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Externally Positive Linear Systems from Transfer Function Properties

Ross Drummond and Matthew C. Turner,

***Abstract*— The characterisation of single-input-single-output externally positive linear systems is considered. A complete characterisation of the class of externally positive second-order and a class of underdamped third-order systems is given and connections to negative-imaginary systems are highlighted. It is shown that negative-imaginary systems have non-negative step responses, leading to a condition for external positivity based on negative imaginary systems theory. Finally, a class of externally positive systems which can be verified using the developed results but which fail a recently developed numerical test for external positivity based upon linear matrix inequalities are introduced. These results extend the class of system for which external positivity can be verified, facilitating large-scale control and less conservative absolute stability analysis.**

***Index Terms*— Externally positive linear systems, negative imaginary systems.**

Notation

REAL vectors of dimension n are denoted $\mathbb{R}^n$, non-negative real vectors are $\mathbb{R}^n_+$ and positive-real vectors are $\mathbb{R}^n_{++}$. $\mathbb{R}^{n_1\times n_2}$ denotes a real $n_1 \times n_2$ matrix and $\mathcal{S}^n_{\succ 0}$ ($\mathcal{S}^n_{\succeq 0}$) denotes a symmetric positive (semi)-definite matrix of size n. The ordered space is defined as $\mathcal{D}^n_{++} = \{(x_1, \dots, x_n) : x_1 \geq \dots \geq x_n > 0\}$. Use $x[1] \geq \dots \geq x[n]$ to denote the components of vector $x \in \mathbb{R}^n$ arranged in decreasing order. Vectors x and y are said to be similarly ordered if there is a permutation π such that $x[i] = x_\pi(i), y[i] = y_\pi(i), i = 1, ..., n$. Equivalently, x and y are similarly ordered if $(x_i - x_j)(y_i - y_j) \geq 0$ for all i, j [27]. Given vectors x, $y \in \mathbb{R}^n$, x is said to *weakly submajorise* y, as in $x \prec_w y$, if $\sum_{k=1}^K x_{[k]} \leq \sum_{k=1}^K y_{[k]}$, $\forall K = 1, \dots, n$. The inner product of two signals $\langle y, u\rangle$ defined on $[0, t]$ is $\langle y, u\rangle_t = \int_0^t y(\tau)u(\tau)\ d\tau$. The Hilbert space of square integrable functions $f : [0, \infty) \to \mathbb{R}$ such that $\langle f, f\rangle_\infty < \infty$ is denoted $\mathcal{L}_2$. A linear system is indicated by its linear operator G, its transfer function $G(s)$, its frequency response $G(j\omega)$ or its impulse response $g(t)$.

R. Drummond is with the Department of Automatic Control and Systems Engineering, University of Sheffield, Sheffield, S1 4DT, UK. Email: ross.drummond@sheffield.ac.uk. R. Drummond was supported by a UKIC Fellowship from the Royal Academy of Engineering.

M. Turner is with the School of Electronics and Computer Science, University of Southampton, Southampton, SO17 1BJ, UK. Email: m.c.turner@soton.ac.uk.

I. Introduction

Consider the single-input-single-output (SISO) linear system with minimal state-space realisation

$$G \sim \begin{cases} \frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$, $x(0) = 0$ and strictly proper transfer function

$$G(s) = C(sI - A)^{-1}B. \tag{2}$$

A system is said to be externally positive if the following definition holds.

Definition 1 (External positivity [15]): System (1) is externally positive if the output $y(t)$, corresponding to a zero initial condition on the state $x(0) = 0$, is non-negative for every non-negative input $u(t)$ for all $t \geq 0$.

Lemma 1 ([15]): System (1) is externally positive iff its impulse response, $g(t)$, is non-negative.

Determining external positivity (or equivalently, determining whether the impulse response is non-negative) for general classes of systems is challenging, and is NP-hard for the specific case of A being self-adjoint [3]. At the same time, there has also been a growing interest in this class of system for two main reasons. Firstly, they are used to model several important physical systems, like batteries and supercapacitors [11], and, secondly, their analysis is often simplified. Examples of this simplicity include the fact that designing feedback controllers for them is scalable [31], [33] and they can be analysed using integral linear constraints instead of integral quadratic constraints [20]. Hence, by first identifying a system as externally positive, large scale system analysis and control becomes more feasible.

