ON THE SCOPE OF THE UNIVERSAL-ALGEBRAIC APPROACH TO CONSTRAINT SATISFACTION

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ABSTRACT. The universal-algebraic approach has proved a powerful tool in the study of the computational complexity of constraint satisfaction problems (CSPs). This approach has previously been applied to the study of CSPs with finite or (infinite) ω -categorical templates. Our first result is an exact characterization of those CSPs that can be formulated with (a finite or) an ω -categorical template.

The universal-algebraic approach relies on the fact that in finite or ω -categorical structures \mathfrak{A} , a relation is primitive positive definable if and only if it is preserved by the *polymorphisms* of \mathfrak{A} . In this paper, we present results that can be used to study the computational complexity of CSPs with arbitrary infinite templates. Specifically, we prove that every CSP can be formulated with a template \mathfrak{A} such that a relation is primitive positive definable in \mathfrak{A} if and only if it is first-order definable on \mathfrak{A} and preserved by the *infinitary* polymorphisms of \mathfrak{A} .

We present applications of our general results to the description and analysis of the computational complexity of CSPs. In particular, we present a polymorphism-based description of those CSPs that are first-order definable (and therefore can be solved in polynomial-time), and give general hardness criteria based on the absence of polymorphisms that depend on more than one argument.

1. Introduction

For a relational structure \mathfrak{A} over a finite signature the constraint satisfaction problem $\mathrm{CSP}(\mathfrak{A})$ is the computational problem to decide whether a primitive positive first-order sentence φ – that is, the existential quantification of a conjunction of atomic formulas – is true on \mathfrak{A} . The case where the template \mathfrak{A} is finite has been extensively studied in the literature, and is known to comprise a significant microcosm of the complexity class NP (see, e.g., [19]). The universal-algebraic approach, of studying the invariance properties of relations under the action of polymorphisms, has been particularly powerful in the complexity analysis of finite-domain CSPs (see [18] as a starting point). This approach has also been successfully used in infinite-domain CSPs where the template is ω -categorical, i.e., is the unique countably infinite model of its first-order (fo) theory up to isomorphism – see, e.g., [8].

Many interesting problems can be formulated as infinite CSPs whose template is not ω -categorical. To illustrate the wealth of the class of CSPs studied in this paper, we present

four concrete computational problems that can be formulated as $\text{CSP}(\mathfrak{A})$, for an infinite \mathfrak{A} . Each of these problems is solvable in polynomial time – and the proofs of this are generally non-trivial. The templates $(\mathbb{Z}; +, 1)$ and $(\mathbb{R}; +, 1)$, where + is read as the ternary relation x + y = z, correspond to solutions of Linear Diophantine Equations and Linear Real Equations, respectively. Another template of interest relates to the Unification Problem. Let $\sigma := (f_1, f_2, \ldots)$ be a functional signature, we form the template $(T; F_1, F_2, \ldots)$, where T is the term algebra on σ built over a countably infinite set of variables, and each F_i is the relational form $f_i(t_1, \ldots, t_{r_i}) = t_0$ of f_i over T. A final problem is that of 2-Variable Word Equations [14] – $\text{CSP}(\{0,1\}^*; R_1, R_2, \ldots)$ – where the $R_i(x, y)$ are binary relations defined by all equations of the form $\{x, y, 0, 1\}^* = \{x, y, 0, 1\}^*$.

In the case that \mathfrak{A} is finite or ω -categorical, the relations over \mathfrak{A} that are invariant under the polymorphisms of $\mathfrak{A} - \operatorname{Inv}(\operatorname{Pol}(\mathfrak{A}))$ – are precisely the relations that are pp-definable over $\mathfrak{A} - \langle \mathfrak{A} \rangle_{pp}$. We note that this connection, which we paraphrase "Inv-Pol = pp", holds on some infinite structures which are not ω -categorical (an example is given in [26], also the natural numbers under successor may easily be verified to have this property). Two templates give rise to the same CSP precisely when they agree on all pp-sentences, that is share the same pp-theory. It might be the case that one such template is better behaved than another. For example, (\mathbb{Z} ; <) and (\mathbb{Q} ; <) share the same pp-theory; while the connection Inv-Pol=pp does not hold for the former, the latter is ω -categorical, and therefore the connection subsists. In the present paper we give a necessary and sufficient condition that a template may have an equivalent that is finite or ω -categorical, which is that the number of maximal pp-*n*-types consistent with its theory (equivalently, with its pp-theory) is finite, for all *n*. It follows that none of the four examples of the previous paragraph may be formulated with an ω -categorical template.

