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Complete Axioms for Categorical Fixed-point Operators

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Abstract

We give an axiomatic treatment of fixed-point operators in categories. A notion of iteration operator is defined, embodying the equational properties of iteration theories. We prove a general completeness theorem for iteration operators, relying on a new, purely syntactic characterisation of the free iteration theory.

We then show how iteration operators arise in axiomatic domain theory. One result derives them from the existence of sufficiently many bifree algebras (exploiting the universal property Freyd introduced in his notion of algebraic compactness). Another result shows that, in the presence of a parameterized natural numbers object and an equational lifting monad, any uniform fixed-point operator is necessarily an iteration operator.

1. Introduction

Fixed points play a central rôle in domain theory. Traditionally, one works with a category such as $\mathbf{Cppo}$, the category of ω-continuous functions between ω-complete pointed partial orders. This possesses a least-fixed-point operator, whose properties are well understood. For example, a theorem of Bekič states that least simultaneous fixed points can be found in sequence by a form of Gaussian elimination, see e.g. [33]. More generally, the equational theory between fixed-point terms (μ-terms), induced by the least-fixed-point operator, has been axiomatized as the free *iteration theory* of Bloom and Ésik [3]. (This theory is known to be decidable.) Also, Eilenberg [6] and Plotkin [25] gave an order-free characterisation of the least-fixed-point operator as the unique fixed-point operator satisfying a condition known as *uniformity*, expressed with respect to the subcategory $\mathbf{Cppo}_\perp$ of *strict* maps in $\mathbf{Cppo}$, see e.g. [15].

Nowadays, one appreciates that $\mathbf{Cppo}$ is one of many possible categories of "domain-like" structures, each with an associated fixed-point operator. Not only are there many familiar order-theoretic variations on the notions of complete partial order and continuous function, but there are also many categories of "domains" based on somewhat different principles — for example, categories of games and strategies [21], realizability-based categories [20] and categories of abstract geometric structures [12]. Thus one needs generally applicable methods for establishing properties of the associated fixed-point operators.

In this paper, we analyse the equational properties of fixed-point operators in arbitrary categories of "domain-like" structures. In Section 2, we consider the basic notions of *(parameterized) fixed-point operator*, *Conway operator* and *iteration operator*, developed from analogous notions in Bloom and Ésik's study of iteration theories [3]. Our definitions are straightforward adaptations of Bloom and Ésik's to the general setting of a category with finite products. In particular, the notion of *iteration operator* is intended to capture all desirable equational properties of a fixed-point operator, as exemplified by the many completeness results for the free iteration theory in [3].

As in the case of the fixed-point operator on $\mathbf{Cppo}$, we also consider a notion of *(parameterized) uniformity* for (parameterized) fixed-point operators. We define this in general assuming a suitable functor $J : \mathcal{S} \to \mathcal{D}$ from a category $\mathcal{S}$ of "strict" maps. In practice, (parameterized) uniformity serves two purposes. First, it is often satisfied by a unique (parameterized) fixed-point operator, and so characterises that operator. Second, any parametrically uniform Conway operator is an iteration operator, so parameterized uniformity is a convenient tool for establishing that the equations of an iteration operator are satisfied.

In Section 3, we examine the equational theory of iteration operators. We use a syntax of multisorted fixed-point terms (μ-terms), which can be interpreted in any category with an iteration operator. In any such category, Bloom and Ésik's axioms for iteration theories [3] are sound. Bloom and Ésik provide numerous completeness theorems, demonstrating that the iteration theory axioms are also complete for deriving the valid equations in many familiar cat-

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egories with iteration operators. The first main contribution of this paper is a precise characterisation of the circumstances in which the iteration theory axioms are complete (Theorem 1). This result accounts for all the examples in [3]. It shows that, in non-degenerate categories, the soundness of the iteration theory axioms implies their completeness. This explains the ubiquity of completeness results for the free iteration theory.

Our completeness theorem follows from a new, purely syntactic characterisation of the free iteration theory as a maximal theory satisfying two properties: *closed consistency* and *typical ambiguity* (Theorem 2). This result, which is of interest in its own right, was inspired by Statman's characterisation of $\beta\eta$-equality in the simply-typed λ-calculus [31].

The remainder of the paper is devoted to providing conditions for establishing the existence (and uniqueness) of parametrically uniform Conway operators (hence iteration operators). In one common setting, which arises in axiomatic domain theory [13, 10, 12], one has that the category $\mathcal{D}$ of "domains" is obtained as the co-Kleisli category of a comonad on the category of strict maps $\mathcal{S}$. (For example, $\mathbf{Cppo}$ is the co-Kleisli category of the lifting comonad on $\mathbf{Cppo}_\perp$.) In axiomatic domain theory, $\mathcal{S}$ satisfies a curious property, first identified by Freyd [13, 14]: a wide class of endofunctors on $\mathcal{S}$ have initial algebras whose inverses are final coalgebras (in Freyd's terminology, $\mathcal{S}$ is *algebraically compact*). Following [7], we call such initial/final algebras/coalgebras *bifree algebras*. (In the example of $\mathbf{Cppo}_\perp$, every $\mathbf{Cppo}$-enriched endofunctor has a bifree algebra [10].)

In Section 5, we give a quick overview of initial algebras, final coalgebras and bifree algebras, including a couple of minor new propositions. Then, in Section 6, we show how bifree algebras in $\mathcal{S}$ can induce properties of fixed-point operators in $\mathcal{D}$. This programme was begun by Freyd and others [13, 5, 24, 28]. A further step was taken by Moggi, who, in unpublished work, gave a direct verification of the Bekič equality. Here, we give the complete story, showing how the presence of sufficiently many bifree algebras determines a unique parametrically uniform Conway operator (hence iteration operator).

In Section 7 we show how the Conway operator identities can be established without assuming the existence of the bifree algebras used in Section 6. This is possible when the category $\mathcal{S}$ of "strict" maps arises as the category of algebras for a "lifting monad" on a suitable category of "predomains" $\mathcal{C}$. (For example, $\mathbf{Cppo}_\perp$ is the category of algebras for the usual lifting monad on the category $\mathbf{Cpo}$ of, not necessarily pointed, ω-complete partial orders.) Axiomatically, we assume that $\mathcal{C}$ is a category with finite products, a monad embodying the equational properties of partial map classifiers (an *equational lifting monad* [4]), partial function spaces (*Kleisli exponentials* [22, 28]), and a (parameterized) natural numbers object. These conditions are always satisfied by the categories of predomains that arise in axiomatic and synthetic domain theory [10, 12, 20, 11, 26, 30]. Theorem 4 states that such categories support at most one uniform recursion operator (a *T-fixed-point operator*), and moreover it determines a unique parametrically uniform Conway operator on the associated category of domains. Thus, in the presence of a lifting monad and a parameterized natural numbers object, uniformity alone implies all equational properties of fixed points.

2. Fixed-point operators

In this section we give an overview of the various notions of fixed-point operator we shall be concerned with. We work with a category, $\mathcal{D}$, with distinguished finite products, to be thought of as a category of "domains". We write $\mathbf{1}$ for the terminal object.

Definition 2.1 (Fixed-point operator) A *fixed-point operator* is a family of functions $(\cdot)^* : \mathcal{D}(A, A) \to \mathcal{D}(\mathbf{1}, A)$ such that, for any $f : A \longrightarrow A$, $f \circ f^* = f^*$.

Definition 2.2 (Parameterized fixed-pt. op.) A *parameterized fixed-point operator* is a family of functions $(\cdot)^\dagger : \mathcal{D}(X \times A, A) \to \mathcal{D}(X, A)$ satisfying:

1. (Naturality.)
   For any $g : X \longrightarrow Y$ and $f : Y \times A \longrightarrow A$, $f^\dagger \circ g = (f \circ (g \times \mathsf{id}_A))^\dagger : X \longrightarrow A$.
2. (Parameterized fixed-point property.)
   For any $f : X \times A \longrightarrow A$, $f \circ \langle \mathsf{id}_X, f^\dagger \rangle = f^\dagger : X \longrightarrow A$.

Observe that a parameterized fixed-point operator corresponds to a family of fixed-point operators $(\cdot)^{*_X}$ in the co-Kleisli categories $\mathcal{D}_{X \times (-)}$ of $X \times (-)$ comonads. In this formulation, naturality states that, for any $g : X \longrightarrow Y$, the induced functor $H_g : \mathcal{D}_{Y \times (-)} \to \mathcal{D}_{X \times (-)}$ preserves the fixed-point operators in the sense that, for any endomorphism in $\mathcal{D}_{Y \times (-)}$, given by a morphism $f : Y \times A \longrightarrow A$ in $\mathcal{D}$, it holds that $H_g(f^{*_Y}) = (H_g f)^{*_X}$.

In practice, well-behaved fixed-point operators satisfy many other equations that do not follow from the fixed-point property alone.

Definition 2.3 (Dinaturality) A fixed-point operator is said to be *dinatural* if, for every $f : A \longrightarrow B$ and $g : B \longrightarrow A$, it holds that $(f \circ g)^* = f \circ (g \circ f)^*$.

Definition 2.4 (Conway operator) A *Conway operator* is a parameterized fixed-point operator that, in addition, satisfies:

3. (Parameterized dinaturality.)
   For any $f : X \times B \longrightarrow A$ and $g : X \times A \longrightarrow B$, $f \circ \langle \mathsf{id}_X, (g \circ \langle \pi_1, f\rangle)^\dagger\rangle = (f \circ \langle \pi_1, g\rangle)^\dagger : X \longrightarrow A$.
4. (Diagonal property.)
   For any $f : X \times A \times A \longrightarrow A$, $(f \circ (\mathsf{id}_X \times \Delta))^\dagger = (f^\dagger)^\dagger : X \longrightarrow A$ (where $\Delta : A \longrightarrow A \times A$ is the diagonal map).