To fully exploit these simplifying results, a general characterisation of the class of externally positive systems is required. Several seminal results have been obtained for positive systems, especially on the positive-realisation problem, whereby a state-space realisation with a positivity preserving input-state and state-output map is sought [4]. A complete solution to this problem was given in [14] which built upon the earlier results of [1] and [28] where observable and reachable cones were introduced. However, many questions still remain unresolved, such as how to efficiently compute the realisation matrices for general classes of systems.

Another recent condition for external positivity was developed in [16], [17] for the purpose of model order reduction, with the characterisation defined from internal positivity and cone-invariance. Also, it was shown in [13, Corollary 1] that, by applying a Kronecker product transformation on the state-space matrices, a generic linear system can be transformed into an externally positive one which allows the $\mathcal{H}_2$ norm to be bounded analytically.

The authors' interest in this class of systems lies in their close association with Zames-Falb multipliers [7], [34], [35], [39]. These multipliers are the least conservative known method for verifying the stability of a class of Lurie system: the feedback interconnection of a linear system and a static, non-odd monotonic nonlineartity [21]. A key condition on the multipliers is a non-negative impulse response, and hence they should be externally positive. For this reason, finding appropriate multipliers remains an open issue. Therefore, amongst other applications, the development of simple and complete tests for externally positive systems promises to reduce conservatism in the absolute stability problem [35].

Contributions: This paper collects and proposes several conditions for external positivity of linear systems. The main results are iff conditions for second order systems and a class of underdamped third order system as well as the characterisations from negative-imaginary systems developed in Section III-B. Emphasis is placed on realisation-independent characterisations, as the many similarity transformations involved in the Zames-Falb multiplier convex search will break the fragile internally positive-realisation structure [35]. Connections to the notion of positivity in input-output systems theory are also explored, with a non-equivalence shown via counter-examples.

II. Characterisations from the Transfer Function

As external positivity represents a restriction on the input-output response, it is desirable to provide characterisations in terms of the parameters of the transfer function (being independent of the state-space realisation). As observed by several authors, including [8], [9], [30], [37], seemingly independently, such a characterisation follows from Post's Inversion Formula for the impulse response [30], [37] and Bernstein's Theorem on completely monotonic functions [32], [37]. External positivity is then equivalent to $G(s)$ being completely monotonic, as in

$$(-1)^k \frac{d^k\, G(s)}{ds^k} \geq 0, \quad \forall s \in \mathbb{R}_+,\, k \in \mathbb{N} \cup 0. \tag{3}$$

Whilst complete monotonicity of $G(s)$ represents a full characterisation of external positivity, determining when a generic rational polynomial $G(s)$ is completely monotonic is challenging in practise, as it requires computing an infinite number of derivatives. For minimum phase systems with real poles and zeros, [2] proposed complete monotonicity conditions based upon the weak majorisation of the poles by the zeros. This restriction is less restrictive than the later results from the control literature [5], [18], [25], and is close to necessary, with necessity requiring that the sum of the zeros exceeds that of the poles [2].

Weak majorisation based sufficient conditions for external positivity of other system types were developed in [12] by exploiting the properties of *relaxation systems* [38]

$$G_{\text{re}}(s) = \sum_{k=1}^{K} \frac{a_k}{s + p_k} \tag{4}$$

with $a_k > 0$, $p_k > 0$, to characterise *mixed relaxation systems*

$$G_{\text{mre}}(s) = \sum_{k=1}^{K} \frac{a_k}{s + p_k} - \sum_{k=1}^{K} \frac{b_k}{s + q_k} \tag{5}$$

with a_k, b_k, p_k, $q_k \in \mathbb{R}_{++}$. One of the criteria introduced in [12] was the following.

Theorem 1 ([12]): For $a, b, p, q \in \mathcal{D}_{++}^K$, if there exists a similar ordering between a and p as well as b and q and if

$$\ln(b) \prec_w \ln(a), \tag{6a}$$
$$-q \prec_w -p, \tag{6b}$$

then the mixed relaxation system (5) is externally positive.

Theorem 1 is extended here to obtain iff conditions for second order systems. Even though second order externally positive systems are fully characterised by internally positive realisations [15], the presented results offer an alternative perspective posed in terms of an input-output analysis.

