# Third Order Matching is Decidable 

Gilles Dowek<br>INRIA-Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex, France<br>Gilles.Dowek@inria.fr


#### Abstract

The higher order matching problem is the problem of determining whether a term is an instance of another in the simply typed $\lambda$-calculus, i.e. to solve the equation $a=b$ where $a$ and $b$ are simply typed $\lambda$-terms and $b$ is ground. The decidability of this problem is still open. We prove the decidability of the particular case in which the variables occurring in the problem are at most third order.


## Introduction

The higher order matching problem is the problem of determining whether a term is an instance of another in the simply typed $\lambda$-calculus i.e. to solve the equation $a=b$ where $a$ and $b$ are simply typed $\lambda$-terms and $b$ is ground.

Pattern matching algorithms are used to check if a proposition can be deduced from another by elimination of universal quantifiers or by introduction of existential quantifiers. In automated theorem proving, elimination of universal quantifiers and introduction of existential quantifiers are mixed and full unification is required, but in proof-checking and semi-automated theorem proving, these rules can be applied separately and thus pattern matching can be used instead of unification.

Higher order matching is conjectured decidable in [6] and the problem is still open. In [5] [6] [7] Huet has given a semi-decision algorithm and shown that in the particular case in which the variables occurring in the term $a$ are at most second order this algorithm terminates, and thus that second order matching is decidable. In [10] Statman has reduced the conjecture to the $\lambda$-definability conjecture and in [11] Wolfram has given an always terminating algorithm whose completeness is conjectured.

We prove in this paper that third order matching is decidable i.e. we give an algorithm that decides if a matching problem, in which all the variables are at most third order, has a solution. The main idea is that if the problem $a=b$ has a solution then it also has a solution whose depth is bounded by some integer $s$ depending only on the problem $a=b$, so a simple enumeration of the substitutions whose depth is bounded by $s$ gives a decision algorithm. This result can also be used to bound the depth of the search tree in Huet's semi-decision algorithm and thus turn it into a always-terminating decision algorithm. It can also be used to design an algorithm which enumerates a complete set of solutions to a third order matching problem and either terminates if the problem has a finite complete set of solutions or keeps enumerating solutions forever if it the
problem admits no such set. At last we discuss the problems that occur when we try to generalize the proof given here to higher order matching.

## 1 Trees and Terms

### 1.1 Trees

Definitions 1 (Following [3]) An occurrence is a list of strictly positive integers $\alpha=<s_{1}, \ldots, s_{n}>$. The number $n$ is called the length of the occurrence $\alpha$. A tree domain $D$ is a non empty finite set of occurrences such that if $\alpha<n>\in D$ then $\alpha \in D$ and if also $n \neq 1$ then $\alpha<n-1>\in D$. A tree is a function from a tree domain $D$ to a set $L$, called the set of labels of the tree.

If $T$ is a tree and $D$ its domain, the occurrence $<>$ is called the root of $T$ and the occurrence $\alpha<n>$ is called the $n^{\text {th }}$ son of the occurrence $\alpha$. The number of sons of an occurrence $\alpha$ is the greatest integer $n$ such that $\alpha<n>\in D$. A leaf is an occurrence that has no sons.

Let $T$ be a tree and let $\alpha=<s_{1}, \ldots, s_{n}>$ be an occurrence in this tree, the path of $\alpha$ is the set of occurrences $\left\{<s_{1}, \ldots, s_{p}>\mid p \leq n\right\}$. The number of elements of this path is the length of $\alpha$ plus one.

The depth of the tree $T$ is the length of the longest occurrence in $D$. This occurrence is, of course, a leaf.

If $T$ is a tree of domain $D$ and $\alpha$ is an occurrence of $D$, the subtree $T / \alpha$ is the tree $T^{\prime}$ whose domain is $D^{\prime}=\{\beta \mid \alpha \beta \in D\}$ and such that

$$
T^{\prime}(\beta)=T(\alpha \beta)
$$

By an abuse of language, if $\alpha<n>$ is an occurrence of a tree $T$, the subtree $T / \alpha<n>$ is also called the $n^{\text {th }}$ son of the occurrence $\alpha$.

If $a$ is a label and $T_{1}, \ldots, T_{n}$ are trees (of domains $D_{1}, \ldots, D_{n}$ ) then the tree of root $a$ and sons $T_{1}, \ldots, T_{n}$ is the tree $T$ of domain $D=\{<>\} \cup \bigcup_{i}\left\{<i>\alpha \mid \alpha \in D_{i}\right\}$ such that

$$
T(<>)=a
$$

and

$$
T(<i>\alpha)=T_{i}(\alpha)
$$

If $T$ is a tree of domain $D, \alpha$ an occurrence of $D$ and $T^{\prime}$ a tree of domain $D^{\prime}$ then the graft of $T^{\prime}$ in $T$ at the occurrence $\alpha\left(T\left[\alpha \leftarrow T^{\prime}\right]\right)$ is the tree $T^{\prime \prime}$ of domain $D^{\prime \prime}=D-\{\alpha \beta \mid \alpha \beta \in D\} \cup\left\{\alpha \beta \mid \beta \in D^{\prime}\right\}$ and such that

$$
T^{\prime \prime}(\gamma)=T^{\prime}(\beta) \text { if } \gamma=\alpha \beta
$$

and

$$
T^{\prime \prime}(\gamma)=T(\gamma) \text { otherwise }
$$

Let $T$ and $T^{\prime}$ be trees and let $a$ be a label such that all the occurrences of $a$ in $T$ are leaves $\alpha_{1}, \ldots, \alpha_{n}$ then the substitution of $T^{\prime}$ for $a$ in $T\left(T\left[a \leftarrow T^{\prime}\right]\right)$ is defined as $T\left[\alpha_{1} \leftarrow T^{\prime}\right] \ldots\left[\alpha_{n} \leftarrow T^{\prime}\right]$. Note that since $\alpha_{1}, \ldots, \alpha_{n}$ are leaves, the order in which the grafts are performed is insignificant.

### 1.2 Types

## Definition 2 (Type)

Let us consider a finite set $\mathcal{T}$. The elements of $\mathcal{T}$ are called atomic types. A type is a tree whose labels are either the elements of $\mathcal{T}$ or $\rightarrow$ and such that the occurrences labeled with an element of $\mathcal{T}$ are leaves and the ones labeled with $\rightarrow$ have two sons.

Let $T$ be a type, if the root of $T$ is labeled with an atomic type $U$ then $T$ is written $U$, if the root of $T$ is labeled with $\rightarrow$ and its sons are written $T_{1}$ and $T_{2}$ then $T$ is written ( $T_{1} \rightarrow T_{2}$ ). By convention $T_{1} \rightarrow T_{2} \rightarrow T_{3}$ is an abbreviation for $\left(T_{1} \rightarrow\left(T_{2} \rightarrow T_{3}\right)\right.$ ).

Definition 3 (Order of a Type)
If $T$ is a type, the order of $T$ is defined by

- $o(T)=1$ if $T$ is atomic,
- $o\left(T_{1} \rightarrow T_{2}\right)=\max \left\{1+o\left(T_{1}\right), o\left(T_{2}\right)\right\}$.


### 1.3 Typed $\lambda$-terms

Definitions 4 For each type $T$ we consider three sets $\mathcal{C}_{T}, \mathcal{I}_{T}, \mathcal{L}_{T}$. The elements of $\mathcal{C}_{T}$ are called constants of type $T$, those of $\mathcal{I}_{T}$ instantiable variables of type $T$ and those of $\mathcal{L}_{T}$ local variables of type $T$.

We assume that we have in each atomic type at least a constant and that there is a finite number of constants i.e. that the set $\bigcup_{T} \mathcal{C}_{T}$ is finite ${ }^{1}$. We assume also that we have an infinite number of instantiable and local variables of each type.

A typed $\lambda$-term is a tree whose labels are either $A p p$, or $\langle\operatorname{Lam}, x\rangle$ where $x$ is a local variable, or $\langle\operatorname{Var}, x\rangle$ where $x$ is a constant, an instantiable variable or a local variable such that the occurrences labeled with App have two sons, the occurrences labeled with $<\operatorname{Lam}, x>$ have one son and the occurrences labeled with $\langle V a r, x\rangle$ are leaves.