It is easily seen that (parameterized) dinaturality implies the (parameterized) fixed-point property, so 2 of Definition 2.2 is redundant in the axiomatization of Conway operators.

The reason for singling out dinaturality is that it is a concept that makes sense for unparameterized fixed-point operators. It is also a powerful property. In special circumstances, it alone characterises a unique well-behaved fixed-point operator [29].

Mainly, however, we shall be interested in well behaved parameterized fixed-point operators, and the notion of Conway operator is appropriate. Conway operators are so named because their axioms correspond to those of the *Conway theories* of Bloom and Ésik [3]. They have also arisen independently in work of M. Hasegawa [16] and Hyland, who established a connection with Joyal, Street and Verity's notion of *trace* [18]. The definition of trace makes sense in any braided monoidal category. Hasegawa and Hyland showed that, in the special case that the monoidal product is cartesian, traces are in one-to-one correspondence with Conway operators.

There are many alternative axiomatizations for Conway operators. The axioms for a trace provide one possibility. Other options are discussed in [3, 16]. The following important property often appears in variant axiomatizations.

Proposition 2.5 (Bekič property) *For any Conway operator, given $\langle f, g\rangle : X \times A \times B \longrightarrow A \times B$, it holds that $\langle f, g\rangle^\dagger = \langle h, (g \circ (\langle \mathsf{id}_X, h\rangle \times \mathsf{id}_B))^\dagger\rangle : X \longrightarrow A \times B$, where $h = (f \circ \langle \mathsf{id}_{X\times A}, g^\dagger\rangle)^\dagger : X \longrightarrow A$.*

In spite of such consequences, there are basic equalities that Conway operators do not necessarily satisfy; for example, it is not true in general that $f^* = (f \circ f)^*$ for an endomorphism f. The *commutative identities* of Bloom and Ésik [3] ensure that such "missing" equalities do hold.

Definition 2.6 (Iteration operator) An *iteration operator* is a Conway operator that, in addition, satisfies:

5. (Commutative identities.)
   Given $f : X \times A^m \longrightarrow A$ and morphisms $\rho_1, \ldots, \rho_m : A^m \longrightarrow A^m$ such that each $\rho_i = \langle p_{i1}, \ldots, p_{im}\rangle$ is a tuple of projections (i.e. each p_{ij} is one of the m projections $\pi_1, \ldots, \pi_m : A^m \longrightarrow A$), it holds that
   $$\langle f \circ (\mathsf{id}_X \times \rho_1), \ldots, f \circ (\mathsf{id}_X \times \rho_m)\rangle^\dagger = \Delta_m \circ (f \circ (\mathsf{id}_X \times \Delta_m))^\dagger : X \longrightarrow A^m$$
   (Here A^m is the m-fold power $A \times \ldots \times A$, and $\Delta_m : A \longrightarrow A^m$ is the diagonal.)[1]

The complex formulation of the commutative identities means that they can be hard to establish in practice. One way of reducing the complexity is to look for simpler equational axiomatizations. For example, Ésik [9] has recently proved that it suffices to consider certain instances of the commutative identities generated, in an appropriate sense, by finite groups. However, in many situations, it is more convenient to derive the commutative identities from (more easily established) non-equational properties that imply them. Many examples of such properties are given by Bloom and Ésik [3]. In this paper we shall be concerned with one such property: *(parameterized) uniformity*.

To define (parameterized) uniformity, we suppose given a category $\mathcal{S}$ with finite products and the same objects as $\mathcal{D}$, together with a functor $J : \mathcal{S} \to \mathcal{D}$ that strictly preserves finite products and is the identity on objects. We use the symbol $\multimap$ for maps in $\mathcal{S}$, and we call morphisms in $\mathcal{D}$ in the image of J *strict*. (We really just need the subcategory of $\mathcal{D}$ consisting of strict maps, but the functorial formulation will be helpful in Section 6. Observe that morphisms given purely by the finite product structure on $\mathcal{D}$ are strict.)

Definition 2.7 (Uniformity) A fixed-point operator is said to be *uniform* (with respect to J) if, for any $f : A \longrightarrow A$, $g : B \longrightarrow B$ and $h : A \multimap B$, $Jh \circ f = g \circ Jh$ implies $g^* = Jh \circ f^*$.

Definition 2.8 (Parameterized uniformity) A parameterized fixed-point operator is said to be *parametrically uniform* if, for any $f : X \times A \longrightarrow A$, $g : X \times B \longrightarrow B$ and $h : A \multimap B$, $Jh \circ f = g \circ (\mathsf{id}_X \times Jh)$ implies $g^\dagger = Jh \circ f^\dagger$.

Observe that parameterized uniformity is just the statement that the fixed-point operator $(\cdot)^{*X}$ in each co-Kleisli category $\mathcal{D}_{X\times(-)}$ is uniform with respect to the composite functor $\mathcal{S} \to \mathcal{D} \to \mathcal{D}_{X\times(-)}$. Hasegawa gives an interesting reformulation of parameterized uniformity directly in terms of a trace [16]. If $\mathcal{S}$ is defined to be the subcategory of morphisms given purely by the finite product structure on $\mathcal{D}$, then parameterized uniformity is exactly the *functorial dagger implication for base morphisms* of [3].

Proposition 2.9 *Any parametrically uniform Conway operator is an iteration operator.*

The proof is an easy application of the strictness of all diagonals $\Delta_m : A \longrightarrow A^m$.

The converse to proposition 2.9 does not hold in general, see [8].

[1] Strictly speaking, we consider only instances of the commutative identities of [3] in which their "surjective base morphism" ρ is a diagonal Δ_m. The general commutative identities of [3] follow from such instances, using properties of Conway operators.

3. Completeness

In this section we introduce Bloom and Ésik's *iteration theories* [3], using a syntax of multisorted fixed-point terms (μ-terms). We prove a very general completeness theorem (Theorem 1) for the free iteration theory relative to interpretations in categories with iteration operators. The completeness theorem follows from a new syntactic characterisation of the free iteration theory (Theorem 2).

We assume given a nonempty collection of base types (or sorts), over which $\alpha, \beta, \ldots$ range. Types $\sigma, \tau, \ldots$ are either base types or product types $\sigma_1 \times \ldots \times \sigma_n$ (for $n \geq 0$). We use σ^n as an abbreviation for the n-fold power $\sigma \times \ldots \times \sigma$. We assume also a signature given by a set Σ of function symbols, each with an associated typing information of the form $(\alpha_1, \ldots, \alpha_n; \beta)$ (there is no loss of generality in considering only base types here). We loosely refer to both $(\alpha_1, \ldots, \alpha_n; \beta)$ and n as the *arity* of the function symbol. Constants are considered as function symbols with arity 0. We assume a countably infinite set of variable symbols $x, y, \ldots$ A variable is a pair, written x^σ, consisting of a variable symbol and a type (we omit the type superscript when convenient). Terms and their types are given by: each variable x^σ is a term of type σ; if $t_1, \ldots, t_n$ are terms of (base) types $\alpha_1, \ldots, \alpha_n$ and f is a function symbol of arity $(\alpha_1, \ldots, \alpha_n; \beta)$ then $f(t_1, \ldots, t_n)$ is a term of (base) type β; if $t_1, \ldots, t_n$ are terms of types $\sigma_1, \ldots, \sigma_n$ then $\langle t_1, \ldots, t_n \rangle$ is a term of type $\sigma_1 \times \ldots \times \sigma_n$; if t is a term of type $\sigma_1 \times \ldots \times \sigma_n$ then $\pi_i t$, where $1 \leq i \leq n$, is a term of type σ_i; if t is a term of type σ then $\mu x^\sigma. t$ is a term of type σ. As usual, the variable x is bound by μ in $\mu x. t$. We identify terms up to α-equivalence, writing $t \equiv t'$ for the identity of terms. We write $t(x_1^{\sigma_1}, \ldots, x_k^{\sigma_k}) : \tau$ for a term of type τ all of whose free variables are contained in $x_1^{\sigma_1}, \ldots, x_k^{\sigma_k}$. We call a term with no free variables *closed*. We write the substitution of n terms $t_1, \ldots, t_n$ for n distinct free variables $x_1, \ldots, x_n$ (of the correct types) in a term t as $t[t_1, \ldots, t_n \,/\, x_1, \ldots, x_n]$. Given $t(\vec{y}, x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}) : \sigma_1 \times \ldots \times \sigma_n$, we use the convenient notation $\mu\langle x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}\rangle.\, t$ to represent the term $\mu x^{\sigma_1 \times \ldots \times \sigma_n}.\, t[\pi_1 x, \ldots, \pi_n x \,/\, x_1, \ldots, x_n]$.

