ISOVECTOR FIELDS AND SIMILARITY SOLUTIONS FOR 1-D LINEAR POROELASTICITY

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Isovector fields and similarity solutions for 1-D linear poroelasticity

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Abstract

The system of isovector fields of the Biot's equations for one-dimensional linear poroelasticity is calculated in this paper by using exterior calculus. Similarity solutions for some special cases are also presented in this paper.

1 Introduction

A poroelastic material consists of an elastic skeleton and pores filled with fluid. Porous seabed, saturated rock and bone are examples of this type of material. Acoustic wave propagation in poroelastic materials is described by the Biot's equations of poroelasticity [4], [3]:

$$\begin{cases} \nabla \cdot \sigma_s - b(x)\dfrac{\partial}{\partial t}(u_s - u_l) = \dfrac{\partial^2}{\partial t^2}\left(\rho_{11}(x)u_s + \rho_{12}(x)u_l\right), \\ \nabla\sigma + b(x)\dfrac{\partial}{\partial t}(u_s - u_l) = \dfrac{\partial^2}{\partial t^2}\left(\rho_{12}(x)u_s + \rho_{22}(x)u_l\right). \end{cases}$$

Here u_s is the displacement in the solid part, u_l the displacement in the fluid part; σ_s, σ are the stress applied to the solid part and the fluid part, respectively. The physical parameters $\rho_{ij}(x)$, $i, j = 1, 2$ are the mass coupling of the fluid part and the solid part whereas $b(x)$ is the energy dissipation term of the system. Following Biot [4], we assume there is a strain energy function $W(x, e_s, \zeta)$ such that the bulk stress τ and the pore fluid pressure p can be written as

$$\tau = \frac{\partial W}{\partial e_s},\ p = \frac{\partial W}{\partial \zeta}$$

, where e_s is the symmetric strain tensor of the solid part, $\zeta := f\dfrac{\partial}{\partial x}(u_s - u_l)$ the increment of fluid content and f the porosity. In order to put the equations into divergence form, we assume f to be a constant. The relation between the bulk stress τ, σ and p is:

$$\tau = \sigma_s + \sigma,\ \sigma = -f\,p.$$

Note that in the 1D case which we consider here, $e_s = \frac{\partial u_s}{\partial_x}$. Define a new variable $u(x,t) := u_s(x,t) - u_l(x,t)$. Then the divergence form of the governing equations is:

$$\text{(PE)}\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial W}{\partial e_s} + \dfrac{\partial W}{\partial u_x}\right) + \dfrac{\partial}{\partial t}\left[-b(x)u - (\rho_{11}(x) + \rho_{12}(x))\dfrac{\partial u_s}{\partial t} + \rho_{12}\dfrac{\partial u}{\partial t}\right] = 0, \\ \dfrac{\partial}{\partial x}\left(-\dfrac{\partial W}{\partial u_x}\right) + \dfrac{\partial}{\partial t}\left[b(x)u - (\rho_{12}(x) + \rho_{22}(x))\dfrac{\partial u_s}{\partial t} + \rho_{22}(x)\dfrac{\partial u}{\partial t}\right] = 0. \end{cases}$$

This paper is organized as follows. In Section 2, we will adapt Şuhubi's notation in [2] to solve for the isovector fields of the balance ideal which is determined by the governing equations. This will involve solving an over-determined system of partial differential equations. We will then solve for the similarity solutions generated by the isovector fields for some special cases. This involves solving a system of first order quasilinear partial differential equations.
We list the definition of some terminology which will be used in this paper.

Definition 1.1 (Kinematic space K). *Given a system of second order partial differential equations with N dependent variables $u_1, \dots u_N$, and n independent variables $x_1, \dots x_n$, the kinematic space $K := \mathcal{G} \times \mathcal{R}_{nN}$ with the global coordinate cover $\{x^i, u^j, \nu^i_j\}$, $i = 1 \sim n$, $j = 1 \sim N$. Here $\mathcal{G}$ is the graph space of the solution and $\mathcal{R}_{nN}$ is an nN-fold Cartesian product of the real number line $\mathcal{R}$.*

Definition 1.2 (Exterior algebra $\Lambda(E_n)$). *Let $\{x^1, x^2, ..., x^n\}$ be a coordinate cover of the vector space E_n. The exterior algebra $\Lambda(E_n)$ is defined as the direct sum:*

$$\Lambda(E_n) := \Lambda^0(E_n) \oplus \Lambda^1(E_n) \oplus ... \oplus \Lambda^n(E_n),$$

where $\Lambda^0(E_n)$ is the vector space of all real-valued C^∞ functions on E_n and $\Lambda^k(E_n)$, $1 \le k \le n$ is the vector space of all exterior forms of degree k over $\Lambda^0(E_n)$ with the natural basis $\{dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}, \ i_1 < i_2 < ... < i_k\}$. Here $\wedge$ is the operation of exterior multiplication.

Definition 1.3 (Solution Map). *A map Φ:$B_n \to K$ is a solution map of a given system $\{\omega^\alpha\}$ of n-form on K iff Φ is a regular map from B_n to K such that the induced pullback map Φ^* satisfies*

$$\Phi^* \omega^\alpha = 0, \ \alpha = 1, \dots, N.$$

2 Isovector Field

Following the notations in [2], we rewrite (PE) as

$$\begin{cases} \dfrac{\partial \Sigma^{11}}{\partial x^1} + \dfrac{\partial \Sigma^{12}}{\partial x^2} = 0, \\ \dfrac{\partial \Sigma^{21}}{\partial x^1} + \dfrac{\partial \Sigma^{22}}{\partial x^2} = 0. \end{cases}$$

We also define new variables:

$$\begin{aligned} & u^1 := u_s; \ u^2 := u, \ x^1 := x, \ x^2 := t, \\ & \nu^i_j := \frac{\partial u^i}{\partial x^j}, \\ & \Sigma^{11} := \frac{\partial W}{\partial \nu^1_1} + \frac{\partial W}{\partial \nu^2_1}, \\ & \Sigma^{12} := -b(x^1)u^2 - \left(\rho_{11}(x^1) + \rho_{12}(x^1)\right) \nu^1_2 + \rho_{12}(x^1)\nu^2_2, \\ & \Sigma^{21} := -\frac{\partial W}{\partial \nu^2_1}, \\ & \Sigma^{22} := b(x^1)u^2 - \left(\rho_{12}(x^1) + \rho_{22}(x^1)\right) \nu^1_2 + \rho_{22}(x^1)\nu^2_2. \end{aligned}$$

There are two independent variables and two dependent variables in this system, so $n = 2$ and $N = 2$ and the kinematic space is $K = R^2 \times R^2 \times R^4$. The two contact 1-forms of the exterior algebra $\Lambda(K)$ are

$$C^i := du^i - \nu^i_j dx^j, \ i, \ j = 1, \ 2, \tag{1}$$

and two balance forms are

$$\omega^i := d\Sigma^{ij} \wedge \mu_j = \frac{\partial \Sigma^{ij}}{\partial x^j}\mu + \frac{\partial \Sigma^{ij}}{\partial u^k} du^k \wedge \mu_j + \frac{\partial \Sigma^{ij}}{\partial \nu_l^k} d\nu_l^k \wedge \mu_j,\, i,\, j = 1,\, 2, \tag{2}$$

where $\mu_j := \partial_j \rfloor \mu$ and $\mu := dx_1 \wedge dx_2$, i.e. $\mu_1 = dx_2$ and $\mu_2 = -dx_1$. Here $\wedge$ is the exterior product and $\{\mu_1,\, \mu_2\}$ is the natural top-down basis of the 1-forms in the exterior algebra $\Lambda(R^2)$.

The idea of solving partial differential equations with a geometric approach is due to Cartan: a solution to (PE) can be regarded as a regular map $\Phi : R^2 \to K$ which solves in parametric form the system of exterior equations $\omega^i = 0$, $i = 1, 2$. In other words, a solution to (PE) is a map $\Phi : R^2 \to K$ which solves the balance ideal

$$B := I\{C^1,\, C^2,\, dC^1,\, dC^2,\, \omega^1,\, \omega^2\},$$

and $\Phi^*\mu \neq 0$. By adding dC^1 and dC^2 into the ideal generator, we "close" the ideal with respect to exterior differentiation without changing its isovector fields. The advantage of doing this is that we can have a representation formula for the elements of the ideal.

Recall that an isovector field V of an ideal I is a vector field in the tangent space $T(K)$ such that

$$\boldsymbol{\mathcal{L}}_V I \subset I,$$

where $\boldsymbol{\mathcal{L}}_V$ is the Lie derivative operator with respect to V. We denote the unknown $V \in T(K)$ by

$$V = V^i \frac{\partial}{\partial x^i} + V_0^i \frac{\partial}{\partial u^i} + V_j^i \frac{\partial}{\partial \nu_j^i}, i,\, j = 1 \sim 2. \tag{3}$$

We only consider materials whose energy density function has the following positive definite quadratic form:

$$W = w_1 \nu_1^1 \nu_1^1 - 2w_2 \nu_1^1 \nu_1^2 + w_3 \nu_1^2 \nu_1^2,$$

where $w_1 > 0$, $w_2 > 0$, $w_3 > 0$ and $w_1 w_3 - w_2^2 > 0$. We will **exclude** the case of special coupling between the elastic and dynamic constant such that $w_3 = \dfrac{\rho_{22} w_2}{\rho_{12} + \rho_{22}} = \dfrac{\rho_{22} w_1}{\rho_{11} + 2\rho_{12} + \rho_{22}}$. We will call this **Assump 1**. The goal in this section is to find all the V's such that $\boldsymbol{\mathcal{L}}_V B \subset B$.