Let $t$ be a term, if the root of $t$ is labeled with $<\operatorname{Var}, x>$ we write it $x$, if the root of $t$ is labeled with $<L a m, x>$ and its son is written $u$ then we write it $\lambda x: T . u$ where $T$ is the type of $x$, if the root of $t$ is labeled with $A p p$ and its sons are written $u$ and $v$ then we write it ( $u v$ ). By convention $(u v w)$ is an abbreviation for $((u v) w)$.

In a term $t$, an occurrence $\alpha$ labeled with $\langle V a r, x\rangle$ is bound if there exists an occurrence $\beta$ in the path of $\alpha$ labeled with $\langle L a m, x\rangle$, it is free otherwise.

A term is ground if no occurrence is labeled with a pair $\langle\operatorname{Var}, x\rangle$ with $x$ instantiable.
Let $t$ and $t^{\prime}$ be terms and $x$ be a variable, the substitution of $t^{\prime}$ for $x$ in $t\left(t\left[x \leftarrow t^{\prime}\right]\right)$ is defined as $t\left[<\operatorname{Var}, x>\leftarrow t^{\prime}\right]$.

Definition 5 (Type of a Term)
A term $t$ is said to have the type $T$ if either:

- $t$ is a constant, an instantiable variable or a local variable of type $T$.
- $t=(u v)$ and $u$ has type $U \rightarrow T$ and $v$ type $U$ for some type $U$,
- $t=\lambda x: U . u$, the term $u$ has type $V$ and $T=U \rightarrow V$.

A term $t$ is said to be well-typed if there exists a type $T$ such that $t$ has type $T$. In this case $T$ is unique and is called the type of $t$.

[^0]Definition 6 ( $\beta \eta$-reduction)
The $\beta \eta$-reduction relation, written $\triangleright$, is defined as the smallest transitive relation compatible with term structure such that

$$
\begin{gathered}
(\lambda x: T . t u) \triangleright t[x \leftarrow u] \\
\lambda x: T .(t x) \triangleright t \quad \text { if } x \text { is not free in } t
\end{gathered}
$$

We adopt the usual convention of considering terms up to $\alpha$-conversion (i.e. bound variable renaming) and we consider that bound variables are renamed to avoid capture during substitutions. A rigorous presentation would use, for instance, de Bruijn indices [2].

Obviously, if $t$ is a term of type $T, x$ is a variable of type $U$ and $u$ a term of type $U$ then the term $t[x \leftarrow u]$ has type $T$. In the same way if a term $t$ has type $T$ and $t$ reduces to $u$ then $u$ has type $T$.

Proposition 1 The $\beta \eta$-reduction relation is strongly normalizable and confluent on typed terms, and thus each term has a unique normal form.
Proof See, for instance, [4].
Proposition 2 Let $t$ be a normal well-typed term of type $U_{1} \rightarrow \ldots \rightarrow U_{n} \rightarrow U$ ( $U$ atomic), the term $t$ has the form

$$
t=\lambda y_{1}: U_{1} . \ldots \lambda y_{m}: U_{m} \cdot\left(x u_{1} \ldots u_{p}\right)
$$

where $m \leq n$ and $x$ is a constant, an instantiable variable or a local variable.
Proof The term $t$ can be written in a unique way $t=\lambda y_{1}: V_{1} . . . \lambda y_{m}: V_{m} . u$ where $u$ is not an abstraction. The term $u$ can be written in a unique way $u=\left(v u_{1} \ldots u_{p}\right)$ where $v$ is not an application. The term $v$ is not an application by definition, it is not an abstraction (if $p=0$ because $u$ is not an abstraction and if $p \neq 0$ because $t$ is normal), it is therefore a constant, an instantiable variable or a local variable. Then since $t$ has type $U_{1} \rightarrow \ldots \rightarrow U_{n} \rightarrow U$, we have $m \leq n$ and for all $i, V_{i}=U_{i}$.

Definition 7 (Head of a Term, Atomic Term)
Let $t=\lambda y_{1}: T_{1} . \ldots \lambda y_{m}: T_{m} .\left(x u_{1} \ldots u_{p}\right)$ be a normal term. The symbol $x$ is called the head of the term. If $m=0$ then $t$ is said to be atomic, it is an abstraction otherwise.

Definition 8 ( $\eta$-long Form)
If $t=\lambda y_{1}: U_{1} . \ldots \lambda y_{m}: U_{m} .\left(x u_{1} \ldots u_{p}\right)$ is a term of type $T=U_{1} \rightarrow \ldots \rightarrow U_{n} \rightarrow U$ ( $U$ atomic) ( $m \leq n$ ) which is in $\beta \eta$-normal form then we define its $\beta$-normal $\eta$-long form as the term

$$
t^{\prime}=\lambda y_{1}: U_{1} . \ldots \lambda y_{m}: U_{m} \cdot \lambda y_{m+1}: U_{m+1} \cdot \ldots \lambda y_{n}: U_{n} \cdot\left(x u_{1}^{\prime} \ldots u_{p}^{\prime} y_{m+1}^{\prime} \ldots y_{n}^{\prime}\right)
$$

where $u_{i}^{\prime}$ is the $\beta$-normal $\eta$-long form of $u_{i}$ and $y_{i}^{\prime}$ is the $\beta$-normal $\eta$-long form of $y_{i}$.
This definition is by induction on the pair $\left\langle c_{1}, c_{2}\right\rangle$ where $c_{1}$ is the number of occurrences in $t$ and $c_{2}$ the number of occurrences in $T$

In the following all the terms are assumed to be in $\beta$-normal $\eta$-long form.

### 1.4 Böhm Trees

Definition 9 (Böhm Tree)
A (finite) Böhm tree is a tree whose occurrences are labeled with pairs $\langle l, x\rangle$ such that $l$ is a list of local variables $<y_{1}, \ldots, y_{n}>$ and $x$ is a constant, an instantiable variable or a local variable.

Definition 10 (Type of a Böhm Tree)
Let $t$ be a Böhm tree whose root is labeled with the pair $\left.\left.\ll y_{1}, \ldots, y_{n}\right\rangle, x\right\rangle$ and whose sons are $u_{1}, \ldots, u_{p}$. The Böhm tree $t$ is said to have the type $T$ if the Böhm trees $u_{1}, \ldots, u_{p}$ have type $U_{1}, \ldots, U_{p}$ the symbol $x$ has type $U_{1} \rightarrow \ldots \rightarrow U_{p} \rightarrow U$ ( $U$ atomic) and $T=T_{1} \rightarrow \ldots \rightarrow T_{n} \rightarrow U$ where $T_{1}, \ldots, T_{n}$ are the types of the variables $y_{1}, \ldots, y_{n}$.

A Böhm tree $t$ is said to be well-typed if there exists a type $T$ such that $t$ has type $T$. In this case $T$ is unique and is called the type of $t$.

Definition 11 (Böhm Tree of a Normal Term)
Let $t=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} .\left(x u_{1} \ldots u_{p}\right)$ be a $\lambda$-term in normal ( $\eta$-long) form. The Böhm tree of $t$ is inductively defined as the tree whose root is the pair $\langle l, x\rangle$ where $\left.l=<y_{1}, \ldots, y_{n}\right\rangle$ is the list of the variables bound at the top of this term, $x$ is the head symbol of $t$ and sons are the Böhm trees of $u_{1}, \ldots, u_{p}$.

Remark Normal ( $\eta$-long) well-typed terms and well-typed Böhm trees are in one-to-one correspondence. Moreover if $t$ is a normal ( $\eta$-long) term and $\tilde{t}$ is its Böhm tree then occurrences in $t$ labeled with a constant, an instantiable variable or a local variable and occurrences in $\tilde{t}$ are in one-to-one correspondence. So we will use the following abuse of notation: if $\alpha$ is an occurrence in the Böhm tree of $t$ we write $(t / \alpha)$ for the normal ( $\eta$-long) term corresponding to the Böhm tree $(\tilde{t} / \alpha)$ and $t[\alpha \leftarrow u]$ for the term $t\left[\alpha^{\prime} \leftarrow u\right]$ where $\alpha^{\prime}$ is the occurrence of a variable or a constant in $t$ corresponding to $\alpha$.

Notation Let $t$ be a term, we write $|t|$ for the depth of the Böhm tree of the normal ( $\eta$-long) form of $t$.