A *theory*, $\mathcal{T}$, is a typed congruence relation on terms that: contains the product equations, i.e. $\mathcal{T} \vdash \pi_i\langle t_1 \ldots, t_k\rangle = t_i$ and $\mathcal{T} \vdash t = \langle \pi_1 t, \ldots, \pi_k t\rangle$ (for $t : \sigma_1 \times \ldots \times \sigma_k$); and is closed under substitution (i.e. if $\mathcal{T} \vdash t = t'$ and $s : \sigma$ then $\mathcal{T} \vdash t[s/x^\sigma] = t'[s/x^\sigma]$). For any theory, $\mathcal{T} \vdash t = t'$ if and only if $\mathcal{T} \vdash (t = t')[\langle x_1{}^{\sigma_1}, \ldots, x_n{}^{\sigma_n}\rangle / x^{\sigma_1 \times \cdots \times \sigma_n}]$ where $x_1, \ldots, x_n$ are fresh variables. Thus a theory is determined by its equations between terms whose only free variables are of base type. We say that $\mathcal{T}$ is *consistent* if there are two terms t, t' of the same type such that $\mathcal{T} \not\vdash t = t'$. We say that $\mathcal{T}$ is *closed-consistent* if there are two such terms that are closed.

We now axiomatize *Conway theories*, in which μ corresponds to a Conway operator, and *iteration theories*, identifying the equational properties of an iteration operator.

Definition 3.1 (Conway theory) A theory $\mathcal{T}$ is said to be a *Conway theory* if it satisfies two axioms:

1. (Dinaturality.)
   For any $t(\vec{z}, y^\tau) : \sigma$ and $t'(\vec{z}, x^\sigma) : \tau$,
   $\mathcal{T} \vdash \mu x.\, t[t'/y] = t[\mu y.\, t'[t/x]/y] : \sigma$.
2. (Diagonal property.)
   For $t(\vec{z}, x^\sigma, y^\sigma) : \sigma$, $\mathcal{T} \vdash \mu x.\, t[x/y] = \mu x.\, \mu y.\, t : \sigma$.

These axioms are just the multisorted version of the axiomatization given by Corollary 6.2.5 of [3], where dinaturality and the diagonal property are called the *composition identity* and the *double dagger identity* respectively.

Definition 3.2 (Iteration theory) We say that a Conway theory $\mathcal{T}$ is an *iteration theory* if it satisfies the following axiom schema.

3. (Amalgamation.)
   For any terms $t_1, \ldots, t_n(\vec{z}, x_1^\sigma, \ldots, x_n^\sigma) : \sigma$ and $s(\vec{z}, y^\sigma) : \sigma$, suppose $t_i[y, \ldots, y \,/\, x_1, \ldots, x_n] \equiv s$, for all i with $1 \leq i \leq n$, then it follows that $\mathcal{T} \vdash \mu\langle x_1, \ldots, x_n\rangle.\, \langle t_1, \ldots, t_n\rangle = \langle \mu y.\, s, \ldots, \mu y.\, s\rangle : \sigma^n$.

Amalgamation is very close to the commutative identities of [3] (as in Definition 2.6). An alternative formulation is employed in [17], whose *alphabetic identification identity* is equivalent to amalgamation.

We write $\mathcal{F}$ for the smallest Conway theory (generated by the given base types and signature), and $\mathcal{I}$ for the smallest iteration theory. As is shown in [3], $\mathcal{I}$ completely captures the valid identities in a wide class of models, including $\mathbf{Cppo}$. We write $\mathcal{I}^*$ for the smallest iteration theory in which all closed terms (with identical types) are equated. Although $\mathcal{I}^*$ is not a closed-consistent theory, it is nonetheless consistent. In fact, $\mathcal{I}^*$ exactly captures the valid identities in $\mathbf{Cppo}_\perp$. Our aim in this section is to prove a general completeness theorem, accounting for all such completeness results for $\mathcal{I}$ and $\mathcal{I}^*$.

First, we consider the interpretation of μ-terms in any category $\mathcal{D}$ with finite products and a family of functions $(\cdot)^\dagger : \mathcal{D}(X \times A, A) \to \mathcal{D}(X, A)$ satisfying the naturality property of Definition 2.2. An *interpretation* is determined by a function $[\![\cdot]\!]$ from base types to objects of $\mathcal{D}$, together with a function mapping each function symbol f, of arity $(\alpha_1, \ldots, \alpha_k; \beta)$, to a map $[\![f]\!] : [\![\alpha_1]\!] \times \ldots \times [\![\alpha_k]\!] \longrightarrow [\![\beta]\!]$. The first function extends (using products) to one from arbitrary types to objects of $\mathcal{D}$, and the second extends to one mapping any term $t(x_1^{\sigma_1}, \ldots, x_k^{\sigma_k}) : \tau$ to a morphism

$[\![t]\!] : [\![\sigma_1]\!] \times \ldots \times [\![\sigma_k]\!] \longrightarrow [\![\tau]\!]$. Such an interpretation determines a theory $\mathcal{T}_{[\![\cdot]\!]}$ defined by $\mathcal{T}_{[\![\cdot]\!]} \vdash t(\vec{x}) = t'(\vec{x})$ iff $[\![t]\!] = [\![t']\!]$. We have no reason to favour one interpretation in $\mathcal{D}$ over another, so we shall be more interested in the theory determined by the category itself. This theory, $\mathcal{T}_{\mathcal{D}}$, is defined by $\mathcal{T}_{\mathcal{D}} \vdash t(\vec{x}) = t'(\vec{x})$ iff, for all interpretations $[\![\cdot]\!]$ in $\mathcal{D}$, it holds that $[\![t]\!] = [\![t']\!]$.

It is immediate from the definitions that if $(\cdot)^\dagger$ is a Conway operator then $\mathcal{T}_{\mathcal{D}}$ is a Conway theory (i.e. $\mathcal{F} \subseteq \mathcal{T}_{\mathcal{D}}$). It is also straightforward (although not quite immediate) that if $(\cdot)^\dagger$ is an iteration operator then $\mathcal{T}_{\mathcal{D}}$ is an iteration theory (i.e. $\mathcal{I} \subseteq \mathcal{T}_{\mathcal{D}}$). Further, if every hom-set $\mathcal{D}(\mathbf{1}, X)$ is a singleton[2] (i.e. if $\mathbf{1}$ is a zero object in $\mathcal{D}$) then $\mathcal{I}^* \subseteq \mathcal{T}_{\mathcal{D}}$.

Theorem 1 (Completeness) *If $(\cdot)^\dagger$ on $\mathcal{D}$ is an iteration operator then:*

1. *If there exists a hom-set $\mathcal{D}(\mathbf{1}, X)$ containing at at least two distinct morphisms then $\mathcal{T}_{\mathcal{D}} = \mathcal{I}$.*

2. *If $\mathbf{1}$ is a zero object and there exists a hom-set $\mathcal{D}(X, Y)$ containing two distinct morphisms then $\mathcal{T}_{\mathcal{D}} = \mathcal{I}^*$.*

The only examples not captured by one of the conditions above are categories $\mathcal{D}$ equivalent to the terminal category, in which case $\mathcal{T}_{\mathcal{D}}$ is the inconsistent theory.

We prove the theorem by obtaining a syntactic characterisation of the free iteration theory. Suppose that $(\cdot)^*$ is any function from base types to types. Suppose also that θ is a function mapping each function symbol f of arity $(\alpha_1, \ldots, \alpha_n; \beta)$ to a term $f\theta(z_1^{\alpha_1^*}, \ldots, z_n^{\alpha_n^*}) : \beta^*$. Then $(\cdot)^*$ extends (by substitution) to an endofunction on types. Similarly, θ extends to a function on terms, mapping each $t(x_1^{\sigma_1}, \ldots, x_k^{\sigma_k}) : \tau$ to a term $t\theta(x_1^{\sigma_1^*}, \ldots, x_k^{\sigma_k^*}) : \tau^*$, defined inductively on the structure of t by:

$$
\begin{array}{lcl}
x^\sigma\theta & = & x^{\sigma^*} \\
f(t_1, \ldots, t_n)\theta & = & f\theta[t_1\theta, \ldots, t_n\theta \,/\, z_1, \ldots, z_n] \\
(\mu x^\sigma.\, t)\theta & = & \mu x^{\sigma^*}.\, (t\theta) \\
\langle t_1, \ldots, t_n \rangle\theta & = & \langle t_1\theta, \ldots, t_n\theta \rangle \\
(\pi_i t)\theta & = & \pi_i(t\theta).
\end{array}
$$

We say that $\mathcal{T}$ is *typically ambiguous* if $\mathcal{T} \vdash t = t' : \sigma$ implies, for all $(\cdot)^*$ and θ as above, $\mathcal{T} \vdash t\theta = t'\theta : \sigma^*$. In the terminology of [9], $\mathcal{T}$ is typically ambiguous iff it contains the *vector forms* of all its equations. $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{I}^*$ are all examples of typically ambiguous theories. In each case, one proves that $t = t'$ implies $t\theta = t'\theta$ by a straightforward induction on the derivation of $t = t'$.

Theorem 2

1. *The only consistent typically ambiguous iteration theories are $\mathcal{I}$ and $\mathcal{I}^*$.*

2. *If $\mathcal{T}$ is a closed-consistent typically ambiguous Conway theory then $\mathcal{T} \subseteq \mathcal{I}$.*

3. *If $\mathcal{T}$ is a consistent typically ambiguous Conway theory then $\mathcal{T} \subseteq \mathcal{I}^*$.*

The proof is outlined in the next section.

The theorem characterises $\mathcal{I}^*$ as the greatest element in the partially ordered set of consistent typically ambiguous Conway theories (ordered by inclusion). Also, when Σ is nonempty, $\mathcal{I}$ is the maximum closed-consistent typically ambiguous Conway theory. (If Σ is empty then $\mathcal{I}$ is not closed-consistent, indeed $\mathcal{I} = \mathcal{I}^*$.) For countable Σ, there are $2^{\aleph_0}$ typically ambiguous Conway theories (this follows from the analysis of the group identities in [9]).

