Iterative Learning Control of Discrete Systems with Actuator Backlash using a Weighted Sum of Previous Trial Control Signals

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***Abstract*— This paper considers iterative learning control design for discrete dynamics in the presence of backlash in the actuators. A new control design for this problem is developed based on the stability theory for nonlinear repetitive processes. An example of this design's effectiveness is where the dynamics model was obtained using data collected from frequency response tests on a physical system.**

I. Introduction

Iterative learning control (ILC) emerged from the problem of how to better the operation of robots that complete the same finite-duration task over and over again [1]. The key feature is that the dynamics repetitively operate over a finite duration, where one example is a pick-and-place robot undertaking the following tasks in sequence, i) collect the payload from a specified location, ii) transfer the payload over a fixed time interval, iii) place the payload on a moving conveyor under synchronization, iv) return to the starting location, and v) repeat i)-iv) as many times as required or until a halt is needed for maintenance or other reasons.

In the literature, each repetition is termed a trial (iteration or pass are also used), and the finite duration is known as the trial length. Suppose that a reference trajectory is specified representing the desired behavior of the output on any trial. Then the error on each trial is defined as the difference between this trajectory and the output of this trial. Also the control problem can be specified as the construction of a sequence of trial inputs that force the sequence of trial errors to converge, under an appropriate norm, with the trial number either to zero (the ideal case) or within a specified tolerance.

A prevalent form of ILC law constructs the input for the subsequent trial as the sum of the previous trial input plus a correction. Once a trial is complete, all information generated during its execution is available, at the cost of storage, to update the control input to be applied on the subsequent trial. Consider discrete dynamics at sample p on trial k: Then the construction of the following trial input at this sample can, as one example, use information from sample $p+\lambda$ on the previous trial. Using such information is the distinguishing feature of ILC over alternative forms of control action.

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Early results on ILC research can be found, e.g., in the survey papers [2], [3]. Since then, ILC has remained an active area of research both in developing new theoretical results and design methods and experimental validation and implementation. More recent developments include applications to additive manufacturing, e.g. high-precision multilayer laser deposition systems [4], robotic-assisted stroke rehabilitation, where the initial results are in [5] with more recent work in, e.g., [6]. Supporting clinical trial results have also been reported. Also, an application to heart ventricular support devices, e.g., [7] has been reported.

This paper reports new results on two aspects of ILC design. The first is the problem of actuator backlash arising in an implementation, which introduces nonlinearity into a linear design. The appearance of nonlinearities in the actuators can have, at the very least, a detrimental effect on the control signal applied to the system. Typical effects of nonlinear behavior in the actuators include reducing the achievable accuracy, slowing down the ILC law convergence from trial to trial or result in complete failure. Consequently, control design in the presence of such implementation nonlinearities is a critical issue in some applications.

The second aspect of the ILC design considered in this paper relates to using data from previous trials. In many ILC designs, the input for the subsequent trial is constructed as the sum of the input used on the previous trial plus a correction term that uses information from the error on the previous trial. In general, however, information (at the cost of data storage) from all previously completed trials is available to compute the input for the subsequent trial. In this paper, the interest is in using a law (sometimes called higher-order) that uses a static and dynamic combination of previous input vectors, the current trial error, and the errors on a finite number of previous trials. Particularly, using a finite number of control inputs on previous trials for design in the presence of input backlash is considered.

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Previous research on ILC with backlash includes [8], where a Timoshenko beam system described by a second order distributed parameter model was considered, and the backlash term is divided into a linear term and an unknown bounded term, which is estimated. Also, in [9], a model of a two-link rigid-flexible manipulator with backlash is considered, where the analysis is in an identical manner to [8], and the effects of an external disturbance are also considered. Both of these designs apply only to the specific systems they consider. In [10], an adaptive ILC scheme for a particular class of nonlinear systems with unknown time-varying delays and control direction preceded by an unknown nonlinear backlash-like hysteresis is considered.