The isovector field (3) in general depends on all the eight variables in K, but for a system with more then one equation, i.e. $N > 1$, it can be shown that [2]

$$V^i = V^i(x^1,\, x^2,\, u^1,\, u^2), \tag{4}$$
$$V_0^i = V_0^i(x^1,\, x^2,\, u^1,\, u^2), \tag{5}$$

i.e. V^i and V_j^i are independent of ν_i^j. Because of the closedness of the balance ideal B and the commutability of the two operators $\boldsymbol{\mathcal{L}}_V$ and d, we have

$$\boldsymbol{\mathcal{L}}_V B \subset B \text{ iff } \boldsymbol{\mathcal{L}}_V C^i \in B \text{ and } \boldsymbol{\mathcal{L}}_V \omega^i \in B.$$

These two equivalent conditions will be used to calculate the isovector fields of B. The first condition $\boldsymbol{\mathcal{L}}_V C^i \in B$ implies there exists $\lambda_j^i \in \Lambda^0(K) \equiv C^\infty(K)$ such that

$$\boldsymbol{\mathcal{L}}_V C^i = \lambda_j^i C^j. \tag{6}$$

Combining this with the well-known formula

$$\boldsymbol{\mathcal{L}}_V \alpha = V \rfloor d\alpha + d(V \rfloor \alpha),\, \forall \alpha \in \Lambda(K), \tag{7}$$

leads to the following set of equations:

$$\begin{aligned}
\lambda^k_j &= \frac{\partial V^k_0}{\partial u^j} - \nu^k_i \frac{\partial V^i}{\partial u^j}, &(8)\\
V^k_j &= \frac{\partial V^k_0}{\partial x^j} - \nu^k_i \frac{\partial V^i}{\partial x^j} + \nu^i_j \left(\frac{\partial V^k_0}{\partial u^i} - \nu^k_j \frac{\partial V^j}{\partial u^i}\right). &(9)
\end{aligned}$$

Equations (8) and (9) show that λ^i_j and V^i_j can be calculated from V^i and V^i_0, $i, j = 1 \sim 2$. Similarly, the condition $\boldsymbol{\mathcal{L}}_V \omega^i \in B$ implies there exists A^α, $A^{\alpha\gamma}_\beta$, $A^{\alpha ij}_{\beta\gamma}$, $A^{\alpha ij}_\beta$, $A^{\alpha ijk}_{\beta\gamma} \in \Lambda^0(K)$ such that

$$\boldsymbol{\mathcal{L}}_V \omega^i = A^\alpha \mu + A^{\alpha\gamma}_\beta du^\beta \wedge \mu_\gamma + A^{\alpha ij}_{\beta\gamma} du^\beta \wedge du^\gamma \wedge \mu_{ji} + A^{\alpha ij}_\beta d\nu^\beta_j \wedge \mu_i + A^{\alpha ijk}_{\beta\gamma} d\nu^\beta_k \wedge du^\gamma \wedge \mu_{ji}. \quad (10)$$

Applying the definition of the balance forms (2) to the left-hand side of (10) leads to the following relations between these coefficient functions and the isovector V:

$$\begin{aligned}
A^\alpha &= \frac{\partial V < \Sigma^{\alpha i} >}{\partial x^i} + \frac{\partial \Sigma^{\alpha i}}{\partial x^i}\frac{\partial V^j}{\partial x^j} - \frac{\partial \Sigma^{\alpha i}}{\partial x^j}\frac{\partial V^j}{\partial x^i}, &(11)\\
A^{\alpha\gamma}_\beta &= \frac{\partial V < \Sigma^{\alpha\gamma} >}{\partial u^\beta} + \frac{\partial \Sigma^{\alpha i}}{\partial x^i}\frac{\partial V^\gamma}{\partial u^\beta} - \frac{\partial \Sigma^{\alpha\gamma}}{\partial x^i}\frac{\partial V^i}{\partial u^\beta}\\
&+ \frac{\partial \Sigma^{\alpha\gamma}}{\partial u^\beta}\frac{\partial V^i}{\partial x^i} - \frac{\partial \Sigma^{\alpha i}}{\partial u^\beta}\frac{\partial V^\gamma}{\partial x^i}, &(12)\\
A^{\alpha ij}_{\beta\gamma} &= -A^{\alpha ji}_{\beta\gamma} = A^{\alpha ij}_{\gamma\beta}\\
&= \frac{1}{4}\left(\frac{\partial \Sigma^{\alpha i}}{\partial u^\beta}\frac{\partial V^j}{\partial u^\gamma} - \frac{\partial \Sigma^{\alpha j}}{\partial u^\beta}\frac{\partial V^i}{\partial u^\gamma} + \frac{\partial \Sigma^{\alpha j}}{\partial u^\gamma}\frac{\partial V^i}{\partial u^\beta} - \frac{\partial \Sigma^{\alpha i}}{\partial u^\gamma}\frac{\partial V^j}{\partial u^\beta}\right), &(13)\\
A^{\alpha ik}_\beta &= \frac{\partial V < \Sigma^{\alpha i} >}{\partial \nu^\beta_k} + \frac{\partial \Sigma^{\alpha i}}{\partial \nu^\beta_k}\frac{\partial V^j}{\partial x^j} - \frac{\partial \Sigma^{\alpha j}}{\partial \nu^\beta_k}\frac{\partial V^i}{\partial x^j}, &(14)\\
A^{\alpha ijk}_{\beta\gamma} &= -A^{\alpha jik}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial \Sigma^{\alpha i}}{\partial \nu^\beta_k}\frac{\partial V^j}{\partial u^\gamma} - \frac{\partial \Sigma^{\alpha j}}{\partial \nu^\beta_k}\frac{\partial V^i}{\partial u^\gamma}\right), &(15)
\end{aligned}$$

where $V < \cdot >$ is a linear functional on $\Lambda^0(K)$, i.e.

$$V < g > := v^i \frac{\partial g}{\partial y^i}, \text{ for } V = v^i \partial_i \in T(K),\ g \in \Lambda^0(K), \text{ where } \{y_i\} \text{ is a coordinate cover of } K.$$

On the other hand, the balance forms ω^i, $i = 1, 2$ in (2) can be decomposed into two parts such that one part contains the contact forms while the other part doesn't, i.e.

$$\begin{aligned}
\omega^i &= \frac{\partial \Sigma^{ij}}{\partial u^k} C^k \wedge \mu_j + \left\{\left(\frac{\partial \Sigma^{ij}}{\partial x^j} - \nu^k_j \frac{\partial \Sigma^{ij}}{\partial u^k}\right)\mu + \frac{\partial \Sigma^{ij}}{\partial \nu^k_l} d\nu^k_l \wedge \mu_j\right\}\\
&=: \frac{\partial \Sigma^{ij}}{\partial u^k} C^k \wedge \mu_j + \omega^{i\prime}.
\end{aligned}$$

Therefore, we have $I\{C^1, C^2, dC^1, dC^2, \omega^1, \omega^2\} = I\{C^1, C^2, dC^1, dC^2, \omega^{1\prime}, \omega^{2\prime}\}$. Replacing the balance form in (10) with this decomposition and considering (6), followed by collecting terms, we obtain these equations:

$$\lambda^\alpha_\beta \left(\frac{\partial \Sigma^{\beta i}}{\partial x^i} + \frac{\partial \Sigma^{\beta i}}{\partial u^\gamma}\nu^\gamma_i\right) = A^\alpha + \left(A^{\alpha i}_\beta + 2A^{\alpha ij}_{\beta\gamma}\nu^\gamma_j\right)\nu^\beta_i, \quad (16)$$

$$\lambda^\alpha_\gamma \left(\frac{\partial \Sigma^{\gamma i}}{\partial \nu^\beta_j} + \frac{\partial \Sigma^{\gamma j}}{\partial \nu^\beta_i}\right) = A^{\alpha ij}_\beta + A^{\alpha ji}_\beta + 2\left(A^{\alpha ikj}_{\beta\gamma} + A^{\alpha jki}_{\beta\gamma}\right)\nu^\gamma_k. \quad (17)$$

The above system contains 18 equations for the 8 unknowns V^i, V^i_0 and λ^α_β. These equations are analyzed in the following section.