Theorem 1 follows easily from Theorem 2. First observe that for any $\mathcal{D}$ and $(\cdot)^\dagger$ (natural in X), the theory $\mathcal{T}_{\mathcal{D}}$ is always typically ambiguous (because any $(\cdot)^*$ and θ determine a mapping between interpretations). Also $\mathcal{T}_{\mathcal{D}}$ is consistent if and only if $\mathcal{D}$ is non-trivial (i.e. not equivalent to the terminal category); and it is closed-consistent if and only if Σ is nonempty and there exists a hom-set $\mathcal{D}(\mathbf{1}, X)$ with at least two distinct elements. With these observations, Theorem 1 is immediate.

4. Proof of Theorem 2

The first part of the proof follows the standard proof of the completeness of the free iteration theory relative to an interpretation in regular trees, see, in particular, Sections 5.4 and 6.4–5 of [3] (see also the recent [17]). We outline the main steps, because the associated notion of normal form and its properties are needed for Lemma 4.3.

The notion of normal form is defined for terms $s(x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}) : \sigma$ (with only free variables of base type), by induction on the structure of σ. For base types β, the normal forms are terms $\pi_1(\mu\langle y_1^{\beta_1}, \ldots, y_m^{\beta_m}\rangle.\langle u_1, \ldots, u_m\rangle)$ where β_1 is β such that, for each i with $1 \leq i \leq m$, one of three possibilities holds for u_i: (i) u_i is y_i (note it cannot be y_j for $j \neq i$); (ii) u_i is a free variable from $x_1, \ldots, x_k$; (iii) u_i is of the form $f_i(y_{p_i(1)}, \ldots, y_{p_i(a_i)})$ for some function symbol f_i (with arity a_i) and function $p_i : \{1, \ldots, a_i\} \to \{1, \ldots, m\}$. For product types $\sigma_1 \times \ldots \times \sigma_n$ the normal forms are terms $\langle s_1, \ldots, s_n\rangle$ where each s_i is in normal form. The restriction to free variables of base type is just a syntactic convenience to avoid considering additional subterms (such as projections on variables of product type).

Lemma 4.1 *For any term $t(x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}) : \sigma$, there exists a term s in normal form such that $\mathcal{F} \vdash t = s : \sigma$.*

We next define two interpretations of normal forms of base type as (regular) trees. Consider a term $u(x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}) : \beta_1 \times \ldots \times \beta_m$ of the form

[2] Note that the existence of $(\cdot)^\dagger$ implies that $\mathcal{D}(\mathbf{1}, X)$ is always nonempty.

$\mu\langle y_1, \ldots, y_m\rangle.\, \langle u_1, \ldots, u_m\rangle$ subject to the restrictions imposed in the definition of normal form. For mnemonic benefit, we henceforth use y_i (where $1 \leq i \leq m$) to represent the term $\pi_i(u)$. To each such term y_i we assign trees, $[\![y_i]\!]$ and $[\![y_i]\!]^*$, whose nodes are labelled by elements from the set $\Sigma^+ = \Sigma \cup \{\perp^{\beta_1}, \ldots, \perp^{\beta_m}\} \cup \{x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}\}$.

Formally, such a tree, $\mathbf{t}$, is a partial function from $\mathbb{N}^*$ (finite sequences of natural numbers) to Σ^+ whose domain of definition is nonempty, closed under prefixes and satisfies: for any $w \in \mathbb{N}^*$, $\mathbf{t}(wn)$ is defined if and only if $\mathbf{t}(w) \in \Sigma$, $1 \leq n \leq \mathit{arity}(\mathbf{t}(w))$ and the result type of $\mathbf{t}(wn)$ is the n-th argument type of $\mathbf{t}(w)$. Given such a tree, $\mathbf{t}$, and an element $w \in \mathbb{N}^*$ such that $\mathbf{t}(w)$ is defined, we write $\mathbf{t}@w$ for the subtree $w' \mapsto \mathbf{t}(ww')$. We say that $\mathbf{t}$ is *regular* if the set of all its subtrees, $\{\mathbf{t}@w \mid \mathbf{t}(w) \text{ is defined}\}$, is finite.

To define $[\![y_i]\!]$, we first define a partial function ρ_i from $\mathbb{N}^*$ to $\{1, \ldots, m\}$. On the empty sequence, ϵ, define $\rho_i(\epsilon) = i$. On a nonempty sequence wn, $\rho_i(wn)$ is defined iff $\rho_i(w)$ is defined, $u_{\rho_i(w)}$ is $f_j(y_{p_j(1)}, \ldots, y_{p_j(a_j)})$ and $1 \leq n \leq a_j$; in which case $\rho_i(wn) = p_j(n)$. Finally, we define a function $\lambda : \{1, \ldots, m\} \to \Sigma^+$ by: if u_i is y_i then $\lambda(i) = \perp^{\beta_i}$; if u_i is a free variable x then $\lambda(i) = x$; and if u_i is $f_i(y_{p_i(1)}, \ldots, y_{p_i(a_i)})$ then $\lambda(i) = f_i$. The desired tree $[\![y_i]\!]$ is defined as the composition $[\![y_i]\!] = \lambda \circ \rho_i$.

We define $[\![y_i]\!]^*$ from $[\![y_i]\!]$ as follows. Say that w is *open* in $[\![y_i]\!]$ if there exists w' such that $[\![y_i]\!](ww') \in \{x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}\}$. Say that w is *closed* in $[\![y_i]\!]$ if $[\![y_i]\!](w)$ is defined and w is not open. Say that w is *minimally closed* if it is closed, but no proper prefix is. Then $[\![y_i]\!]^*(w)$ is defined if and only if w is either open in $[\![y_i]\!]$ or minimally closed. In the case that w is open, define $[\![y_i]\!]^*(w) = [\![y_i]\!](w)$. In the case that w is minimally closed, define $[\![y_i]\!]^*(w) = \perp^{\beta_{\rho_i(w)}}$.

Given a normal form of base type, $s(x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}) : \beta$, we have that $s \equiv \pi_1(u)$ for some u of the form above. Accordingly, we have trees $[\![s]\!] = [\![y_1]\!]$ and $[\![s]\!]^* = [\![y_1]\!]^*$ associated to s itself.

Lemma 4.2 *For normal forms $s, s'(x_1{}^{\alpha_1}, \ldots, x_k{}^{\alpha_k}) : \beta$:*

1. *If $[\![s]\!] = [\![s']\!]$ then $\mathcal{I} \vdash s = s'$.*
2. *If $[\![s]\!]^* = [\![s']\!]^*$ then $\mathcal{I}^* \vdash s = s'$.*

The lemma is proved by defining, for every regular tree $\mathbf{t}$, a canonical normal form $t_{\mathbf{t}}$. One then proves that, for every normal form s, $\mathcal{I} \vdash s = t_{[\![s]\!]}$ and $\mathcal{I}^* \vdash s = t_{[\![s]\!]^*}$. This requires various consequences of amalgamation. Similar arguments can be found in Sections 5.4 and 6.4–5 of [3] and also in [17]. The lemma follows.

The argument thus far has established the known completeness of the iteration theory axioms relative to regular trees. To prove Theorem 2, we show that distinct regular trees can never be identified in a typically ambiguous way without losing (closed) consistency. The proof adapts the "Böhm-out" method from the λ-calculus [1].

Lemma 4.3 *Suppose $\mathcal{T}$ is a typically-ambiguous Conway Theory, $s, s'(x_1{}^{\alpha_1}, \ldots, x_k{}^{\alpha_k}) : \beta$ are normal forms and $\mathcal{T} \vdash s = s'$.*

1. *If $\mathcal{T}$ is closed-consistent then $[\![s]\!] = [\![s']\!]$.*
2. *If $\mathcal{T}$ is consistent then $[\![s]\!]^* = [\![s']\!]^*$.*

PROOF For statement 1, suppose $[\![s]\!] \neq [\![s']\!]$. Let $r : \sigma$ be any closed term. We show that $\mathcal{T} \vdash r = \mu y^\sigma.\, y$. Thus $\mathcal{T}$ is not closed-consistent.

Let $w \in \mathbb{N}^*$ be a sequence of minimum length such that $[\![s]\!](w) \neq [\![s']\!](w)$ (both $[\![s]\!](w)$ and $[\![s']\!](w)$ are therefore defined). Without loss of generality, we assume that $[\![s]\!](w) \neq \perp$. We shall define a suitable $(\cdot)^*$ and θ (as in the definition of typical ambiguity) allowing us to bring the disagreement between $[\![s]\!]$ and $[\![s']\!]$ up to the top level.