This paper develops a new design for ILC in the presence of backlash in the actuator, where the control law includes the weighted sum of the control signals on a finite number of previous trials The approach is based on representing the dynamics in the form of a repetitive process, where such processes are a a distinct class of 2D systems and on the further development of the vector Lyapunov functions approach to the stability of nonlinear repetitive processes, see, e.g. [11]. A simulation-based case study using a model of a physical process constructed from measured frequency response data highlights the benefits of the new design.

Throughout this paper, the notation for variables is of the form $h_k(p)$, $0 \leq p \leq N-1$, $k \geq 0$, where h denotes the scalar or vector-valued variable under consideration, N denotes the number of samples along a trial (N times the sampling period gives the trial length) and the integer k denotes the trial number. Moreover, $\succ 0$ and $\prec 0$ denote a symmetric positive definite and a symmetric negative definite matrix. Also, $\succeq 0$ and $\preceq 0$ denote, respectively, a symmetric positive semi-definite and a symmetric positive semi-definite matrix.

II. Problem Description

Consider a single-input single-output discrete-time system operating in a repetitive mode, where on trial k the dynamics are described by the state-space model

$$\begin{aligned} x_k(p+1) &= Ax_k(p) + B\psi_k(p), \\ \psi_k(p) &= \mathrm{back}(u_k(p)), \\ y_k(p) &= Cx_k(p) \quad p \in [0, N-1], \quad k \geq 1, \end{aligned} \tag{1}$$

where $x_k(p) \in \mathbb{R}^{n_x}$ is the state vector, $u_k(p)$ is the control input, $y_k(p)$ is the trial profile (or output), and $\psi_k(p)$ denotes the backlash function. This last function is illustrated in Fig. 1), and described [12] by

$$\psi_k(p) = \begin{cases} m_l(u_k(p) - c_l), & \text{if } u_k(p) \leq \underline{u}_k(p), \\ m_r(u_k(p) - c_r), & \text{if } u_k(p) \geq \overline{u}_k(p), \\ \psi_k(p-1), & \text{if } \underline{u}_k(p) < u_k(p) < \overline{u}_k(p), \end{cases} \tag{2}$$

where $m_l,\ m_r,\ c_r$ are positive constants, c_l is a negative constant, and

$$\begin{aligned} \underline{u}_k(p) &= \frac{1}{m_l}\psi_k(p-1) + c_l, \\ \overline{u}_k(p) &= \frac{1}{m_r}\psi_k(p-1)) + c_r. \end{aligned}$$

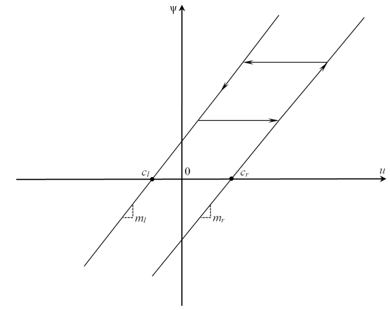

Fig. 1. Backlash model (shown for input u and output y.

This paper considers the case when $m_r = m_l = m$, and $cl = c_r$ which is of application relevance (the other cases follow by appropriate amendments to the analysis in this paper). Also, no loss of generality results from assuming that the boundary conditions are $x_k(0) = 0$ and $y_k(0) = f(p)$, where $f(p)$ is known scalar functions of p, $p \in [0,\ N-1]$. It is also assumed that the pair $\{A, B\}$ is controllable and $CB \neq 0$.

Let $y_{\mathrm{ref}}(p) \in \mathbb{R}$, $0 \leq p \leq N-1$ denote the supplied reference signal and then

$$e_k(p) = y_{\mathrm{ref}}(p) - y_k(p) \tag{3}$$

is the error on trial k. The control design problem is to construct a control input sequence $\{u_k\}$, such that

$$||e_k(p)|| \leq \kappa \varrho^k + \mu,\ \kappa > 0,\ \mu \geq 0,\ 0 < \varrho < 1, \tag{4}$$

$$\lim_{k\to\infty} ||u_k(p)|| = ||u_\infty(p)|| < \infty, \tag{5}$$

where the bounded variable $u_\infty(p)$ is termed the learned control, $||\cdot||$ denotes the chosen norm (which is the absolute value for scalar functions). If there is no backlash present, the design developed in this paper reduces to the case for linear dynamics, and $\lim_{k\to\infty} ||e_k(p)|| = 0$ is ensured.