Proposition 1: A second-order strictly proper, linear system $G(s)$ with poles in $\mathfrak{Re}[s] \leq 0$ is externally positive iff it is either i) a relaxation system of the form (4); ii) a mixed relaxation system of the form (5) with $K = 1$, $a_1 \geq b_1 > 0$ and $p_1 < q_1$; or iii) a system of the form

$$G_{\text{rep}}(s) = \frac{a_1}{s + p_1} + \frac{b_1}{(s + p_1)^2} \tag{7}$$

with $a_1, p_1 \geq 0$ and $b_1 > 0$.

Proof: Firstly, if the poles of the second order system are a pair of complex conjugates, then the system is not externally positive from Perron-Frobenius theorem. This also follows from a simple inspection of the impulse response. Therefore, attention is confined to systems containing real poles only.

Secondly, consider systems with only distinct poles. These systems are either relaxation systems of the form (4) (which are trivially externally positive), mixed relaxation systems of the form (5), or *negative relaxation systems* of the form (4) except with $a_k < 0$. Trivially, the impulse response of negative relaxation systems are non-positive for all time and hence are not externally positive.

Thus, consider the case when $G(s)$ is a second-order mixed relaxation system: it has the structure of (5) with $K = 1$, and the impulse response is simply

$$g(t) = a_1 e^{-p_1 t} - b_1 e^{-q_1 t}. \tag{8}$$

External positivity requires $g(0) \geq 0$, hence $a_1 \geq b_1 \geq 0$. For the restriction on the poles, if $g(0) \geq 0$ then, because $e^{-p_1 t}$ and $e^{-q_1 t}$ are both monotonically decreasing, $g(t) \geq 0$ for all $t \in [0, \infty)$ if and only if the only time for which $g(t) = 0$ is non-positive. The solution to $g(t) = 0$ in (8) satisfies $e^{\ln(a_1) - p_1 t} = e^{\ln(b_1) - q_1 t}$ and so, from the exponential being injective, the time t^* when $g(t) = 0$ is $t^* = \frac{\ln(a_1) - \ln(b_1)}{p_1 - q_1}$. Monotonicity of the logarithm means $\ln(a_1) \geq \ln(b_1)$, under the necessary conditions $a_1 \geq b_1 > 0$, and so $t^* \geq 0$ when $p_1 < q_1$. Sufficiency follows from $a_1 e^{-p_1 t} \geq b_1 e^{-p_1 t} > b_1 e^{-q_1 t}$.

Thirdly, note that if the poles are repeated, as in $\mathfrak{Re}[s] \geq 0$, and are not complex, the system has the form

$$G_{\text{rep}}(s) = \frac{a_1}{s+p_1} + \frac{b_1}{(s+p_1)^2}, \; p_1 \geq 0, b_1 \neq 0. \tag{9}$$

It is easy to see that the necessary condition $g(0) \geq 0$ requires $a_1 > 0$. Similarly to above, a necessary condition for $g(t) \geq 0$ for all $t \in [0, \infty)$ is for $g(t) = 0$ for only non-positive times. The impulse response of $G_{\text{rep}}(s)$ is

$$g(t) = (a_1 + b_1 t)e^{-p_1 t} \tag{10}$$

and thus $g(t)$ becomes zero at $t = -a_1/b_1$ and thus for $a_1 > 0$, it follows that $b_1 > 0$. Sufficiency is then clear. In the case that $a_1 = 0$ (no zeros in the transfer function), it is trivial to see that $b_1 > 0$ for external positivity. ■

A. Third-order underdamped systems

In [12], Theorem 1 was generalised to transfer functions with complex poles and numerators using the crude bound $\min \cos(t) = \min \sin(t) = -1 \;\forall t \in \mathbb{R}_+$. However, this approach did not take into account the periodically positive contribution of the sinusoidal response, resulting in loose external positivity criteria, as observed in [17]. In particular, for systems of the form

$$G_{\text{cd}}(s) = \sum_{k=1}^{n} \frac{a_1}{s+p_1} + \frac{b_1 j}{s+q_1+\omega j} + \frac{-b_1 j}{s+q_1-\omega j}, \tag{11}$$

the test of [12] requires $a_1 \geq b_1$ but [17] showed the existence of an externally positive system with $b_1 > a_1$. Motivated by this observation, the following external positivity conditions are obtained for a class of third-order systems with complex poles that exploit the fact that the periodic term is positive in some time intervals.

