

Minimax Linear Optimal Control of Positive Systems

Alba Gurpegui, Emma Tegling and Anders Rantzer

Abstract—We present a novel class of minimax optimal control problems with positive dynamics, linear objective function and homogeneous constraints. The proposed problem class can be analyzed with dynamic programming and an explicit solution to the Bellman equation can be obtained, revealing that the optimal control policy (among all possible policies) is linear. This policy can in turn be computed through standard value iterations. Moreover, the feedback matrix of the optimal controller inherits the sparsity structure from the constraint matrix of the problem statement. This permits structural controller constraints in the problem design and simplifies the application to large-scale systems. We use a simple example of voltage control in an electric network to illustrate the problem setup.

Index Terms—Dynamic Programming, Large-scale systems, Minimax, Optimal Control, Positive Systems.

I. INTRODUCTION

A. Motivation

Optimal control problems with minimax objectives are ubiquitous in control theory and engineering. They provide a powerful framework for modeling and solving problems that involve competition and uncertainty [1], [2]. These problems appear in contexts such as robust control, game theory, and multi-agent systems. In our particular case, they are used to design control systems that are robust to uncertainties and disturbances, with the objective to minimize the worst-case performance of the system. Finding solutions to these types of problems can be a challenging task, especially when dealing with large-scale systems.

In this paper, we present a novel class of minimax optimal control problems, with positive dynamics, linear objective function, and linear, homogeneous, constraints. Inspired by the explicit solution presented in [3] for the minimization case, the solution of this class of problems is based on dynamic programming, with an explicit solution to the problem’s Bellman equation. Thus, it is possible to find the optimal control policy, which, for this class of problems, is manifested as a linear feedback policy that minimizes the objective function over the system’s trajectory when subjected to the worst-case disturbance or uncertainty that is homogeneous in the system state.

To understand the relevance of our problem class, recall that a linear system is positive if the state and output remain

nonnegative as long as the initial state and the inputs are nonnegative. This type of dynamics have gained attention in control theory literature because of all the technological and physical phenomena that can be captured by positive dynamics. Classical books on the topic are [4] and [5]. In the latter, David Luenberger in 1979 devotes a chapter to positive dynamical systems, which is considered by many the initiation of “Positive System Theory.” Significant research has been conducted to represent natural extensions of the class of positive systems, for instance positive systems with delays [6], positive switched systems [7] and monotone systems [8]. Positive systems theory has been useful to describe dynamical systems in a wide range of applications, such as, biology, ecology, physiology and pharmacology [9]–[14], thermodynamics [12], [15], epidemiology [16]–[18], econometrics [19], filtering and charge routing networks or power systems [20]–[22].

One of the main advantages of positive systems is that stability can be verified using linear Lyapunov functions [23], making this class of systems more tractable in a large scale setting because of their computational scalability [24], [25]. Another well known advantage is that linear controllers $u = -Kx$ can be designed with sparsity constraints on K . See for example [26], [27]. This paper is optimizing sparse controllers of the same form, but with one important difference compared to past literature: *Dynamic programming is carried out without a priori constraints on linearity or sparsity. Instead these properties are a consequence of the optimization criteria and constraints.* Hence it is possible to conclude that no nonlinear nonsparse controller can ever achieve a lower value of the cost. The minimax optimization is also related to past work on gain minimization [26], [28], but again the dynamic programming approach is different. More powerful conclusions are obtained at the expense of more restrictive assumptions.

B. Problem Setup

We present the optimal control problem of this paper as a discrete-time, infinite-horizon, minimax optimal control problem with nonnegative cost and positive dynamics with linear, continuous objective function and homogeneous constraints,

$$\inf_{\mu} \max_w \sum_{t=0}^{\infty} [s^{\top} x(t) + r^{\top} u(t) - \gamma^{\top} w(t)] \quad (1)$$

subject to

$$x(t+1) = Ax(t) + Bu(t) + Fw(t),$$

$$u(t) = \mu(x(t)); \quad x(0) = x_0$$

$$|u| \leq Ex; \quad |w| \leq Gx$$

The authors A. Gurpegui, E. Tegling and A. Rantzer are with the Department of Automatic Control and the ELLIIT Strategic Research Area at Lund University, Lund, Sweden. Email: alba.gurpegui@control.lth.se, emma.tegling@control.lth.se, anders.rantzer@control.lth.se. This work is partially funded by the Wallenberg AI, Autonomous Systems and Software Program (WASP) and the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme under grant agreement No 834142 (ScalableControl).