A commonly used ILC law constructs the input for the subsequent trial as the sum of the previous trial input plus a correction term that uses previous trial data. This approach is considered in this paper, and the control law has the structure

$$\psi_{k+1}(p) = \mathrm{back}(\psi_k(p) + \delta u_{k+1}(p)), \tag{6}$$

where $\delta u_{k+1}(p)$ is the control update that is designed using previous trial information. As discussed in the previous section information (at the cost of data storage) from all previously completed trials is available to compute the input for the subsequent trial. In this paper, the interest is in a law (sometimes called higher-order) uses a finite number of control inputs from $d > 1$ previous trials for design. The premise is that the use of such information will assist in overcoming the effects of the backlash and progress in this respect providing motivation for considering other higher-order laws.

In the remainder of this paper, $\psi_k(p)$ is replaced by

$$\Psi_k(p) = \sum_{i=0}^{d} \tau_i \psi_{k-i}(p), \tag{7}$$

where d is the number of previous trials whose control input directly contributes to the computation of the new trial control and $\tau_i,\ 1 \le i \le d$, is a non-negative scalars, and and $\psi_{k-i} = 0$ if $k - i < 0$. Next, the formulation of the dynamics as a nonlinear repetitive process is detailed. Note also from this point onwards $\psi_k(p)$ in (6) denotes this new structure.

III. Representation as a Nonlinear Repetitive Process

Introduce the variables $\check{x}_{k,1}(p) = \psi_k(p),\ \check{x}_{k,2}(p) = \psi_{k-1}(p), \dots, \check{x}_{k,d}(p) = \psi_{k-d+1}(p),\ \check{x}_{k,d+1}(p) = \psi_{k-d}(p)$ and the vector $\check{x}_k = [\check{x}_{k,1}^\top \dots \check{x}_{k,d+1}^\top]^\top$. Then by construction

$$\check{x}_k(p) = A_d \check{x}_{k-1}(p) + B_d \psi_k(p), \tag{8}$$

where

$$A_d = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, B_d = [1\ 0\ 0 \dots 0]^\top.$$

Using (8), the first equation in (1) can be written as

$$x_k(p+1) = Ax_k(p) + BC_d \check{x}_k(p), \tag{9}$$

where $C_d = [\tau_0\ 0 \dots 0\ \tau_d]$.

Introduce, for the design purpose only, the auxiliary vectors

$$\begin{aligned} \eta_k(p) &= x_k(p) - x_{k-1}(p), \\ \check{\eta}_k(p) &= \check{x}_k(p) - \check{x}_{k-1}(p). \end{aligned} \tag{10}$$

Then using (8) and (10) it follows that

$$\check{\eta}_k(p) = A_d \check{\eta}_{k-1}(p) + B_d \Delta\psi_k(p), \tag{11}$$

where $\Delta\psi_k(p) = \psi_k(p) - \psi_{k-1}(p)$. Also

$$\begin{aligned} \eta_k(p+1) &= A\eta_k(p) + BC_d A_d \check{\eta}_{k-1}(p) \\ &+ BC_d B_d \Delta\psi_k(p). \end{aligned} \tag{12}$$

Using (3) $e_k(p) = y_{ref}(p) - Cx_k(p)$, and then using (12) gives the following system of equations in terms of the incremental variables

$$\begin{aligned} \eta_k(p+1) &= A\eta_k(p) + BC_d A_d \check{\eta}_{k-1}(p) \\ &\quad + BC_d B_d \Delta\psi_k(p), \\ \check{\eta}_k(p) &= A_d \check{\eta}_{k-1}(p) + B_d \Delta\psi_k(p), \\ \bar{e}_k(p) &= -CA\eta_k(p) - CBC_d A_d \check{\eta}_{k-1}(p) \\ &\quad + \bar{e}_{k-1}(p) - CBC_d B_d \Delta\psi_k(p), \end{aligned} \tag{13}$$

