Synthesis of All Known Analytical Permittivity Reconstruction Techniques of Nonmagnetic Materials From Reflection and Transmission Measurements

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***Abstract*—This letter shows that all published analytical reconstruction formulas for the permittivity of a nonmagnetic material from the measured total reflection and transmission responses—the Nicolson–Ross, Weir, and Stuchly–Matuszewsky equations—are equal to either the impedance or propagation methods. We also show that the measured total reflection and transmission are independent measurements and hence allow for simultaneous and unique reconstruction of the complex electric permittivity, including conductivity, and the complex magnetic permeability.**

***Index Terms*—Microwave measurements, permittivity, permittivity reconstruction techniques, reflection and transmission.**

I. Introduction

MEASURING permittivity is a very broad field of research. Here we focus on broadband measurements from the megahertz region up to the gigahertz region of the spectrum. These are commonly done through analytical reconstruction from the measured total reflection and transmission of a transmission line where the material is placed, e.g., see [1]–[7], among others. The coaxial line theory assumes a transverse electromagnetic (TEM) propagation mode only. To ensure it, the line has to be operated below the cutoff frequency of the transverse magnetic (TM) mode (the only one that can be generated in a coaxial geometry [8]). This limits the range of applications for a given sample holder size and material under test.

The reflection and transmission measurements have to be compensated (moved from the measuring planes to the interface of the material), in one way or an other [2], [5]–[7], [9], and then different reconstruction formulas can be used to reconstruct the permittivity: the Nicolson–Ross, Weir, and Stuchly–Matuszewsky equations [1], [3], [4]. When a calibrated model of the line is available, it can be computed directly from the measured S-parameters with the propagation matrices method [10].

As reported in the literature, all these reconstruction techniques suffer from inaccuracies and instability for low-loss materials at resonant frequencies, and further, mathematical algorithms are used to reduce them [11]. As we show in this letter, this is understandable since all published analytical reconstruction formulas are in fact equal to one of the two fundamental and independent solutions. One is related to the impedance and the other to the propagation constant of the sample under test. The existence of two independent solutions is solely due to the assumption that the material is nonmagnetic. When this assumption is not made, then both solutions are needed to simultaneously reconstruct both the complex electric permittivity, including conductivity, and the magnetic permeability. They are as follows:

$\varepsilon^* = \varepsilon' - j\left(\varepsilon'' + (\sigma_{dc}/w)\right)$	complex permittivity;
$\varepsilon' = \varepsilon_0\varepsilon'_r$	real part of the permittivity, or dielectric constant;
$\varepsilon'' = \varepsilon_0\varepsilon''_r$	imaginary part of the permittivity, or dielectric loss, and the term $\varepsilon''+(\sigma_{dc}/w)$ accounts for dielectric and conductive losses;
$\mu^* = \mu' - j\mu''$	is the complex permeability,

The simultaneous reconstruction of both ε^* and μ^* requires both the reflection and transmission responses. This shows, that for nonshorted lines, these responses are independent of each other, as we prove in this letter.

II. Propagation Matrices: Representation and Inversion

In [10], we introduced the propagation matrices method as a representation of transmission lines and inverted for both the electric permittivity (ε^*) and the magnetic permeability (μ^*). In this letter, we show this schematically and only for ε^*.

The total reflection (Γ) and transmission (Υ) response of a multisectional transmission line (see Fig. 1) can be expressed as

$$\begin{bmatrix} 1 \\ \Gamma \end{bmatrix} = \boldsymbol{LPR} \begin{bmatrix} \Upsilon \\ 0 \end{bmatrix} \tag{1}$$

where $\boldsymbol{P}$ represents the propagation through the sample holder, whereas $\boldsymbol{L}$ and $\boldsymbol{R}$ represent the propagation at the left and right of the sample holder for arbitrary extensions. The propagation of TEM waves through the sections of the line is determined by products involving the following two matrices, each corresponding to a single section:

$$\boldsymbol{M}_n^L = \begin{bmatrix} e^{-\hat{\gamma}_n d_n} & 1 \\ \frac{1}{Z_n} e^{-\hat{\gamma}_n d_n} & -\frac{1}{Z_n} \end{bmatrix}$$
$$\boldsymbol{M}_n^R = \begin{bmatrix} 1 & e^{-\hat{\gamma}_n d_n} \\ \frac{1}{Z_n} & -\frac{1}{Z_n} e^{-\hat{\gamma}_n d_n} \end{bmatrix} \tag{2}$$

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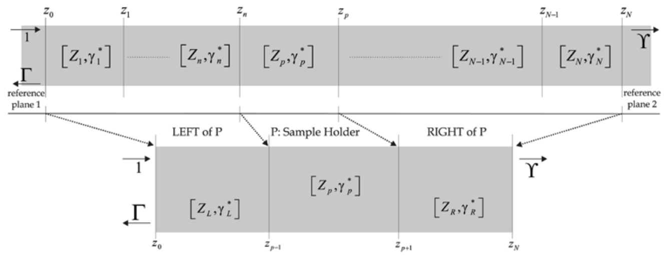

Fig. 1. Representation of a multisectional transmission line, and its simplified three-section model.

where γ_n and Z_n are the propagation constant and impedance of section n, while d_n is its length, for $n = 1$ to N sections. We assume here an $\exp(-j\omega t)$ time dependence. For a nonresistive line and nonmagnetic materials they read

$$\gamma_n = \frac{j\omega}{c_0}\sqrt{\varepsilon_r^*} \tag{3}$$

$$Z_n = \frac{Z_0}{\sqrt{\varepsilon_r^*}} \tag{4}$$

where $j^2 = -1$, ω is the angular frequency, c_0 is the speed of light in free space, and $Z_0 = \ln(a/b)\sqrt{\mu_0/\varepsilon_0}/2\pi$ is the characteristic impedance for a coaxial line of outer radius a and inner radius b. Then, the propagation matrices through the three different groups of sections are

$$\boldsymbol{P} = \boldsymbol{M}_p^R(\boldsymbol{M}_p^L)^{-1} = \begin{bmatrix} \cosh(\hat{\gamma}_p d_p) & Z_p \sinh(\hat{\gamma}_p d_p) \\ \frac{\sinh(\hat{\gamma}_p d_p)}{Z_p} & \cosh(\hat{\gamma}_p d_p) \end{bmatrix} \tag{5}$$

where p stands for the sampe holder, and

$$\boldsymbol{L} = (\boldsymbol{M}_0^L)^{-1}\left[\prod_{n=1}^{n=p-1} \boldsymbol{M}_n^R(\boldsymbol{M}_n^L)^{-1}\right] \tag{6}$$

$$\boldsymbol{R} = \left[\prod_{n=p+1}^{n=N-1} (\boldsymbol{M}_n^L)^{-1}\boldsymbol{M}_n^R\right]\boldsymbol{M}_N^R. \tag{7}$$

With the restriction of nonmagnetic materials, it is now possible to obtain two solutions for ε_r^*. Equation (1) can be rewritten into a more convenient form as

$$\begin{bmatrix} A \\ B \end{bmatrix} = \mathbf{P}\begin{bmatrix} C \\ D \end{bmatrix} \tag{8}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix} = \mathbf{L}^{-1}\begin{bmatrix} 1 \\ \Gamma \end{bmatrix} \quad \begin{bmatrix} C \\ D \end{bmatrix} = \mathbf{R}\begin{bmatrix} \Upsilon \\ 0 \end{bmatrix}. \tag{9}$$