3 Analysis of equations (16) and (17)

The equation of $(\alpha, i, j, \beta) = (1, 2, 1, 2)$ in (17) is

$$2\,\rho_{12}\frac{\partial V^1}{\partial x^2} + 4(w_3 - w_2)\frac{\partial V^2}{\partial x^1} + 2(w_3 - w_2)\frac{\partial V^2}{\partial u^1}\nu_1^1 + 2(w_3 - w_2)\frac{\partial V^2}{\partial u^2}\nu_1^2 + \rho_{12}\frac{\partial V^1}{\partial u^1}\nu_2^1 + \rho_{12}\frac{\partial V^1}{\partial u^2}\nu_2^2 = 0.$$

Because of (5) and the fact that [5] $\rho_{12} < 0$, the coefficients of ν_2^i gives

$$\frac{\partial V^1}{\partial u^1} = 0, \;\; \frac{\partial V^1}{\partial u^2} = 0.$$

Applying the above conclusion to the equation of $(\alpha, i, j, \beta) = (2, 2, 1, 2)$ in (17) leads to:

$$2\,\rho_{22}\frac{\partial V^1}{\partial x^2} - 4w_3\frac{\partial V^2}{\partial x^1} - 2w_3\frac{\partial V^2}{\partial u^1}\nu_1^1 - 2w_3\frac{\partial V^2}{\partial u^2}\nu_1^2 = 0.$$

Similarly, $w_3 > 0$ leads to the conclusion that

$$\frac{\partial V^2}{\partial u^1} = 0, \;\; \frac{\partial V^2}{\partial u^2} = 0.$$

Applying the above results to the equations of $(\alpha, i, j, \beta) = (2, 2, 1, 2)$ and $(\alpha, i, j, \beta) = (1, 1, 2, 2)$, we get the following system of equations for $\frac{\partial V^1}{\partial x^2}$ and $\frac{\partial V^2}{\partial x^1}$:

$$\begin{aligned} &\rho_{22}\frac{\partial V^1}{\partial x^2} - 2w_3(x^1)\frac{\partial V^2}{\partial x^1} = 0, \\ &\rho_{12}\frac{\partial V^1}{\partial x^2} + 2\left(w_3(x^1) - w_2(x^1)\right)\frac{\partial V^2}{\partial x^1} = 0. \end{aligned}$$

Under **Assump 1**, we have

$$\frac{\partial V^1}{\partial x^2} = 0, \;\; \frac{\partial V^2}{\partial x^1} = 0.$$

That is, $V^1 = V^1(x^1)$ and $V^2 = V^2(x^2)$. Applying these conclusions, (17) reduces to the following 8 equations

$$
\begin{aligned}
(2,2,2,1) \qquad & (\rho_{11}+\rho_{12})\,\lambda_1^2 + (\rho_{12}+\rho_{22})\,\lambda_2^2 - (\rho_{12}+\rho_{22})\,V^{1\prime} - V^1\,(\rho_{12}'+\rho_{22}') \\
& - (\rho_{12}+\rho_{22})\left(\frac{\partial V_0^1}{\partial u^1} - \dot{V}^2\right) + \rho_{22}\frac{\partial V_0^2}{\partial u^1} = 0, \qquad (18)
\end{aligned}
$$

$$
(2,2,2,2) \qquad \rho_{12}\lambda_1^2 + \rho_{22}\lambda_2^2 - \rho_{22}V^{1\prime} - \rho_{22}'V^1 + (\rho_{11}+\rho_{22})\,\frac{\partial V_0^1}{\partial u^2} - \rho_{22}\left(\frac{\partial V_0^2}{\partial u^2} - \dot{V}^2\right) = 0, \qquad (19)
$$

$$
(1,2,2,2) \qquad \rho_{12}\lambda_1^1 + \rho_{22}\lambda_2^1 - \rho_{12}V^{1\prime} - V^1\rho_{12}' + (\rho_{11}+\rho_{12})\frac{\partial V_0^1}{\partial u^2} - \rho_{12}(\frac{\partial V_0^2}{\partial u^2} - \dot{V}^2) = 0, \qquad (20)
$$

$$
\begin{aligned}
(1,2,2,1) \qquad & (\rho_{11}+\rho_{12})\,\lambda_1^1 + (\rho_{12}+\rho_{22})\,\lambda_2^1 - (\rho_{11}+\rho_{12})\,V^{1\prime} - (\rho_{11}'+\rho_{12}')\,V^1 \\
& - (\rho_{11}+\rho_{12})\left(\frac{\partial V_0^1}{\partial u^1} - \dot{V}^2\right) + \rho_{12}\frac{\partial V_0^2}{\partial u^2} = 0, \qquad (21)
\end{aligned}
$$

$$
(2,1,1,2) \qquad (w_3-w_2)\lambda_1^2 - w_3\lambda_2^2 + w_3\dot{V}^2 + V^1w_3' - w_2\frac{\partial V_0^1}{\partial u^2} + w_3\left(\frac{\partial V_0^2}{\partial u^2} - V^{1\prime}\right) = 0, \qquad (22)
$$

$$
\begin{aligned}
(2,1,1,1) \qquad & (w_1-w_2)\,\lambda_1^2 + w_2\lambda_2^2 - w_2\,\dot{V}^2 - V^1\,w_2' \\
& -w_2\left(\frac{\partial V_0^1}{\partial u^1} - V^{1\prime}\right) + w_3\frac{\partial V_0^2}{\partial u^1} = 0, \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
(1,1,1,1) \qquad & (w_1-w_2)\,\lambda_1^1 + w_2\lambda_2^1 - (w_1-w_2)\,\dot{V}^2 - V^1(w_1'-w_2') \\
& -\left(\frac{\partial V_0^1}{\partial u^1} - V^{1\prime}\right)(w_1-w_2) + (w_2-w_3)\frac{\partial V_0^2}{\partial u^1} = 0, \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
(1,1,1,2) \qquad & (w_3-w_2)\lambda_1^1 - w_3\lambda_2^1 - (w_3-w_2)\dot{V}^2 - V^1(w_3'-w_2') + \frac{\partial V_0^1}{\partial u^2}(w_2-w_1) \\
& -(w_3-w_2)(\frac{\partial V_0^2}{\partial u^2} - V^{1\prime}) = 0. \qquad (25)
\end{aligned}
$$

The two equations in (16) are

$$
\begin{aligned}
& \lambda_1^1\left(\frac{\partial^2 W}{\partial\,\nu_1^1\partial x^1} + \frac{\partial^2 W}{\partial\,\nu_1^2\partial x^1}\right) - \lambda_2^1\frac{\partial^2 W}{\partial\,\nu_1^2\partial x^1} + \left(\lambda_2^1 - \lambda_1^1\right)b(x^1)\,\nu_2^2 - A^1 \\
& -A_1^{11}\,\nu_1^1 - A_1^{12}\,\nu_2^1 - A_2^{11}\,\nu_1^2 - A_2^{12}\,\nu_2^2 = 0, \qquad (26) \\
& \lambda_1^2\left(\frac{\partial^2 W}{\partial\,\nu_1^1\partial x^1} + \frac{\partial^2 W}{\partial\,\nu_1^2\partial x^1}\right) - \lambda_2^2\frac{\partial^2 W}{\partial\,\nu_1^2\partial x^1} + \left(\lambda_2^2 - \lambda_1^2\right)b(x^1)\,\nu_2^2 - A^2 \\
& -A_1^{21}\,\nu_1^1 - A_1^{22}\,\nu_2^1 - A_2^{21}\,\nu_1^2 - A_2^{22}\,\nu_2^2 = 0. \qquad (27)
\end{aligned}
$$

Note that equations (18)-(21) imply that λ_j^i, $i,j=1,2$ are not functions of ν_j^i, $i,j=1,2$ because V^i and V_0^i are not. Now we will extract from (26) and (27) some information of V_0^i. Collecting the $(\nu_2^2)^2$ terms in these two equations and setting them equal to zero, combined with the fact that λ_j^i are not functions of ν_j^i, we get the following system of equations:

$$
\begin{aligned}
(\rho_{11}+\rho_{12})\,\frac{\partial^2 V_0^1}{\partial u^1\partial u^1} + \rho_{12}\frac{\partial^2 V_0^2}{\partial u^1\partial u^1} = 0, \\
(\rho_{12}+\rho_{22})\,\frac{\partial^2 V_0^1}{\partial u^1\partial u^1} + \rho_{22}\frac{\partial^2 V_0^2}{\partial u^1\partial u^1} = 0.
\end{aligned}
$$

Because $\triangle := \rho_{11}\rho_{22} - \rho_{12}^2 > 0$ [5], we conclude from these two equations that

$$
\frac{\partial^2 V_0^1}{\partial u^2\partial u^2} = 0, \;\; \frac{\partial^2 V_0^2}{\partial u^2\partial u^2} = 0. \qquad (28)
$$

Similarly, by examining the coefficients of $\nu_2^1\,\nu_2^2$ and those of $(\nu_2^1)^2$ in (26) and (27), it can be shown that

$$\frac{\partial^2 V_0^1}{\partial u^1 \partial u^2} = 0, \;\; \frac{\partial^2 V_0^2}{\partial u^1 \partial u^2} = 0 \tag{29}$$

and

$$\frac{\partial^2 V_0^1}{\partial u^1 \partial u^1} = 0, \;\; \frac{\partial^2 V_0^2}{\partial u^1 \partial u^1} = 0. \tag{30}$$

Notice that these conditions imply that $\dfrac{\partial V_0^i}{\partial u^j}$ depends on x^1 and x^2 only. Therefore, V_0^1 and V_0^2 must have the following forms:

$$V_0^1(x^1, x^2, u^1, u^2) = g_1^1(x^1, x^2)u^1 + g_2^1(x^1, x^2)u^2 + g_3^1(x^1, x^2), \tag{31}$$
$$V_0^2(x^1, x^2, u^1, u^2) = g_1^2(x^1, x^2)u^1 + g_2^2(x^1, x^2)u^2 + g_3^2(x^1, x^2), \tag{32}$$

where g_j^i are arbitrary functions of x^1 and x^2. The condition that the ν_2^1 term in both (26) and (27) must vanish result in the following system of equations:

$$\begin{aligned}
&\left(\frac{\partial^2 V_0^1}{\partial u^1 \partial x^2} - \frac{d^2 V^2}{dx^2 dx^2}\right)\left(\rho_{11}\left(x^1\right) + \rho_{12}\left(x_1\right)\right) - 2\left(\frac{\partial^2 V_0^2}{\partial u^1 \partial x^2}\right)\rho_{12}\left(x^1\right) \\
&+ \left(\frac{\partial V_0^2}{\partial u^1}\right) b\left(x^1\right) + \left(\frac{\partial^2 V_0^1}{\partial u^1 \partial x^2}\right)\left(\rho_{11}\left(x^1\right) + \rho_{12}\left(x^1\right)\right) = 0,
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\left(\frac{\partial^2 V_0^1}{\partial u^1 \partial x^2} - \frac{d^2 V^2}{dx^2 dx^2}\right)\left(\rho_{12}\left(x^1\right) + \rho_{22}\left(x_1\right)\right) - 2\left(\frac{\partial^2 V_0^2}{\partial u^1 \partial x^2}\right)\rho_{22}\left(x^1\right) \\
&- \left(\frac{\partial V_0^2}{\partial u^1}\right) b\left(x^1\right) + \left(\frac{\partial^2 V_0^1}{\partial u^1 \partial x^2}\right)\left(\rho_{12}\left(x^1\right) + \rho_{22}\left(x^1\right)\right) = 0.
\end{aligned} \tag{34}$$