where $\bar{e}_k(p) = e_k(p+1)$. Also, consider the case of (6)

$$\delta u_k(p) = K_1 \eta_k(p) + K_2 \bar{e}_{k-1}(p), \tag{14}$$

where K_1 and K_2 are matrices of compatible dimensions to be designed. Then, using (13) and (14), the model of the controlled dynamics can be written as

$$\begin{aligned} \eta_k(p+1) &= A_c \eta_k(p) + BC_d A_d \check{\eta}_{k-1}(p) \\ &\quad + BC_d B_d K_2 \bar{e}_{k-1}(p) + BC_d B_d \varphi_k(p), \\ \check{\eta}_k(p) &= B_d K_1 \eta_k(p) + A_d \check{\eta}_{k-1}(p) \\ &\quad + B_d K_2 \bar{e}_{k-1}(p) + B_d \varphi_k(p), \\ \bar{e}_k(p) &= -CA_c \eta_k(p) - CBC_d A_d \check{\eta}_{k-1}(p) \\ &\quad + (1 - CBC_d B_d K_{2i}) \bar{e}_{k-1}(p) \\ &\quad - CBC_d B_d \varphi_k(p), \end{aligned} \tag{15}$$

where $A_c = A + BC_d B_d K_1$, $\varphi_k(p) = \Delta\psi_k(p) - \delta u_k(p)$. Also, it follows from (2) and Fig. 1 that $\Delta\psi_k(p) = \psi_k(p) - \psi(u_{k-1}(p)$ satisfies the constraints

$$m\delta u_k(p) - m\Delta c \le \Delta\psi_k(p) \le m\delta u_k(p) + m\Delta c,$$

where $\Delta c = c_r - c_l$. Also $\varphi_k(p)$ satisfies the constraints

$$m_1 \delta u_k(p) - m\Delta c \le \varphi_k(p) \le m_1 \delta u_k(p) + m\Delta c,$$

or

$$m^2(\Delta c)^2 - [\varphi_k(p) - m_1 \delta u_k(p)]^2 \ge 0, \tag{16}$$

where $m_1 = m - 1$. If $m_r = m_l = m$, $\mathrm{back}(mu) = m\mathrm{back}(u)$ and hence, without the loss of generality, the case $m = 1$ will be considered and the quadratic constraint (16) can then be written as

$$(\Delta c)^2 - \varphi_k(p)^2 \ge 0. \tag{17}$$

The model (15) is in the form of a discrete repetitive process, a particular class of 2D systems. In the presence of backlash, the ILC dynamics are nonlinear, and there has been recent work on developing a stability theory for nonlinear repetitive processes. One approach is based on the use of vector Lyapunov functions [11]. This theory is used for ILC design in this paper, starting from the convergence conditions given in the next section.

IV. Analysis and Design

Introduce the vector $\epsilon_k(p) = [\check{\eta}_k^\top(p) \bar{e}_{k-1}(p)]^\top$ and the vector Lyapunov function for dynamics described by (15) as

$$V(\eta_k(p), \epsilon_k(p)) = \begin{bmatrix} V_1(\eta_k(p)) \\ V_2(\epsilon_k(p)) \end{bmatrix}, \tag{18}$$

where $V_1(\eta_k(p)) > 0,\ \eta_k(p)) \ne 0,\ \ V_2(\epsilon_k(p)) > 0,\ \epsilon_k(p) \ne 0,\ \ V_1(0) = 0,\ V_{2i}(0) = 0$ and define the counterpart of the divergence operator along the trajectories of (15) as

$$\begin{aligned} \mathcal{D}V(\eta_k(p), \epsilon_k(p)) &= V_1(\eta_k(p+1)) - V_1(\eta_k(p)) \\ &+ V_2(\epsilon_{k+1}(p)) - V_2(\epsilon_k(p)). \end{aligned} \tag{19}$$

For ease of presentation this last property will be referred to as the divergence operator in the remainder of this paper.