Proposition 3 In each type $T$ there is a ground term $t$ such that $|t|=0$.
Proof Let $T=U_{1} \rightarrow \ldots \rightarrow U_{n} \rightarrow U$ with $U$ atomic and let $c$ be a constant of type $U$. The term $t=\lambda x_{1}: U_{1} . \ldots \lambda x_{n}: U_{n} . c$ has type $T$ and $|t|=0$.

### 1.5 Substitution

Definition 12 (Substitution)
A substitution is a finite set of pairs $\left\langle x_{i}, t_{i}\right\rangle$ where $x_{i}$ is an instantiable variable and $t_{i}$ a term of the same type in which no local variable occurs free such that if $\langle x, t\rangle$ and $\left\langle x, t^{\prime}\right\rangle$ are both in this set then $t=t^{\prime}$. The variables $x_{i}$ are said to be bound by the substitution.

Definition 13 (Substitution applied to a Term)
If $\sigma$ is a substitution and $t$ a term then we let

$$
\sigma t=t\left[\alpha_{1}^{1} \leftarrow t_{1}\right] \ldots\left[\alpha_{1}^{p_{1}} \leftarrow t_{1}\right] \ldots\left[\alpha_{n}^{1} \leftarrow t_{n}\right] \ldots\left[\alpha_{n}^{p_{n}} \leftarrow t_{n}\right]
$$

where $\alpha_{i}^{1}, \ldots, \alpha_{i}^{p_{i}}$ are the occurrences of $x_{i}$ in $t$.
Note that since the $\alpha_{i}^{j}$ are leaves, the order in which the grafts are performed is insignificant.

Definition 14 (Composition of Substitutions)
Let $\sigma$ and $\tau$ be two substitutions the substitution $\tau \circ \sigma$ is defined by

$$
\tau \circ \sigma=\{<x, \tau t>\mid<x, t>\in \sigma\} \cup\{<x, t>\mid<x, t>\in \tau \text { and } x \text { not bound by } \sigma\}
$$

Proposition 4 Let $\sigma$ and $\tau$ be two substitutions and $t$ is a term, we have

$$
(\tau \circ \sigma) t=\tau(\sigma t)
$$

Proof By decreasing induction on the depth of an occurrence $\alpha$ in $t$ we prove that we have

$$
(\tau \circ \sigma)(t / \alpha)=\tau(\sigma(t / \alpha))
$$

## 2 Pattern Matching

Definition 15 (Matching Problem)
A matching problem is a set $\left.\left.\Phi=\left\{<a_{1}, b_{1}\right\rangle, \ldots,<a_{n}, b_{n}\right\rangle\right\}$ of pairs of terms of the same type such that the terms $b_{1}, \ldots, b_{n}$ are ground. A pair $\langle a, b\rangle$ is frequently written as an equation $a=b$.

Definition 16 (Third Order Matching Problem)
A third order matching problem is a matching problem $\Phi=\left\{a_{1}=b_{1}, \ldots, a_{n}=b_{n}\right\}$ such that the types of the instantiable variables that occur in $a_{1}, \ldots, a_{n}$ are of order at most three.

Definition 17 (Solution)
Let $\Phi=\left\{a_{1}=b_{1}, \ldots, a_{n}=b_{n}\right\}$ be a matching problem. A substitution $\sigma$ is a solution of this problem if and only if for every $i$, the normal form of the terms $\sigma a_{i}$ and $b_{i}$ are identical up to $\alpha$-conversion.

Remark Usual unification terminology distinguishes variables (here instantiable variables) and constants. The need for local variables comes from the fact that we want to transform the problem $\lambda y: T . x=\lambda y: T . y$ (where $x$ is an instantiable variable of type $T$ ) into the problem $x=y$ by dropping the common abstraction. The symbol $y$ cannot be an instantiable variable (because it cannot be instantiated by a substitution), it cannot be a constant because, if it were, we would have the solution $x \leftarrow y$ to the second problem which is not a solution to the first. So we let $y$ be a local variable and the solution $x \leftarrow y$ is now forbidden in both problems because no local variable can occur free in the terms substituted for variables in a substitution.

In Huet's unification algorithm [5] [6] these local variables are always kept in the head of the terms in common abstractions. In Miller's mixed prefixes terminology [8], constants are universal variables declared to the left hand side of the instantiable variables and local variables are universal variables declared to the right hand side of all the instantiable variables.

Remark In an alternative definition of matching problems, the terms $b_{1}, \ldots, b_{n}$ do not need to be ground. The method of this paper can be adapted to such problems using the standard technique of variable freezing [6].

Definition 18 (Ground Solution)
Let $\Phi=\left\{a_{1}=b_{1}, \ldots, a_{n}=b_{n}\right\}$ be a problem and let $\sigma$ be a solution to $\Phi$. The solution $\sigma$ is said to be ground if for each instantiable variable that has an occurrence in some $a_{i}$, the term $\sigma x$ is ground.

Proposition 5 If a matching problem has a solution then it has a ground solution.
Proof Let $\Phi=\left\{a_{1}=b_{1}, \ldots, a_{n}=b_{n}\right\}$ be a matching problem and let $\sigma$ be a solution to this problem. Let $y_{1}: T_{1}, \ldots, y_{n}: T_{n}$ be the instantiable variables occurring in the term $\sigma x$ for some $x$ instantiable variable occurring in some $a_{i}$. Let $u_{1}, \ldots, u_{n}$ be ground terms of the types $T_{1}, \ldots, T_{n}$. Let $\tau=\left\{<y_{1}, u_{1}>, \ldots,<y_{n}, u_{n}>\right\}$, and $\sigma^{\prime}=\tau \circ \sigma$. Obviously, for each instantiable variable $x$ of $a$, the term $\sigma^{\prime} x$ is ground and $\sigma^{\prime}$ is a solution to $\Phi$.

Definition 19 (Complete Set of Solutions)
Obviously if $\sigma$ is a solution to a problem $\Phi$ then for any substitution $\tau, \tau \circ \sigma$ is also a solution to $\Phi$. A set $S$ of solutions to a problem $\Phi$ is said to be complete if for every substitution $\theta$ that is a solution to this problem there exists a substitution $\sigma \in S$ and a substitution $\tau$ such that $\theta=\tau \circ \sigma$.

Lemma 1 Some problems have no finite complete set of solutions.
Proof (Example 1) Consider an atomic type $T$ and an instantiable variable $x: T \rightarrow(T \rightarrow T) \rightarrow T$ and the problem

$$
\lambda a: T .(x a \lambda z: T . z)=\lambda a: T . a
$$

The substitutions

$$
x \leftarrow \lambda o: T . \lambda s: T \rightarrow T .(s \ldots(s o) \ldots)
$$

are solutions to this problem and they cannot be obtained as instances of a finite number of solutions.

Remark In [6] [12], the similar examples $(x \lambda z: T . z)=a$ and $(x \lambda z: T . z)=b(a)$ are considered.
So in contrast with second order matching [6] [7] there is no (always terminating) algorithm that enumerates a complete set of solutions to a third order matching problem.

We consider now algorithms that take as an input a matching problem and either give one solution to the problem or fail if it does not have any.

## 3 A Bound on the Depth of Solutions

All the problems considered in the rest of the paper are third order.
To prove the decidability of third order matching we are going to prove that the depth of the term $t$ substituted to a variable $x$ by a solution $\sigma$ to a problem $\Phi$ can be bounded by an integer $s$ depending only on the problem $\Phi$. Of course the previous example shows that a matching problem may have solutions of arbitrary depth, but to design a decision algorithm we do not need to prove that all the solutions are bounded by $s$ but only that at least one is. To show this result we take a problem $\Phi$ that has a solution $\sigma$ (by proposition 5 , we can consider without loss of generality that this solution is ground) and we build another solution $\sigma^{\prime}$ whose depth is bounded by an integer $s$ depending only on the problem $\Phi$.

The proof is divided into two parts. In the first part,we focus on a particular case in which the problem $\Phi$ is a an interpolation problem i.e. set of equations of the form $\left(x c_{1} \ldots c_{n}\right)=b$ such that $x$ is an instantiable variable and $c_{1}, \ldots, c_{n}$ and $b$ are ground terms. Then, in the second part, we reduce the general case to this particular case.