Replacing V_0^1 and V_0^2 in the above equations with (31) and (32), we get the following system of PDEs for $g_1^1(x^1, x^2)$ and $g_1^2(x^1, x^2)$:

$$2\left(\rho_{11}(x^1) + \rho_{12}(x^1)\right)\frac{\partial g_1^1}{\partial x^2} - 2\rho_{12}(x^1)\frac{\partial g_1^2}{\partial x^2} + b(x^1)g_1^2 = \left(\rho_{11}(x^1) + \rho_{12}(x^1)\right)\ddot{V}^2,$$
$$2\left(\rho_{12}(x^1) + \rho_{22}(x^1)\right)\frac{\partial g_1^1}{\partial x^2} - 2\rho_{22}(x^1)\frac{\partial g_1^2}{\partial x^2} - b(x^1)g_1^2 = \left(\rho_{12}(x^1) + \rho_{22}(x^1)\right)\ddot{V}^2,$$

where the dot $\cdot$ denotes the derivative with respect to x^2. The general solution to this system is:

$$\begin{aligned}
g_1^2(x^1, x^2) &= f_1(x^1)\, e^{-F(x^1)\, x^2}, && (35) \\
g_1^1(x^1, x^2) &= \frac{1}{2}\,\dot{V}^2 + \left[\rho_{12}(x^1) + \frac{b(x^1)}{2\,F(x^1)}\right]\left[\frac{f_1(x^1)}{(\rho_{11}(x^1) + \rho_{12}(x^1))}\right]\, e^{-F(x^1)\, x^2} \\
&\quad + f_2(x^1). && (36)
\end{aligned}$$

Here $F(x^1) := \dfrac{b(x^1)\left[2\rho_{12}(x^1) + \rho_{11}(x^1) + \rho_{22}(x^1)\right]}{2\triangle}$, whereas f_1 and f_2 are arbitrary functions of x^1. Hereafter, for simplicity of notations, we will omit the x^1 in the material parameters $\rho_{ij}(x^1)$ and $b(x^1)$. Following Biot's notation [5], new material variables ρ_1, ρ_2 and ρ are defined in the following:

$$\begin{aligned}
\rho_1 &:= \rho_{11} + \rho_{12}, \\
\rho_2 &:= \rho_{12} + \rho_{22}, \\
\rho &:= \rho_{11} + 2\,\rho_{12} + \rho_{22}
\end{aligned}$$

Similarly, the requirement that the ν_2^2 term must vanish in both (26) and (27) leads to a system of PDEs for g_2^1 and g_2^2 after replacing the λ_j^i terms in the coefficients of ν_2^2 with expressions solved from the system of (18) to (21). The general solution to this system of PDEs is:

$$
\begin{aligned}
g_2^2(x^1,x^2) &= \frac{1}{2}\dot{V}^2 - F(x^1)\,V^2 - G(x^1)\,V^1(x^1)\,x^2 \\
&\quad -\frac{\rho_{12}+\rho_{22}}{\rho}\,f_1(x^1)\,e^{-F(x^1)\,x^2} + f_3(x^1), && (37)\\
g_2^1(x^1,x^2) &= C_1(x^1)\,V^2(x^2) + C_2(x^1)\,V^1(x^1)\,x^2 + C_3(x^1)\,V^1(x^1) \\
&\quad + C_4(x^1)\,f_1(x^1)\,e^{-F(x^1)\,x^2} + C_5(x^1)\,(f_3(x^1) - f_2(x^1)) \\
&\quad + f_4(x^1)\,e^{F(x^1)\,x^2}, && (38)
\end{aligned}
$$

Here f_3 and f_4 are arbitrary functions of x^1 and new material parameter functions are denoted by upper case Latin letters:

$$
\begin{aligned}
G(x^1) &:= F', \\
C_5(x^1) &:= \frac{(\rho_{12}+\rho_{22})}{\rho}, \\
C_1(x^1) &:= -\frac{2\,\rho_{12}\,F + b}{2\,(\rho_{11}+\rho_{12})} = -FC_5, \\
C_2(x^1) &:= -C_5\,G, \\
C_3(x^1) &:= \frac{-(\rho_{12}+\rho_{22})\,\rho_{11}' + (\rho_{11}-\rho_{22})\,\rho_{12}' + (\rho_{12}+\rho_{11})\,\rho_{22}'}{\rho^2} = C_5', \\
C_4(x^1) &:= -(C_5)^2,
\end{aligned}
$$

where the prime $'$ denotes derivatives with respect to variable x^1.
At this point, (26) and (27) reduced to

$$c_{121}\,\nu_1^2 + c_{111}\,\nu_1^1 + c_1 = 0, \tag{39}$$
$$c_{221}\,\nu_1^2 + c_{211}\,\nu_1^1 + c_2 = 0, \tag{40}$$

where the coefficients are:

$$
\begin{aligned}
c_{121} &:= \lambda_1^1(w_3' - w_2') - \lambda_2^1 w_3' - (w_3' - w_2')\dot{V}_2 - 2\left(\frac{\partial^2 V_0^1}{\partial x_1 \partial u_2}\right)(w_1 - w_2) \\
&\quad - \left(\frac{\partial V_0^1}{\partial u_2}\right)(w_1' - w_2') - \frac{\partial}{\partial x^1}\left[\left(-V^{1\prime} + \frac{\partial V_0^2}{\partial u_2}\right)(w_3 - w_2)\right] - \left(\frac{\partial^2 V_0^2}{\partial x_1 \partial u_2}\right)(w_3 - w_2) \\
&\quad - V_1(w_3^{''} - w_2^{''}) - V^{1\prime}(w_3' - w_2'), \\
c_{111} &:= \lambda_1^1(w_1' - w_2') + \lambda_2^1 w_2' - [V_1\,(w_1' - w_2')]' \\
&\quad - \frac{\partial}{\partial x}\left[\left(-V_1{}' + \frac{\partial V_0^1}{\partial u_1}\right)(w_1 - w_2)\right] - (w_1' - w_2')\,\dot{V}^2 \\
&\quad - \left(\frac{\partial^2 V_0^1}{\partial x_1 \partial u_1}\right)(w_1 - w_2) - 2\left(\frac{\partial^2 V_0^2}{\partial x_1 \partial u_1}\right)(w_3 - w_2) - \left(\frac{\partial V_0^2}{\partial u_1}\right)(w_3' - w_2'), \\
c_1 &:= -\frac{\partial^2 V_0^1}{\partial {x_1}^2}(w_1 - w_2) - \left(\frac{\partial^2 V_0^2}{\partial {x_1}^2}\right)(w_3 - w_2) - \left(\frac{\partial^2 V_0^2}{\partial {x_2}^2}\right)\frac{\rho_{12}}{2} \\
&\quad - \left(\frac{\partial^2 V_0^1}{\partial {x_2}^2}\right)\frac{-\rho_{11} - \rho_{12}}{2} - \left(\frac{\partial V_0^1}{\partial x_1}\right)(w_1' - w_2') + \frac{b}{2}\left(\frac{\partial V_0^2}{\partial x_2}\right) - \left(\frac{\partial V_0^2}{\partial x_1}\right)(w_3' - w_2'), \\
c_{221} &:= \lambda_1^2\,(w_3' - w_2') - \lambda_2^2 w_3' + w_3\left(\frac{\partial^2 V_0^2}{\partial x_1 \partial u_2}\right) + \left[V^1\, w_3'\right]' \\
&\quad - 2w_2\left(\frac{\partial^2 V_0^1}{\partial x_1 \partial u_2}\right) - w_2'\left(\frac{\partial V_0^1}{\partial u_2}\right) + \frac{\partial}{\partial x^1}\left[\left(-V^{1\prime} + \frac{\partial V_0^2}{\partial u_2}\right) w_3\right] + w_3'\dot{V}^2, \\
c_{211} &:= \lambda_2^2 w_2' + \lambda_1^2\,(w_1' - w_2') - \frac{\partial}{\partial x^1}\left[\left(-V^{1\prime} + \frac{V_0^1}{u^1}\right) w_2\right] - \left[V^1\, w_2'\right]' \\
&\quad - w_2'\,\dot{V}^2 - w_2\left(\frac{\partial^2 V_0^1}{\partial x_1 \partial u_1}\right) + w_3'\left(\frac{\partial V_0^2}{\partial u_1}\right) + 2w_3\left(\frac{\partial^2 V_0^2}{\partial x_1 \partial u_1}\right), \\
c_2 &:= \frac{\rho_{12} + \rho_{22}}{2}\left(\frac{\partial^2 V_0^1}{\partial {x_2}^2}\right) - \frac{\partial}{\partial x^1}\left[w_2\left(\frac{\partial V_0^1}{\partial x_1}\right)\right] + \frac{\partial}{\partial x^1}\left(\frac{\partial V_0^2}{\partial x_1}\, w_3\right) \\
&\quad - \frac{\rho_{22}}{2}\left(\frac{\partial^2 V_0^2}{\partial (x^2)^2}\right) - \frac{b}{2}\left(\frac{\partial V_0^2}{\partial x^2}\right).
\end{aligned}
$$

The system of equations (18) to (25) and the 6 equations $c_{121} = 0$, $c_{111} = 0$, $c_{221} = 0$, $c_{211} = 0$, $c_1 = 0$, $c_2 = 0$ contains equations which must be satisfied simultaneously by λ_1^1, λ_2^1, λ_1^2, λ_2^2, f_2, f_3, V^1, V^2, w_1, w_2, w_3, ρ_{11}, ρ_{12}, ρ_{22} and b.