where x represents the n -dimensional vector of state variables, u the m -dimensional control variable, w the l -dimensional disturbance, μ is any, potentially nonlinear, control policy, E prescribes the structure of the control action and G is assumed to determine the linear dependency of the disturbance and the state. The objective is to minimize the worst-case cost over all possible control strategies.

C. Notation

Let \mathbb{R}_+ denote the set of nonnegative real numbers. The inequality $X > Y$ ($X \geq Y$) mean that all the elements of the matrix $(X - Y)$ are positive (nonnegative). A matrix X is called positive if all the elements of X are nonnegative but at least one element is nonzero. The notation $|X|$ means elementwise absolute value.

II. MAIN RESULT

In this section we state and prove the main theorem of the paper, which gives an explicit solution to the presented class of linear minimax optimal control problems (1).

Theorem 1. *Let $A \in \mathbb{R}^{n \times n}$, $B = [B_1^\top, \dots, B_m^\top]^\top \in \mathbb{R}_+^{m \times n}$, $F \in \mathbb{R}^{n \times l}$, $E = [E_1^\top, \dots, E_m^\top]^\top \in \mathbb{R}_+^{m \times n}$, $G \in \mathbb{R}_+^{l \times n}$, $s \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $\gamma \in \mathbb{R}^l$. Suppose that*

$$A \geq |B|E + |F|G \quad (2)$$

$$s \geq E^\top |r| - G^\top |\gamma|. \quad (3)$$

Then the following statements are equivalent:

(i) *The optimal control problem (1), has a finite value for every $x_0 \in \mathbb{R}_+^n$.*

(ii) *The recursive sequence $\{p_k\}_{k=0}^\infty$ with $p_0 = 0$ and*

$$p_k = s + A^\top p_{k-1} - E^\top |r + B^\top p_{k-1}| + G^\top |-\gamma + F^\top p_{k-1}| \quad (4)$$

has a finite limit.

(iii) *There exists $p \in \mathbb{R}_+^n$ such that*

$$p = s + A^\top p - E^\top |r + B^\top p| + G^\top |-\gamma + F^\top p|. \quad (5)$$

If (iii) is true then (1) has the minimal, finite, nonnegative value $p^\top x_0$, with p being the limit of the recursive sequence $\{p_k\}_{k=0}^\infty$ in (ii). Moreover, the control law $u(t) = -Kx(t)$, is optimal when

$$K := \begin{bmatrix} \text{sign}(r_1 + p^\top B_1)E_1 \\ \vdots \\ \text{sign}(r_m + p^\top B_m)E_m \end{bmatrix}. \quad (6)$$

Remark 1. *Even though the optimization is done without any a priori assumption of linearity of the control policy, the resulting optimal controller is linear.*

Remark 2. *The condition (2) ensures the invariance of the positive orthant under the system dynamics. The second condition (3) will be needed when applying Proposition 3*

in the Appendix to our objective function $g(x, u, w) = s^\top x + r^\top u - \gamma^\top w$.