Theorem 2: For $a, b, p, q, \omega \in \mathbb{R}_{++}$ and $q > p$, consider systems with impulse responses

$$g^-_{\text{sin}}(t) = ae^{-pt} - b\sin(\omega t)e^{-qt}, \tag{12a}$$
$$g^+_{\text{sin}}(t) = ae^{-pt} + b\sin(\omega t)e^{-qt}, \tag{12b}$$
$$g^-_{\text{cos}}(t) = ae^{-pt} - b\cos(\omega t)e^{-qt}, \tag{12c}$$
$$g^+_{\text{cos}}(t) = ae^{-pt} + b\cos(\omega t)e^{-qt}, \tag{12d}$$

and define

$$t^* = \frac{1}{\omega}\cot^{-1}\left(\frac{q-p}{\omega}\right). \tag{13}$$

Then the following statements are true

i) $g^-_{\text{sin}}(t) \geq 0 \quad \forall t \geq 0$ iff

$$\ln(a) - \ln(b) \geq \ln(\sin(\omega t^*)) - (q-p)t^* \tag{14}$$

ii) $g^+_{\text{sin}}(t) \geq 0 \quad \forall t \geq 0$ iff

$$\ln(a) - \ln(b) + \frac{(q-p)\pi}{\omega} \geq \ln(\sin(\omega t^*)) - (q-p)t^* \tag{15}$$

iii) $g^-_{\text{cos}}(t) \geq 0 \quad \forall t \geq 0$ iff $a \geq b$

iv) $g^+_{\text{cos}}(t) \geq 0 \quad \forall t \geq 0$ iff

$$\ln(a) - \ln(b) + \frac{(q-p)\pi}{2\omega} \geq \ln(\sin(\omega t^*)) - (q-p)t^* \tag{16}$$

Proof: Consider first the impulse response $g^-_{\text{sin}}(t)$. Since $q > p$, it follows that $g^-_{\text{sin}}(t) \geq 0$ for all time as long as it is non-negative in the interval $0 < t < \frac{\pi}{\omega}$. This is because if $g^-_{\text{sin}}(t)$ is positive during the first period of the sine wave, $t \in [0, \frac{2\pi}{\omega}]$, then since $q > p$, it will be positive in subsequent periods. Moreover, the only interval during this first period in which the sinusoidal term is negative is in the first half-period, $t \in (0, \frac{\pi}{\omega})$. In this interval, $g^-_{\text{sin}}(t)$ can be written

$$g^-_{\text{sin}}(t) = e^{\ln(a)-pt} - e^{\ln(b\sin(\omega t))-qt}, \tag{17}$$

which is non-negative iff

$$\ln(a) - \ln(b) \geq \ln(\sin(\omega t)) - (q-p)t. \tag{18}$$

When $0 < t < \frac{\pi}{\omega}$, the maximum value of the right hand side of (18) is obtained when

$$\frac{d}{dt}(\ln(\sin(\omega t)) - (q-p)t) = \omega\cot(\omega t) - (q-p) = 0, \tag{19}$$

as in the point t^* of (13). Condition (14) then holds iff (18) is satisfied, and hence $g^-_{\text{sin}}(t)$ is non-negative. Condition (15) for $g^+_{\text{sin}}(t)$ is obtained in a similar way, except that in this case, it is only necessary to verify non-negativity of the impulse response in $\frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$, and so there is a time shift of $\frac{\pi}{\omega}$. Similarly, the non-negativity of $g^-_{\text{cos}}(t)$ is satisfied if the impulse response is non-negative at $t = 0$. Condition (16) for $g^+_{\text{cos}}(t)$ follows from the same argument as above, except in this case the time region is $\frac{\pi}{2\omega} < t < \frac{3\pi}{2\omega}$. ■

Remark 1: These conditions build upon existing results for externally positive third order systems. Notably, [23] derived necessary and sufficient conditions for third-order systems with real poles. The results of [23] were simplified in [36] by developing interesting connections between externally positive systems and probability density functions. The most pertinent result to Theorem 2 is [19] which extended the earlier conditions of [23] to systems with complex poles. Whilst being in the same spirit as Theorem 2, the formulation of the results of [19] are different– with [19] involving the coefficients of the transfer function whereas Theorem 2 uses those of the impulse response. ⋆

III. Connections to Input-Output Systems Theory

The term "positivity" is frequently used in the analysis of input-output systems theory as a generalisation of *passivity*. In that context, positivity refers to the non-negativity of the inner product $\langle u, Gu \rangle_t > 0$, not of the output at each time instant. This section discusses the similarities and differences between these two different notions of positivity.