Consider now an equation ( $x c_{1} \ldots c_{n}$ ) $=b$ and a substitution $\sigma$ solution to this equation. Let us write $t=\sigma x=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} . u$ ( $u$ atomic). We have

$$
\sigma\left(x c_{1} \ldots c_{n}\right)=\left(\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} . u c_{1} \ldots c_{n}\right)
$$

This term reduces to $u\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]$ whose normal form is $b$.
The terms $c_{i}$ are at most second order. In the key lemma, we prove that, in the general case, when we substitute a second order term $c$ to a variable $y$ in a term $u$ and we normalize the term $u[y \leftarrow c]$, we get a term with a depth larger than or equal to the one of $u$. If this were true in all the cases, we would know that the depth of $t$ (the solution) has to be less than or equal to the depth of $b$ (the right hand side of the equation). A simple enumeration of the terms $t$ whose depth is less than or equal to $|b|$ would give a decision procedure.

Actually, the key lemma shows that the depth of the normal form of $u[y \leftarrow c]$ can be less than the depth of $u$ in two cases : when $c$ is a non relevant term and when $|c|=0$. When such cases happen, solutions may have an arbitrary depth. In these cases, we show that if the problem $\Phi$ has a solution $\sigma$ then it has also another solution $\sigma^{\prime}$ whose depth is bounded by some integer $s$ depending only on the problem $\Phi$.

### 3.1 Interpolation Problems

Definition 20 (Interpolation Problem)
An interpolation problem is a set of equations of the form $\left(\begin{array}{llll}x & c_{1} & \ldots & c_{n}\end{array}\right)=b$ such that $x$ is an instantiable variable and $c_{1}, \ldots, c_{n}$ and $b$ are ground terms.

### 3.1.1 Key Lemma

## Definition 21 (Relevant Term)

Let $c=\lambda z_{1}: U_{1} . \ldots \lambda z_{p}: U_{p} . d$ ( $d$ atomic) be a normal term and $i$ an integer, $i \leq p$. We say that $c$ is relevant in its $i^{\text {th }}$ argument if $z_{i}$ has an occurrence in the term $d$.

Lemma 2 (Key Lemma) Let us consider a normal term $u$, a variable $y$ of type $T$ of order at most two and a normal ground term $c$ of type $T$.
(1) If $y$ has an occurrence in $u$ then $|c| \leq|u[y \leftarrow c]|$.
(2) If $\alpha$ is an occurrence in the Böhm tree of $u$ such that no occurrence in the path of $\alpha$ is labeled with $y$, then $\alpha$ is also an occurrence in the normal form of $u[y \leftarrow c]$ and has the same label in the Böhm tree of $u$ and in the Böhm tree of the normal form of $u[y \leftarrow c]$.
(3) If $\alpha=<s_{1}, \ldots, s_{n}>$ is an occurrence in the Böhm tree of $u$ such that for each occurrence $\beta=<s_{1}, \ldots, s_{k}>$ in the path of $\alpha, \beta \neq \alpha$, labeled with $y$, the term $c$ is relevant in its $r^{\text {th }}$ argument where $r$ is the position of the son of $\beta$ in the path of $\alpha$ i.e. $r=s_{k+1}$, then there exists an occurrence $\alpha^{\prime}$ of the Böhm tree of the normal form of $u[y \leftarrow c]$ such that all the labels occurring in the path of $\alpha$, except $y$, occur in the path of $\alpha^{\prime}$ and the number of times they occur in the path of $\alpha^{\prime}$ is greater than or equal to the number of times they occur in the path of $\alpha$. Moreover if the occurrence $\alpha$ is labeled with a symbol different from $y$, then the occurrence $\alpha^{\prime}$ is labeled with this same symbol.
(4) Moreover if $|c| \neq 0$ then the length of $\alpha^{\prime}$ is greater than or equal to the length of $\alpha$.

Proof By induction on the number of occurrences of $y$ in $u$. We substitute these occurrences one by one and we normalize the term. Let $\beta$ be the occurrence in the Böhm tree of $u$ corresponding
to the occurrence of $y$ in $u$ we substitute. Let us write

$$
c=\lambda z_{1}: U_{1} . \ldots \lambda z_{p}: U_{p} . d
$$

The term $(u / \beta)$ has the form $\lambda v_{1}: V_{1} . \ldots \lambda v_{q}: V_{q} \cdot\left(y e_{1} \ldots e_{p}\right)$. When we substitute $y$ by the term $c$ in ( $\left.\begin{array}{llll}y & e_{1} & \ldots & e_{p}\end{array}\right)$ we get ( $c e_{1} \ldots e_{p}$ ) and when we normalize this term we get the term $d\left[z_{1} \leftarrow e_{1}, \ldots, z_{p} \leftarrow e_{p}\right]$ which is normal because the type of the $e_{i}$ are first order.

Let us consider the occurrences in the Böhm tree of $u$, while substituting the occurrence of $y$ corresponding to $\beta$, we have removed all the occurrences $\beta<i>\gamma$ where $i$ is an integer ( $i \leq p$ ) and $\gamma$ is an occurrence in the Böhm tree of $e_{i}$. We have added all the occurrences $\beta \delta$ where $\delta$ is an occurrence of the Böhm tree of $c$ labeled with a symbol different from $z_{1}, \ldots, z_{p}$ and all the occurrences $\beta \delta \gamma$ where $\delta$ is a leaf occurrence in the Böhm tree of $c$ labeled with a $z_{i}$ and $\gamma$ is an occurrence of the Böhm tree of $e_{i}$.

(1) Let $\beta$ be an outermost occurrence of $y$ in the Böhm tree of $u$. For each occurrence $\delta$ in the Böhm tree of $c, \beta \delta$ is an occurrence in the Böhm tree of the normal form of $u[y \leftarrow c]$. So $|c| \leq|u[y \leftarrow c]|$.
(2) When an occurrence $\beta$ of $y$ is substituted by $c$ all the occurrences removed have the form $\beta<i>\gamma$. So if no occurrence in the path of $\alpha$ is labeled with $y$, the occurrence $\alpha$ remains in the normal form of $u[y \leftarrow c]$.
(3) If the occurrence $\beta$ is not in the path of $\alpha$ then the occurrence $\alpha$ is still an occurrence in the normal form of $u[y \leftarrow c]$, we take $\alpha^{\prime}=\alpha$.

If $\beta=\alpha$ then the occurrence $\beta$ is an occurrence of the Böhm tree of the normal form of $u[y \leftarrow c]$. We take $\alpha^{\prime}=\beta=\alpha$.

If $\beta$ is in the path of $\alpha$ and $\beta \neq \alpha, \beta=<s_{1}, \ldots, s_{k}>$ then let $r$ be the position of the son of $\beta$ in the path of $\alpha$ i.e. $r=s_{k+1}$. Let $\gamma$ be such that $\alpha=\beta<r>\gamma$. By hypothesis $z_{r}$ has an occurrence in $d$, let $\delta$ be such an occurrence. The occurrence $\beta \delta \gamma$ is an occurrence in the Böhm tree of the normal form of $u[y \leftarrow c]$. We take $\alpha^{\prime}=\beta \delta \gamma$.

In all the cases, all the labels occurring in the path of $\alpha$, except $y$, occur in the path of $\alpha^{\prime}$ and the number of times they occur in the path of $\alpha^{\prime}$ is greater than or equal to the number of times they occur in the path of $\alpha$.

If the occurrence $\alpha$ is labeled with a symbol different from $y$, then the occurrence $\alpha^{\prime}$ is labeled with the same symbol as $\alpha$.
(4) If $\delta=<>$ then $c=\lambda z_{1}: U_{1} . \ldots \lambda z_{p}: U_{p} . z_{r}$ and $|c|=0$. So if $|c| \neq 0$ then $\delta \neq<>$ and the length of $\alpha^{\prime}$ is greater than or equal to the length of $\alpha$.

Corollary Let us consider a normal term $u$, a variable $y$ of type $T$ of order at most two and a ground term $c$ of type $T$. If $c$ is relevant in all its arguments and $|c| \neq 0$ then $|u| \leq|u[y \leftarrow c]|$.
Proof We take for $\alpha$ the longest occurrence in the Böhm tree of $u$. When we substitute one by one the occurrences of $y$, by part (4) of the key lemma, we get occurrences that are at least long. So there is an occurrence in the Böhm tree of the normal form of $u[y \leftarrow c]$ which is at least long as $\alpha$. So $|u| \leq|u[y \leftarrow c]|$.