Remark 3. *In (6), it can be observed that the sparsity structure of the control gain K is directly determined by the E matrix. The sparsity of E is in turn determined by the problem designer and may capture limitations in actuation and sensing.*

Proof: The general nonlinear minimax optimal control problem (14) presented in the Appendix V reduces to our problem set up (1) if

$$\begin{aligned} f(x, u, w) &:= Ax + Bu + Fw \\ g(x, u, w) &:= s^\top x + r^\top u - \gamma^\top w. \end{aligned}$$

Furthermore, under condition (3) we observe that

$$\begin{aligned} &\max_{|w| \leq Gx} [s^\top x + r^\top u - \gamma^\top w] \\ &= s^\top x + r^\top u + |\gamma|^\top Gx \\ &\geq (E^\top |r| - G^\top |\gamma|)^\top x + r^\top u + |\gamma|^\top Gx \\ &\geq -(G^\top |\gamma|)^\top x + |\gamma|^\top Gx = 0. \end{aligned}$$

Therefore,

$$\max_{|w| \leq Gx} [g(x, u, w)] = \max_{|w| \leq Gx} [s^\top x + r^\top u - \gamma^\top w] \geq 0$$

as required in Lemma 2 in the Appendix V. Now, it is clear that (i) in Theorem 1 is equivalent to (i) in Lemma 2 in the Appendix V. Next, we will verify that the recursive sequence in (ii) is equivalent to (ii) in Lemma 2. To prove this we use induction over $p_k^\top x = J_k(x)$. By definition, it is direct that $p_0^\top x = 0 = J_0(x)$ for all x . For the induction step we assume that $p_k^\top x = J_k(x)$. Now we want to prove that $p_{k+1}^\top x = J_{k+1}(x)$. From (15) and the induction hypothesis we have that

$$\begin{aligned} J_{k+1}(x) &= \min_u \max_w [g(x, u, w) + J_k(f(x, u, w))] \\ &= \min_u \max_w [s^\top x + r^\top u - \gamma^\top w + p_k^\top (Ax + Bu + Fw)] \\ &= s^\top x + p_k^\top Ax + \min_{|u| \leq Ex} [r^\top u + p_k^\top Bu] \\ &\quad + \max_{|w| \leq Gx} [-\gamma^\top w + p_k^\top Fw] \\ &= s^\top x + p_k^\top Ax - |r + B^\top p_k|^\top Ex + |-\gamma + F^\top p_k|^\top Gx \\ &= p_{k+1}^\top x. \end{aligned}$$

Therefore, $p_k^\top x = J_k(x)$ for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}_+^n$, and $p^\top x = J^*(x)$ for all x . Hence, (ii) and (iii) both in Theorem 1 and Lemma 2 are equivalent. Furthermore, note that under homogeneous constraints the linearity of J_k is preserved during value iteration.

Because (i), (ii) and (iii) in this theorem and in Lemma 2 are equivalent, the proof of equivalence between (i), (ii) and (iii) in Theorem 1 follows from the proof of Lemma 2 in the Appendix V.

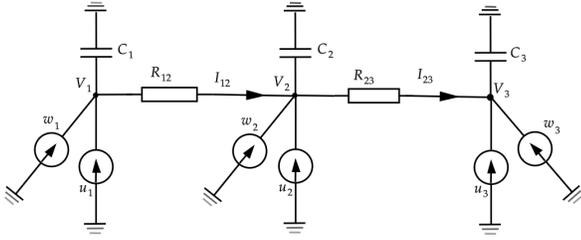


Fig. 1. Example of a DC network consisting of 3 terminals (buses) and 5 lines. The controls u_i are used to control the voltage when the system is subjected to the disturbances w_i .

To finish this proof it is just left to give an expression for the optimal control policy $u(t) = \mu(x(t))$,

$$\begin{aligned} \mu(x) &= \arg \min_{|u| \leq Ex} [s^\top x + r^\top u - \gamma^\top w + p^\top (Ax + Bu + Fw)] \\ &= \arg \min_{|u| \leq Ex} \sum_{i=1}^m [(r_i + p^\top B_i) u_i]. \end{aligned}$$

Finally, since for all $i = 1 \dots m$ the inequality $|u| \leq Ex$ restricts u_i to the interval $[-E_i x, E_i x]$, the minimum is attained when $(r_i + p^\top B_i)$ and u_i have opposite signs. Thus, $u_i = -\text{sign}(r_i + p^\top B_i) E_i$ for all $i = 1 \dots m$. ■

Next, we present an example of a simple scenario where the main theorem is applied to a network problem – a DC power network – to illustrate the role of the assumptions and constraints.