Substituting $\boldsymbol{P}$ into (8) and eliminating the exponential terms, we obtain a solution for the impedance Z_p of the sample under test as

$$Z_p^2 = \frac{A^2 - C^2}{B^2 - D^2}. \tag{10}$$

If, instead, the impedance terms are eliminated, a solution in terms of the exponentials is found

$$\cosh \gamma_p d_p = \frac{AB + CD}{AD + BC}. \tag{11}$$

We hence obtain an expression for the propagation factor as

$$\gamma_p = \pm\frac{\operatorname{acosh}(\frac{AB+CD}{AD+BC})}{d_p}. \tag{12}$$

For strictly passive media, $\varepsilon' > 0$ and $\varepsilon'' \geq 0$ and choosing the positive branch of the square roots leads to the physical constraints $\Re\{\gamma_p\} \geq 0$, $\Im\{\gamma_p\} \geq 0$; hence the sign for the acosh function is determined.

The electric permittivity of the sample is related to the impedance and propagation factor via (3) and (4). As mentioned earlier, we then obtain two solutions for ε_r^*

$$\varepsilon_r^* = \left(\frac{Z_0}{Z_p}\right)^2 \text{ the impedance method} \tag{13}$$

$$\varepsilon_r^* = \left(\frac{c_0\gamma_p}{j\omega}\right)^2 \text{ the propagation method} \tag{14}$$

with Z_p and γ_p defined in (10) and (12), respectively.

One can now reconstruct the permittivity [(13) and (14)] directly from the measurements at the reference planes of the tool, not at the end levels of the sample holder. Previous techniques required the measurements to be moved from the reference planes to the end levels of the sample holder, removing the contribution of transition sections [1], [3]–[6], [12]. In the reviewed literature, there are at least six reported analytical equations to reconstruct the permittivity from measurements of the total reflection and transmission responses. They appear to be different solutions; however, it can be proven that these solutions are equal to either the impedance method of (13) or the propagation method of (14). To prove it, it is instructive to first consider the propagation of TEM waves along an ideal line of a sample in free space.

III. Permittivity Reconstruction From an Ideal Line

Let us consider the simplest configuration possible, where the material of study is placed at the sample holder of the coaxial transmission line, and the transition sections between the reference planes and the interfaces of the sample holder have been eliminated. Then, the measurement planes are at the interfaces of the material under study. This is the configuration that most of the published methods consider [3], [4], [13]. The input on reference plane 1 is equal to the product of three propagation matrices: first to go from reference plane 1 into the sample holder, then the propagation in the sample holder itself, after that, from the sample holder into the reference plane 2, and finally, the output at reference plane 2. As the reference planes coincide with the sample holder interfaces, $\boldsymbol{L}$ and $\boldsymbol{R}$ have no exponential terms but only free-space impedances. For such a configuration, we can write the extended form of (1) as

$$\begin{bmatrix}1\\ \Gamma\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & Z_0\\ 1 & -Z_0\end{bmatrix}\begin{bmatrix}\cosh(\hat{\gamma}_p d_p) & Z_p\sinh(\hat{\gamma}_p d_p)\\ \frac{\sinh(\hat{\gamma}_p d_p)}{Z_p} & \cosh(\hat{\gamma}_p d_p)\end{bmatrix} \times \begin{bmatrix}1 & 1\\ \frac{1}{Z_0} & -\frac{1}{Z_0}\end{bmatrix}\begin{bmatrix}\Upsilon\\ 0\end{bmatrix}. \quad (15)$$

Taking into consideration that $r = (Z_p - Z_0/Z_p + Z_0)$ is the local reflection coefficient, and introducing $\zeta = e^{-\hat{\gamma}_p d_p}$ we can rewrite (15) as

$$\begin{bmatrix}1\\ \Gamma\end{bmatrix} = \frac{1}{\zeta(1-r^2)}\begin{bmatrix}1-r^2\zeta^2 & r(\zeta^2-1)\\ r(1-\zeta^2) & \zeta^2-r^2\end{bmatrix}\begin{bmatrix}\Upsilon\\ 0\end{bmatrix}. \quad (16)$$