Substitute (31) and (32), together with the solutions of g_1^1, g_2^1, g_1^2 and g_2^2 in (36)-(37) into $c_1 = 0$ and $c_2 = 0$. The fact that the u_1-term in each these two equations must vanish in K leads to a set two equations. We then eliminate all $V^2(x^2)$ terms to get the following equation:

$$
\begin{aligned}
&\left[\frac{(\rho_{12} + \rho_{22})^2}{\rho} w_1 - 2(\rho_{12} + \rho_{22})w_2 + \rho w_3\right]\left[f_1\, e^{-Fx^2}(F')^2\right](x^2)^2 \\
&+ \left[A_1(x^1) f_1\, F^{''} + A_2(x^1) f_1\,(F')^2 + \left(A_3(x^1) f_1' + A_4(x^1) f_1\right) F'\right] e^{-Fx^2} x^2 \\
&\qquad + A_5(x^1) e^{-Fx^2} + A_6(x^1) = 0,
\end{aligned}
$$

where A_i, $i = 1 \sim 6$ are functions of x^1 only and none of them contain terms with f_1 or derivatives of F. From the $(x^2)^2$-term of the above equation, we conclude that $f_1(x^1)(F')^2 = 0$ **provided** $\frac{(\rho_{12}+\rho_{22})^2}{\rho} w_1 - 2(\rho_{12} + \rho_{22})w_2 + \rho w_3 \neq 0$. **We will consider non-constant case first**.

3.1 $F'(x^1) \neq 0$

For this case, we must have $f_1 = 0$. Noticing the similarity between (22) and $c_{211} = 0$, we may use the former to simplify the latter and then sub. $f_1 = 0$ into the result to get

$$f_2' w_2 = 0.$$

Since $w_2 > 0$, we must have $f_2' = 0$, ie. $f_2 = f_{2c}$ for some constant f_{2c}. Applying this information to the u^1-term in $c_1 = 0$ and c_2=0, we conclude that

$$\dddot{V}^2 = 0.$$

Let $V^2 := a_2\,(x^2)^2 + a_1\,x^2 + a_0$, with constants a_2, a_1 and a_0. Sub. this and $\{f_1 = 0,\ f_2 = f_{2c}\}$ into the u^2-term in $c_2 = 0$, the $(x^2)^2$-term gives

$$a_2\ (C_1' w_2 + F' w_3)' + f_4 (F')^2 w_2 e^{F x^2} = 0.$$

Therefore, for $F' \neq 0$, it must be that $f_4 = 0$. Note that (18) and (19) constitute a system of linear equations for λ_1^2 and λ_2^2 with non-zero Jacobian $\triangle$. We solve this system for λ_1^2 and λ_2^2 and substitute the results into (22). After replacing V_0^1 and V_0^2 with (31) and (32) and applying $\{f_1 = 0, f_4 = 0\}$, we obtain the following equations:

$$\begin{aligned}
(2112) \qquad & 2w_3 \dot{V}^2 - 2w_3 {V^1}' + \left\{ \left(\frac{-1}{\rho\Delta}\right) [(\rho_{12}+\rho_{22})(-\rho_{22}w_2 + (\rho_{12}+\rho_{22})w_3)\rho_{11}' \right. \\
& + \left((\rho_{22}\rho_{11} - \rho_{12}\rho_{11} - 2\rho_{12}^2)w_2 + (\rho_{11}+\rho_{12})^2 w_3\right)\rho_{22}' \\
& \left. +2(\rho_{11}+\rho_{12})(\rho_{22}w_2 - (\rho_{12}+\rho_{22})w_3)\rho_{12}'] + w_3' \right\} V^1 = 0.
\end{aligned}$$

Note that in this equation, only V^2 is a function of x^2. Therefore, $\dot{V}^2$ must be a constant. This implies that $a_2 = 0$,i.e. $V^2 = a_1 x^2 + a_0$. Consequently, the system of $c_1 = 0$ and $c_2 = 0$ reduces to the following equations:

$$\begin{aligned}
& \left\{ \left[(w_2 - w_3)\left((GV^1)' + a_1 F'\right)'\right]' + \left[(w_1 - w_2)(a_1 C_1' + (C_2 V^1)')'\right]' \right\} u^2 x^2 \\
+ \quad & \left\{ \left[(w_1 - w_2)(a_0 C_1 + C_3 V^1 + C_5 f_3 - f_{2c} C_5)'\right]' \right. \\
& \left. + \left[(a_0 F' - f_3')(w_2 - w_3)\right]' + (a_1 F + G V^1) b/2 \right\} u^2 \\
+ \quad & \left\{ (w_1 - w_2)\, \frac{\partial^2 g_3^1}{\partial {x_1}^2} - (w_2 - w_3)\left(\frac{\partial^2 g_3^2}{\partial {x_1}^2}\right) + \frac{\rho_{12}}{2}\left(\frac{\partial^2 g_3^2}{\partial {x_2}^2}\right) \right. \\
& - \frac{(\rho_{11}+\rho_{12})}{2}\left(\frac{\partial^2 g_3^1}{\partial {x_2}^2}\right) + \left(\frac{\partial g_3^1}{\partial x_1}\right)(w_1' - w_2') + \frac{b}{2}\left(\frac{\partial g_3^2}{\partial x_2}\right) \\
& \left. -(w_2' - w_3')\left(\frac{\partial g_3^2}{\partial x_1}\right) \right\} = 0, \qquad (41) \\
& \left\{ \left[w_3\left((GV^1)' + a_1 F'\right)'\right]' + \left[w_2(a_1 C_1' + (C_2 V^1)')'\right]' \right\} u^2 x^2 \\
+ \quad & \left\{ \left[w_2(a_0 C_1 + C_3 V^1 + C_5 f_3 - f_{2c} C_5)'\right]' \right. \\
& \left. - \left[(a_0 F' - f_3') w_3\right]' - (a_1 F + G V^1) b/2 \right\} u^2 \\
+ \quad & \left\{ w_2\, \frac{\partial^2 g_3^1}{\partial {x_1}^2} - w_3\left(\frac{\partial^2 g_3^2}{\partial {x_1}^2}\right) + \frac{\rho_{22}}{2}\left(\frac{\partial^2 g_3^2}{\partial {x_2}^2}\right) - \frac{(\rho_{12}+\rho_{22})}{2}\left(\frac{\partial^2 g_3^1}{\partial {x_2}^2}\right) \right. \\
& \left. + w_2'\left(\frac{\partial g_3^1}{\partial x_1}\right) + \frac{b}{2}\left(\frac{\partial g_3^2}{\partial x_2}\right) - w_3'\left(\frac{\partial g_3^2}{\partial x_1}\right) \right\} = 0. \qquad (42)
\end{aligned}$$

After using (22), (23), (24) and (25) to simplify $c_{221}=0$, $c_{211}=0$, $c_{111}=0$ and $c_{121}=0$, respectively, and substituting in $\{f_1=0,\ f_2=f_{2c},\ f_4=0,\ V^2=a_1x^2+a_0\}$, $c_{211}=0$ and $c_{111}=0$ become $0=0$ whereas $c_{221}=0$ and $c_{121}=0$ become:

$$\begin{aligned}&\left\{[a_1F'+(GV^1)']w_3+[a_1C_1'+(C_2V^1)']w_2\right\}x^2\\&+\left\{[a_0F'-f_3']w_3+[a_0C_1+C_3V^1+C_5f_3-f_{2c}C_5]'w_2\right\}=0,\end{aligned} \tag{43}$$

$$\begin{aligned}&\left\{\left[a_1F'+(GV^1)'\right](w_2-w_3)+\left[a_1C_1'+(C_2V^1)'\right](w_1-w_2)\right\}x^2\\&+\{[a_0F'-f_3'](w_2-w_3)\\&+\left[a_0C_1+C_3V^1+C_5f_3-f_{2c}C_5\right]'(w_1-w_2)\Big\}.\end{aligned} \tag{44}$$

Noting that by the independence condition of the coordinates of K, the above four equations in K give a system of ten equations. The two equations resulting from the constant term in (41) and (42) constitute a system of second-order PDEs for g_3^1 and g_3^2. We will refer to this system of equations as the **$\mathbf{g_3}$-equations** and analyze the other eight equations first.

Because of the assumption $w_3w_1-w_2^2>0$, the x^2 terms in (44) and (43) imply the following two equations:

$$a_1F'+(GV^1)'=0, \tag{45}$$
$$a_1C_1'+(C_2V^1)'=0,. \tag{46}$$

Similarly, the constant terms in (44) and (43) give

$$a_0F'-f_3'=0, \tag{47}$$
$$(a_0C_1+C_3V^1+C_5f_3-f_{2c}C_5)'=0. \tag{48}$$

Applying (45)-(48) to (41) and (42), it can be easily seen that the two equations given by the u^2x^2 terms are satisfied automatically whereas the two equations given by the u^2 term reduce to

$$(a_1F+GV^1)b=0.$$

Since the dissipation function $b(x^1)\neq 0$, we must have

$$a_1F+GV^1=0, \tag{49}$$

and this equation in turns imply (45) and (46) because of the relations $C_1(x^1)=-F(x^1)C_5(x^1)$ and $C_2(x^1)=-F'(x^1)C_5(x^1)$. Therefore, the ten equations from (41), (42), (43) and (44) reduce to the $\mathbf{g_3}$-equations and the equations of (47), (48) and (49).