III. EXAMPLE: OPTIMAL VOLTAGE CONTROL IN A DC POWER NETWORK

The optimal control problem (1) admits sparsity constraints on the controller, making it particularly useful for problems defined over network graphs. Here, we consider a simple example of voltage control in a DC (i.e., direct current) power network. Here, the nodes represent voltage source converters with positive voltage dynamics, interconnected through resistive lines. The model can, for example, capture an envisioned multi-terminal high-voltage DC network, whose design aims to transmit power over long distances while maintaining low losses [29], [30] or a simplified DC distribution network [31]. The (continuous) voltage dynamics at the DC bus i (node i) is given by:

$$\begin{aligned} C_i \dot{V}_i(t) &= - \sum_{j=1}^n I_{ij} + u_i(t) + w_i(t) \\ &= - \sum_{j=1}^n \frac{1}{R_{ij}} (V_i(t) - V_j(t)) + u_i(t) + w_i(t), \quad (7) \end{aligned}$$

for all $i = 1, 2, \dots, n$, where u_i denotes the controlled injected current, R_{ij} the resistance of transmission line (i, j) (with $R_{i,j} = \infty$ if there exists no line connecting nodes i and j), and C_i is the total capacitance at bus i .¹ We

¹Any line capacitances can for the purpose of this example be absorbed in to the buses.

have also included the disturbance current w_i , arising from variations in local generation and load. Defining the vector $V = [V_1(t), \dots, V_n(t)]^\top$, u and w analogously, and $C = \text{diag}([C_1, \dots, C_n])$, we may write (7) on vector form as

$$C \dot{V}(t) = -\mathcal{L}_R V(t) + u(t) + w(t). \quad (8)$$

Here, \mathcal{L}_R is the weighted Laplacian matrix of the graph representing the transmission lines, whose edge weights are given by the conductances $\frac{1}{R_{ij}}$, i.e.,

$$[\mathcal{L}_R]_{i,j} = \begin{cases} -\frac{1}{R_{i,j}} & \text{if } i \neq j \\ \sum_{j=1}^n \frac{1}{R_{i,j}} & \text{if } i = j \end{cases}.$$

Note that $-\mathcal{L}_R$ is Metzler, and the system (8) thus positive. The dynamics in 8 can be discretized as

$$C(V(\tau h + h) - V(\tau h))/h = -\mathcal{L}_R V(\tau h) + u(\tau h) + w(\tau h).$$

Setting $t = \tau h$ and re-defining the state $x(t) = V(\tau h)$ gives the discrete-time dynamics

$$\begin{aligned} x(t+1) &= [I - hC^{-1}\mathcal{L}_R] x(t) + hC^{-1}u(t) \\ &\quad + hC^{-1}w(t). \quad (9) \end{aligned}$$

Now, we formulate the optimal control problem (1) for the dynamics (9):

$$\inf_{\mu} \max_w \sum_{t=0}^{\infty} [s^\top x(t) + r^\top u(t) - \gamma^\top w(t)] \quad (10)$$

Subject to

$$\begin{aligned} x(t+1) &= [I - hC^{-1}\mathcal{L}_R] x(t) + hC^{-1}u(t) + hC^{-1}w(t) \\ u(t) &= \mu(x(t)); \quad x(0) = x_0 \\ |u| &\leq Ex; \quad |w| \leq Gx \end{aligned} \quad (11)$$

Identifying A and B , condition (2) reads

$$[I - hC^{-1} \cdot \mathcal{L}_R] \geq hC^{-1}E + hC^{-1}G. \quad (12)$$

Clearly, the right hand side of the inequality must inherit the zero pattern of \mathcal{L}_R , i.e. its sparsity pattern. In other words, the disturbances and control signal must be compatible with the physical network structure and depend only on connected nodes. E , or G can, however be more sparse than \mathcal{L}_R . Furthermore, (12) reveals that the diagonals of the left hand side must satisfy

$$(e_{ii} + g_{ii}) + \sum_{j=1}^N \frac{1}{R_{ij}} \leq \frac{C_i}{h}$$

This can always be satisfied by making h sufficiently small. However, the off-diagonals reveal conditions on e_{ij} , g_{ij} that depend on the line resistances R_{ij} in a manner best illustrated by (13).