Now, we obtain the more common format for the reflection and transmission from (16)

$$\Gamma = \frac{r(1-\zeta^2)}{1-r^2\zeta^2} \quad \Upsilon = \frac{\zeta(1-r^2)}{1-r^2\zeta^2}. \quad (17)$$

From (17), it is clear that for low-loss materials $(\varepsilon_r'' < 10^{-2}\varepsilon_r')$, resonant frequencies are likely to occur in the frequency range of interest. This happens when $e^{-2\gamma_p d_p} = 1$, $|\Gamma| = 0$, and $|\Upsilon| = 1$.

Using the method of solution of Section II, (9) for an ideal line is given by

$$\begin{bmatrix}A\\ B\end{bmatrix} = \begin{bmatrix}1+\Gamma\\ \frac{1}{Z_0} - \frac{\Gamma}{Z_0}\end{bmatrix} \text{ and } \begin{bmatrix}C\\ D\end{bmatrix} = \begin{bmatrix}\Upsilon\\ \frac{\Upsilon}{Z_0}\end{bmatrix}. \quad (18)$$

So that

$$\varepsilon_r^* = \frac{(1-\Gamma)^2 - \Upsilon^2}{(1+\Gamma)^2 - \Upsilon^2}, \text{ impedance method} \quad (19)$$

$$\varepsilon_r^* = -\frac{c_0^2}{\omega^2 d_p^2}\left[\operatorname{acosh}\left(\frac{1+\Upsilon^2-\Gamma^2}{2\Upsilon}\right)\right]^2, \text{ propagation method.} \quad (20)$$

These are the two fundamental and independent reconstruction methods for a nonmagnetic material filling an ideal line. In Section IV, we show how other published analytical solutions are equivalent to either of these two methods.

IV. Analytical Solutions for the Ideal Case

As mentioned before, in the reviewed literature, there are at least six reported analytical equations to reconstruct the permittivity from measurements of the total reflection and transmission responses. They appear to be different solutions however; in this section, we prove that these solutions are equal to either the impedance method of (19) or the propagation method of (20).

The impedance and the propagation reconstruction methods for an ideal line, (19) and (20), are already equivalent to those derived by Stuchly and Matuszewsky [4]. They claimed that the propagation equation is ambiguous for certain sample lengths. As a matter of fact, the acosh of a complex number z is defined as

$$\operatorname{acosh}(z) = \log\left[z + \sqrt{z+1}\sqrt{z-1}\right] \quad (21)$$

and physical constraints determine the sign for the complex logarithm, as expressed for (11). However, the imaginary part of the logarithm of a complex number is equal to the angle of the complex value plus $2n\pi$, with n being an integer, and hence the $\operatorname{acosh}(z)$ would have infinite number of roots. Weir [3] presents a thorough explanation of this ambiguity and resolves it with the use of automatic measurements, where discrete frequency steps, small enough so that the phase of z changes less than 360° from one measurement frequency to the next.

From (21), it is interesting to note that for $z = \pm 1$ (resonance) the logarithm is zero, and the permittivity will not be accurately determined. This occurs with low-loss materials so that almost no reflection, and therefore, total transmission occur at resonant frequencies.

Palaith and Chang [12] present a comparison of three methods; their $1/Z^2$ method is the first equation presented in Stuchly and Matuszewsky [4] and therefore the impedance method. The second is their so-called K^2 method that they claim to be new, while it is equal to the second equation from Stuchly and Matuszewsky [4] and the same as the propagation method, since they write

$$\zeta = e^{-\gamma_p d_p} = \frac{1}{2\Upsilon}\left[(1+\Upsilon^2-\Gamma^2) \pm \left[(1+\Upsilon^2-\Gamma^2)^2 - 4\Upsilon^2\right]^{1/2}\right]. \quad (22)$$