A. Positive-real functions

A property of central importance in the analysis of linear systems is that of $G(s)$ being strictly positive-real [10], [21].

Definition 2 (Strictly positive-real function [10]): A SISO transfer function $G(s)$ is strictly positive-real ($\mathcal{SPR}$) if

- The poles of $G(s)$ satisfy $\mathfrak{Re}[s] < 0$.
- $G(j\omega) + G(-j\omega) > 0$ for all $\omega \in \mathbb{R}$.

A consequence of G being strictly positive-real is that it can be realised by (A, B, B^T) since any realisation (A, B, C) of a positive-real G satisfies $PB = C^T$ for some $P \in \mathcal{S}^n_{\succ 0}$ from the KYP Lemma.

The importance of positive-realness follows from its characterisation of passive systems.

Definition 3 (Passive Systems): If $G(s)$ is strictly positive-real then the linear system $y = Gu$ mapping $u \in \mathcal{L}_2$ to $y \in \mathcal{L}_2$ is strictly passive and satisfies $\langle y, u\rangle_t > 0$.

Definition 3 implies that all externally positive systems are passive when the input is restricted to be positive, but not all passive systems are externally positive. This non-equivalence between passive (i.e. $G \in \mathcal{SPR}$) systems and external positivity was discussed extensively by De la Sen in [9] who gave the following counter-example

$$G_{\mathcal{SPR}}(s) = \frac{s + a}{s + b}. \tag{20}$$

When evaluated on $s = j\omega$, $G_{\mathcal{SPR}}$ is simply a Möbius transformation on the imaginary axis and positive-real when $a > 0$ and $b > 0$. However, the impulse response is

$$g(t) = 1\delta(t) + (a - b)e^{-bt} \tag{21}$$

which takes negative values when $b > a$. There then exists a range in the parameter space where $G_{\mathcal{SPR}}$ is positive-real but not externally positive. Similarly, the strictly proper plant

$$G_{\mathcal{SPR}}(s) = \frac{s}{s^2 + 0.1s + 1} \tag{22}$$

is positive-real but not externally positive, as its poles are a complex-conjugate pair (noting proof of Proposition 1).

B. Negative-imaginary transfer functions

Another transfer function property that would appear on first glance to be connected to external positivity is that of $G(s)$ being negative-imaginary ($\mathcal{NI}$).

Definition 4 (Negative-imaginary function [26]): A transfer function $G(s)$ is said to be negative-imaginary if

- $G(s)$ has no pole in $\mathfrak{Re}[s] > 0$.
- For all $\omega \geq 0$ such that $j\omega$ is not a pole of $G(s)$ then $j[G(j\omega) - G(-j\omega)] \geq 0$.
- If $s = j\omega_0$, with $\omega_0 > 0$, is a pole of $G(s)$,then it is a simple pole and the residue matrix $K = \lim_{s\to j\omega_0}(s - j\omega_0)jG(s)$ is Hermitian and positive semidefinite.
- If $s = 0$ is a pole of $G(s)$, then $\lim_{s\to 0} s^k G(s) = 0$ for all $k \geq 3$ and $\lim_{s\to 0} s^2 G(s)$ is Hermitian and positive semidefinite.

Negative imaginary systems can be characterised in state-space terms using the following result from [29].

