_\delta) \begin{bmatrix} \mathbf{\Theta} \otimes_{\text{col}} \mathbf{X}_- \\ \mathbf{U} \end{bmatrix}. \tag{24b}$$

All conditions are satisfied, so Theorem 3.2 is proven. □

Theorem 3.3 (Discrete-Time)**.** *Under assumptions A1-A5, QMI* (21) *is satisfied* $\forall(\{A_\ell\}, B) \in \Sigma_{\mathcal{D}}(\Phi)$ *if and only if* $\exists\alpha_v \geq 0\beta_v > 0$ *such that*

$$\begin{bmatrix} P - \beta_v I_n & * & * \\ \mathbf{0} & -(\omega_v\omega_v^T) \otimes P & * \\ \mathbf{0} & -(\omega_v^T) \otimes (K_v P) & -K_v P K_v^T \end{bmatrix} - \alpha_v \Psi \\ \in \mathbb{S}^{(L+2)n+m}_+. \tag{25}$$

Proof. This proof follows the same pattern as in the above Theorem 3.2. The only modification required is demonstrating that the negative of the lower right-corner matrix in (21) is PSD, which holds by

$$\begin{bmatrix} (\omega_v\omega_v^T) \otimes P & * \\ (\omega_v^T) \otimes (K_v P) & K_v P K_v^T \end{bmatrix} = \begin{bmatrix} \omega_v \otimes I_n \\ K_v P \end{bmatrix} P \begin{bmatrix} \omega_v \otimes I_n \\ K_v P \end{bmatrix}^T. \tag{26}$$

All other conditions are valid, completing the proof. □

3.4 Controller Generation Program

This subsection will pose a pair of Semidefinite Programs (SDPs) to solve data-driven LPV stabilization under continuous-time and discrete-time, as introduced by Remark 3 under assumptions A1-A5. In the language of [30], the tuple $(\mathcal{D}, \Phi, \Omega)$ is *informative* for LPV quadratic stabilization if the respective LMI is feasible.

3.4.1 Continuous-Time

The first matrix of (22) admits the substitution $P = Y^{-1},\ S_v = K_v P$ to form the LMI

$$\begin{bmatrix} -\beta_v I_n & * & * \\ -\omega_v \otimes_{\text{col}} P & \mathbf{0} & * \\ -S_v & \mathbf{0} & \mathbf{0} \end{bmatrix} - \alpha_v \Psi \in \mathbb{S}^{(L+1)n+m}_+. \tag{27}$$

The continuous-time stabilization SDP with gain-scheduled control matrices $\{K_v = S_v P^{-1}\}_{v=1}^{N_v}$ is

$$\text{find } P \in \mathbb{S}^n_{++} \tag{28a}$$

$$\alpha \in \mathbb{R}^{N_v}_{\geq 0},\ \beta \in \mathbb{R}^{N_v}_{>0}, \tag{28b}$$

$$S_v \in \mathbb{R}^{m \times n} \qquad \forall v = 1..N_v \tag{28c}$$

$$\text{LMI (27) holds} \qquad \forall v = 1..N_v. \tag{28d}$$

3.4.2 Discrete-Time

The first matrix in (25) may be expressed using a substitution $S_v = K_v P$ (with $K_v P K_v^T = S_v P^{-1} S_v^T$)

$$\begin{bmatrix} P - \beta_v I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\omega_v \omega_v^T) \otimes P & -\omega_v \otimes (S_v^T) \\ \mathbf{0} & -(\omega_v^T) \otimes S_v & S_v P^{-1} S_v^T \end{bmatrix}, \tag{29}$$

followed by a Schur Complement

$$\rightarrow \begin{bmatrix} P - \beta_v I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\omega_v \omega_v^T) \otimes P & -\omega_v \otimes (S_v^T) & \mathbf{0} \\ \mathbf{0} & -(\omega_v^T) \otimes S_v & \mathbf{0} & S_v \\ \mathbf{0} & \mathbf{0} & S_v & P \end{bmatrix}. \tag{30}$$

Letting $\Gamma_v(\beta_v)$ be the matrix in (30), the LMI (25) from Theorem 3.3 may be restated as,

$$\Gamma_v(\beta_v) - \alpha_v \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n \times n} \end{bmatrix} \in \mathbb{S}_+^{n(L+2)+m}. \tag{31}$$

The discrete-time stabilization SDP with gain-scheduled control matrices $\{K_v = S_v P^{-1}\}_{v=1}^{N_v}$ is

$$\begin{aligned}
&\text{find } P \in \mathbb{S}_{++}^n && && \text{(32a)} \\
&\quad \alpha \in \mathbb{R}_{\geq 0}^{N_v}, \ \beta \in \mathbb{R}_{>0}^{N_v}, && && \text{(32b)} \\
&\quad S_v \in \mathbb{R}^{m \times n} && \forall v = 1..N_v && \text{(32c)} \\
&\quad \text{LMI (31) holds} && \forall v = 1..N_v. && \text{(32d)}
\end{aligned}$$