In the system of equations (18) to (25), note that (18) and (19) constitute a system of linear equations for λ_1^2 and λ_2^2 whereas (20) and (21)for λ_1^1 and λ_2^1 with non-zero Jacobian $\triangle$. We solve this system for λ_j^i and substitute the results into (22)-(25). Replacing V_0^1 and V_0^2 with (31) and (32), followed by applying

$\{f_1=0, f_2=f_{2c}, f_3=a_0F+f_{3c}, f_4=0, V^2=a_1x^2+a_0\}$ to these equations, (22)-(25) become

$$(2112): \quad -2w_3\frac{dV^1}{dx^1}+\left[w_3'+\left(\frac{C_3\rho}{\triangle}\right)(\rho_2w_3-\rho_{22}w_2)+\left(\frac{\rho_{22}'}{\triangle}\right)(\rho_{12}w_2-\rho_1w_3)\right.$$
$$\left.+\left(\frac{\rho_{12}'}{\triangle}\right)(\rho_2w_3-\rho_{22}w_2)\right]V^1+2a_1w_3=0, \tag{50}$$

$$(2111): \quad 2w_2\frac{dV^1}{dx^1}+\left[-w_2'+\left(\frac{C_3\rho_2}{\triangle}\right)R_w+\left(\frac{\rho_{22}'}{\triangle}\right)(\rho_1w_2-\rho_{12}w_1)\right.$$
$$\left.+\left(\frac{\rho_{12}'}{\triangle}\right)(\rho_{22}w_1-\rho_2w_2)\right]V^1-2a_1w_2+\left(\frac{R_w}{\rho}\right)(f_{2c}-f_{3c})=0, \tag{51}$$

$$(1111): \quad 2w_1\frac{dV^1}{dx^1}+\left[-w_1'-\left(\frac{C_3\rho}{\triangle}\right)(R_w)+\left(\frac{\rho_{11}'}{\triangle}\right)(\rho_2w_2-\rho_{22}w_1)+\left(\frac{\rho_{12}'}{\triangle}\right)[(\rho_{22}-\rho_{11})w_2\right.$$
$$\left.+(\rho_{12}-\rho_{22})w_1]+\left(\frac{\rho_{22}'}{\triangle}\right)(\rho_{12}w_1-\rho_1w_2)\right]V^1-2a_1w_1=0, \tag{52}$$

$$(1112): \quad (2w_3-2w_2)\frac{dV^1}{dx^1}+\left[w_2'-w_3'+\left(\frac{C_3}{\triangle}\right)(-\triangle w_1-\rho\rho_{12}w_2+\rho\rho_1w_3)\right.$$
$$\left.+\left(\frac{\rho_{11}'}{\triangle}\right)(\rho_2w_3-\rho_{22}w_2)+\left(\frac{\rho_{12}'}{\triangle}\right)(\rho_{12}w_2-\rho_1w_3)\right]V^1+2a_1(w_2-w_3)$$
$$+\frac{R_w}{\rho}(f_{2c}-f_{3c})=0, \tag{53}$$

where $R_w:=\rho_2w_1-\rho w_2$, which is non-zero because of **Assump 1**. The corresponding V_0^1 and V_0^2 are:

$$\begin{aligned}
V_0^1 &= (a_1C_1(x^1)+C_2(x^1)V^1)u^2x^2+\left(\frac{a_1}{2}+f_{2c}\right)u^1 \\
&\quad+\left[a_0C_1(x^1)+C_3(x^1)V^1+C_5(x^1)(a_0F_cb(x^1)+f_{3c}-f_{2c})\right]u^2+g_3^1(x^1,x^2), \\
V_0^2 &= \left[-a_1F(x^1)-G(x^1)V^1\right]u^2x^2+\left(\frac{a_1}{2}\right)u^2+g_3^2(x^1,x^2),
\end{aligned}$$

where g_3^1 and g_3^2 satisfies the $\mathbf{g_3}$-equations, which is the same as the equations in (PE), i.e. $\{g_3^1, g_3^2\}$ is any solution to (PE). Therefore, the g_3^i terms in V_0^i arise from the fact that the equations in (PE) are linear and hence satisfies the superposition principle (ie. the translation of any solution by a solution gives a solution). The isovector field for this case solves the system which consists of the $\mathbf{g_3}$-equations and equations (48)-(53).

3.2 $F'(x^1)=0$

In this case, we have $F(x^1)=F_*$ for some constant F_* and $G(x^1)=C_2(x^1)=0$. Adding (23) into the equation $c_{221}=0$, followed by substituting into it (31), (32) and $\{F(x^1)=F_*, G(x^1)=C_2(x^1)=0\}$, equation $c_{221}=0$ becomes

$$\left[(f_1C_5)'w_2+f_1'w_3\right]e^{-F_*x^2}-f_2'w_2=0. \tag{54}$$

Because $F_*,\ w_2\neq 0$, (54) implies

$$\begin{aligned}
&(f_1C_5)'w_2+f_1'w_3=0, \\
&f_2(x^1)=f_{2c},
\end{aligned}$$

for some constant f_{2c}. Similarly, adding (24) to $c_{111}=0$ and considering $f_2=f_{2c}$ result in another equations for f_1:

$$(w_1-w_2)(f_1C_5)'w_2+f_1'(w_2-w_3)=0. \tag{55}$$

Equations (55) and (54) imply that $f_1(x^1) = f_{1c}$ for some constant f_{1c} because $w_1w_3 - w_2^2 \neq 0$. Furthermore, subtracting $1/2$ times the derivative of the equation $c_{211} = 0$ from the u^1 term in $c_2 = 0$, together with $\{f_1(x^1) = f_{1c}, f_2(x^1) = f_{2c}\}$, we get

$$f_{1c}F_* \left(F_*\rho_2(x^1)C_5(x^1) + b(x^1) - F_*\rho_{22}\right) e^{-F_*x^2} + \frac{\rho_2}{2}\dddot{V}^2 = 0. \tag{56}$$

Similarly, from the u^1 term in $c_1 = 0$ and the equation $c_{111} = 0$, we obtain

$$f_cF_*b(x^1)e^{-F_*x^2} - \rho_1(x^1)\dddot{V}^2 = 0. \tag{57}$$

Eliminating the $\dddot{V}^2$ term in the above two equations result in

$$\frac{f_{1c}\rho(x^1)^2b(x^1)e^{-F_*x^2}}{\triangle} = 0.$$

This implies $f_{1c} = 0$. Consequently, we have $\dddot{V}^2 = 0$, ie.

$$V^2(x^2) = a_2(x^2)^2 + a_1x^2 + a_0,\ \text{for some constants } a_2,\, a_1 \text{ and } a_0. \tag{58}$$

Substituting $\{f_1 = 0, f_2 = f_{2c}, V^2(x^2) = a_2(x^2)^2 + a_1x^2 + a_0\}$ into the four equations $c_{111} = 0$, $c_{121} = 0$, $c_{211} = 0$ and $c_{221} = 0$, both $c_{111} = 0$ and $c_{211} = 0$ are satisfied automatically whereas $c_{121} = 0$ $c_{221} = 0$ reduce to the following equations:

$$\begin{aligned}
(c_{121} = 0): \quad & \left[a_2(w_1 - w_2)C_1'\right](x^2)^2 - \left[a_1C_1'(w_1 - w_2)\right]x^2 - f_4(x^1)'(w_1 - w_2)e^{F_*x^2} \\
& - \left\{a_0C_1' + \left[C_5(f_3(x^1) - f_{2c})\right]' + (C_3V^1)'\right\}(w_1 - w_2) + f_3'(w_2 - w_3) = 0, \qquad (59) \\
(c_{221} = 0): \quad & \left[a_2w_2C_1'\right](x^2)^2 + \left[a_1C_1'w_2\right]x^2 + (f_4(x^1)'w_2)e^{F_*x^2} \\
& + \left\{a_0C_1' + \left[C_5(f_3(x^1) - f_{2c})\right]' + (C_3V^1)'\right\}w_2 - f_3'w_3 = 0., \qquad (60)
\end{aligned}$$

From (59) and (60), it can be shown that

$$f_4 = f_{4c},\ f_3 = f_{3c},$$

for some constant f_{4c} and f_{3c}.