For the general case, in which there may be no equivalent ω -categorical template, we are able to prove the existence of an equivalent, but uncountable template over which a restricted connection holds. Given any \mathfrak{A} , we prove the existence of a highly saturated "monster" elementary extension \mathfrak{M} such that a relation is pp-definable on \mathfrak{M} iff it is fodefinable on \mathfrak{M} and invariant under the polymorphisms of (countably) infinite arity of \mathfrak{M} . In fact, we also prove this weaker connection, which we may paraphrase "Inv-Pol $^{\omega}$ \cap fo = pp", for all saturated structures of cardinality at least 2^{\u03c0}. The "monster" construction obviates the need for the set-theoretic assumptions usually required to assert the existence of a saturated elementary extension of an arbitrary structure. However, in many concrete cases, such as for structures that are uncountably categorical, such saturated models can be exhibited directly. We go on to prove that each of the three assumptions – high saturation, infinitary (and not finitary) polymorphism and fo-intersection – is necessary. That is, we exhibit structures for which any two of these is insufficient for the respective connection. We note an alternative "global" view of our weak connection states, for any \mathfrak{A} , that a fosentence φ is pp-definable in \mathfrak{A} iff it is preserved by the ω -polymorphisms of all elementary extensions of \mathfrak{A} .

There are several extant works on notions of pp-definability over infinite structures, including those involving infinitary polymorphisms and infinitary relations [23, 25, 29]. Relational operations transcending normal pp-definitions are usually permitted, for example: infinite conjunction, infinite projection and various forms of monotone disjunction. In order for our results to be applicable to the (finite!) instances of CSPs, we are not able to sacrifice anything on the relational side, and so pp-definability must remain in its most basic form. This represents the principle difference between our work and those that have come before.

We note that this is the first time that infinitary polymorphisms have been considered in connection with the complexity of CSPs.

We go on to consider the repercussions of our weak connection for the complexity of CSPs. We show that existential positive (ep-) and pp-definability coincide on a structure \mathfrak{A} iff all ω -polymorphisms of all elementary extensions of \mathfrak{A} are essentially unary. We demonstrate that the move to elementary extension is necessary by giving a structure whose ω -polymorphisms include only projections but for which $(x = y \lor u = v)$ is not primitive positive definable. Using the notion of local refutability in [5], we note that if a structure \mathfrak{A} is not locally refutable, and all ω -polymorphisms of all elementary extensions of \mathfrak{A} are essentially unary, then $CSP(\mathfrak{A})$ is NP-hard. Introducing our philosophy to the work of [21], we present a polymorphism-based description of those CSPs that are first-order definable. We show that $CSP(\mathfrak{A})$ is first-order definable if and only if \mathfrak{A} has an elementary extension which has a type near unanimity-polymorphism. It follows that such CSPs are polynomialtime solvable. Finally, we recall a known relationship between certain binary injective polymorphisms and Horn definability (given in the context of ω -categorical structures in [3]). Considering as a polymorphism an embedding e of $(\mathbb{R}; +, 1)^2$ into $(\mathbb{R}; +, 1)$, we show that the recent complexity classification of [6] may be given a natural algebraic specification. Assuming $P \neq NP$, the presence of the polymorphism e separates those fo-expansions of $(\mathbb{R}; +, 1)$ whose CSP is in P from those whose CSP is NP-complete. Thus we demonstrate that the presence of certain polymorphisms can delineate complexity even outside of the realm of ω -categoricity.

For reasons of space, the majority of proofs are deferred to the appendix.