Remark 4. *In the specific discrete-time case where $L = 1$ and $\Theta = \{\theta = 1\}$, Eq.* (32) *is identical to Thm. 14 of [14].*

Remark 5. *Programs* (28) *and* (32) *can be normalized by constraining $Tr(P) = 1$.*

3.5 Computational Considerations

The per-iteration complexity of solving an SDP using an interior point method up to arbitrary (nonzero) accuracy with a single PSD variable of size N with M affine constraints is $O(N^3 M + M^2 N^2)$ [31]. The continuous-time SDP in (28) has 1 PSD constraint of size n (28a) and N_v PSD constraints of size $n(L+1)+m$ (28d). The discrete-time SDP in (32) has 1 PSD constraint of size n (32a) and N_v PSD constraints of size $n(L+2)+m$ (32d).

The performance of SDPs (28) and (32) therefore scales linearly in N_v, polynomially in (n, L, m), and independently of T. Linear dependence on N_v may result in an exponential scaling on L (e.g. a hypercube with $N_v = 2^L$).

4 H2 Optimal Control

A continuous-time LPVA state-space system with external input $\xi \in \mathbb{R}^e$ and regulated output $z \in \mathbb{R}^r$ given matrices $C \in \mathbb{R}^{r \times n}$, $D \in \mathbb{R}^{r \times m}$, $F \in \mathbb{R}^{n \times e}$ is

$$\dot{x} = \textstyle\sum_{\ell=1}^{L} \theta_\ell A_\ell x + Bu + F\xi, \qquad z = Cx + Du. \tag{33}$$

The recorded data in $\mathcal{D}$ has $\xi = 0$ while $\mathbf{W} \in \mathrm{QBP}(\Phi)$. The input ξ is applied during system execution.

Define the H_2 norm of (33) as the worst-case (over all parameter trajectories) expected root-mean-square value of $\|z\|_2$ when the input ξ is a white noise process with identity covariance. Then we have the following bound:

Proposition 4.1. *There exists a gain-scheduled controller $u = K(\theta)x$ such that the closed-loop H_2 norm of the LPVA system* (33) *is bounded above by $\gamma \in \mathbb{R}_+$ if for all $v = 1..N_v$ the following LMI is feasible [32]*

$$\begin{aligned}
\underset{P,Z,S}{\text{find}} \quad & -2\ \boldsymbol{sym}(A_v P + B S_v) - F F^T \in \mathbb{S}^n_{++} && (34a)\\
& \begin{bmatrix} Z & CP + DS_v \\ * & P \end{bmatrix} \in \mathbb{S}^{n+r}_{++} && (34b)\\
& Tr(Z) \leq \gamma^2 && (34c)\\
& P \in \mathbb{S}^n_{++},\ Z \in \mathbb{S}^r_{+},\ S_v \in \mathbb{R}^{m \times n}. && (34d)
\end{aligned}$$

The gain-scheduled controller $K(\theta)$ may be recovered from $\{\forall v : K_v = S_v P^{-1}\}$ and Eq. (4). The variables (Z, P) and given entries (C, D, F) are independent of $(A, B) \in \Sigma_{\mathcal{D}}$.

$$\begin{bmatrix} -\beta_v I_n - F F^T & * & * \\ -\omega_v \otimes_{\text{col}} P & \mathbf{0} & * \\ -S_v & \mathbf{0} & \mathbf{0} \end{bmatrix} - \alpha_v \Psi \in \mathbb{S}^{(L+1)n+m}_{+}. \tag{35}$$

Constraint (35) is equal to (27) when $F = \mathbf{0}_{n \times e}$, given that conditions (34a) and (9a) are identical under this restriction.

Worst-case H_2 control of (33) for all $(A(\theta), B) \in \Sigma_{\mathcal{D}}$ given (C, D, F) may be conducted by solving

$$\begin{aligned}
\gamma^2 = \inf \quad & \text{Tr}(Z) && (36a)\\
& P \in \mathbb{S}^n_{++},\ Z \in \mathbb{S}^r_{+} && (36b)\\
& \alpha \in \mathbb{R}^{N_v}_{\geq 0},\ \beta \in \mathbb{R}^{N_v}_{>0}, && (36c)\\
& \text{LMIs (34b) and (35) hold} \qquad \forall v = 1..N_v. && (36d)
\end{aligned}$$

The resultant H_2 norm is upper-bounded by $\gamma = \sqrt{\text{Tr}(Z)}$ when using gain-scheduled control matrices $\{K_v = S_v P^{-1}\}_{v=1}^{N_v}$. All results in this section may be extended to discrete-time H_2 control with appropriate LMIs.