### 3.1.2 Computing the Substitution $\sigma^{\prime}$

Let us consider an equation $\left(x c_{1} \ldots c_{n}\right)=b$. Let $\sigma$ be a solution to this equation and let $t=\sigma x$. Let us write $t=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} . u$. The normal form of the term $\sigma\left(x c_{1} \ldots c_{n}\right)$ is the normal form of $u\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]$. If all the $c_{i}$ are relevant in their arguments and $\left|c_{i}\right| \neq 0$ then using the corollary of the key lemma we have $|t| \leq\left|\left(t c_{1} \ldots c_{n}\right)\right|$, so $|t| \leq|b|$ and this gives a bound on the depth of $t$. But the depth of $t$ may decrease when applied to the terms $c_{i}$ and normalized in two cases:

- if one of the terms $c_{i}$ is not relevant in one of its arguments,
- if one of the terms $c_{i}$ is such that $\left|c_{i}\right|=0$.

So solutions may have an arbitrary depth. When this happens, we compute another solution to the problem whose depth is bounded by an integer $s$ depending only on the initial problem.

This new substitution is constructed in two steps. In the first step we deal with non relevant terms and in the second with terms of depth 0 .

Example 2 Let $x$ be an instantiable variable of type $T \rightarrow(T \rightarrow T) \rightarrow T$. Consider the problem

$$
(x a \lambda z: T . b)=b
$$

The variable $z$ has no occurrence in $b$ so this problem has solutions of arbitrary depth

$$
x \leftarrow \lambda o: T . \lambda s: T \rightarrow T .(s t)
$$

where $t$ is an arbitrary term of type $T$. In this example we will compute the substitution

$$
x \leftarrow \lambda o: T . \lambda s: T \rightarrow T .(s c)
$$

where $c$ is a constant.
Example 1 (continued) The term $\lambda z: T . z$ has depth 0, so we have solutions of an arbitrary depth. In this example we will compute the substitution

$$
x \leftarrow \lambda o: T . \lambda s: T \rightarrow T .(s o o)
$$

Definition 22 (Occurrence Accessible with Respect to an Equation of the Form ( $x c_{1} \ldots c_{n}$ ) $=b$ ) Let us consider an equation

$$
\left(x c_{1} \ldots c_{n}\right)=b
$$

and the term

$$
t=\sigma x=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} \cdot u
$$

Let us consider the Böhm tree of $t$. The set of the occurrences of the Böhm tree of $t$ accessible with respect to the equation $\left(x c_{1} \ldots c_{n}\right)=b$ is inductively defined as:

- the root of the Böhm tree of $t$ is accessible,
- if $\alpha$ is an accessible occurrence labeled with $y_{i}$ and $c_{i}$ is relevant in its $j^{\text {th }}$ argument then the occurrence $\alpha<j>\left(\right.$ the $j^{\text {th }}$ son of $\alpha$ ) is accessible,
- if $\alpha$ is an accessible occurrence labeled with a symbol different from all the $y_{i}$ then all the sons of $\alpha$ are accessible.

Definition 23 (Occurrence Accessible with Respect to an Interpolation Problem)
An occurrence is accessible with respect to an interpolation problem if it is accessible with respect to one of the equations of this problem.

Definition 24 (Term Accessible with Respect to an Interpolation Problem)
A term is accessible with respect to an interpolation problem if all the occurrences of its Böhm tree which are not leaves are accessible with respect to this problem.

Definition 25 (Accessible Solution Built from a Solution)
Let $\Phi$ be an interpolation problem and let $\sigma$ be a solution to this problem. For each instantiable variable $x$ occurring in the equations of $\Phi$ we consider the term $t=\sigma x$. In the Böhm tree of $t$, we prune all the occurrences non accessible with respect to the equations of $\Phi$ in which $x$ has an occurrence and put Böhm trees of ground terms of depth 0 of the expected type as leaves. The tree obtained that way is the Böhm tree of some term $t^{\prime}$. We let $\hat{\sigma} x=t^{\prime}$.

Example 2 (continued) From the solution

$$
x \leftarrow \lambda o: T . \lambda s: T \rightarrow T .(s t)
$$

where $t$ is an arbitrary term, we compute the substitution

$$
x \leftarrow \lambda o: T . \lambda s: T \rightarrow T .(s c)
$$

where $c$ is a constant.
Proposition 6 Let $\Phi$ be an interpolation problem and let $\sigma$ be a solution to $\Phi$, then the accessible solution $\hat{\sigma}$ built from $\sigma$ is a solution to $\Phi$.
Proof Let us consider an equation $\left(x c_{1} \ldots c_{n}\right)=b$ of $\Phi$ and the terms

$$
\sigma x=t=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} . u
$$

and

$$
\hat{\sigma} x=t^{\prime}=\lambda y_{1}: T_{1} \ldots \lambda y_{n}: T_{n} \cdot u^{\prime}
$$

We prove by decreasing induction on the depth of the occurrence $\alpha$ of the Böhm tree of $u$ that if $\alpha$ is accessible with respect to the equation $\left(x c_{1} \ldots c_{n}\right)=b$ then $\alpha$ is also an occurrence of the Böhm tree of $u^{\prime}$ and

$$
\left(u^{\prime} / \alpha\right)\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]=(u / \alpha)\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]
$$

and then since the root of $u$ is accessible with respect to this equation we have

$$
u^{\prime}\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]=u\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]
$$

i.e.

$$
\left((\hat{\sigma} x) c_{1} \ldots c_{n}\right)=b
$$

So $\hat{\sigma}$ is a solution to $\Phi$.

Proposition 7 Let $\Phi$ be an interpolation problem and let $\sigma$ be a solution to $\Phi$. Let $h$ be the maximum depth of the right hand side of the equations of $\Phi$. Let $\hat{\sigma}$ the accessible solution built from $\sigma$. Let

$$
t=\hat{\sigma} x=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} . u
$$

( $u$ atomic). There are at most $h+1$ occurrences of symbols not in $\left\{y_{1}, \ldots, y_{n}\right\}$ on a path of the Böhm tree of $t$.
Proof Let $\alpha$ be an occurrence in the Böhm tree of $t$ such that there are more than $h+1$ occurrences of symbols not in $\left\{y_{1}, \ldots, y_{n}\right\}$ in the path of $\alpha$.

Let $\beta$ be the $(h+1)-t h$ occurrence of such a symbol. Since there are more that $h+1$ occurrences of symbols not in $\left\{y_{1}, \ldots, y_{n}\right\}$ in the path of $\alpha$, the occurrence $\beta$ is not a leaf, so it is accessible with respect to some equation $\left(x c_{1} \ldots c_{n}\right)=b$ of $\Phi$. Also, since this occurrence is not a leaf, it is labeled with a symbol $f$ whose type is not first order.

For each occurrence $\gamma=<s_{1}, \ldots, s_{k}>$ in the path of $\beta$ labeled with $y_{i}$, let $r$ be the position of the son of this occurrence in this path (i.e. $r=s_{k+1}$ ). Since the occurrence $\beta$ is accessible with respect to the equation $\left(x c_{1} \ldots c_{n}\right)=b$, the term $c_{i}$ is relevant in its $r^{t h}$ argument. So using $n$ times the part (3) of the key lemma there exists an occurrence $\beta^{\prime}$ in the Böhm tree of the normal form of the term $b=\left(\begin{array}{llll}\hat{\sigma} x & c_{1} & \ldots & c_{n}\end{array}\right)$ such that the path of $\beta^{\prime}$ contains at least $h+1$ occurrences. Thus, the length of this occurrence is at least $h$. This occurrence is labeled with the symbol $f$ whose type is not first order, so it has a son $\beta^{\prime \prime}$ whose length is at least $h+1$.

So the depth of $b$ is greater than or equal to $h+1$ which is contradictory.
Definition 26 (Compact Term)
A term $t=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} . u(u$ atomic $)$ is compact with respect to an interpolation problem $\Phi$ if no variable $y_{i}$ has more than $h+1$ occurrences in a path of its Böhm tree, where $h$ is the maximum depth of the right hand side of the equations of $\Phi$.