We write w as $n_1 \ldots n_d$ (where $d \geq 0$), and w_i for its prefix $n_1 \ldots n_i$ (where $0 \leq i \leq d$). For each base type α, define $\alpha^* = \sigma^{d+1}$, where σ is the type of the closed term r. For each function symbol f, of arity $(\alpha_1, \ldots, \alpha_n; \beta')$, we define $f\theta(z_1{}^{\sigma^{d+1}}, \ldots, z_n{}^{\sigma^{d+1}}) : \sigma^{d+1}$ by $f\theta = \langle u_1^f, \ldots, u_{d+1}^f\rangle$ where:

$$u_i^f = \begin{cases} \pi_{i+1}(z_{n_i}) & \text{if } 1 \leq i \leq d \text{ and } [\![s]\!](w_{i-1}) = f \\ r & \text{if } i = d+1 \text{ and } [\![s]\!](w) = f \\ \mu y^\sigma.\, y & \text{otherwise.} \end{cases}$$

Then, as in the definition of typical ambiguity, θ determines terms $s\theta, s'\theta(x_1{}^{\sigma^{d+1}}, \ldots, x_k{}^{\sigma^{d+1}})$ of type σ^{d+1}. By typical ambiguity, $\mathcal{T} \vdash s\theta = s'\theta$. Define terms $t_1, \ldots, t_k : \sigma^{d+1}$ by: t_i is $\langle \mu y^\sigma.\, y, \ldots, \mu y^\sigma.\, y\rangle$ if $[\![s]\!](w) \neq x_i$, and t_i is $\langle r, \ldots, r\rangle$ if $[\![s]\!](w) = x_i$ (only the last component of these tuples is important). We write $[\vec{t}/\vec{x}]$ for the substitution $[t_1, \ldots, t_k/x_1{}^{\sigma^{d+1}}, \ldots, x_k{}^{\sigma^{d+1}}]$. By the substitution property of theories, $\mathcal{T} \vdash s\theta[\vec{t}/\vec{x}] = s'\theta[\vec{t}/\vec{x}]$. However, we prove below that $\mathcal{F} \vdash \pi_1(s\theta[\vec{t}/\vec{x}]) = r$ and $\mathcal{F} \vdash \pi_1(s'\theta[\vec{t}/\vec{x}]) = \mu y^\sigma.\, y$. Thus indeed $\mathcal{T} \vdash r = \mu y^\sigma.\, y$.

For the proof that $\mathcal{F} \vdash \pi_1(s\theta[\vec{t}/\vec{x}]) = r$, suppose s is $\pi_1(u)$ where u is a term of the form $\mu\langle y_1, \ldots, y_m\rangle.\, \langle u_1, \ldots, u_m\rangle$. As earlier, we write y_i for the term $\pi_i(u)$. We write ρ for the partial function ρ_1 from $\mathbb{N}^*$ to $\{1, \ldots, m\}$ used in the definition of $[\![s]\!]$. By working from $i = d+1$ down to $i = 1$, one calculates that, for any i with $1 \leq i \leq d+1$, it holds that

$$\mathcal{F} \vdash \pi_i(y_{\rho(w_{i-1})}\theta[\vec{t}/\vec{x}]) = r.$$

The desired equality is just the special case $i = 1$.

The proof that $\mathcal{F} \vdash \pi_1(s'\theta[\vec{t}/\vec{x}]) = \mu y^\sigma.\, y$ is very similar, again proving an equality for $i = d+1$ which propagates down to the desired case for $i = 1$, using the normal form expansion of s'. The crucial fact required in the argument is that $[\![s']\!](w_{i-1}) = [\![s]\!](w_{i-1})$ if and only if $i \leq d$, which follows from the choice of w.

For statement 2, suppose $[\![s]\!]^* \neq [\![s']\!]^*$. Let x^σ be any variable. We show that $\mathcal{T} \vdash x^\sigma = \mu y^\sigma . y$. Thus $\mathcal{T}$ is not consistent.

Let $w \in \mathbb{N}^*$ be a sequence such that: (i) $[\![s]\!]^*(w) \in \{x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}\}$; and (ii) $[\![s']\!]^*(w)$ is undefined or $[\![s']\!]^*(w) \neq [\![s]\!]^*(w)$. (Such a sequence w can always be found, by swapping s and s' if necessary.)

As before, write w as $n_1 \ldots n_d$ (where $d \geq 0$), and w_i for $n_1 \ldots n_i$ (where $0 \leq i \leq d$). For each base type α, define $\alpha^* = \sigma^{d+1}$. For each function symbol f, of arity $(\alpha_1, \ldots, \alpha_n; \beta')$, we define $f\theta({z_1}^{\sigma^{d+1}}, \ldots, {z_n}^{\sigma^{d+1}}) : \sigma^{d+1}$ by $f\theta = \langle u_1^f, \ldots, u_{d+1}^f \rangle$ where:

$$u_i^f \;=\; \begin{cases} \pi_{i+1}(z_{n_i}) & \text{if } [\![s]\!]^*(w_{i-1}) = f \\ \mu y^\sigma . y & \text{otherwise.} \end{cases}$$

θ determines terms $s\theta, s'\theta({x_1}^{\sigma^{d+1}}, \ldots, {x_k}^{\sigma^{d+1}}) : \sigma^{d+1}$. By typical ambiguity, $\mathcal{T} \vdash s\theta = s'\theta$. Define $t_1, \ldots, t_k : \sigma^{d+1}$ by: t_i is $\langle \mu y^\sigma . y, \ldots, \mu y^\sigma . y \rangle$ if $[\![s]\!]^*(w) \neq x_i$, and t_i is $\langle \mu y^\sigma . y, \ldots, \mu y^\sigma . y, x^\sigma \rangle$ if $[\![s]\!]^*(w) = x_i$. By substitution, $\mathcal{T} \vdash s\theta[\vec{t}/\vec{x}] = s'\theta[\vec{t}/\vec{x}]$. Then, much as above, $\mathcal{F} \vdash \pi_1(s\theta[\vec{t}/\vec{x}]) = x^\sigma$ and $\mathcal{F} \vdash \pi_1(s'\theta[\vec{t}/\vec{x}]) = \mu y^\sigma . y$. Thus $\mathcal{T} \vdash x^\sigma = \mu y^\sigma . y$ as required. $\square$

We now complete the proof of Theorem 2. We have already seen that $\mathcal{I}$ and $\mathcal{I}^*$ are typically ambiguous Conway theories. Consistency can be shown easily by giving a non-trivial semantics. To show that $\mathcal{I}$ contains any closed-consistent typically-ambiguous Conway theory, let $\mathcal{T}$ be any such theory. We must show that $\mathcal{T} \vdash t = t'$ implies $\mathcal{I} \vdash t = t'$. As $\mathcal{T}$ is determined by its equations between terms whose only free variables are of base type, it suffices to show the implication for such terms t, t'. Suppose then that $\mathcal{T} \vdash t = t' : \sigma$. By Lemma 4.1, there exist normal forms s, s' such that $\mathcal{F} \vdash t = s$ and $\mathcal{F} \vdash t' = s'$, so also $\mathcal{T} \vdash s = s'$. One shows, by induction on σ, that $\mathcal{I} \vdash s = s'$ hence $\mathcal{I} \vdash t = t'$. If σ is a base type then this follows from Lemmas 4.3 and 4.2. For product types, the induction step is easy. The maximality of $\mathcal{I}^*$ amongst consistent typically-ambiguous Conway theories is established similarly.

5. Initial and bifree algebras

In the remainder of the paper, we show that parametrically uniform Conway operators arise from universal properties in axiomatic approaches to semantics. A principal tool we use is the notion of *bifree algebra*, embodying the fundamental universal property introduced by Freyd in his work on algebraic compactness [13], which combines the properties of initial algebras and final coalgebras. In this section, we briefly review the relevant concepts.

Given an endofunctor F on a category $\mathcal{C}$, an *F-algebra* is a morphism $a : FA \longrightarrow A$. An *F-algebra homomorphism* from $a : FA \longrightarrow A$ to $b : FB \longrightarrow B$ is a morphism $f : A \longrightarrow B$ such that $f \circ a = b \circ Ff$. An *initial F-algebra* is an initial object in the category of F-algebras and homomorphisms.

The results below summarise some properties of initial algebras. Propositions 5.3 and 5.4 appear to be new.

Proposition 5.1 (Lambek) *If $a : FA \longrightarrow A$ is an initial F-algebra then a is an isomorphism.*

Proposition 5.2 (Freyd [13]) *If F^2 has an initial algebra then F has an initial algebra, $a : FA \longrightarrow A$ say, and $a \circ Fa$ is an initial F^2-algebra. Conversely, if $\mathcal{C}$ has binary products and F has an initial algebra then so does F^2.*

Proposition 5.3 *For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ between any two categories $\mathcal{C}$ and $\mathcal{D}$, if $a : GFA \longrightarrow A$ is an initial GF-algebra in $\mathcal{C}$ then Fa is an initial FG-algebra in $\mathcal{D}$.*

Proposition 5.4 *Suppose idempotents in $\mathcal{C}$ split, F is a retract of G in the category of endofunctors on $\mathcal{C}$, and G has an initial algebra $b : GB \longrightarrow B$. Then F also has an initial algebra $a : FA \longrightarrow A$ where A is a retract of B.*

An *F-coalgebra* is a morphism $a' : A \longrightarrow FA$, and a *final* F-coalgebra is a terminal object in the evident category of F-coalgebras and homomorphisms. A *bifree* algebra for F is a morphism $a : FA \longrightarrow A$ such that a is an initial F-algebra and a^{-1} is a final F-coalgebra (a is an isomorphism by Lambek's result above). All the propositions above have evident analogues applying to final coalgebras and bifree algebras.

One word of warning, in addition to algebras and coalgebras for endofunctors (as discussed above), we shall also make considerable use of algebras for monads and coalgebras for comonads. To avoid ambiguity, we shall always refer to such (co)algebras as (co)monad (co)algebras.

6. Fixed points from bifree algebras

In this section we show how parametrically uniform Conway operators arise in axiomatic domain theory. We work in a very general setting in which the category $\mathcal{D}$ of "domains" arises as the co-Kleisli category of a comonad on the category $\mathcal{S}$ of "strict maps".