5 Numerical Examples

Experiments were written in Matlab R2021a and are available at `https://github.com/jarmill/lpv_qmi` in the folder `experiments`. Dependencies include Mosek [33] and YALMIP [34]. For both examples, the problem of finding a θ-independent controller $K^c \in \mathbb{R}^{m \times n}$ with $\forall v :\ K_v = K^c$ that stabilizes all plants $(\{A_\ell\}, B)$ in the consistency set $\Sigma_{\mathcal{D}}$ is infeasible.

5.1 Two-Parameter, Two-State

The experiment ground truth with $\Theta = [0, 2] \times [-1, 1]$ is

$$A_1^{\text{true}} = \begin{bmatrix} -0.2396 & -0.5845 \\ 0.5845 & -0.2396 \end{bmatrix} \qquad A_2^{\text{true}} = \begin{bmatrix} -0.1696 & 0.8434 \\ 0.8434 & 0.4140 \end{bmatrix}$$
$$B^{\text{true}} = \begin{bmatrix} 0 & -1.0072 \\ 0.4848 & 0 \end{bmatrix} \tag{37}$$

The plant A_2 in (37) is open-loop unstable for both continuous-time and discrete-time with eigenvalues of $-0.7703, 1.0146$. Data $\mathcal{D}$ with $T = 35$ was collected under an individual-sample noise bound of $\epsilon = 0.1$.

5.1.1 Continuous-Time

Eq. (28) synthesizes the following continuous-time vertex-controllers

$$\begin{aligned}
K_{(0,1)} &= \begin{bmatrix} -4.5348 & -10.0625 \\ 9.9319 & 6.7597 \end{bmatrix} \\
K_{(0,-1)} &= \begin{bmatrix} -4.7998 & -10.5553 \\ 10.7794 & 7.1231 \end{bmatrix} \\
K_{(2,1)} &= \begin{bmatrix} -4.7566 & -9.8257 \\ 9.5553 & 6.4091 \end{bmatrix} \\
K_{(2,-1)} &= \begin{bmatrix} -4.7646 & -9.7462 \\ 9.8597 & 6.4104 \end{bmatrix}.
\end{aligned} \tag{38}$$

The LMI parameters associated with K in (38) are

$$\begin{aligned}
P &= \begin{bmatrix} 0.0738 & -0.0149 \\ -0.0149 & 0.0361 \end{bmatrix} \in \mathbb{S}^2_{++} \\
\alpha &= [0.0535, 0.0577, 0.0518, 0.0530] \in \mathbb{R}^4_{\geq 0} \\
\beta &= [10^{-5}, 10^{-5}, 10^{-5}, 10^{-5}] \in \mathbb{R}^4_{>0}.
\end{aligned} \tag{39}$$

The blue trajectories in Figure 1 are system executions from 15 plants in the set $([A_1, A_2], B) \in \Sigma_{\mathcal{D}}$ starting from the point $x(0) = [-2; 1.5]$. The parameter values θ are drawn uniformly from $[0, 2] \times [-1, 1]$ with exponentially distributed switching times (mean switching time is 0.05). The red dotted-line in the top plot is the ground truth system from (37) given the fixed parameter sequence. The bottom plot contains system trajectories for 30 parameter sequences on the ground truth and each of the 15 sampled plants.

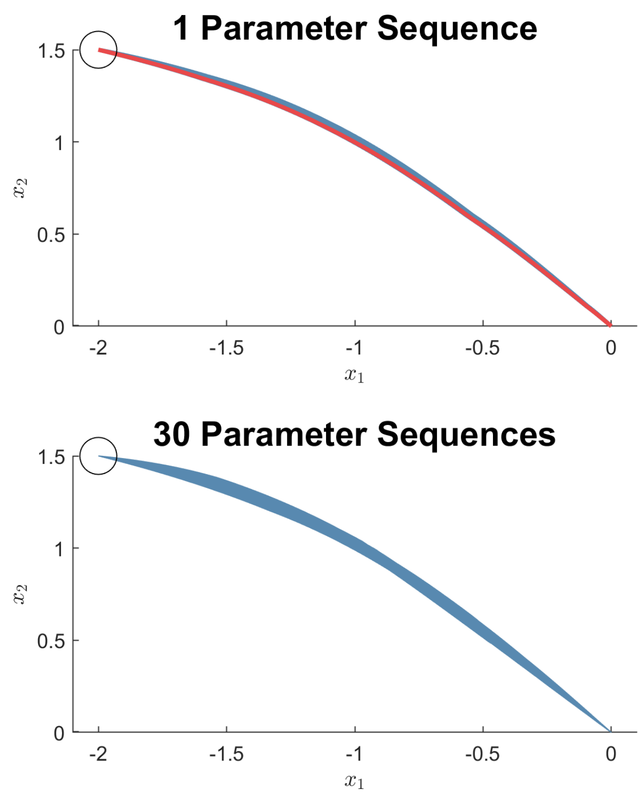

Figure 1: Plots of controlled trajectories using (38)