Proposition 8 Let $\Phi$ be an interpolation problem and let $\hat{\sigma}$ be an accessible solution to $\Phi$. Let $h$ be the maximum depth of the right hand side of the equations of $\Phi$. Let us consider an instantiable variable $x$ and

$$
t=\hat{\sigma} x=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} . u
$$

( $u$ atomic). Let us consider a variable $y_{i}$ and an occurrence $\alpha$ of the Böhm tree of $t$ such that there are more than $h+1$ occurrences on the path of $\alpha$ labeled with the variable $y_{i}$.

We consider all the equations $\left(x c_{1} \ldots c_{n}\right)=b$ of $\Phi$ such that the $(h+2)-t h$ occurrence of $y_{i}$ is accessible with respect to this equation. Then there exists an integer $j$ such that for every such equation we have

$$
c_{i}=\lambda z_{1}: U_{1} . \ldots \lambda z_{p}: U_{p} \cdot z_{j}
$$

Proof Let $\beta$ be the first occurrence of $y_{i}$ in the path of $\alpha$. Let $j$ be the integer such that $\alpha=\beta<j>\beta^{\prime}$.

Let $\left(\begin{array}{llll}x & c_{1} & \ldots & c_{n}\end{array}\right)=b$ be an equation of $\Phi$ such that the $(h+2)-t h$ occurrence of $y_{i}$ on the considered path is accessible with respect to this equation.

If the head of $c_{i}$ is a symbol different from a $z_{k}$ then $\left|c_{i}\right| \neq 0$. Using part (3) of the key lemma when we substitute $c_{1}, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{n}$ we have an occurrence $\alpha^{\prime}$ that has more than $h+1$ occurrences of $y_{i}$ on its path. Then using part (4) of the key lemma, when we substitute $c_{i}$ we have an occurrence $\alpha^{\prime \prime}$ whose length is greater than or equal to $h+1$ so

$$
h+1 \leq\left|u\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]\right|
$$

i.e. $h+1 \leq|b|$ which is contradictory. So we have

$$
c_{i}=\lambda z_{1}: U_{1} . \ldots \lambda z_{p}: U_{p} . z_{k}
$$

Since $h+2>1$ the occurrence $\beta<j>$ is accessible with respect to the equation $\left(x c_{1} \ldots c_{n}\right)=b$. Thus as the occurrence $\beta$ is labeled with $y_{i}$ and the occurrence $\beta<j>$ is accessible with respect to this equation, the term $c_{i}$ is relevant in its $j^{\text {th }}$ argument. Therefore $k=j$ and

$$
c_{i}=\lambda z_{1}: U_{1} . \ldots \lambda z_{p}: U_{p} . z_{j}
$$

Definition 27 (Compact Accessible Solution Built from an Accessible Solution)
Let $\Phi$ be an interpolation problem and let $\hat{\sigma}$ be an accessible solution to this problem. Let $h$ be the maximum depth of a right hand side of the equations of $\Phi$. We let

$$
\hat{\sigma} x=t=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} . u
$$

For each $\alpha$, occurrence in $t$ labeled with $y_{i}$ such that the corresponding occurrence $\alpha^{\prime}$ in the Böhm tree of $t$ has more than $h+1$ occurrences labeled with $y_{i}$ in its path, we have $c_{i}=\lambda z_{1}: U_{1} . \ldots \lambda z_{p}$ : $U_{p} . z_{j}$ in all the equations $\left(\begin{array}{llll}x & c_{1} & \ldots & c_{n}\end{array}\right)=b$ of $\Phi$ such that $\alpha^{\prime}$ is accessible with respect to this equation. We substitute the occurrence $\alpha$ by the term $\lambda z_{1}: U_{1} . \ldots \lambda z_{p}: U_{p} . z_{j}$. We get that way a term $t^{\prime}$. We let $\sigma^{\prime} x=t^{\prime}$.

Example 1 (continued) We build the substitution

$$
x \leftarrow \lambda o: T . \lambda s: T \rightarrow T .\left(\begin{array}{ll}
s & o
\end{array}\right)
$$

Example 3 Consider an instantiable variable $x$ of type $(T \rightarrow T \rightarrow T) \rightarrow T$. And the problem

$$
\begin{aligned}
& (x \lambda y: T . \lambda z: T . y)=a \\
& (x \lambda y: T \cdot \lambda z: T . z)=b
\end{aligned}
$$

We have the solution

$$
x \leftarrow \lambda f: T \rightarrow T \rightarrow T .(f a(f c(f d b)))
$$

This solution is accessible but not compact. The first occurrence of $f$ is accessible with respect to both equations, but the second and third occurrences are accessible only with respect to the second one. We have $h=0$, so we substitute the second and third occurrences of $f$ by the term $\lambda y: T . \lambda z: T . z$ and we get the substitution

$$
x \leftarrow \lambda f: T \rightarrow T \rightarrow T .(f a b)
$$

Note that we must not substitute the first occurrence of $f$ by $\lambda y: T . \lambda z: T . z$, because we would get the substitution $x \leftarrow \lambda f: T \rightarrow T \rightarrow T . b$ which is not a solution to the first equation.

Proposition 9 Let $\Phi$ be an interpolation problem and let $\sigma$ be a solution to $\Phi$. Let $\hat{\sigma}$ the accessible solution built from $\sigma$ and $\sigma^{\prime}$ the compact accessible solution built from $\hat{\sigma}$. Then $\sigma^{\prime}$ is a solution to $\Phi$.
Proof We consider an equation $\left(x c_{1} \ldots c_{n}\right)=b$ and we let

$$
\hat{\sigma} x=t=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} . u
$$

and

$$
\sigma^{\prime} x=t=\lambda y_{1}: T_{1} . \ldots \lambda y_{n}: T_{n} \cdot u^{\prime}
$$

The term $u^{\prime}$ is obtained by substituting in the term $u$ some occurrences (say $\beta_{1}, \ldots, \beta_{k}$ ) by some terms (say $e_{1}, \ldots, e_{k}$ ). If $\alpha$ is an occurrence of $u$ then we define $u_{\alpha}^{\prime}$ as the term obtained by substituting in the term $u / \alpha$ the occurrence $\gamma_{i}$ by the term $e_{i}$ if $\beta_{i}=\alpha \gamma_{i}$.

We prove by decreasing induction on the depth of the occurrence $\alpha$ of the Böhm tree of $u$ that if $\alpha$ is accessible with respect to the equation $\left(x c_{1} \ldots c_{n}\right)=b$ then

$$
\left(u_{\alpha}^{\prime}\right)\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]=(u / \alpha)\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]
$$

Thus for the root we get

$$
u^{\prime}\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]=u\left[y_{1} \leftarrow c_{1}, \ldots, y_{n} \leftarrow c_{n}\right]
$$

i.e.

$$
\left(\left(\sigma^{\prime} x\right) c_{1} \ldots c_{n}\right)=b
$$

So $\sigma^{\prime}$ is a solution to all the equations of $\Phi$.
Proposition 10 Let $\Phi$ be an interpolation problem and let $\sigma$ be a solution to $\Phi$. Let $\hat{\sigma}$ be the accessible solution built from $\sigma$ and $\sigma^{\prime}$ the compact accessible solution built from $\hat{\sigma}$. Let $h$ be the maximum depth of the right hand side of the equations of $\Phi$. For every instantiable variable $x$ of arity $n, \sigma^{\prime} x$ has a depth less than or equal to $(n+1)(h+1)-1$.
Proof In a path of the Böhm tree of $\sigma^{\prime} x$ each $y_{i}$ has at most $h+1$ occurrences and there are at most $h+1$ occurrences of other symbols, so there are at most $(n+1)(h+1)$ occurrences. Therefore the depth of $\sigma^{\prime} x$ is bounded by $(n+1)(h+1)-1$.

Lemma 3 Let $\Phi$ be a third order interpolation problem. If $\Phi$ has a solution $\sigma$ then it also has a solution $\sigma^{\prime}$ such that for every instantiable variable $x, \sigma x$ has a depth less than or equal to $(n+1)(h+1)-1$, where $h$ is maximum of the depths of the right hand side of the equations and $n$ the arity of $x$.
Proof The compact accessible solution $\sigma^{\prime}$ built from the accessible solution built from the solution $\sigma$ is a solution and for every instantiable variable $x, \sigma^{\prime} x$ has a depth less than or equal to ( $n+$ 1) $(h+1)-1$.