To find γ_p we would take the natural logarithm of the right-hand side. This can be written as a acosh in virtue of (21). Then, it is the same expression for ε_r^* as (20). As a third solution, Palaith and Chang [12] analyze the Nicolson–Ross–Weir method, published by Nicolson and Ross [1] and by Weir [3] for both magnetic permeability and electrical permittivity. We restrict ourselves to the solution for the permittivity. Their equation is as follows:

$$\varepsilon_r^* = j\frac{c_0}{\omega d_p}\left[\frac{1-r}{1+r}\right]\ln(\zeta) \quad (23)$$

where r is the local reflection coefficient, and ζ is expressed in terms of Γ and Υ by (22). Again we obtain the propagation method. Note that the expression of (23) is somewhat misleading because the local reflection coefficient in the right-hand side contains the unknown complex permittivity of the sample under test: $\sqrt{\varepsilon_r^*} = [(1-r)/(1+r)]$, so that, in fact, it is not another equation but the same as the square root of (20). These methods are obviously the same, and they naturally all suffer from the same problems. They are not well behaved for low-loss materials, especially at frequencies corresponding to integer multiples of one-half wavelength in the sample; see [4], or in other words at resonant frequencies. Ligthart [13] and Baker-Javis *et al.* [11] have tried to bypass this ill behavior in two different ways but using the same equations. Ligthart [13] presented a method for shorted line measurements where the scattering equations for the permittivity were solved over a calculated uncertainty region and the results were then averaged, but he could not avoid the low-loss problem. Finally, Baker-Javis *et al.* [11] minimized the instability by an iterative procedure.

From the fact there are two independent methods to solve for the complex electric permittivity, including conductivity, for nonmagnetic materials, from single reflection and transmission response measurements, it can be assumed that the reflections and transmission responses are independent for a general, nonshorted, transmission line. This will be shown in Section V.

V. Independence of the Reflection and Transmission Responses

To prove that Γ and Υ are independent measurements, we make use of basic linear algebra. A system of two linear equations, such as (1), has a unique solution only when the determinant $|\boldsymbol{LPR}|$ is not zero. Rewriting $\boldsymbol{LPR}$ in its extended form

$$\boldsymbol{LPR} = (\boldsymbol{M}_0^L)^{-1} \left[\prod_{n=1}^{n=p-1} \boldsymbol{M}_n^R (\boldsymbol{M}_n^L)^{-1} \right] \left[\boldsymbol{M}_p^R (\boldsymbol{M}_p^L)^{-1} \right] \times \left[\prod_{n=p+1}^{n=N-1} (\boldsymbol{M}_n^L)^{-1} \boldsymbol{M}_n^R \right] \boldsymbol{M}_N^R \quad (24)$$

and taking into consideration that

$$|(\boldsymbol{M}_n^L)^{-1}||\boldsymbol{M}_n^R| = |\boldsymbol{M}_n^R||(\boldsymbol{M}_n^L)^{-1}| = 1 \quad (25)$$

it is then clear that

$$|\boldsymbol{LPR}| = 1. \quad (26)$$

Therefore Γ and Υ are independent, and two solutions (permittivity and permeability) can be found from (1).

VI. Conclusion

Writing the propagation of TEM waves along an ideal transmission line (which is used to measure the permittivity of a certain material) in terms of propagation matrices has proven very useful. It has enabled us to show how the seemingly different published analytical reconstruction formulas for the permittivity of nonmagnetic materials: the Nicolson–Ross, Weir, Stuchly–Matuszewsky, and Palaith–Chang equations are the same as the solutions derived here, based on either the impedance or propagation methods. In the future, researchers will not need to investigate the performance of the different equations for their experiments. The range of choices has been reduced to only the two fundamental methods.

We have also demonstrated that the measured total reflection and transmission responses are independent measurements. This shows the possibility to uniquely determine two parameters, the complex electric permittivity, including conductivity, and the complex magnetic permeability for a single frequency from these measurements assuming that the sample under test can be regarded homogeneous.

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