Lemma 2 ([29]): Let (A, B, C, D) be a minimal realisation of $H(s)$. Then $H(s)$ is $\mathcal{NI}$ if and only if $D = D^T$ and there exists a matrix $Q \in \mathcal{S}^n_{\succeq 0}$ such that the following matrix inequality is satisfied

$$\begin{bmatrix} A^TQ + QA & QB - A^TC^T \\ (QB - A^TC^T)^T & -CB - B^TC^T \end{bmatrix} \preceq 0. \tag{23}$$

Negative-imaginary and externally positive systems share several similar features, e.g. verifying their stability in feedback is scalable [6], [22]. However, there exists a gap between these two classes of system. For example the negative imaginary system

$$G_{\mathcal{NI}}(s) = \frac{1}{s^2 + 0.1s + 1} \tag{24}$$

is clearly not externally positive since it has a pair of complex poles (Proposition 1).

In general, $\mathcal{NI}$ systems do not have non-negative impulse responses but *do* have non-negative step responses. To show this, first consider the intermediate result below.

Lemma 3: Consider a system $G(s)$ as in (1). If it is negative-imaginary and strictly proper, then $\langle \dot{y}, u\rangle_t \geq 0$ for all inputs $u \in \mathcal{L}_2$.

Proof: Strict properness of $G(s)$ implies $G(s) \sim (A, B, C, 0)$ and guarantees that the output is at least once differentiable. Consider the storage function

$$S(x) = x^TQx \tag{25}$$

with $Q \in \mathbb{S}^n_{\succ 0}$. If the matrix inequality (23) is satisfied, then

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} A^TQ + QA & QB - A^TC^T \\ (QB - A^TC^T)^T & -CB - (CB)^T \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \leq 0 \tag{26}$$

and the dissipitivity inequality

$$\dot{S}(x, u) \leq 2\dot{y}(\tau)u(\tau) \tag{27}$$

is then verified. Integrating (27) in time gives

$$S(x) \leq 2\langle \dot{y}, u\rangle_t \tag{28}$$

which implies $\langle \dot{y}, u\rangle_t \geq 0$. ■

The step response can then be bounded from below.

Proposition 2: If $G(s)$ is $\mathcal{NI}$ and such that $G(\infty) \geq 0$, then its unit step response is non-negative for all $t \geq 0$.

Proof: The system $G(s)$ can be written as

$$G(s) = \tilde{G}(s) + G(\infty) = \tilde{G}(s) + D,$$

where $\tilde{G}(s) \sim (A, B, C, 0)$ is strictly proper. By assumption $G(\infty) = D \geq 0$ so a sufficient condition for the unit step response of $G(s)$ to be non-negative is for the unit step response of $\tilde{G}(s)$ to be non-negative.

Defining the output of $\tilde{G}(s)$ to be $\tilde{y}(t)$ and the input to be $u(t)$, by Lemma 3, we have $\langle \dot{\tilde{y}}, u\rangle_t \geq 0$. Thus, with the step input $u(t) = 1$ for all $t \in [0, \infty)$ and $u(t) = 0$ for $t < 0$, we have

$$0 \leq \int_0^t \dot{\tilde{y}}(\tau)u(\tau)d\tau = \int_0^t \dot{\tilde{y}}(\tau)d\tau = \tilde{y}(t).$$

■

This result can be leveraged into a characterisation of external positivity.

Proposition 3: If $G(s) \sim (A, B, C, 0)$ is such that $CB \geq 0$ and $sG(s)$ is negative imaginary, then it is externally positive.

Proof: Since $G(s) \sim (A, B, C, 0)$ then $H(s) = sG(s)$ admits a state-space realisation

$$H(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$$

where $\tilde{C} = CA$ and $\tilde{D} = CB$ are defined for convenience. Since $H(s)$ is negative imaginary, Proposition 2 implies the step response of $H(s)$ is positive. With $y(t)$ the output of $H(s)$ and $u(t)$ the input, this positive step response implies

$$y(t) = \tilde{C}\int_0^t e^{A\tau}Bu(t-\tau)d\tau + \tilde{D}u(t) \geq 0. \tag{29}$$

By noting that $u(t) = 1\ \forall t \geq 0$, this becomes

$$y(t) = \tilde{C}\int_0^t e^{A\tau}Bd\tau + \tilde{D} \geq 0. \tag{30a}$$

Now, since $\frac{d}{dt}\left(e^{A}t\right) = A\ e^{At}$, the above expression can be simplified, viz,

$$\begin{aligned} y(t) &= CA\int_0^t e^{A\tau}Bd\tau + CB, && (31a)\\ &= C\int_0^t \frac{d}{d\tau}\left(e^{A\tau}\right)Bd\tau + CB, && (31b)\\ &= [Ce^{A}\tau B]_0^t + CB, && (31c)\\ &= Ce^{At}B \geq 0. && (31d)\end{aligned}$$

This last expression is the impulse response of $G(s)$[1].