This bound is met, for instance by the example 3 .

### 3.2 General Case

Let $a=b$ be an equation and let $\sigma$ be a solution to this equation. We construct an interpolation problem $\Phi(a=b, \sigma)$ such that for every equation $\left(x c_{1} \ldots c_{n}\right)=b^{\prime}$ of $\Phi(a=b, \sigma)$ we have $\left|b^{\prime}\right| \leq|b|$, $\sigma$ is a solution to $\Phi(a=b, \sigma)$ and every solution to $\Phi(a=b, \sigma)$ is a solution to $a=b$.

Definition 28 Let $a=b$ be an equation and let $\sigma$ be a (ground) solution to this equation. By induction on the number of occurrences of $a$ we construct an interpolation problem $\Phi(a=b, \sigma)$.

- If $a=\lambda x: T . d$ then since $\sigma$ is a solution to the problem $a=b$ we have $b=\lambda x: T$.e and $\sigma$ is a solution to the problem $d=e$. We let

$$
\Phi(a=b, \sigma)=\Phi(d=e, \sigma)
$$

- If $a=\left(\begin{array}{lll}f & d_{1} \ldots d_{n}\end{array}\right)$ with $f$ a constant or a local variable then since $\sigma$ is a solution to $a=b$ we have $b=\left(\begin{array}{llll}f & e_{1} & \ldots & e_{n}\end{array}\right)$ and $\sigma$ is a solution to the problems $d_{i}=e_{i}$. We let

$$
\Phi(a=b, \sigma)=\bigcup_{i} \Phi\left(d_{i}=e_{i}, \sigma\right)
$$

- If $a=\left(\begin{array}{llll}x & d_{1} \ldots & d_{n}\end{array}\right)$ with $x$ instantiable then for all $i$ such that $z$ has an occurrence in the normal form of the term $\left(\sigma x \sigma d_{1} \ldots \sigma d_{i-1} z \sigma d_{i+1} \ldots \sigma d_{n}\right)$ we let $c_{i}=\sigma d_{i}$ and $H_{i}=\Phi\left(d_{i}=\sigma d_{i}, \sigma\right)$ (obviously $\sigma$ is a solution to $d_{i}=\sigma d_{i}$ ). Otherwise we let $c_{i}=z_{i}$ where $z_{i}$ is a new local variable and $H_{i}=\emptyset$. We let

$$
\Phi(a=b, \sigma)=\left\{\left(x c_{1} \ldots c_{n}\right)=b\right\} \cup \bigcup_{i} H_{i}
$$

Proposition 11 Let $t=\left(\begin{array}{lll}x & d_{1} \ldots & d_{n}\end{array}\right)$ be a term and let $\sigma$ be a substitution. Let $c_{i}=\sigma d_{i}$ if $z$ has an occurrence in $\left(\sigma x \sigma d_{1} \ldots \sigma d_{i-1} \quad z \sigma d_{i+1} \ldots \sigma d_{n}\right)$ and $c_{i}=z_{i}$ where $z_{i}$ is a new local variable of the same type as $d_{i}$ otherwise. The variables $z_{i}$ do not occur in the normal form of ( $\sigma x c_{1} \ldots c_{n}$ ). Proof Let us assume that some of these variables have an occurrence in the normal form of $\left(\sigma x c_{1} \ldots c_{n}\right)$ and consider an outermost occurrence of such a variable $z_{i}$ in the Böhm tree of the normal form of $\left(\sigma x c_{1} \ldots c_{n}\right)$. By part (2) of the key lemma, the variable $z_{i}$ has also an occurrence in the normal form of term $\left(\sigma x c_{1} \ldots c_{n}\right)\left[z_{j} \leftarrow \sigma d_{j} \mid j \neq i\right]$ i.e. in the normal form of the term $\left(\sigma x \sigma d_{1} \ldots \sigma d_{i-1} z_{i} \sigma d_{i+1} \ldots \sigma d_{n}\right)$, which is contradictory.

Proposition 12 Let $a=b$ be an equation and let $\sigma$ be a solution to this equation,

- the substitution $\sigma$ is a solution to $\Phi(a=b, \sigma)$,
- conversely, if $\sigma^{\prime}$ is a solution to $\Phi(a=b, \sigma)$ then $\sigma^{\prime}$ is also a solution to the equation $a=b$.


## Proof

- By induction on the number of occurrences of $a$. When $a$ is an abstraction $a=\lambda x: T$.d (resp. an atomic term whose head is a constant or local variable $a=\left(f d_{1} \ldots d_{n}\right)$ ) then $b$ is also an abstraction $b=\lambda x: T . e$ (reps. an atomic term with the same head $\left.b=\left(f e_{1} \ldots e_{n}\right)\right)$ and by induction hypothesis $\sigma$ is a solution to all the equations of the set $\Phi(d=e, \sigma)\left(\operatorname{resp} . \Phi\left(d_{i}=e_{i}, \sigma\right)\right)$, so it is a solution to all the equations of $\Phi(a=b, \sigma)$.

When $a=\left(\begin{array}{llll}x & d_{1} & \ldots & d_{n}\end{array}\right)$ then by induction hypothesis $\sigma$ is a solution to all the equations of the sets $H_{i}$ and using the previous proposition the variables $z_{i}$ have no occurrences in the term $\left(\sigma x c_{1} \ldots c_{n}\right)$ so we have

$$
\left.\begin{array}{c}
(\sigma x \\
\sigma
\end{array} c_{1} \ldots c_{n}\right)=\left(\begin{array}{llll}
\sigma x & c_{1} & \ldots & c_{n}
\end{array}\right)\left[z_{i} \leftarrow \sigma d_{i}\right] ~\left(\begin{array}{llll}
\sigma x & c_{1} & \ldots & c_{n}
\end{array}\right)=\left(\begin{array}{llll}
\sigma x & \sigma d_{1} & \ldots & \left.\sigma d_{n}\right)=b
\end{array}\right.
$$

So $\sigma$ is a solution to the equation $\left(x c_{1} \ldots c_{n}\right)=b$.

- By induction on the number of occurrences of $a$. Let $\sigma^{\prime}$ be a substitution solution to $\Phi(a=$ $b, \sigma)$. If $a$ is an abstraction $a=\lambda x: T . d$ (resp. an atomic term whose head is a constant or a local
variable $\left.a=\left(f d_{1} \ldots d_{n}\right)\right)$ then $b$ is also an abstraction $b=\lambda x: T . e$ (reps. an atomic term with the same head $b=\left(\begin{array}{lll}f & e_{1} \ldots e_{n}\end{array}\right)$ ) and by induction hypothesis we have $\sigma^{\prime} d=e\left(\right.$ resp. $\left.\sigma^{\prime} d_{i}=e_{i}\right)$ and so $\sigma^{\prime} a=b$.

If $a=\left(\begin{array}{llll}x & d_{1} & \ldots & d_{n}\end{array}\right)$ then we have

$$
\left(\sigma^{\prime} x c_{1} \ldots c_{n}\right)=b
$$

and for all $i$ such that $z$ has an occurrence in ( $\sigma x \sigma d_{1} \ldots \sigma d_{i-1} z \sigma d_{i+1} \ldots \sigma d_{n}$ ) by induction hypothesis we have $\sigma^{\prime} d_{i}=\sigma d_{i}$, so $c_{i}=\sigma^{\prime} d_{i}$. Therefore

$$
\begin{gathered}
\left(\sigma^{\prime} x c_{1} \ldots c_{n}\right)\left[z_{i} \leftarrow \sigma^{\prime} d_{i}\right]=b\left[z_{i} \leftarrow \sigma^{\prime} d_{i}\right] \\
\left(\sigma^{\prime} x c_{1} \ldots c_{n}\right)\left[z_{i} \leftarrow \sigma^{\prime} d_{i}\right]=b \\
\left(\sigma^{\prime} x \sigma^{\prime} d_{1} \ldots \sigma^{\prime} d_{n}\right)=b \\
\sigma^{\prime} a=b
\end{gathered}
$$

Proposition 13 Let $a=b$ be an equation and let $\sigma$ be a solution to this equation, if $a^{\prime}=b^{\prime}$ is an equation of $\Phi(a=b, \sigma)$ then $\left|b^{\prime}\right| \leq|b|$.
Proof By induction on the number of occurrences of $a$. When $a$ is an abstraction $a=\lambda x: T . d$ (reps. an atomic term whose head is a constant or a local variable $\left.a=\left(f d_{1} \ldots d_{n}\right)\right)$ then $b$ is also an abstraction $b=\lambda x: T . e$ (reps. an atomic term with the same head $\left.b=\left(f e_{1} \ldots e_{n}\right)\right)$ and by induction hypothesis $\left|b^{\prime}\right| \leq|e|$ (resp. $\left.\left|b^{\prime}\right| \leq\left|e_{i}\right|\right)$ so $\left|b^{\prime}\right| \leq|b|$.