Suppose then that $\mathcal{S}$ is a category with finite products, and (L, ε, δ) is a comonad on $\mathcal{S}$. We write $\mathcal{D}$ for the co-Kleisli category of the comonad, $J : \mathcal{S} \to \mathcal{D}$ for the functor which is the right-adjoint part of the pair of adjoint functors determined by (and determining) the comonad, and $K : \mathcal{D} \to \mathcal{S}$ for the left-adjoint part. Then J is the identity on objects and, as it is a right adjoint, $\mathcal{D}$ has finite products and J preserves them. Thus we are in a situation to apply the concepts introduced in Section 2. As there, we use the symbol $\multimap$ for morphisms in $\mathcal{S}$. We shall find conditions on $\mathcal{S}$ and L such that $\mathcal{D}$ has a parametrically uniform Conway operator, indeed a unique one.

We begin with a fundamental proposition relating bifree algebras and fixed-point operators.

Proposition 6.1 *If $L : \mathcal{S} \to \mathcal{S}$ has a bifree algebra then $\mathcal{D}$ has a unique uniform fixed-point operator. If L^2 has a bifree algebra then this fixed-point operator is dinatural.*

Although a similar result is proved in [13], here, in stating the proposition with respect to an arbitrary comonad on $\mathcal{S}$, we are placing the result in what we believe to be its natural general setting. The possibilities this provides are thoroughly exploited in the proofs of Proposition 6.5 and Theorem 3 below. That said, the proof of Proposition 6.1, outlined below, is essentially due to Freyd [13].

Let $s : L\Phi \multimap \Phi$ be the bifree algebra for L on $\mathcal{S}$. The L-algebra $s : L\Phi \multimap \Phi$ corresponds to an endomorphism $s : \Phi \longrightarrow \Phi$ in $\mathcal{D}$. The next two lemmas give the critical properties of s in $\mathcal{D}$.

Lemma 6.2 *For any $f : A \longrightarrow A$ there exists a unique $u_f : \Phi \multimap A$ such that $f \circ Ju_f = Ju_f \circ s$.*

Lemma 6.3 *There exists a unique map $\infty : \mathbf{1} \longrightarrow \Phi$ such that $s \circ \infty = \infty$.*

The proof that $\mathcal{D}$ has a unique uniform fixed-point operator follows swiftly from Lemmas 6.2 and 6.3. The argument is carried out directly in $\mathcal{D}$. Define $(\cdot)^* : \mathcal{D}(A, A) \to \mathcal{D}(\mathbf{1}, A)$ by $f^* = Ju_f \circ \infty$. Then $(\cdot)^*$ is a fixed-point operator by: $f \circ f^* = f \circ Ju_f \circ \infty = Ju_f \circ s \circ \infty = Ju_f \circ \infty = f^*$. For uniformity, suppose we have h, g with $Jh \circ f = g \circ Jh$, as in Definition 2.7 Then $J(h \circ u_f) \circ s = Jh \circ f \circ Ju_f = g \circ J(h \circ u_f)$, so $h \circ u_f = u_g$ (by the uniqueness of u_g). Thus indeed $Jh \circ f^* = Jh \circ Ju_f \circ \infty = Ju_g \circ \infty = g^*$. For uniqueness, suppose $(\cdot)^{*'}$ is any fixed-point operator. Then $s^{*'} = \infty$ by the uniqueness part of Lemma 6.3. If, further, $(\cdot)^{*'}$ is uniform then, because $f \circ Ju_f = Ju_f \circ s$, we have $f^{*'} = Ju_f \circ s^{*'} = Ju_f \circ \infty = f^*$.

It remains to prove the dinaturality claim of Proposition 6.1. Suppose that L^2 has a bifree algebra. (By Proposition 5.2, this itself implies that L has a bifree algebra.) Again, the argument follows Freyd [13].

Lemma 6.4 *For any $f : A \longrightarrow A$ in $\mathcal{D}$, it holds that $f^* = (f \circ f)^*$.*

The lemma is first proved for $s : \Phi \longrightarrow \Phi$, using the fact that $s \circ Ls : L^2\Phi \multimap \Phi$ is a bifree L^2-algebra, as given by Proposition 5.2. It follows for arbitrary f by uniformity.

Dinaturality is an easy consequence of the lemma. Given $f : A \longrightarrow B$ and $g : B \longrightarrow A$, consider the map $q = c \circ (f \times g) : A \times B \longrightarrow A \times B$, where c is the symmetry map for product. Then $q^* = (q \circ q)^* = \langle (g \circ f)^*, (f \circ g)^* \rangle$, where the last equality is obtained by uniformity. It follows that $q \circ \langle (g \circ f)^*, (f \circ g)^* \rangle = \langle (g \circ f)^*, (f \circ g)^* \rangle$, which implies the desired equality.

In the remainder of the section, we obtain conditions that imply that $\mathcal{D}$ has a parametrically uniform Conway operator. The main strategy is to instantiate Proposition 6.1 by varying the comonad. This will allow us to derive all the properties of Conway operators by assuming the existence of enough bifree algebras. The first example of such an application is to obtain a parameterized fixed-point operator.

Proposition 6.5 *If every endofunctor $L(X \times (-))$ on $\mathcal{S}$ has a bifree algebra then $\mathcal{D}$ has a unique parametrically uniform parameterized fixed-point operator.*

The proof is by instantiating Proposition 6.1 using the endofunctors $L(X \times (-))$ as comonads. The comonad structure is most easily seen by considering the composite functor $\mathcal{S} \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}_{X \times (-)}$, which has a left adjoint, thereby exhibiting $\mathcal{D}_{X \times (-)}$ as the co-Kleisli category of the composite comonad on $\mathcal{S}$. Indeed, by the remarks after Definitions 2.2 and 2.8, the parameterized fixed-point property, parameterized uniformity, and uniqueness all follow directly from Proposition 6.1 when interpreted in the appropriate co-Kleisli categories. It only remains to verify the naturality of the induced $(\cdot)^\dagger$ operator in its parameter.

A *comonad morphism* $\nu : (L_1, \varepsilon_1, \delta_1) \Rightarrow (L_2, \varepsilon_2, \delta_2)$, between two comonads on $\mathcal{S}$, is a natural transformation $\nu : L_1 \Rightarrow L_2$ that preserves the counit and comultiplication in the evident way (see [2, §3.6] for the dual notion of monad morphism). Any such comonad morphism induces a functor $H : \mathcal{D}_2 \to \mathcal{D}_1$ between the associated co-Kleisli categories (it is the identity on objects, and a morphism from A to B in $\mathcal{D}_2$, given by $f : L_2A \multimap B$ in $\mathcal{S}$, gets mapped to the morphism from A to B in $\mathcal{D}_1$ given by $f \circ \nu_A : L_1A \multimap B$ in $\mathcal{S}$).

Lemma 6.6 *Suppose $\nu : (L_1, \varepsilon_1, \delta_1) \Rightarrow (L_2, \varepsilon_2, \delta_2)$ is a comonad morphism, where both L_1 and L_2 have bifree algebras in $\mathcal{S}$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the associated co-Kleisli categories, let $H : \mathcal{D}_2 \to \mathcal{D}_1$ be the ν-induced functor, and let $(\cdot)^{*_1}$ and $(\cdot)^{*_2}$ be the unique uniform fixed-point operators in $\mathcal{D}_1$ and $\mathcal{D}_2$. Then, for any $f : A \longrightarrow A$ in $\mathcal{D}_2$, it holds that $H(f^{*_2}) = (Hf)^{*_1}$.*

The proof, which uses the construction of the uniform fixed-point operators given after Lemmas 6.2 and 6.3, is routine.

To derive the naturality of $(\cdot)^\dagger$ from lemma 6.6, consider any $g : X \longrightarrow Y$ in $\mathcal{D}$ (the co-Kleisli category of L) as in Definition 2.2. Then $(g \times (-)) : X \times (-) \Rightarrow Y \times (-)$ is a comonad morphism between comonads on $\mathcal{D}$, and $K(g \times J(-)) : L(X \times (-)) \Rightarrow L(Y \times (-))$ is a corresponding comonad morphism, between comonads on $\mathcal{S}$, that induces the functor $H_g : \mathcal{D}_{Y\times(-)} \to \mathcal{D}_{X\times(-)}$ between its co-Kleisli categories. Thus, by the discussion after Definition 2.2, Lemma 6.6 does indeed imply naturality.

In order to obtain that the unique parametrically uniform $(\cdot)^\dagger$ is also a Conway operator, we require yet more bifree algebras. We say that $\mathcal{S}$ has *sufficiently many bifree algebras* if all endofunctors $L(X \times L(X \times (-)))$ and $L(X \times (-) \times (-))$ on $\mathcal{S}$ have bifree algebras.

Theorem 3 *If $\mathcal{S}$ has sufficiently many bifree algebras then $\mathcal{D}$ has a unique parametrically uniform parameterized fixed-point operator, and it is a Conway operator.*

To prove the theorem, suppose that $\mathcal{S}$ has sufficiently many bifree algebras. By Proposition 5.2, all endofunctors $L(X \times (-))$ on $\mathcal{S}$ have bifree algebras. So by Proposition 6.5, $\mathcal{D}$ has a unique parametrically uniform parameterized fixed-point operator. Moreover, parameterized dinaturality follows from ordinary dinaturality given by Proposition 6.1, when the comonad is instantiated to $L(X \times (-))$.