Substituting into (22) the λ_1^2 and λ_2^2 solved from the system of (18) and (19) and (31), (32), followed by $\{F = F_*, G = 0, C_2 = 0, f_1 = 0, f_2 = f_{2c}, f_3 = f_{3c}, f_4 = f_{4c}, V^2 = a_2(x^2)^2 + a_1x^2 + a_0\}$, the coefficient of x^2 in (22) is $16w_3a_2$. Therefore, we must have $a_2 = 0$. Consequently, equations (41) and

(42) become:

$$
\begin{aligned}
(c_1=0): \quad & -2\left[a_1(w_1-w_2)C_1'\right]' u^2x^2 \\
+ \quad & \left\{ f_{4c}F_*^2\rho_1 e^{F_*x^2} - 2\left\{(w_1-w_2)\left[C_5(f_{3c}-f_{2c})+C_3V^1+a_0C_1\right]'\right\}' - a_1F_*b\right\} u^2 \\
- \quad & 2\left\{(w_1-w_2)\,\frac{\partial^2 g_3^1}{\partial {x_1}^2} - (w_2-w_3)\,\left(\frac{\partial^2 g_3^2}{\partial {x_1}^2}\right) + \frac{\rho_{12}}{2}\left(\frac{\partial^2 g_3^2}{\partial {x_2}^2}\right)\right. \\
& -\frac{(\rho_{11}+\rho_{12})}{2}\left(\frac{\partial^2 g_3^1}{\partial {x_2}^2}\right) + \left(\frac{\partial g_3^1}{\partial x_1}\right)(w_1'-w_2') + \frac{b}{2}\,\left(\frac{\partial g_3^2}{\partial x_2}\right) \\
& \left. -(w_2'-w_3')\left(\frac{\partial g_3^2}{\partial x_1}\right)\right\} = 0, \qquad (61) \\
(c_2=0): \quad & -2\left[a_1w_2C_1'\right]' u^2x^2 \\
+ \quad & \left\{ f_{4c}F_*^2\rho_2 e^{F_*x^2} - 2\left\{w_2\left[C_5(f_{3c}-f_{2c})+C_3V^1+a_0C_1\right]'\right\}' + a_1F_*b\right\} u^2 \\
- \quad & 2\left\{w_2\,\frac{\partial^2 g_3^1}{\partial {x_1}^2} - w_3\,\left(\frac{\partial^2 g_3^2}{\partial {x_1}^2}\right) + \frac{\rho_{22}}{2}\,\left(\frac{\partial^2 g_3^2}{\partial {x_2}^2}\right) - \frac{(\rho_{12}+\rho_{22})}{2}\left(\frac{\partial^2 g_3^1}{\partial {x_2}^2}\right)\right. \\
& \left. +w_2'\,\left(\frac{\partial g_3^1}{\partial x_1}\right) + \frac{b}{2}\,\left(\frac{\partial g_3^2}{\partial x_2}\right) - w_3'\,\left(\frac{\partial g_3^2}{\partial x_1}\right)\right\} = 0. \qquad (62)
\end{aligned}
$$

From (61) and (62), it can be seen that

$$f_{4c}=0,\ a_1=0.$$

Therefore, the system of (39) and (40) reduce to the system of $\mathbf{g_3}$-equations and the following equation:

$$\left[-C_5(a_0F_*+f_{2c}-f_{3c})+C_3V^1\right]'=0, \qquad (63)$$

and the equations (18)-(25) are equivalent with the following four equations:

$$
\begin{aligned}
(2112): \quad & -2w_3\frac{dV^1}{dx^1} + \left[w_3' + \left(\frac{C_3\rho}{\triangle}\right)(\rho_2w_3-\rho_{22}w_2) + \left(\frac{\rho_{22}'}{\triangle}\right)(\rho_{12}w_2-\rho_1w_3)\right. \\
& \left. + \left(\frac{\rho_{12}'}{\triangle}\right)(\rho_2w_3-\rho_{22}w_2)\right] V^1 = 0, \qquad (64) \\
(2111): \quad & 2w_2\frac{dV^1}{dx^1} + \left[-w_2' + \left(\frac{C_3\rho_2}{\triangle}\right)R_w + \left(\frac{\rho_{22}'}{\triangle}\right)(\rho_1w_2-\rho_{12}w_1)\right. \\
& \left. + \left(\frac{\rho_{12}'}{\triangle}\right)(\rho_{22}w_1-\rho_2w_2)\right] V^1 + \left(\frac{R_w}{\rho}\right)(a_0F_*+f_{2c}-f_{3c}) = 0, \qquad (65) \\
(1111): \quad & 2(w_1-w_2)\frac{dV^1}{dx^1} + \left[(w_2'-w_1') + \left(\frac{C_3\rho_1}{\triangle}\right)(R_w) + \left(\frac{\rho_{11}'}{\triangle}\right)(\rho_2w_2-\rho_{22}w_1)\right. \\
& \left. + \left(\frac{\rho_{12}'}{\triangle}\right)(\rho_1w_2-\rho_{12}w_1)+\right] V^1 - \left(\frac{R_w}{\rho}\right)(a_0F_*+f_{2c}-f_{3c}) = 0, \qquad (66) \\
(1112): \quad & (2w_3-2w_2)\frac{dV^1}{dx^1} + \left[w_2'-w_3' + \left(\frac{C_3}{\triangle}\right)(-w_1\triangle-\rho\rho_{12}w_2+\rho\rho_1w_3)\right. \\
& \left. + \left(\frac{\rho_{11}'}{\triangle}\right)(\rho_2w_3-\rho_{22}w_2) + \left(\frac{\rho_{12}'}{\triangle}\right)(\rho_{12}w_2-\rho_1w_3)\right] V^1 \\
& +\frac{R_w}{\rho}(a_0F_*+f_{2c}-f_{3c}) = 0. \qquad (67)
\end{aligned}
$$

where $R_w := \rho_2w_1-\rho w_2$, which is non-zero because of **Assump 1**. The isovector field for this case solves the system which consists of the $\mathbf{g_3}$-equations, and equations (63)-(67).

4 Similarity solutions of special cases

Definition 4.1 (Similarity Solution). *A map $\Phi{:}B_n \to K$ is a similarity solution generated by an isovector field V of the balance ideal iff Φ is a solution map of the balance ideal that satisfies the first-order differential constraints*

$$\Phi^*(V\rfloor C^\alpha) = 0,\ \alpha = 1, \ldots, N.$$

A similarity solution generated by V satisfies

$$\Phi^* B = 0,\ \Phi^*\mu \neq 0,\ \Phi^*(V\rfloor C^\alpha) = 0,\ \alpha = 1, \ldots, N.$$

Note that in terms of this representation of the isovector field (3), the conditions $\phi^*(V\rfloor C^\alpha) = 0$, $\alpha = 1,\ 2$, in the above definition give the following system of quasi-linear first-order PDEs:

$$V^1 \frac{\partial u^1}{\partial x} + V^2 \frac{\partial u^1}{\partial t} = V_0^1,$$
$$V^1 \frac{\partial u^2}{\partial x} + V^2 \frac{\partial u^2}{\partial t} = V_0^2.$$

The above equations were used to derive the system of differential equations in terms of the similarity variable ξ which is found by solving the system of $\frac{d\bar{x}}{d\xi} = V^1$ and $\frac{d\bar{t}}{d\xi} = V^2$.

4.1 Special cases of constant $\rho_{ij}(x)$

4.1.1 Nonconstant $b(x)$

Here, we consider the case when the inertia coupling functions $\rho_{11}(x)$, $\rho_{12}(x)$ and $\rho_{22}(x)$ are constant functions. In this case, we have $C_3 = 0$ and $F(x^1) = F_c b(x^1)$ for some constant F_c. This corresponds to the case of $F'(x) \neq 0$ in the previous section. It can be deduced from (50) to (53) that $f_{2c} = f_{3c}$ and consequently equation (48) is satisfied identically. Therefore, the projection of the isovector field onto the graph space is:

$$b'(x)V^1 = -a_1 b(x), \tag{68}$$
$$V^2 = a_1 t + a_0, \tag{69}$$
$$V_0^1 = \left[a_1 C_1(x) + C_2(x)V^1\right] tu^2 + \left(\frac{a_1}{2} + f_{2c}\right) u^1 + a_0 C_1(x) u^2 + g_3^1(x,t), \tag{70}$$
$$V_0^2 = \left(\frac{a_1}{2} - a_0 F_c b(x) + f_{3c}\right) u^2 + g_3^2(x,t), \tag{71}$$

and the equations (50)-(53) reduce to the following three equations:

$$2w_3(V^1)' - w_3' V^1 - 2a_1 w_3 = 0, \tag{72}$$
$$2w_2(V^1)' - w_2' V^1 - 2a_1 w_2 = 0, \tag{73}$$
$$2w_1(V^1)' - w_1' V^1 - 2a_1 w_1 = 0. \tag{74}$$

It can be derived from these three equations that

$$\left(\frac{w_2}{w_3}\right)' V^1 = \left(\frac{w_1}{w_2}\right)' V^1 = \left(\frac{w_1}{w_3}\right)' V^1 = 0.$$

Therefore, V^1 is non-zero only if $w_1(x) = k_1 w_3(x)$ and $w_2(x) = k_2 w_3(x)$ for some positive constants k_1 and k_2. We will discuss these two cases in the following.