When $a=\left(\begin{array}{llll}x & d_{1} & \ldots & d_{n}\end{array}\right)$ and the considered equation is $\left(\begin{array}{llll}x & c_{1} & \ldots & c_{n}\end{array}\right)=b$ then we have $b^{\prime}=b$ so $\left|b^{\prime}\right| \leq|b|$. When the considered equation is in one of the sets $H_{i}$, the set $H_{i}$ is non empty so $z$ has an occurrence in the normal form of the term ( $\sigma x \sigma d_{1} \ldots \sigma d_{i-1} \quad z \quad \sigma d_{i+1} \ldots \sigma d_{n}$ ) and $\left.\left(\begin{array}{llllll}\sigma x & \sigma d_{1} \ldots & \ldots d_{i-1} & z & \sigma d_{i+1} \ldots & \left.\ldots d_{n}\right)\end{array}\right] \leftarrow \sigma d_{i}\right]=b$ so using part (1) of the key lemma we have $\left|\sigma d_{i}\right| \leq|b|$ and by induction hypothesis $\left|b^{\prime}\right| \leq\left|\sigma d_{i}\right|$ so $\left|b^{\prime}\right| \leq|b|$.
Definition 29 Let $\Psi$ be a third order matching problem and let $\sigma$ be a solution to $\Psi$. We let $\Phi(\Psi, \sigma)$ be the following third order interpolation problem:

$$
\Phi(\Psi, \sigma)=\bigcup_{a=b \in \Psi} \Phi(a=b, \sigma)
$$

Proposition 14 Let $\Psi$ be a third order matching problem and let $\sigma$ be a solution to $\Psi$. Let $h$ be the maximum of the depth of the right hand side of the equations of $\Psi$. Then $\sigma$ is a solution to the problem $\Phi(\Psi, \sigma)$, each substitution $\sigma^{\prime}$ solution to the problem $\Phi(\Psi, \sigma)$ is a solution to $\Psi$ and if $a^{\prime}=b^{\prime} \in \Phi(\Psi, \sigma)$ then $\left|b^{\prime}\right| \leq h$.
Proof By propositions 12 and 13.
Lemma 4 Let $\Psi$ be third order matching problem. Let $h$ be the maximum of the depth of the the right hand side of the equations of $\Psi$. If this problem has a solution $\sigma$ then it also has a solution $\sigma^{\prime}$ such that for every instantiable variable $x, \sigma x$ has a depth less than or equal to $(n+1)(h+1)-1$ where $n$ the arity of $x$.
Proof The substitution $\sigma$ is a solution to the problem $\Phi(\Psi, \sigma)$, thus, by lemma 3, this problem has a solution $\sigma^{\prime}$ such that for every instantiable variable $x, \sigma^{\prime} x$ has a depth less than or equal to $(n+1)(h+1)-1$. This solution $\sigma^{\prime}$ is a solution to the problem $\Psi$.
Remark This method, in which an interpolation problem $\Phi(\Psi, \sigma)$ is constructed from a pair $\langle\Psi, \sigma\rangle$ where $\Psi$ is an arbitrary problem and $\sigma$ a solution to $\Psi$, can be compared to the one used in the completeness proof of [9] in which a problem in solved form is constructed from such a pair.

## 4 A Decision Procedure

Theorem Third Order Matching is Decidable
Proof A decision procedure is obtained by considering the problem $\Phi$ and enumerating all the ground substitutions such that the term substituted for $x$ has a depth less than or equal to ( $n+$ 1) $(h+1)-1$, where $h$ maximum depth of $b$ for $a=b \in \Phi$ and $n$ is the arity of $x$. If one of these substitutions is a solution then success else failure. This decision procedure is obviously sound. By lemma 4, it is complete.

Remark A more efficient decision algorithm is obtained by enumerating the nodes of the tree obtained by pruning Huet's search tree [5] [6] at each node corresponding to a substitution whose depth is larger than $(n+1)(h+1)-1$. This tree is obviously finite and thus this algorithm terminates. It is obviously sound. By lemma 4 , it is complete.

Remark This result can be used to design an algorithm which enumerates a complete set of solutions to a third order matching problem and either terminates if the problem has a finite complete set of solutions or keeps enumerating solutions forever if it the problem admits no such set. Such an algorithm is got by enumerating the nodes of the tree obtained by pruning Huet's search tree [5] [6] at each node labeled with a problem that has no solution (by the theorem above, it is decidable if such a problem has a solution or not). Obviously, this algorithms still produces a complete set of solutions.

Let us show now that when a matching problem has a finite complete set of solutions then this algorithm terminates. Recall that a set of substitutions is called minimal if no substitution of this set is an instance of another and that Huet's algorithm applied to a matching problem produces a minimal complete set of solutions [6]. It is routine to verify that if a a problem has a finite complete set of solutions then any minimal complete set of solutions is also finite. So, if a problem has a finite complete set of solutions then Huet's tree for this problem has a finite number of success nodes and thus a finite number of nodes labeled with a problem that has a solution. The pruned tree is therefore finite and the algorithm obtained by enumerating its nodes terminates.

Remark This decidability result can be compared with the decidability of the equations of the form $P\left(x_{1}, \ldots, x_{n}\right)=b$ where $P$ is a polynomial whose coefficients are natural numbers and $b$ is a natural number.

If this equation has a solution $<a_{1}, \ldots, a_{n}>$ then it has a solution $\left\langle a_{1}^{\prime}, \ldots, a_{n}^{\prime}\right\rangle$ such that $a_{1}^{\prime} \leq b$. Indeed either $Q(X)=P\left(X, a_{2}, \ldots, a_{n}\right)$ is not a constant polynomial and for all $n, Q(n) \geq n$, so $a_{1} \leq b$, or the polynomial $Q$ is identically equal to $b$ and $<0, a_{2}, \ldots, a_{n}>$ is also a solution. So a simple induction on $n$ proves that if the equation has a solution then it also has a solution in $\{0, \ldots, b\}^{n}$ and an enumeration of this set gives a decision procedure.

## Conclusion: Towards Higher Order Matching

The proof given here is based on the fact that if $t$ is a third order term then when we reduce the term $\left(t c_{1} \ldots c_{n}\right)$, in the general case, we get a term deeper than $t$ (or, at least, if it is not, the depth loss can bounded). This gives a bound (in terms of the depth of $b$ ) on the depth of the solutions of the equation $\left(x c_{1} \ldots c_{n}\right)=b$. In the particular cases in which the depth loss is greater than the
bound, some part of the term $t$ is superfluous and that we can construct a smaller term $t^{\prime}$ such that $\left(t^{\prime} c_{1} \ldots c_{n}\right)=\left(\begin{array}{llll}c_{1} & \ldots & c_{n}\end{array}\right)$.

Generalizing this property of reduction to the full $\lambda$-calculus would give the decidability of higher order matching. To get the normal form of the term $\left(t c_{1} \ldots c_{n}\right)$ we have followed the strategy hinted by the weak normalization theorem and reduced first all the second order redexes, then all the first order redexes. So a generalization of this proof to higher order should require an induction on the maximal order of a redex. In the proof for the third order case, we quickly get the normal form of the term $\left(t c_{1} \ldots c_{n}\right)$ and we do not need to define the depth of a non-normal term. It seems that the generalization of this result to higher order requires such a definition.

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[^0]:    ${ }^{1}$ This technical restriction is in fact superfluous, because a matching problem expressed in a language with an infinite number of constants can always be reduced to one expressed in the language with a finite number of constants obtained by considering only the constants occurring in the problem and one constant in each atomic type.