In Fig. 1 a 3-terminal DC power network system is introduced. For this network, the condition (12) reads:

$$\begin{bmatrix} 1 - \sum_{j=1}^3 \frac{h}{R_{1,j} \cdot C_1} & & 0 \\ \frac{h}{R_{2,1} \cdot C_2} & 1 - \sum_{j=1}^3 \frac{h}{R_{2,j} \cdot C_2} & \\ 0 & \frac{h}{R_{3,2} \cdot C_3} & 1 - \sum_{j=1}^3 \frac{h}{R_{3,j} \cdot C_3} \end{bmatrix} \geq \begin{bmatrix} \frac{h}{C_1}(e_{1,1} + g_{1,1}) & \frac{h}{C_1}(e_{1,2} + g_{1,2}) & 0 \\ \frac{h}{C_2}(e_{2,1} + g_{2,1}) & \frac{h}{C_2}(e_{2,2} + g_{2,2}) & \frac{h}{C_2}(e_{2,3} + g_{2,3}) \\ 0 & \frac{h}{C_3}(e_{3,2} + g_{3,2}) & \frac{h}{C_3}(e_{3,3} + g_{3,3}) \end{bmatrix}. \quad (13)$$

This element-wise matrix inequality shows necessary constraints on the elements of E and G .

In parallel, condition (3) means that E and G need to satisfy

$$s \geq E^T |r| - G^T |\gamma|.$$

Here, the structure of E determines the states available to the local current controllers and G the manner in which disturbances enter the system.

Particularly, in our 3 terminal DC power network we need the problem design to satisfy

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \geq \begin{bmatrix} e_{1,1} & e_{2,1} & 0 \\ e_{1,2} & e_{2,2} & e_{3,2} \\ 0 & e_{2,3} & e_{3,3} \end{bmatrix} \begin{bmatrix} |r_1| \\ |r_2| \\ |r_3| \end{bmatrix} - \begin{bmatrix} g_{1,1} & g_{2,1} & 0 \\ g_{1,2} & g_{2,2} & g_{3,2} \\ 0 & g_{2,3} & g_{3,3} \end{bmatrix} \begin{bmatrix} |\gamma_1| \\ |\gamma_2| \\ |\gamma_3| \end{bmatrix}.$$

In this example, the resulting optimal controller (6) can take 8 different configurations depending on the sign of each element of the parameter r . If r is positive, because in this problem $B = hC^{-1}$ is positive, all the signs of the rows in K are positive so that $u(t) = -Ex(t)$ becomes optimal. However, if r is not positive, it is possible to use this parameter to modify the signs of the rows in the resulting control action.

IV. CONCLUSIONS

In this paper, we have extended the optimal control problem class presented in [3] by exploring the minimax worst-case. Specifically, we have derived a solution p for this novel class of optimal control problems using value iteration. Our resulting optimal controller bears resemblance to the one derived in [32] for a continuous-time problem, however, the homogenous constraints we impose allow for a prescribed controller structure. We believe that our approach is an interesting first step in analyzing a new class of optimal control problems, but that there is still room for improving our methodology. Considering the high computational cost of value iteration, the exploration of alternatives such as policy iteration [33], offers a compelling direction for future research.

Our results demonstrate that this class of problems can be scaled to large dynamical systems efficiently, partly since the sparsity of the optimal feedback is directly related to the constraints imposed in the problem statement. We demonstrated the problem setup on a simple example of a 3-node power network, but the same method can be applied when the network scales.

The explicit solution we obtain to the Bellman equation appears to rely on the homogeneity of the constraints on u and w . We believe, however, that constant upper bounds on the signals can be accounted for, and this is the subject of ongoing research.

ACKNOWLEDGMENTS

We would like to thank Richard Pates and anonymous reviewers for their insightful comments.