5.1.2 Discrete-Time

Eq. (32) with the same data X_δ creates the following discrete-time vertex-controllers

$$\begin{aligned} K_{\theta=(0,1)} &= \begin{bmatrix} -1.2258 & -0.6755 \\ -0.1672 & 0.7948 \end{bmatrix} \\ K_{\theta=(0,-1)} &= \begin{bmatrix} 1.2258 & 0.6755 \\ 0.1672 & -0.7948 \end{bmatrix} \\ K_{\theta=(2,1)} &= \begin{bmatrix} -3.4132 & 0.1113 \\ -0.6730 & -0.3555 \end{bmatrix} \\ K_{\theta=(2,-1)} &= \begin{bmatrix} -0.5723 & 1.4858 \\ -0.3528 & -1.9440 \end{bmatrix}. \end{aligned} \tag{40}$$

The resultant P matrix is $[0.0588, 0.0014; 0.0014, 0.1022]$.

Figure 2 visualizes a discrete-time trajectory of the ground truth ground-truth and 15 sample plants when the controller (40) is applied to a single parameter sequence starting at $x(0) = [-2, 1.5]$. The bottom plot displays a sampled reachable set attained from 30 parameter sequences and all plants (15 sample plants plus ground truth).

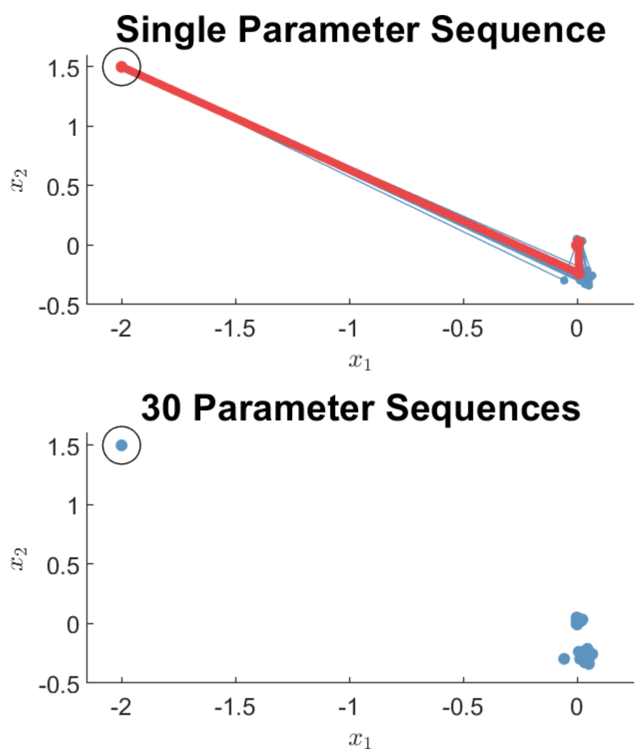

Figure 2: Plots of controlled trajectories using (38)

The discrete-time worst-case controlled H_2 norm with $C = [I_2; \mathbf{0}_2],\ D = [\mathbf{0}_2; \sqrt{2}I_2],\ F = I_2$ is bounded by $\gamma = 9.334$ by Eq. (36).

5.2 Three-Parameter, Five-State

The second experiment involves a system with $n = 5, m = 3, L = 3$. The parametric set is $\Theta = [-0.3, 0.3] \times [0.2, 0.8] \times [0.5, 1.5]$ with $N_v = 8$. A trajectory is recorded with a time horizon of $T = 50$ and an individual-sample noise bound of $\epsilon = 0.1$. The associated P matrix to the controller is

$$P = \begin{bmatrix} 0.268 & 0.141 & -0.112 & 0.113 & -0.171 \\ 0.141 & 0.322 & -0.151 & 0.250 & -0.277 \\ -0.112 & -0.151 & 0.410 & 0.210 & -0.130 \\ 0.113 & 0.250 & 0.210 & 1.226 & -1.138 \\ -0.171 & -0.277 & -0.123 & -1.138 & 1.191 \end{bmatrix}.$$

6 Conclusion

This work considered quadratic stabilization of all LPV systems $(\{A_\ell\}, B) \in \Sigma_{\mathcal{D}}(\Phi)$. SDPs (28) and (32) perform this task by solving a set of $N_v + 1$ LMIs in order to recover a gain-scheduled controller. The unknown LPVA plants may be regulated using a worst-case H_2-optimal controller. Sparsity of the LMIs may be employed to speed up computation of these controllers. Future work involves finding $K(\theta)$ policies using methods that scale based on the number of faces of Θ rather than on N_v and reducing the conservatism of $K(\theta)$-controllers by letting P depend on θ.

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