2. Preliminaries

2.1. Models, operations, theories and closure

A relational signature (with constants) τ is a set of relation symbols R_i , each of which has an associated finite arity k_i , and a set of constants c_i . We consider only countable, relational signatures (with constants) in this paper. A (relational) structure \mathfrak{A} over the signature τ (also called τ -structure) consists of a set A (the domain) together with a relation $R^{\mathfrak{A}} \subseteq A^k$ for each relation symbol R of arity k from τ and a constant $c^{\mathfrak{A}} \in A$ for each constant symbol c.

Let \mathfrak{A} be a τ -structure, and let \mathfrak{A}' be a τ' -structure with $\tau \subseteq \tau'$. If \mathfrak{A} and \mathfrak{A}' have the same domain and $R^{\mathfrak{A}} = R^{\mathfrak{A}'}$ for all $R \in \tau$, then \mathfrak{A} is called the τ -reduct (or simply reduct) of \mathfrak{A}' , and \mathfrak{A}' is called a τ' -expansion (or simply expansion) of \mathfrak{A} . If \mathfrak{A} is a τ -structure and $\langle a_{\alpha} \rangle_{\alpha < \beta}$ is a sequence of elements of A, then $(\mathfrak{A}; \langle a_{\alpha} \rangle_{\alpha < \beta})$ is the natural $\tau \cup \{c_{\alpha} : \alpha < \beta\}$ expansion of \mathfrak{A} with β new constants, where c_{α} is interpreted by a_{α} , in the natural way. \mathfrak{A} is an extension of \mathfrak{B} if $B \subseteq A$. Let $\langle b_{\alpha} \rangle_{\alpha < |B|}$ well-order the elements of \mathfrak{B} . \mathfrak{A} is an elementary extension of \mathfrak{B} , denoted $\mathfrak{B} \preceq \mathfrak{A}$, if it is an extension and, for each first-order $\tau \cup \{c_{\alpha} : \alpha < |B|\}$ -sentence φ , $(\mathfrak{B}, \langle b_{\alpha} \rangle_{\alpha < |B|}) \models \varphi$ iff $(\mathfrak{A}, \langle b_{\alpha} \rangle_{\alpha < |B|}) \models \varphi$.

A first-order (fo) formula is existential positive (ep) if it involves no instances of universal quantification or negation. Furthermore, if it involves no instances of disjunction, then it is termed primitive positive (pp). Suppose \mathfrak{A} is a finite structure over a finite signature with domain $A := \{a_1, \ldots, a_s\}$, and let $\overline{a} := (a_1, \ldots, a_r)$ be a tuple of distinct elements corresponding to the subset $\{a_1, \ldots, a_r\}$ of that domain. Let $\theta(x_1, \ldots, x_s)$ be the conjunction of the positive facts of \mathfrak{A} , where the variables x_1, \ldots, x_s correspond to the elements a_1, \ldots, a_s . That is, $R(x_{\lambda_1}, \ldots, x_{\lambda_k})$ appears as an atom in θ iff $(a_{\lambda_1}, \ldots, a_{\lambda_k}) \in \mathbb{R}^{\mathfrak{A}}$. Define the ppformula $\varphi[\mathfrak{A}|\overline{a}]$ to be $\exists x_{r+1} \ldots x_s . \theta(x_1, \ldots, x_s)$. The pp-formula $\varphi[\mathfrak{A}|\epsilon]$, where ϵ is the empty tuple, is better known as the *canonical query* of \mathfrak{A} . A set of formulas $\Phi := \Phi(x_1, \ldots, x_n)$ with free variables x_1, \ldots, x_n is called *satisfiable in* \mathfrak{A} if there are elements a_1, \ldots, a_n from A such that for all sentences $\varphi \in \Phi$ we have $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$. We say that Φ is *satisfiable* if there exists a structure \mathfrak{A} such that Φ is satisfiable in \mathfrak{A} .

A τ -theory is a set of τ -sentences; two theories are *equivalent* if they share the same models. For a τ -structure \mathfrak{A} , define the *theory* of \mathfrak{A} , Th(\mathfrak{A}), to be the set of τ -sentences true on \mathfrak{A} . Note that $\mathfrak{A} \preceq \mathfrak{B}$ implies that Th(\mathfrak{A}) = Th(\mathfrak{B}). Define the *primitive positive theory* of \mathfrak{A} ,

pp-Th(\mathfrak{A}) := { $\varphi : \mathfrak{A} \models \varphi$ and φ is a primitive positive τ -sentence or its negation}.