Theorem 1: Suppose that there exists a vector Lyapunov function (18) and positive scalars c_1, c_2, c_3 and γ for dynamics described by (15) such that

$$c_1||\eta_k(p)||^2 \le V_1(\eta_k(p)) \le c_2||\eta_k(p)||^2, \tag{20}$$
$$c_1||\epsilon_k(p)||^2 \le V_2(\epsilon_k(p)) \le c_2||\epsilon_k(p)||^2, \tag{21}$$

$$\begin{aligned}\mathcal{D}V(\eta_{k+1}(p), e_k(p)) \le \gamma \\ - c_3(||\eta_{k+1}(p)||^2 + ||\epsilon_k(p)||^2).\end{aligned} \tag{22}$$

Then the error convergence conditions of (4) hold under the ILC law (6) where $\delta u_{k+1}(p)$ is given by (14).

Proof: Calculating divergence along the trajectories (15) and following the same steps as in the proof of Theorem 1 in [13] gives

$$||e_k(p)||^2 \le ||\epsilon_k(p)||^2 \le \lambda^k \sum_{q=0}^{p} \lambda^{p-q}||\epsilon_0(q)||^2 + \frac{\gamma}{c_1(1-\lambda)^2},$$

where $\lambda = 1 - \frac{\bar{c}_3}{c_2}$ and $c_3 \le \bar{c}_3 < c_2$, which implies that (4) holds for

$$\varrho = \lambda,\ \kappa = \frac{\alpha}{1-\lambda},\ \alpha = \max_q ||\epsilon_0(q)||^2,\ \delta = \frac{\gamma}{c_1(1-\lambda)^2}$$

. ■

Two ILC designs are developed in this paper, termed one step and two step, respectively.

A. One Step Method

Consider the vector Lyapunov function (18) along the trajectories of (15) in the case when

$$\begin{aligned} V_1(\eta_k(p)) &= \eta_k^\top(p) P_1 \eta_k(p), \\ V_2(\epsilon_k(p)) &= \epsilon_k^\top(p) P_2 \epsilon_k(p), \end{aligned}$$

where $P_1 \succ 0$ and $P_2 \succ 0$ and set $P = \mathrm{diag}[P_1\ P_2]$. Also, introduce $\xi_k(p) = [\eta_k(p)^\top\ \check{\eta}_k(p)^\top\ \bar{e}_k(p)]^\top$ (where in what follows the dependence of some variables on k and p is omitted). Computing (19) along trajectories of (15), gives

$$\begin{aligned} \mathcal{D}V_i(\eta, \epsilon) &= [(\bar{A}_+\bar{B}KH)\xi + \bar{B}\varphi]^\top P[(\bar{A} \\ &+ \bar{B}KH)\xi + \bar{B}\varphi] - \xi^\top P \xi, \end{aligned} \tag{23}$$

where $K = [K_1\ K_2]$,

$$\bar{A} = \begin{bmatrix} A & BC_dA_d & 0 \\ 0 & A_d & 0 \\ -CA & -CBC_dA_d & 1 \end{bmatrix},$$
$$\bar{B}_i = \begin{bmatrix} BC_dB_d \\ B_d \\ -CBC_dB_d \end{bmatrix}\ H = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

Since $V_1(\eta) \succ 0$ and $V_2(\epsilon) \succ 0$, (20) and (21) of Theorem 1 hold.

A sufficient condition for (22) to hold under the constraints (17) is that

$$\begin{aligned} \mathcal{D}V(\eta, \epsilon) &+ \tau((\Delta c)^2 - \varphi^2) \\ &\le \gamma - \xi^\top[Q + (KH)^\top RKH]\xi \end{aligned} \tag{25}$$

holds for all φ and ξ, where $Q \succ 0$ and $R \succ 0$ are matrices of compatible dimensions and $\tau > 0$ (see also [14]). The inequality (25) holds if $\gamma = \tau(\Delta c)^2$ and

$$\begin{bmatrix} (\bar{A}+\bar{B}KH)^\top P(\bar{A}+\bar{B}KH) - P + M & (\bar{A}+\bar{B}KH)^\top P\bar{B} \\ \bar{B}^T P(\bar{A}+\bar{B}KH) & \bar{B}^\top P\bar{B} - \tau \end{bmatrix} \preceq 0,$$