REFERENCES

- [1] T. Başar and P. Bernhard. *H_∞-Optimal Control and Related Minimax Design Problems*. Birkhäuser Boston, MA, 2008.
- [2] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series. Princeton University Press, Princeton, N. J., 1970.
- [3] A. Rantzer. Explicit solution to Bellman equation for positive systems with linear cost. *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 6154–6155, 2022.
- [4] A. Berman and R.J. Plemmons. *Nonnegative matrices in the mathematical sciences*. Academic Press, 1979.
- [5] D.G. Luenberger. *Introduction to Dynamic Systems. Theory, Models, and Applications*. Academic Press, 1979.
- [6] Y. Ebihara, D. Peaucelle, and D. Arzelier. Steady-state analysis of delay interconnected positive systems and its application to formation control. *IET Control Theory and Applications*, 11, 2017.
- [7] F. Blanchini, P. Colaneri, and M.E. Valcher. Switched linear positive systems. *Foundations and trends in systems and control*, 2015.
- [8] Hal L. Smith. *Monotone Dynamical Systems: An Introduction To The Theory Of Competitive And Cooperative Systems*. AMS, Mathematical Surveys And Monographs, 1995.
- [9] E. Carson and C. Cobelli. Modelling methodology for physiology and medicine. *Academic Press, San Diego*, 2001.
- [10] P.G. Coxson and H. Shapiro. Positive reachability and controllability of positive systems. *Linear Algebra and its Appl.*, 94, page 35–53, 1987.
- [11] W. Haddad and V. Chellaboina. Stability and dissipativity theory for nonnegative dynamical systems: A unified analysis framework for biological and physiological systems. *Nonlinear Analysis: Real World Applications*, 6:35–65, 02 2005.
- [12] W. Haddad, V. Chellaboina, and Q. Hui. Nonnegative and compartmental dynamical systems. *Princeton, NJ: Princeton Univ. Press*, 2010.
- [13] E. Hernandez-Vargas, R. Middleton, P. Colaneri, and F. Blanchini. Discrete-time control for switched positive systems with application to mitigating viral escape. *Int. J. Robust Nonlinear Control*, 21, page (10):1093–1111, 2011.
- [14] J.A. Jacquez. Compartmental analysis. Biology and medicine. *Biometrische Zeitschrift*, 16(8):537–537, 1974.
- [15] F. Blanchini, P. Colaneri, and M.E. Valcher. Switched linear positive systems. *Foundations and Trends in Systems and Control*, 2, page (2):101– 273, 2015.
- [16] M. Ait Rami, V. Bokharaie, O. Mason, and F. Wirth. Stability criteria for sis epidemiological models under switching policies. *Discrete and Continuous Dynamical Systems - Series B*, 19, 2013.
- [17] E.A. Hernandez-Vargas and R.H. Middleton. Modelling the three stages in HIV infection. *Journal of Theoretical Biology*, 320:33–40, 2013.
- [18] Y. Moreno, R. Pastor-Satorras, and A. Vespignani. Epidemic outbreaks in complex heterogeneous networks. *The European Physical J. B: Condensed Matter and Complex Systems*, 26(4):521–529, 2002.
- [19] J.W. Nieuwenhuis. Some results about a Leontieff-type model. In *C.I.Byrnes and Lindquist A, editors, Frequency domain and State space methods for Linear Systems*. Elsevier Science, page 213–225, 1986.
- [20] L. Benvenuti and L. Farina. Discrete-time filtering via charge routing networks. *Signal Processing*, 49(3):207–215, 1996.
- [21] L. Benvenuti and L. Farina. The design of fiber-optic filters. *Journal of Lightwave Technology*, 19(9):1366–1375, 2001.
- [22] L. Benvenuti, L. Farina, and B.D.O. Anderson. The positive side of filters: a summary. *IEEE Circuits and Systems Magazine*, 1(3):32–36, 2001.
- [23] F. Blanchini and G. Giordano. Piecewise-linear Lyapunov functions for structural stability of biochemical networks. *Automatica*, 50:2482–2493, 10 2014.