- For arbitrary $w_i(x)$, $i = 1, 2, 3$.
  For this case, from (68) with the assumption that $b(x) \neq 0$ for any x, we conclude that a_1=0. Thus the following isovector field:

$$V^1 = 0, \tag{75}$$
$$V^2 = a_0, \tag{76}$$
$$V_0^1 = f_{3c}u^1 + g_3^1(x,t), \tag{77}$$
$$V_0^2 = f_{3c}u^2 + g_3^2(x,t). \tag{78}$$

  For non-trivial symmetry, we must have $a_0 \neq 0$. Letting $g_3^1 = 0$ and $g_3^2 = 0$, the similarity solutions have the following form:

$$u^1(x,t) = U(x)e^{\frac{f_{3c}}{a_0}t}, \tag{79}$$
$$u^2(x,t) = V(x)e^{\frac{f_{3c}}{a_0}t}, \tag{80}$$

  where $U(x)$ and $V(x)$ satisfy the following set of ODEs:

$$2(w_1 - w_2)U'' + 2(w_3 - w_2)V'' + 2(w_1' - w_2')U' + 2(w_3' - w_2')V'$$
$$-\frac{f_{3c}b(x)}{a_0} - \frac{f_{3c}^2(\rho_{11} + \rho_{12})}{a_0^2}U + \frac{f_{3c}^2\rho_{12}}{a_0^2}V = 0,$$
$$2w_2U'' - 2w_3V'' + 2w_2'U' - 2w_3'V' + \frac{f_{3c}b(x)}{a_0}V - \frac{f_{3c}^3(\rho_{12} + \rho_{22})}{a_0^2}U + \frac{f_{3c}^2\rho_{22}}{a_0^2}V = 0.$$

- For $w_1(x) = k_1w_3(x)$ and $w_2(x) = k_2w_3(x)$.
  In this case, we are left with only two equations:

$$b'(x)V^1 + a_1b(x) = 0, \tag{81}$$
$$2w_3(V^1)' - w_3'V^1 - 2a_1w_3 = 0. \tag{82}$$

  The general solution to (82) is

$$V^1(x) = \sqrt{w_3(x)}\left[\int_0^x \frac{a_1}{\sqrt{w_3(\zeta)}}d\zeta + k\right], \tag{83}$$

  for some constant k. If $w_i(x)$, $i = 1, 2, 3$, are constant functions, it can be seen that $b(x)$ must have the form

$$b(x) = \frac{c^*}{a_1x + c_0}.$$

$$V^1 = a_1x + c_0,$$
$$V^2 = a_1t + a_0,$$
$$V_0^1 = \left(\frac{a_1}{2} + f_{2c}\right)u^1 + g_3^1,$$
$$V_0^2 = \left(\frac{a_1}{2} + f_{2c}\right)u^2 + g_3^2.$$

  The similarity variable is:

$$\xi = \frac{a_1x + c_0}{a_1t + a_0},$$

and the corresponding ODEs are:

$$\left[-2w_2\xi^2+2w_3\xi^2+\rho_{12}\xi^4\right]V_{\xi\xi}+\left[(2a_1+4f_{3c})(w_3-w_2)\xi+\frac{c^*}{a_1}\xi^2+2\rho_{12}\xi^3\right]V_\xi$$
$$+\left[2w_3-2w_2\right]\left[\frac{a_1}{2}+f_{3c}\right]\left[\frac{a_1}{2}+f_{3c}-1\right]V+\left[2w_1\xi^2-2w_2\xi^2-(\rho_{11}+\rho_{12})\xi^4\right]U_{\xi\xi}$$
$$+\left[(2a_1+4f_{3c})(w_1-w_2)\xi-2(\rho_{11}+\rho_{12})\xi^3\right]U_\xi+\left[2w_1-2w_2\right]\left[\frac{a_1}{2}+f_{3c}\right]\left[\frac{a_1}{2}+f_{3c}-1\right]U=0,$$
$$\left[-2w_3\xi^2+\rho_{22}\xi^4\right]V_{\xi\xi}+\left[-(2a_1+4f_{3c})w_3\xi-\frac{c^*}{a_1}\xi^2+2\rho_{22}\xi^3\right]V_\xi$$
$$-2w_3\left[\frac{a_1}{2}+f_{3c}\right]\left[\frac{a_1}{2}+f_{3c}-1\right]V+\left[2w_2\xi^2-(\rho_{12}+\rho_{22})\xi^4\right]U_{\xi\xi}$$
$$+\left[(2a_1+4f_{3c})w_2\xi-2(\rho_{12}+\rho_{22})\xi^3\right]U_\xi+2w_2\left[\frac{a_1}{2}+f_{3c}\right]\left[\frac{a_1}{2}+f_{3c}-1\right]U=0.$$

4.1.2 Constant $b(x)$

In this case, we have $C_3=0$ and $b(x)=b_c$ for some constant b_c. From (64), (65) and (67), it can be seen that we must as well have

$$a_0F_*+f_{2c}-f_{3c}=0.$$

The equations (63) to (67) reduce to the following three equations:

$$\begin{aligned}2w_1(V^1)'-w_1'V^1&=0,\\2w_2(V^1)'-w_2'V^1&=0,\\2w_3(V^1)'-w_3'V^1&=0.\end{aligned}$$

These three equations imply

$$\left(\frac{w_1}{w_2}\right)'V^1=\left(\frac{w_3}{w_2}\right)'V^1=\left(\frac{w_1}{w_3}\right)'V^1=0.$$

- $w_i(x)$, i=1,2,3, are arbitrary.
  In this case, we have

$$\begin{aligned}V^1&=0,\\V^2&=a_0,\\V_0^1&=f_{2c}u^1+g_3^1,\\V_0^2&=f_{2c}u^2+g_3^2.\end{aligned}$$

Letting $g_3^1=0$ and $g_3^2=0$, we obtain similarity solutions of the following form:

$$u^1(x,t)=U(x)e^{\frac{f_{2c}}{a_0}t}, \tag{84}$$
$$u^2(x,t)=V(x)e^{\frac{f_{2c}}{a_0}t}, \tag{85}$$

where $U(x)$ and $V(x)$ satisfy the following ODE's.

$$2(w_1-w_2)U''+2(w_3-w_2)V''+2(w_1'-w_2')U'+2(w_3'-w_2')V'$$
$$+\left(\frac{\rho_{12}f_{2c}^2-a_0f_{2c}b_c}{a_0^2}\right)V-\left[\frac{f_{2c}^2(\rho_{11}+\rho_{12})}{a_0^2}\right]U=0,$$
$$-2w_2U''+2w_3V''-2w_2'U'+2w_3'V'-\left(\frac{\rho_{22}f_{2c}^2+a_0f_{2c}b_c}{a_0^2}\right)V+\left[\frac{f_{2c}^2(\rho_{12}+\rho_{22})}{a_0^2}\right]U=0,$$

- $w_1(x) = k_1 w_3(x)$, $w_2(x) = k_2 w_3(x)$ for some constants k_1 and k_2.
  For this case, we have

$$V^1(x) = b_0 \sqrt{w_3(x)}, \text{ for some constant } b_0.$$

For the case of **constant** w_3, we have

$$V^1 = b_0, \tag{86}$$
$$V^2 = a_0, \tag{87}$$
$$V_0^1 = f_{2c} u^1 + g_3^1, \tag{88}$$
$$V_0^2 = f_{2c} u^2 + g_3^2. \tag{89}$$

Letting $g_3^1 = 0$, $g_2^3 = 0$ and for $a_0 \neq 0$, we have the following similarity solutions:

$$u^1(\xi, t) = U(\xi) e^{(\frac{f_{2c}}{a_0})t}, \tag{90}$$
$$u^2(\xi, t) = V(\xi) e^{(\frac{f_{2c}}{a_0})t}, \tag{91}$$

with the similarity variable ξ being

$$\xi = x - \frac{b_0}{a_0} t,$$

and $U(\xi)$ and $V(\xi)$ satisfying the following set of ODEs:

$$\begin{aligned}
&\left[-2w_2 + 2w_3 + \frac{\rho_{12} b_0^2}{a_0^2}\right] V_{\xi\xi} + \left[\frac{b_c b_0}{a_0} - \frac{2\rho_{12} b_0 f_{2c}}{a_0^2}\right] V_\xi + \left[-\frac{b_c f_{2c}}{a_0} + \frac{\rho_{12} f_{2c}^2}{a_0^2}\right] V \\
&+ \left[-2w_2 + 2w_1 - \frac{(\rho_{11} + \rho_{12}) b_0^2}{a_0^2}\right] U_{\xi\xi} + \left[\frac{2(\rho_{11} + \rho_{12}) b_0 f_{2c}}{a_0^2}\right] U_\xi - \left[\frac{(\rho_{11} + \rho_{12}) f_{2c}^2}{a_0^2}\right] U = 0, \\
&\left[-2w_3 + \frac{\rho_{22} b_0^2}{a_0^2}\right] V_{\xi\xi} + \left[-\frac{b_c b_0}{a_0} - \frac{2\rho_{22} b_0 f_{2c}}{a_0^2}\right] V_\xi + \left[\frac{b_c f_{2c}}{a_0} + \frac{\rho_{22} f_{2c}^2}{a_0^2}\right] V \\
&+ \left[2w_2 - \frac{(\rho_{12} + \rho_{22}) b_0^2}{a_0^2}\right] U_{\xi\xi} + \left[\frac{2(\rho_{12} + \rho_{22}) b_0 f_{2c}}{a_0^2}\right] U_\xi - \left[\frac{(\rho_{12} + \rho_{22}) f_{2c}^2}{a_0^2}\right] U = 0,
\end{aligned}$$

Letting $g_3^1 = 0$, $g_2^3 = 0$ and for $a_0 = 0$, the similarity solutions take the following form:

$$u^1(x, t) = U(t) e^{\frac{f_{2c}}{b_0} x}, \tag{92}$$
$$u^2(x, t) = V(t) e^{\frac{f_{2c}}{b_0} x}, \tag{93}$$

where $U(t)$ and $V(t)$ satisfy the following set of ODE's:

$$\begin{aligned}
&b_0^2(\rho_{11} + \rho_{12})\ddot{U} - b_0^2 \rho_{12} \ddot{V} + b_0^2 b_c \dot{V} + 2 f_{2c}^2 (w_2 - w_1) U + 2 f_{2c}^2 (w_2 - w_3) V = 0, \\
&b_0^2(\rho_{12} + \rho_{22})\ddot{U} - b_0^2 \rho_{22} \ddot{V} - b_0^2 b_c \dot{V} - 2 f_{2c}^2 w_2 U + 2 f_{2c}^2 w_3 V = 0.
\end{aligned}$$

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