It remains to prove the diagonal property. To this end, observe that on any category $\mathcal{C}$ with finite products, the endofunctor $(-) \times (-)$ can be endowed with the structure of a comonad; in fact this can be done in two inequivalent ways. In both cases the counit is $\pi_1 : A \times A \longrightarrow A$. The two possible comultiplications are $\langle \pi_1, \pi_2, \pi_2, \pi_1 \rangle$ and $\langle \pi_1, \pi_2, \pi_2, \pi_2 \rangle : A \times A \longrightarrow A \times A \times A \times A$. (In the first case the coalgebras of the comonad are involutions in $\mathcal{C}$, in the second case the coalgebras are idempotents.) Strangely, the proof below works equally well with either choice of comultiplication.

Consider the comonad $(-) \times (-)$ on the category $\mathcal{D}_{X\times(-)}$, and write $\mathcal{D}_{d_X}$ for its co-Kleisli category. The chain of right adjoints $\mathcal{S} \to \mathcal{D} \to \mathcal{D}_{X\times(-)} \to \mathcal{D}_{d_X}$ shows that $\mathcal{D}_{d_X}$ arises as the co-Kleisli category of a comonad with underlying functor $L(X \times (-) \times (-))$ on $\mathcal{S}$. Because $L(X \times (-) \times (-))$ has a bifree algebra, Proposition 6.1 relativizes to give that $\mathcal{D}_{d_X}$ has a unique uniform fixed-point operator. In terms of $\mathcal{D}$ this says that there exists a unique family $(\cdot)^\ddagger : \mathcal{D}(X \times A \times A, A) \to \mathcal{D}(X, A)$ satisfying:

1. For any $f : X \times A \times A \longrightarrow A$, $f \circ \langle \mathsf{id}_X, f^\ddagger, f^\ddagger \rangle = f^\ddagger : X \longrightarrow A$.
2. For any $f : X \times A \times A \longrightarrow A$, $g : X \times B \times B \longrightarrow B$ and $h : A \multimap B$, $h \circ f = g \circ (\mathsf{id}_X \times Jh \times Jh)$ implies $g^\ddagger = Jh \circ f^\ddagger$.

The diagonal property is proved by showing that the two operations, mapping $f : X \times A \times A \longrightarrow A$ to $(f^\dagger)^\dagger$ and $(f \circ (\mathsf{id}_X \times \Delta))^\dagger : X \longrightarrow A$ respectively (defined using the parametrically uniform parameterized fixed-point operator on $\mathcal{D}$), both satisfy the characterising properties of $(\cdot)^\ddagger$ above. Therefore the two operations are equal, hence $(f \circ (\mathsf{id}_X \times \Delta))^\dagger = (f^\dagger)^\dagger$.

We end this section by observing that it is a simple application of Proposition 5.3 to show that the requirement of the existence of sufficiently many bifree algebras in $\mathcal{S}$ is equivalent to requiring the existence of bifree algebras in $\mathcal{D}$. We write $T : \mathcal{D} \to \mathcal{D}$ for the induced monad (given by the composite JK) on $\mathcal{D}$.

Proposition 6.7 *$\mathcal{S}$ has sufficiently many bifree algebras if and only if all endofunctors of the form $T(X \times T(X \times (-)))$ and $T(X \times (-) \times (-))$ on $\mathcal{D}$ have bifree algebras.*

In spite of the above reformulation, we believe that it is usually more appropriate to consider the bifree algebras as living in $\mathcal{S}$. A common application of the results in this section will involve using a category $\mathcal{S}$ that is *algebraically compact* [13, 14], in which case the existence of sufficiently many bifree algebras in $\mathcal{S}$ is guaranteed. The canonical example of this situation is when $\mathcal{S}$ is $\mathbf{Cppo}_\perp$, which is algebraically compact with respect to $\mathbf{Cppo}$-enriched endofunctors [10]. The results in this section thus apply to the co-Kleisli category of any comonad on $\mathbf{Cppo}_\perp$ whose underlying functor is $\mathbf{Cppo}$-enriched, not just to the lifting comonad. A degenerate case is the identity comonad, showing that it is also possible for $\mathcal{D}$ itself to be algebraically compact (although this implies that $\mathbf{1}$ is a zero object in $\mathcal{D}$).

7. Fixed points and lifting monads

In Section 6, we took a category of "strict" maps as basic, and derived the relevant properties of fixed points in a category of "domains" determined as the co-Kleisli category of a comonad on $\mathcal{S}$. In many examples, however, the category $\mathcal{S}$ is itself obtained as the category of algebras for a "lifting" monad on a category of "predomains". In this section, we investigate such situations in general. Surprisingly, the strong properties of a lifting monad allow all assumptions about the existence of bifree algebras to be dropped. Instead, the mere existence of uniform non-parameterized fixed-points suffices to determine a unique parametrically uniform Conway operator.

Let $\mathcal{C}$ be a category with finite products and a strong monad (T, η, μ, t) (see e.g. [22, 23]). We write $\mathcal{K}$ for the Kleisli category of the monad, and we write $I : \mathcal{C} \to \mathcal{K}$ for the associated (left-adjoint) functor. We assume that $\mathcal{C}$ has *Kleisli exponentials*, i.e. that, for every X in $\mathcal{C}$ the functor $I(X \times (-)) : \mathcal{C} \to \mathcal{K}$ has a right adjoint (see e.g. [22, 28]). These assumptions give the structure required to model Moggi's *computational λ-calculus* [22].

We wish to consider a notion of fixed-point in $\mathcal{C}$ suitable for adding a recursion operator to the computational λ-calculus. Because of the existence of Kleisli exponentials, it suffices to consider a non-parameterized notion.

Definition 7.1 (Uniform T-fixed-pt. op.) A *uniform T-fixed-point operator* is a a family of functions

$(\cdot)^* : \mathcal{C}(TA, TA) \to \mathcal{C}(\mathbf{1}, TA)$ such that:

1. For any $f : TA \longrightarrow TA,\ f \circ f^* = f^*$.

2. For any $f : TA \longrightarrow TA$, $g : TA \longrightarrow TA$ and $h : TA \longrightarrow TB$, if $h \circ \mu = \mu \circ Th$ and $g \circ h = h \circ f$ then $g^* = h \circ f^*$.

One familiar setting in which a (unique) uniform T-fixed-point operator exists is when $\mathcal{C}$ has a *fixpoint object* in the sense of Crole and Pitts [5], see [24, 28]. In this paper, we take the weaker notion of uniform T-fixed-point operator as primitive. However, we shall see circumstances below in which the two notions are equivalent.

In this section, our aim is to show how T-fixed-point operators give rise to fixed-point operators as considered earlier in the paper. To this end, we write $\mathcal{S}$ for the category of algebras of the monad T (the Eilenberg-Moore category) and L for the comonad on $\mathcal{S}$ induced by the adjunction with $\mathcal{C}$. Let $\mathcal{D}$ be the co-Kleisli category of L, and let $J : \mathcal{S} \to \mathcal{D}$ be the induced functor (as in Section 6). Concretely $\mathcal{D}$ can be described as the category whose objects are Eilenberg-Moore algebras for T, with hom-sets: $\mathcal{D}(TA \xrightarrow{a} A, TB \xrightarrow{b} B) = \mathcal{C}(A, B)$.

Proposition 7.2 *There is a one-to-one correspondence between uniform T-fixed-point operators on $\mathcal{C}$ and parametrically uniform parameterized fixed-point operators on $\mathcal{D}$.*

Our aim is to show that, under suitable conditions, there is a unique uniform T-fixed-point operator, and that the unique parametrically uniform parameterized fixed-point operator determined is a Conway operator.

One condition is that $\mathcal{C}$ have a *parameterized natural numbers object* $\mathbf{1} \xrightarrow{0} \mathbf{N} \xrightarrow{s} \mathbf{N}$ (see [19, p. 71, Exercise 4] — this is the appropriate notion of natural numbers object for non-cartesian-closed categories). The other assumption is one on the monad.

Definition 7.3 (Equational lifting monad [4]) A strong monad (T, η, μ, t) is said to be an *equational lifting monad* if it is commutative and also satisfies the equation $t \circ \Delta = T\langle \eta, \mathsf{id}_X \rangle : TX \longrightarrow T(TX \times X)$.

In [4], it is shown that equational lifting monads exactly capture the equational properties of partial map classifiers.

Theorem 4 *Suppose $\mathcal{C}$ has a parameterized natural numbers object and T is an equational lifting monad. Then $\mathcal{C}$ has at most one uniform T-fixed-point operator. Moreover, if such an operator exists then the associated unique parameterized fixed-point operator on $\mathcal{D}$ is a Conway operator.*

The bulk of the work in the proof of Theorem 4 goes into proving the proposition below.

Proposition 7.4 *Under the conditions of Theorem 4, if all idempotents in $\mathcal{C}$ split and $\mathcal{C}$ has a uniform T-fixed-point operator then $\mathcal{S}$ has sufficiently many bifree algebras.*

The proof is outlined in Section 8.

To derive Theorem 4 from Proposition 7.4, one first shows that all the structure on $\mathcal{C}$ (parameterized natural number object, equational lifting monad, Kleisli exponentials) extends to its *Karoubi envelope*, $\mathsf{Split}(\mathcal{C})$, (see [19, p. 100, Exercise 2]). Moreover, if $\mathcal{C}$ has a uniform T-fixed-point operator then so does $\mathsf{Split}(\mathcal{C})$. By definition, all idempotents split in $\mathsf{Split}(\mathcal{C})$. Thus, by Proposition 7.4, $\mathsf{Split}(\mathcal{S})$ has sufficiently many bifree algebras ($\mathsf{Split}(\mathcal{S})$ is indeed the category of algebras for the monad on $\mathsf{Split}(\mathcal{C})$). Hence, by Theorem 3, $\mathsf{Split}(\mathcal{D})$ has a unique parametrically uniform parameterized fixed-point operator, and it is a Conway operator. However, it is easily shown that parametrically uniform parameterized fixed-point operators and Conway operators on $\mathcal{D}$ and on $\mathsf{Split}(\mathcal{D})$ are in one-to-one correspondence. Theorem 4 follows by Proposition 7.2.