Note that we arrive at an equivalent theory if we substitute existential positive for primitive positive in this definition. More generally, a *complete* pp- τ -theory T is a satisfiable set of pp- and negated pp- τ -sentences such that, for all pp- τ -sentences φ , either φ or $\neg \varphi$ is in T.

For $n \ge 0$, an *n*-type of a theory T is a set $p := p(x_1, \ldots, x_n)$ of formulas in the free variables x_1, \ldots, x_n such that $p \cup T$ is satisfiable. In a similar manner, a primitive positive *n-type* (pp-*n*-type) of a theory T is a set of pp-formulas such that $p \cup T$ is satisfiable. A pp-n-type p of T is maximal if $T \cup p \cup \varphi(x_1, \ldots, x_n)$ is unsatisfiable for any pp $\varphi \notin p$. A (pp-) *n*-type of a structure \mathfrak{A} is just a (pp-) *n*-type of the theory Th(\mathfrak{A}). An application of compactness demonstrates, for a set of pp-formulas p, that $p \cup Th(\mathfrak{A})$ is satisfiable iff $p \cup \text{pp-Th}(\mathfrak{A})$ is satisfiable; thus we could equivalently have defined pp-*n*-type with respect to the latter theory. An *n*-type $p(x_1, \ldots, x_n)$ of $(\mathfrak{A}; \langle a_\alpha \rangle_{\alpha < \beta})$ is realised in $(\mathfrak{A}; \langle a_\alpha \rangle_{\alpha < \beta})$ if there exists $a'_1, \ldots, a'_n \in A$ s.t., for each $\varphi \in p$, $(\mathfrak{A}; \langle a_\alpha \rangle_{\alpha < \beta}) \models \varphi(a'_1, \ldots, a'_n)$. For an infinite cardinal κ , a structure \mathfrak{A} is κ -saturated if, for all $\beta < \kappa$ and expansions $(\mathfrak{A}; \langle a_\alpha \rangle_{\alpha < \beta})$ of \mathfrak{A} , every 1-type of $(\mathfrak{A}; \langle a_{\alpha} \rangle_{\alpha < \beta})$ is realised in $(\mathfrak{A}; \langle a_{\alpha} \rangle_{\alpha < \beta})$. We say that an infinite \mathfrak{A} is saturated when it is |A|-saturated. Realisation of pp-types and pp-(κ -)saturation is defined in exactly the analogous way. Note that a structure that is κ -saturated is a fortiori pp- κ saturated. A theory T is said to be κ -categorical, for some cardinal κ , if it has a unique model of cardinality κ , up to isomorphism. It is known that, if T is κ -categorical for one uncountable cardinal κ , then T is κ' -categorical for all uncountable cardinals κ' . A structure \mathfrak{A} , of cardinality κ , is said to be κ -categorical if Th(\mathfrak{A}) is κ -categorical.

Let \mathfrak{A} and \mathfrak{B} be τ -structures. A homomorphism from \mathfrak{A} to \mathfrak{B} is a function f from A to Bsuch that for each k-ary relation symbol R in τ and each k-tuple (a_1, \ldots, a_k) , if $(a_1, \ldots, a_k) \in R^{\mathfrak{A}}$, then $(f(a_1), \ldots, f(a_k)) \in R^{\mathfrak{B}}$. In this case we say that the map f preserves the relation R. Injective homomorphisms that also preserve the complement of each relation are called *embeddings*. Surjective embeddings are called isomorphisms; homomorphisms and isomorphisms from \mathfrak{A} to itself are called *endomorphisms* and *automorphisms*, respectively. We will make use later of the following lemma, a close relative of Theorem 10.7.1 in [17]

Lemma 2.1. Let \mathfrak{A} and \mathfrak{B} be τ -structures, where $|A| \leq |B|$ and \mathfrak{B} is pp-|A|-saturated. Suppose f is a mapping from $\{a_{\alpha} : \alpha < \mu\} \subseteq A$ ($\mu < |A|$) to B such that all $pp-(\tau \cup \{c_{\alpha} : \alpha < \mu\})$ -sentences true on $(\mathfrak{A}; \langle a_{\alpha} \rangle_{\alpha < \mu})$ are true on $(\mathfrak{B}; \langle f(a_{\alpha}) \rangle_{\alpha < \mu})$. Then f can be extended to a homomorphism from \mathfrak{A} to \mathfrak{B} .