- [24] A. Rantzer and M.E. Valcher. Scalable control of positive systems. *Annual Review of Control, Robotics, and Autonomous Systems*, 4(1):319–341, 2021.
- [25] Y. Ebihara, D. Peaucelle, and D. Arzelier. Analysis and synthesis of interconnected positive systems. *IEEE Trans. Automatic Control*, 62(2):652–667, 2017.
- [26] C. Briat. Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: L_1 - and L_∞ -gains characterizations. *Int. J. of Robust and Nonlinear Control*, 23(17):1932–1954, 2013.
- [27] T. Tanaka and C. Langbort. The bounded real lemma for internally positive systems and H_∞ structured static state feedback. *IEEE Trans. Automatic Control*, 56(9):2218–2223, September 2011.
- [28] Y. Ebihara, D. Peaucelle, and D. Arzelier. L_1 gain analysis of linear positive systems and its application. In *2011 50th IEEE Conference on Decision and Control and European Control Conference*, pages 4029–4034, 2011.
- [29] D. Van Hertem and M. Ghandhari. Multi-terminal VSC HVDC for the European supergrid: Obstacles. *Renewable and Sustainable Energy Reviews*, 14(9):3156–3163, 2010.
- [30] M. Andreasson, E. Tegling, H. Sandberg, and K.H. Johansson. Performance and scalability of voltage controllers in Multi-Terminal HVDC Networks. *2017 American Control Conference (ACC), Seattle, WA, USA*, pages 3029–3034, 2017.
- [31] P. Karlsson and J. Svensson. DC bus voltage control for a distributed power system. *IEEE Transactions on Power Electronics*, 18(6):1405–1412, 2003.
- [32] F. Blanchini, P. Bolzern, P. Colaneri, G. De Nicolao, and G. Giordano. Optimal control of compartmental models: The exact solution. *Automatica*, 147:110680, 2023.
- [33] Dimitri P. Bertsekas. Value and policy iterations in optimal control and adaptive dynamic programming. *IEEE Transactions on Neural Networks and Learning Systems*, 28(3):500–509, March 2017.

V. APPENDIX

A. General Problem Set Up

We define, a general discrete time, infinite horizon, min-max optimal control problem with continuous cost function and constraints as

$$\inf_{\mu} \max_w \sum_{t=0}^{\infty} g(x(t), u(t), w(t)) \quad (14)$$

subject to

$$\begin{aligned} x(t+1) &= f(x(t), u(t), w(t)), \\ x(t) &\in X; \quad x(0) = x_0; \quad u(t) = \mu(x(t)) \\ u(t) &\in U(x(t)); \quad w(t) \in W(x(t)) \end{aligned}$$

where $f : X \times U \times W \rightarrow X$, x represents the vector of n -dimensional state variables, u the m -dimensional control variable and w the q -dimensional disturbance.

Lemma 2. *Suppose*

$$\max_{w \in W(x)} [g(x, u, w)] \geq 0$$

$\forall x \in X, \forall u \in U(x)$. Then, the following statements are equivalent.

(i) *The general optimal control problem in (14) has a finite value for every $x_0 \in \mathbb{R}_+^n$.*

(ii) *The recursive sequence $\{J_k\}_{k=0}^{\infty}$ with $J_0 = 0$ and*

$$J_k(x) = \min_u \max_w [g(x, u, w) + J_{k-1}(f(x, u, w))] \quad (15)$$

has a finite limit $\forall x \in X$.

(iii) *The Bellman equation*

$$J^*(x) = \min_u \max_w [g(x, u, w) + J^*(f(x, u, w))] \quad (16)$$

has nonnegative solution $J^(x), \forall x \in X$.*

Proof: Note that, by dynamic programming, the increasing monotone recursive sequence $\{J_k\}_{k=1}^{\infty}$ (see Proposition 3) defined in (15) also satisfies

$$J_k(x) = \inf_{\mu} \max_w \sum_{t=0}^k [g(x(t), u(t), w(t))]. \quad (17)$$

To prove the equivalence we will prove implications (i) \Rightarrow (ii), (ii) \Rightarrow (iii), (iii) \Rightarrow (ii), (iii) \Rightarrow (i).