where $M = Q + (KH)^\top RKH$. Rewriting this last inequality as

$$\begin{bmatrix} -P & 0 \\ 0 & -\tau \end{bmatrix} + \begin{bmatrix} (\bar{A}+\bar{B}KH)^\top & I & (KH)^\top \\ \bar{B}^\top & 0 & 0 \end{bmatrix} \times \begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} (\bar{A}+\bar{B}KH) & \bar{B} \\ I & 0 \\ KH & 0 \end{bmatrix} \preceq 0$$

and applying the Schur's complement lemma gives

$$\begin{bmatrix} -X & 0 & (\bar{A}+\bar{B}YH)^\top & X & (YH)^\top \\ 0 & -\tau & \bar{B}^\top & 0 & 0 \\ (\bar{A}+\bar{B}YH) & \bar{B} & -X & 0 & 0 \\ X & 0 & 0 & -Q^{-1} & 0 \\ YH & 0 & 0 & 0 & -R^{-1} \end{bmatrix} \preceq 0,\ i \in [0,\ d], \tag{26}$$

where $X = P^{-1}$, $Y = KW$ and W is a solution of

$$HX = WH. \tag{27}$$

If the system of linear matrix inequalities (26) (LMIs) and the linear matrix equation (27) are solvable for the variables X, Y and W, then the ILC law (6) ensures that the convergence condition (4) holds where $\delta u_{k+1}(p)$ is given by (14) and

$$K = [K_1\ K_2] = YW^{-1}.$$

To prove the boundedness conditions (5) for this ILC law, first note that the convergence condition (4) and (3) imply that $||Cx_\infty(p)|| = \lim_{k\to\infty}||Cx_k(p)||$ is bounded for all $p \in [0,\ N-1]$. If $\tau_0 \neq 0$ it follows from (1) and using (7) that

$$\begin{aligned} &\psi_k(p-1) = \tau_0^{-1}(CB)^{-1}[Cx_k(p) - CA^p x_k(0) \\ &-\sum_{q=0}^{p-1} CA^{p-1-q}B\sum_{i=1}^{d}\tau_i\psi_{k-i}(q) - \sum_{q=0}^{p-2} CA^{p-1-q}B\tau_0\psi_k(q)]. \end{aligned}$$

If $p = 2$

$$\begin{aligned} &\psi_k(1) = \tau_0^{-1}(CB)^{-1}[Cx_k(2) - CA^2x_k(0) \\ &- \sum_{q=0}^{1} CA^{p-1-q}B\sum_{i=1}^{d}\tau_i\psi_{k-i}(q) - CAB\tau_0\psi_k(0)] \end{aligned} \tag{28}$$

All terms in the right hand side of (28) are bounded on the considered interval, hence $\psi_k(1)$ is also bounded on this interval.

If $p = 3$

$$\psi_k(2) = \tau_0^{-1}(CB)^{-1}[Cx_k(p) - CA^3x_k(0) - \sum_{q=0}^{2} CA^{2-q}B\sum_{i=1}^{d}\tau_i\psi_{k-i}(q) - \sum_{q=0}^{1} CA^{2-q}B\tau_0\psi_k(q)]. \quad (29)$$

Since $\psi_k(1)$ is bounded, all terms in the right-hand side of(29) are bounded, and hence $\psi_k(2)$ is also bounded on this interval. Repeating these derivations for all p gives that $\psi_k(p)$ is bounded for all $p \in [0,\ N-1]$ and $k = 0, 1, 2, \ldots$. Then $||\psi_\infty(p)|| = \lim_{k\to\infty} ||\psi_k(p)|| < \infty$ and by the definition of the inverse backlash function [12] the boundedness condition (5) holds. Finally, in the case of $\tau_0 = 0$, the analysis above should be repeated for $\psi_{k-m}(p)$, where m is minimum number for which $\tau_m \neq 0$.