For τ -structures \mathfrak{A} and \mathfrak{B} , define the *direct* (or categorical) product $\mathfrak{A} \times \mathfrak{B}$ to be the τ -structure on domain $A \times B$ such that $((a_1, b_1), \ldots, (a_r, b_r)) \in R^{\mathfrak{A} \times \mathfrak{B}}$ iff $(a_1, \ldots, a_r) \in R^{\mathfrak{A}}$ and $(b_1, \ldots, b_r) \in R^{\mathfrak{B}}$. A property of pp-sentences φ that we will use later is that $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \varphi$ iff $\mathfrak{A} \times \mathfrak{B} \models \varphi$.

Let $\langle \mathfrak{A} \rangle_{\text{fo}}$ (respectively, $\langle \mathfrak{A} \rangle_{\text{ep}}$ and $\langle \mathfrak{A} \rangle_{\text{pp}}$) be the sets of relations, over domain A, that are fo- (respectively, ep- and pp-) definable over \mathfrak{A} . Let $\operatorname{Aut}(\mathfrak{A})$ and $\operatorname{End}(\mathfrak{A})$ be the sets of automorphisms and endomporphisms, respectively, of \mathfrak{A} . A κ -polymorphism of \mathfrak{A} is a homomorphism from \mathfrak{A}^{κ} to \mathfrak{A} , where the power is with respect to the direct product already defined. Let $\operatorname{Pol}^{\infty}(\mathfrak{A})$, $\operatorname{Pol}^{\omega}(\mathfrak{A})$ and $\operatorname{Pol}(\mathfrak{A})$ be the sets of κ -polymorphisms (for any κ), κ polymorphisms (for $\kappa \leq \omega$) and k-polymorphisms (for each finite k), respectively. Let $\operatorname{Inv}(\operatorname{Aut}(\mathfrak{A}))$ be the set of relations, over domain A, that are preserved by (invariant under) the automorphisms of \mathfrak{A} . Define $\operatorname{Inv}(\operatorname{End}(\mathfrak{A}))$, $\operatorname{Inv}(\operatorname{Pol}^{\infty}(\mathfrak{A}))$, $\operatorname{Inv}(\operatorname{Pol}^{\omega}(\mathfrak{A}))$ and $\operatorname{Inv}(\operatorname{Pol}(\mathfrak{A}))$ in the similar fashion (note that the condition of preservation of an m-ary relation by a κ ary function $f: \mathfrak{A}^{\kappa} \to \mathfrak{A}$ is component-wise, i.e. if $(a_1^{\beta}, \ldots, a_m^{\beta}) \in \mathbb{R}^{\mathfrak{A}}$, for all $\beta < \kappa$, then $(f(\langle a_1^{\beta} \rangle_{\beta < \kappa}), \ldots, f(\langle a_m^{\beta} \rangle_{\beta < \kappa})) \in \mathbb{R}^{\mathfrak{A}})$.

A classical result of model theory holds that \mathfrak{A} is finite or ω -categorical if, and only if, $\operatorname{Inv}(\operatorname{Aut}(\mathfrak{A})) = \langle \mathfrak{A} \rangle_{\text{fo}}$ (follows from the Theorem of Ryll-Nardzewski, see, e.g., [2]). One direction persists in the realm of the primitive positive, as attested to by the following.

Theorem 2.2 (see [9,10,16]). When \mathfrak{A} is finite or ω -categorical, $\operatorname{Inv}(\operatorname{Pol}(\mathfrak{A})) = \langle \mathfrak{A} \rangle_{\operatorname{pp}}$.