One other consequence of Proposition 7.4 is that, by Proposition 5.3, the existence of a bifree L-algebra on $\mathcal{S}$ is equivalent to the existence of a bifree T-algebra on $\mathcal{C}$. Freyd observed that any bifree T-algebra is a *fixpoint object* [5]. We have already mentioned that any fixpoint object determines a uniform T-fixed-point operator. Thus, in the circumstances of Proposition 7.4, the existence of a T-fixed-point operator is equivalent to that of a fixpoint object.

8. Proof of Proposition 7.4

We have a category $\mathcal{C}$, with finite products, parameterized natural numbers object, equational lifting monad and Kleisli exponentials, in which every idempotent splits. One consequence of idempotents splitting is that, for every Y in $\mathcal{C}$, the functor $F(Y \times (-)) : \mathcal{C} \to \mathcal{S}$ (where $F : \mathcal{C} \to \mathcal{S}$ is the standard "free algebra" functor) has a right adjoint $(-)^Y : \mathcal{S} \to \mathcal{C}$. Essentially this means that for any object X of $\mathcal{C}$ that lies in the image of the forgetful $U : \mathcal{S} \to \mathcal{C}$ (we henceforth call such objects *algebra carrying*), and every object Y of $\mathcal{C}$, the exponential X^Y exists in $\mathcal{C}$ (the object X^Y is constructed as a retract of the Kleisli exponential TX^Y). It also implies that $\mathcal{D}$ is cartesian closed.

Lemma 8.1 *For every algebra-carrying X, the endofunctors $X \times (-) : \mathcal{C} \to \mathcal{C}$ and $X \times (-) \times (-) : \mathcal{C} \to \mathcal{C}$ have final coalgebras.*

Lemma 8.2 *For every object X of $\mathcal{S}$, the endofunctors $X \times (-) : \mathcal{S} \to \mathcal{S}$ and $X \times (-) \times (-) : \mathcal{S} \to \mathcal{S}$ have final coalgebras.*

Lemma 8.3 *L is a retract of $(L\mathbf{1}) \times (-)$ in the category of endofunctors on $\mathcal{S}$.*

Lemma 8.4 *All endofunctors $L(X \times L(X \times (-)))$ and $L(X \times (-) \times (-))$ have final coalgebras in $\mathcal{S}$.*

Briefly, the final coalgebras of Lemma 8.1 both have carrier $X^{\mathbf{N}}$, which is used to encode infinite sequences and full binary trees. Lemma 8.2 follows by proving that the forgetful from the category of coalgebras for $X \times (-)$ on $\mathcal{S}$ to the category of coalgebras for $X \times (-)$ on $\mathcal{C}$ is monadic and so creates the terminal object (similarly for $X \times (-) \times (-)$). For Lemma 8.3, the retraction is given by the pair of morphisms $\langle T!, a\rangle : TA \longrightarrow (T\mathbf{1}) \times A$ and $(T\pi_2) \circ t' : (T\mathbf{1}) \times A \longrightarrow TA$ in $\mathcal{C}$ (where $t' : T\mathbf{1} \times A \longrightarrow T(\mathbf{1} \times A)$ is the "costrength"of T). The verification that this pair has the required properties is the only place in which the equational lifting monad equation (Definition 7.3) is used. Finally, Lemma 8.4 follows from Lemmas 8.2, 8.3 and Proposition 5.4.

Now that final coalgebras for the desired functors have been constructed in $\mathcal{S}$, it remains to show that they are bifree. We achieve this by some more manœuvring between categories, using Proposition 5.3 to transfer universal properties from one place to another. In fact, we shall exploit properties of the Kleisli category $\mathcal{K}$.

As T is a commutative strong monad, $\mathcal{K}$ is a symmetric monoidal category and $I : \mathcal{C} \to \mathcal{K}$ is monoidal (where cartesian product is taken as the monoidal product on $\mathcal{C}$). We write $\otimes$ for the monoidal product on $\mathcal{K}$, and L' for the underlying functor of the comonad on $\mathcal{K}$ induced by T.

Proposition 8.5

1. *$\mathcal{K}$ can be construed as a $\mathcal{D}$-enriched category.*
2. *$\otimes$ and L' can be construed as $\mathcal{D}$-functors on $\mathcal{K}$.*
3. *For any $\mathcal{D}$-enriched endofunctor F on $\mathcal{K}$, an isomorphism $a : FA \longrightarrow A$ in $\mathcal{K}$ is an initial F-algebra if and only if a^{-1} is a final F-coalgebra.*

The proof is given in [28]. In outline, 1 is proved using Kleisli exponentials, and 2 is then routine. For 3, the idea is to use the uniform fixed-point operator in $\mathcal{D}$ to establish that the property of being an initial F-algebras is equivalent to a self dual property (called *special F-invariance*), and hence equivalent to the property of being a final F-coalgebra; see Theorem 5.2 of [28].

To finish the proof of Proposition 7.4, consider, for example, the functor $L(X \times L(X \times (-))) : \mathcal{S} \to \mathcal{S}$. We write $K : \mathcal{K} \to \mathcal{S}$ for the "comparison" functor from Kleisli category to Eilenberg-Moore category. We write $H : \mathcal{S} \to \mathcal{S}$ for the composite:

$$\mathcal{S} \xrightarrow{X \times (-)} \mathcal{S} \xrightarrow{U} \mathcal{C} \xrightarrow{I} \mathcal{K}$$

Then $L(X \times L(X \times (-))) = KHKH : \mathcal{S} \to \mathcal{S}$. By Lemma 8.4, $KHKH$ has a final coalgebra in $\mathcal{S}$. Thus, by Proposition 5.3, $HKHK$ has a final coalgebra in $\mathcal{K}$. By Proposition 8.5.2, $HKHK$ is $\mathcal{D}$-enriched. Hence, by Proposition 8.5.3, $\mathcal{K}$ has a bifree $HKHK$-algebra. Thus, again by Proposition 5.3, $\mathcal{S}$ has a bifree $KHKH$-algebra, i.e. there is a bifree algebra for $L(X \times L(X \times (-)))$. The argument for $L(X \times (-) \times (-))$ is similar. This completes the proof of Proposition 7.4.

9. Discussion

Theorem 4 has applications to an axiomatic approach to denotational semantics. The conditions on $\mathcal{C}$ are exactly suited to modelling a call-by-value version, $\mathbf{PCF_v}$, of $\mathbf{PCF}$ with product types (as considered in e.g. [33]). The monad and Kleisli exponentials interpret Moggi's computational λ-calculus [22], which is the core of $\mathbf{PCF_v}$. The natural numbers object is used to interpret the arithmetic operations. Intuitively, the assumption of an equational lifting monad expresses that nontermination is the only computational effect in $\mathbf{PCF_v}$. We suggest that a uniform T-fixed-point operator is the natural structure for interpreting recursion. By Theorem 4, there is at most one such operator, and so the interpretation of recursion is uniquely determined. Moreover, the interpretation of recursion satisfies all desirable equational properties.

An interesting aspect of the proposed notion of model is that all ingredients in the model correspond to syntactic features of the language. Thus the free category with the identified structure corresponds to a term model constructed out of $\mathbf{PCF_v}$ programs quotiented by the equivalence induced by the categorical structure. Then the interpretation of $\mathbf{PCF_v}$ terms in an arbitrary model is given by the unique structure preserving functor from the free model. Thus the denotational semantics of $\mathbf{PCF_v}$ is recast in the framework of Lawvere's functorial semantics.

The categorical structure of the models determines a rudimentary equational logic for proving operational equalities between $\mathbf{PCF_v}$ programs. On the one hand, this logic supports a "denotational" form of reasoning, using categorical universal properties. On the other, by interpreting the equalities in the free model, any argument has a direct "operational" reading as following a chain of equalities between $\mathbf{PCF_v}$ programs. Thus one might argue that the notion of model provides a denotational framework for direct operational reasoning. One wonders how powerful the induced proof principles are.

Another question of power is how far our approach of deriving equational properties of recursion from categorical universal properties can be extended to derive properties of higher-order recursion. A natural syntax for higher-order recursion is given by the simply-typed λ-calculus extended with a typed fixed-point combinator. It can be shown that the desired equational theory between such terms is that induced by a suitable notion of η-expanded typed Böhm trees, that this theory is co-r.e. and satisfies a characterisation as a maximally-consistent typically ambiguous theory (cf. [31] and our Theorem 2). A major open question is whether the theory is decidable. The restricted case of equalities between so-called *recursion schemas* has recently been settled in the positive by the long awaited proof of the decidability of language equivalence for DPDAs [27]. It would be remarkable if the proof rules in Stirling's tableau approach to decidability [32] could be derived from category-theoretic universal properties.

Another interesting (and less ambitious!) direction for research is to investigate the equational theory induced by Hasegawa's notion of *uniform trace* [16], which generalises parametrically uniform Conway operators to symmetric monoidal categories. In particular, Hasegawa considers traced *cartesian-center* monoidal categories as models of *cyclic sharing graphs*. Perhaps there is a completeness theorem for uniform traces with respect to an equational theory induced by suitable unfoldings of such graphs.

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