B. Two Step Method

Depending on the specific choice of the vector Lyapunov function (18) entries, various sufficient convergence conditions can be obtained based on Theorem 1, and it is difficult to evaluate in advance how the level of conservativeness of each of them. For example, it could be that the ILC law constructed using a linear model ensures convergence also ensures convergence for nonlinear dynamics. This section develops an alternative design.

Consider the discrete Riccati inequality

$$\bar{A}^\top \bar{P}\bar{A} - (1-\sigma)\bar{P} - \bar{A}^\top\bar{P}\bar{B}[\bar{B}^\top\bar{P}\bar{B} + R]^{-1}\bar{B}^\top\bar{P}\bar{A} + Q \preceq 0 \quad (30)$$

relative to the matrix $\bar{P}_i = \mathrm{diag}[P_1\ P_2] \succ 0$, where $P_1 \in \mathbb{R}^{n_x\times n_x}$, $P_2 \in \mathbb{R}^{d+2\times d+2}$, $0 < \sigma < 1$. Applying Schur's complement formula gives that if the LMI's

$$\begin{bmatrix} (1-\sigma)\bar{X} & X\bar{A}^\top & \bar{X} \\ \bar{A}\bar{X} & \bar{X} + \bar{B}R^{-1}\bar{B}^\top & 0 \\ \bar{X} & 0 & Q^{-1} \end{bmatrix} \succeq 0,$$

$$X_i \succ 0 \quad (31)$$

are solvable for $X = \mathrm{diag}[X_1\ X_2] \succ 0$, where X_1 and X_2 have the same dimensions as P_1 and P_2, respectively, then $P = X^{-1}$.

Define

$$L = [\underbrace{L_1}_{n_x}\ \underbrace{L_2}_{d+1}\ \underbrace{L_3}_{1}] = -[\bar{B}^\top\bar{P}\bar{B} + R]^{-1}\bar{B}^\top\bar{P}\bar{A} \quad (32)$$

$$F = [\underbrace{F_1}_{n_x}\ \underbrace{0}_{d+1}\ \underbrace{F_3}_{1}] = L\Theta, \quad (33)$$

where

$$\Theta = \begin{bmatrix} \Theta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Theta_3 \end{bmatrix}$$

is a matrix with blocks of compatible dimensions, satisfying the LMI's

$$\begin{bmatrix} M - M\Theta - \Theta M - Q & \Theta\sqrt{M} \\ \sqrt{M}\Theta & -I \end{bmatrix} \preceq 0 \quad (34)$$

and $M = \bar{A}^\top\bar{P}\bar{B}[\bar{B}^\top\bar{P}\bar{B} + R]^{-1}\bar{B}^\top\bar{P}\bar{A}$. The following result can now be established.

Theorem 2: Assume that for weighting matrices $Q \succ 0$ and $R \succ 0$ and scalar $0 < \sigma < 1$ the LMI's (31), (34) and

$$\begin{bmatrix} (\bar{A}+\bar{B}KH)^\top S(\bar{A}+\bar{B}KH) - S & (\bar{A}+\bar{B}KH)^\top S\bar{B} \\ \bar{B}_i^\top S(\bar{A}+\bar{B}KH) & \bar{B}^\top S\bar{B} - \tau \end{bmatrix} \prec 0, \quad (35)$$

where

$$K = [F_1\Theta_1\ F_3\Theta_3] \quad (36)$$

are solvable relative to X, $\Theta, \tau > 0$ and $S = \mathrm{diag}[S_1\ S_2] \succ 0$ with blocks of the same dimensiomn as P_1 and P_2. Then the ILC law that ensures that convergence condition (4) holds and also the boundedness condition (5) is given by (6) where $\delta u_{k+1}(p)$ is given by (14) and $K = [K_1\ K_2]$ by (36).