This characterization is not tight, i.e. there are infinite non- ω -categorical strutures \mathfrak{A} for which $\operatorname{Inv}(\operatorname{Pol}(\mathfrak{A})) = \langle \mathfrak{A} \rangle_{\operatorname{PP}}$ [26].

2.2. The constraint satisfaction problem

For a relational structure \mathfrak{A} over a finite signature, $\operatorname{CSP}(\mathfrak{A})$ is the computational problem to decide whether a given pp-sentence is true in \mathfrak{A} . It is not hard to see that, for any \mathfrak{A} and \mathfrak{A}' with the same domain, such that $\langle \mathfrak{A} \rangle_{pp} \subseteq \langle \mathfrak{A}' \rangle_{pp}$, we have $\operatorname{CSP}(\mathfrak{A}) \leq_{\mathrm{P}} \operatorname{CSP}(\mathfrak{A}')$ (see [18]), where \leq_{P} indicates polynomial-time many-to-one reduction (in fact, logspace reductions may be used, though this is harder to see and requires the celebrated result of [24]). In light of this observation, together with Theorem 2.2, we may use the sets $\operatorname{Pol}(\mathfrak{A})$ to classify the computational complexity of $\operatorname{CSP}(\mathfrak{A})$, and a most successful research program has run in this direction (see [11, 12, 18], and [31] for a survey).

Sets of the form $\operatorname{Pol}(\mathfrak{A})$ are always *clones* (for definitions, see [30]), and the machinery of Clone Theory can be brought to bear on the classification program for CSPs (e.g., the classification of minimal clones of [27]). It often transpires that instances of the CSP with low complexity can be explained by the presence of particular classes of polymorphisms on the template. When \mathfrak{A} is finite, the class of problems $\operatorname{CSP}(\mathfrak{A})$ is conjectured to display complexity dichotomy between those problems that are in P and those that are NP-complete (a remarkable property given the breadth of CSP problems together with the result of Ladner that NP itself does not possess the dichotomy, so long as $P \neq \operatorname{NP}$ [20]). While the *dichotomy conjecture* was formulated independently of the algebraic method [15], a conjecture as to exactly where the boundary sits relies on the algebraic language [13].

In the case where \mathfrak{A} is infinite but ω -categorical, the connection of Theorem 2.2 has been used to good effect in the complexity classification, e.g., of temporal CSPs in [8]. In that case dichotomy between P and NP-complete was again observed. For ω -categorical templates in general, it is known that there are templates whose CSP is undecideable [4] and of various complexities [4] (even coNP-complete). For infinite templates that are not ω -categorical, no algebraic machinery has thus far been developed.

3. Existential-positively closed models

In this section we state some basic concepts and facts about existential-positively closed models. They are the positive analogs of existentially closed models (the latter are treated in great detail in [17], Section 8), and have been studied under the name of existentially closed models in a recent paper on positive model theory by Ben-Yaacov [1].

Definition 3.1. A model \mathfrak{A} of a theory T is existential-positively closed in T – or short epc – iff for any homomorphism h from \mathfrak{A} into another model \mathfrak{B} of T, any tuple \bar{a} from A, and any primitive positive formula φ with $\mathfrak{B} \models \varphi(h(\bar{a}))$ we have that $\mathfrak{A} \models \varphi(\bar{a})$.

Note that we could equivalently have used existential positive formula in the previous definition. To show the existence of certain epc models we apply the direct limit construction (for simplicity – and because it is the only case we need – we give a presentation for countable structures only).

Definition 3.2. Let τ be a relational signature, and let $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ be a sequence of countable τ -structures such that there are homomorphisms $f_{ij}: \mathfrak{A}_i \to \mathfrak{A}_j$ with $f_{jk} \circ f_{ij} = f_{ik}$ for every $i \leq j \leq k$. Then the *direct limit* $\lim_{i < \omega} \mathfrak{A}_i$ is the τ -structure \mathfrak{A} defined as follows. The domain A of \mathfrak{A} comprises the equivalence classes of the equivalence relation \sim defined on $\bigcup_{i < \omega} A_i$ by setting $x_i \sim x_j$ for $x_i \in A_i, x_j \in A_j$ iff there is a k such that $f_{ik}(x_i) = f_{jk}(x_j)$. Let $g_i: A_i \to A$ be the function that maps $a \in A_i$ to the equivalence class of a in A. For $R \in \tau$, define $\mathfrak{A} \models R(\bar{a})$ iff there is a k and $\bar{b} \in A_k$ such that $\mathfrak{A}_k \models R(\bar{b})$ and $\bar{a} = g_k(\bar{b})$.