(i) \Rightarrow (ii) Assume (i). Then, in (17) when $k \rightarrow \infty$ we get $J_k(x) < \alpha < \infty$ for all x , with α representing a finite value for (14). Hence, the recursive sequence (15) has a finite limit.

(ii) \Rightarrow (iii) Assume (ii). Taking the limit when $k \rightarrow \infty$ on both sides of the equation (15) we get (iii).

(iii) \Rightarrow (ii) Assume (iii). We want to prove that $\lim_{k \rightarrow \infty} J_k(x) < \infty$ for all x with $J_k(x)$ defined in (17). To achieve this, we use induction over $J_k(x) \leq J^*(x) < \infty$. It is clear that $0 = J_0(x) \leq J^*(x)$ with J^* nonnegative by definition. For the induction step we assume that $J_k(x) \leq J^*(x)$. We want to prove that $J_{k+1}(x) \leq J^*(x)$. From the induction hypothesis it is direct that

$$\begin{aligned} J_{k+1}(x) &= \min_u \max_w [g(x, u, w) + J_k(f(x, u, w))] \\ &\leq \min_u \max_w [g(x, u, w) + J^*(f(x, u, w))] \\ &= J^*(x). \end{aligned}$$

Thus, $J_{k+1}(x) \leq J^*(x)$ for all x , and $\lim_{k \rightarrow \infty} J_k(x) < \infty$ for all k and for all x , as we wanted to prove.

(iii) \Rightarrow (i) Assume (iii). Define for all x

$$\mu^*(x) = \arg \min_u \max_w \{g(x, u, w) + J^*(f(x, u, w))\}$$

such that

$$J^*(x) = \min_u \max_w [g(x, u, w) + J^*(f(x, u, w))].$$

Indeed, from (17) and implication (iii) \Rightarrow (ii) it can be observed that

$$\begin{aligned} \max_w \sum_{t=0}^k [g(x, \mu^*(x), w)] &\leq \inf_{\mu} \max_w \sum_{t=0}^k [g(x(t), u(t), w(t))] \\ &= J_k(x) \leq J^*(x) < \infty. \end{aligned}$$

Hence,

$$\max_w \sum_{t=0}^k [g(x, \mu^*(x), w)] \leq J^*(x)$$

for all k and for all x . This proves (i). \blacksquare

Proposition 3. *Let*

$$\max_w [g(x, u, w)] \geq 0$$

$\forall x \in X$ and $\forall u \in U(x)$. Then, the recursive sequence $\{J_k(x)\}_{k=0}^{\infty}$ with $J_0(x) = 0$ and

$$J_k(x) = \min_u \max_w [g(x, u, w) + J_{k-1}(f(x, u, w))] \quad (18)$$

satisfies that $0 \leq J_0(x) \leq J_1(x) \leq J_2(x) \leq \dots$ for all $x \in X$ and all $k \in \mathbb{N}$.

Proof: To prove this we use induction over $J_k(x)$. From the proposition statement $J_0(x) = 0$ gives

$$\begin{aligned} J_1(x) &= \min_u \max_w [g(x, u, w) + J_0(f(x, u, w))] = \\ &= \min_u \max_w [g(x, u, w)] \geq 0 = J_0(x) \end{aligned}$$

for all x . For the induction step we assume that $J_k(x) \geq J_{k-1}(x)$. We then want to prove that $J_{k+1} \geq J_k$. From (15) and the induction hypothesis we have that

$$\begin{aligned} J_{k+1}(x) &= \min_{u \in U(x)} \max_{w \in W(x)} [g(x, u, w) + J_k(f(x, u, w))] \\ &\geq \min_{u \in U(x)} \max_{w \in W(x)} [g(x, u, w) + J_{k-1}(f(x, u, w))] \\ &= J_k(x) \end{aligned}$$

Thus, $J_{k+1}(x) \leq J_k(x)$ for all k and for all x . ■