Proof: The inequality (35) implies that for all ξ and φ including those satisfying (17)

$$[(\bar{A}+\bar{B}KH)\xi + \bar{B}\varphi]^\top S[(\bar{A} + \bar{B}KH)\xi + \bar{B}\varphi] - \xi^\top S\xi - \varphi^2 < 0. \quad (37)$$

Since the left hand side is quadratic form relative to ξ and φ and $S > 0$ then all the conditions of Theorem 1 are satisfied and convergence condition (4) holds for $\gamma = \tau(\Delta c)^2$. The boundedness condition (5) is proved by the same way as in previous section. ■

Next, the idea outlined at the beginning of this section can be fully developed. Consider, therefore, the system (1) without backlash. In this case the ILC law has form

$$u_{k+1}(p) = u_k(p) + \delta u_{k+1}(p).$$

If $\delta u_{k+1}(p)$ is obtained as in Theorem 2 then it follows as corollary of theorem 2 from [13] that for this linear case conditions (4), (5) hold with $\gamma = 0$ and by this reason the condition of this Theorem seems less conservative compared to the conditions of the previous section.

V. Case Study

As an example, consider the model of one-axis of the multi-axis gantry robot described in [15]. Frequency response tests (also detailed in [15]) result in the following 3rd order continuous-time transfer-function as an adequate model of the dynamics to use for control law design.

$$G(s) = \frac{23.7356(s + 661.2)}{s(s^2 + 426.7s + 1.744\cdot 10^5)}. \quad (38)$$

The reference trajectory is the same as in [15] with a trial length of 2 secs. For discrete design, a sampling period of 0.01 secs was used and in the backlash nonlinearity (Fig. 1) $m = 1$ and $c_r = -c_l = c$.

As representative results two cases for $d = 1$ are given, and hence in (8)

$$A_d = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C_d = [\tau_0 \ \tau_d].$$

Also in both cases $Q = \mathrm{diag}[1 \ 1 \ 10^4 \ 10 \ 10 \ 0.8 \cdot 10^4]$, $R = 10^{-2}$, $\sigma = 0.0125$.

Case 1 is for $\tau_0 = \tau_d = 1$, and hence on trial $k+1$ information from both trials k and $k-1$ are used. In this case Theorem 2 gives

$$K_1 = [-1.6872 \ -1.3464 \ -551.7177], \ K_2 = 30.1069.$$

Case 2 is when $\tau_0 = 1, \ \tau_d = 0$, and hence on trial $k+1$ only information from trial k is used In this cas Theorem 2 gives

$$K_1 = [-1.1315 \ -0.8714 \ -357.8247], \ K_2 = 36.2096.$$

To measure the performance of this ILC law , the root mean square error for each trial is used, i.e.,

$$\mathrm{RMS}(k) = \sqrt{\frac{1}{N}\sum_{p=0}^{N} ||e_k(p)||^2}, \qquad (39)$$

Figure 2 shows that the Case 1 design accelerates the trial-to-trial error convergence.

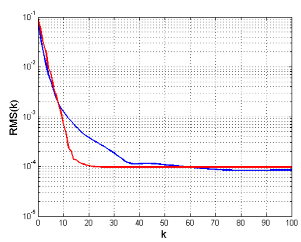

Fig. 2. The RMS_k progression for $c = 0.003$: Case 1 (red line), Case 2 (blue line).

The parameter c in Fig. 1 determines the dead-zone in the system and it is of interest to examine the effects of varying this parameter, where here interest is restricted to the Case 1 design. In this case, Fig. 3 shows progression of the RMS_k progression for two values of c. A greater value of c results in the trial-to-trial error converging to a larger value (recall that if the nonlinearity is present then convergence to zero error may not occur).

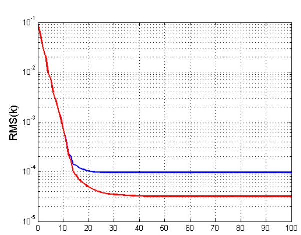

Fig. 3. The RMS_k progression for the Case 1 design with $c = 0.001$ (red line) and $c = 0.003$ (blue line).

VI. Conclusions and Future Work

This paper has developed new results on the effects of actuator backlash on the performance of ILC designs for discrete linear systems. Moreover, the use of a weighted sum of previous trial inputs in the computation the control input for the next trial has been considered. A numerical example confirms the results obtained.

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