The direct limits defined above can be seen as a positive variant of the basic model-theoretic notion of a *union of chains* (see Section 2.4 in [17]); we essentially replace embeddings in chains by homomorphisms. Unions of chains preserve \forall_2 -sentences; the analogous statement for direct limits is as follows. A sentence is called *positively restricted* \forall_2 if it is a universally quantified positive boolean combination of existential positive formulas and negative atomic formulas.

Proposition 3.3 (see Theorem 2.4.6 in [17]). Let \mathfrak{A} be the direct limit of $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$; if φ is positively restricted \forall_2 such that $\mathfrak{A}_i \models \varphi$ for all i, then $\mathfrak{A} \models \varphi$.

Proposition 3.4 (Essentially from [1]). Let \mathfrak{A} be a countable model of a set T of positively restricted \forall_2 sentences. Then there is a homomorphism from \mathfrak{A} to a countable epc model \mathfrak{B} of T.

Proposition 3.5. Let \mathfrak{A} be a countable epc model of a theory T. Each of the pp-types realised in \mathfrak{A} is a maximal pp-type of T.

We conclude this section by noting that epc structures are related to the concept of cores, which play such an important role in the classification program for CSPs when the template is finite or ω -categorical. A structure \mathfrak{A} is a *core* if all its endomorphisms are embeddings.

Proposition 3.6. If \mathfrak{A} is pp-saturated or finite, then \mathfrak{A} is a core iff \mathfrak{A} is epc for pp-Th(\mathfrak{A}).

4. Equivalent ω -categorical templates

A structure is *homogeneous* (sometimes called *ultrahomogeneous* [17]) if every finite partial automorphism can be extended to a full automorphism. It is known that a homogeneous structure \mathfrak{A} on which one can define only a finite number of inequivalent atomic formulas (i.e. through variable substitution on extensional relations) of each arity k is ω -categorical.

For a satisfiable theory T, let \sim_n^T be the equivalence relation defined on pp-formulas with n free variables x_1, \ldots, x_n as follows. For two such formulas φ_1 and φ_2 , let $\varphi_1 \sim_n \varphi_2$ iff for all pp-formulas ψ with free variables x_1, \ldots, x_n we have that $\{\varphi_1, \psi\} \cup T$ is satisfiable if and only if $\{\varphi_2, \psi\} \cup T$ is satisfiable. By proving that an epc model of a certain type of theory is in fact ω -categorical, we will derive the following.

Theorem 4.1. For a complete $pp-\tau$ -theory T, the following are equivalent.

- (i) T has a finite or ω -categorical model.
- (ii) \sim_n^T has finite index for each n.
- (iii) T has finitely many maximal pp-n-types for each n.

We note that, if a pp-theory T has a finite model, then it necessarily has an ω -categorical model (see [9]), thus (i) above could be more concisely stated.

Corollary 4.2. Let \mathfrak{A} be such that the number of maximal pp-n-types consistent with $\operatorname{Th}(\mathfrak{A})$ (equivalently, pp-Th(\mathfrak{A})) is finite for each n. Then there is an ω -categorical template \mathfrak{C} such that pp-Th(\mathfrak{A}) = pp-Th(\mathfrak{B}), i.e. $\operatorname{CSP}(\mathfrak{A}) = \operatorname{CSP}(\mathfrak{C})$.