To the authors' knowledge this is the first result providing a way to verify the time-domain property of external positivity using the frequency domain property of *negative imaginariness*. ∎

C. Positive-realness and negative-imaginariness

The previous subsections have shown the general non-equivalence between external positivity and either positive-realness or negative-imaginariness. Here, it is shown that the additional constraint of G being both negative-imaginary and positive-real is also *not* equivalent to external positivity. The consideration of this class of system was motivated from the results of [24] which showed that if a system admits a symmetric realisation, then its transfer function is both negative-imaginary and positive-real. A simple characterisation of external positivity would then follow if one could go in the reverse direction and show that all negative-imaginary and positive-real transfer functions admit a symmetric realisation. Unfortunately, this can be disproved by the counterexample

$$G_{\mathcal{SPR}+\mathcal{NI}}(s) = \frac{30}{s+75} - \frac{10}{s+45} \tag{32}$$

which is not externally positive. Finally, it is highlighted that the main results of [24] can be derived immediately from the equivalence between symmetric and relaxation systems [38].

IV. Connection to [17]

Finally, we identify a class of externally positive systems (those with relative degree greater than two) which can satisfy the conditions of this paper but fail those of [17, Theorem 1]. The recent result of [17, Theorem 1] provides a convenient LMI-based method for determining externally positivity of linear systems, with this theorem recalled here.

[1]This proof is more elegant than our first attempt and was suggested by one of the anonymous reviewers.

Theorem 3 ([17]): Given $G(s) \sim (A, B, C, D)$ with $D \geq 0$, assume there exists $K = K^T$ with inertia $(n-1, 0, 1)$ and $\gamma,\ \tau \in \mathbb{R}$ such that

$$\begin{aligned} A^TK + KA + 2\gamma K &\preceq 0, && (33a)\\ B^TKB &\leq 0, && (33b)\\ K + \tau C^TC &\succ 0, && (33c)\\ CB &\geq 0. && (33d)\end{aligned}$$

Then $G(s)$ is externally positive.

However, these conditions fail to certify externally positive $G(s)$ with a relative degree 2 or greater. 3. To show this, it is noted that if $G(s) \sim (A, B, C, 0)$ has relative degree 2 or greater, then B (which is assumed to be full column rank to avoid the trivial case when $G(s) = 0$) belongs to the null-space of C, and hence $CB = 0$. Now, multiply the inequality (33b) on the left by B^T and on the right by B to give

$$B^T(K + \tau C^TC)B = B^TKB + \tau B^TC^TCB = B^TKB > 0.$$

This directly contradicts (33b) and hence Theorem 3 cannot be applied if $G(s)$ has relative degree greater than 1. It is then possible to generate externally positive systems which fail the conditions of Theorem 3, such as

$$G(s) = \frac{1}{(s+a)^2} \text{ and } G(s) = \frac{1}{s}G_{\mathcal{NI}}(s), \tag{34}$$

with $G_{\mathcal{NI}}$ from (24). Externally positivity of these systems can be verified from Proposition 1 and 3 respectively, yet since both are relative degree 2, they fail Theorem 3. Since Theorem 3 is only applicable to systems with *strictly* positive impulse responses, it can be argued that relative degree two systems are not applicable to Theorem 3 – and there even exists numerical solutions to work around this limitation [16]. Nevertheless, the above analysis highlights how care has to be taken with a naive application of Theorem 3.

Conclusions

Several characterisations of externally positive linear systems were introduced. Particular emphasis was placed on realisation independent conditions defined by transfer function parameters with a complete characterisation of externally positive second-order and a class of third-order underdamped system developed. Criteria based around a negative-imaginariness property were defined, with negative-imaginary systems also shown to have a non-negative step response. Finally, counter-examples for a recently proposed numerical external positivity test were developed - systems of relative degree greater than two fail this test. Future work will focus on incorporating the external positivity conditions into convex Zames-Falb multiplier searches.

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