5. Primitive positive definability of first-order formulas

To show hardness of $\text{CSP}(\mathfrak{A}')$, we often try to prove that there is a finite signature reduct \mathfrak{A} of $\langle \mathfrak{A}' \rangle_{\text{pp}}$ such that $\text{CSP}(\mathfrak{A})$ is NP-hard. An important set of relations that contains the set of all pp-definable relations $\langle \mathfrak{A}' \rangle_{\text{pp}}$ is the set of all fo-definable relations $\langle \mathfrak{A}' \rangle_{\text{fo}}$. For every structure \mathfrak{A} of cardinality greater than one there are fo-definable relations yielding an NP-hard CSP, and these relations are usually good candidates for proving hardness. Therefore, it is natural and important to understand which fo-definable relations are pp-definable in \mathfrak{A} . In this section we show that, for every problem $\text{CSP}(\mathfrak{A})$, we can find a relational structure \mathfrak{M} for which $\text{CSP}(\mathfrak{A}) = \text{CSP}(\mathfrak{M})$ where infinitary polymorphisms exactly characterize pp-definability of fo-definable relations. We will do this by building a "monster" model of Th(\mathfrak{A}) that is highly saturated.

Definition 5.1. A τ -structure \mathfrak{M} has the homomorphism lifting property if, for any $a_1, \ldots, a_k \in \mathfrak{M}^{\omega}$ and $b_1, \ldots, b_k \in \mathfrak{M}$ s.t. all pp- $(\tau \cup \{c_1, \ldots, c_k\})$ -sentences true in $(\mathfrak{M}^{\omega}; a_1, \ldots, a_k)$ are true in $(\mathfrak{M}; b_1, \ldots, b_k)$, there is a homomorphism $f : (\mathfrak{M}^{\omega}; a_1, \ldots, a_k) \to (\mathfrak{M}; b_1, \ldots, b_k)$.

The most natural of structures with the homomorphism lifting property are those that are of large cardinality and saturated.

Lemma 5.2. If \mathfrak{M} is a saturated structure of cardinality $\kappa = \kappa^{\omega}$, then \mathfrak{M} has the homomorphism lifting property.

We remark that the continuum has the property of Lemma 5.2 – that is $2^{\omega} = (2^{\omega})^{\omega}$. On the assumption of the continuum hypothesis, we could only work with large saturated structures, because we could always assume the existence of an elementary extension to a (countable) model that is of cardinality 2^{ω} and saturated. However, without such a set-theoretic assumption, we need to construct the rather unwieldy "monster" model as follows. **Lemma 5.3.** For every τ -structure \mathfrak{A} there is a "monster" elementary extension $\mathfrak{M} \succeq \mathfrak{A}$ that is ω -saturated and has the homomorphism lifting property.

Let $\langle \mathfrak{A} \rangle_{pp\infty}$ be the set of relations pp-definable on \mathfrak{A} , possibly involving infinitary conjunction (of pp-formulas in a finite number of free variables). Because we will use it again later, we give the following lemma in its strongest form.

Lemma 5.4. For all structures \mathfrak{A} , $\langle \mathfrak{A} \rangle_{pp\infty} \subseteq Inv(Pol^{\infty}(\mathfrak{A}))$.

We are now ready for the main result of this section.

Theorem 5.5. Let \mathfrak{A} have the homomorphism lifting property. Then a fo-definable relation R is preserved by the ω -polymorphisms of \mathfrak{A} if and only if R is pp-definable in \mathfrak{A} , i.e.

$$\operatorname{Inv}(\operatorname{Pol}^{\omega}(\mathfrak{A})) \cap \langle \mathfrak{A} \rangle_{\operatorname{fo}} = \langle \mathfrak{A} \rangle_{\operatorname{pp}}.$$

Corollary 5.6. Let \mathfrak{A} be any structure with finite relational signature. Then there exists a structure \mathfrak{M} such that $CSP(\mathfrak{A}) = CSP(\mathfrak{M})$, and such that an fo-definable relation R is pp-definable in \mathfrak{M} if and only if R is preserved by all ω -polymorphisms of \mathfrak{M} .

Proof. By Lemma 5.3, there is an elementary extension of $\mathfrak{M} \succeq \mathfrak{A}$ with the homomorphism lifting property. We now apply Theorem 5.5.

In the parlance of [23], the following may be seen as the "global" analog of Theorem 5.5.

Corollary 5.7. An fo-formula φ is preserved by the ω -polymorphisms of all elementary extensions of \mathfrak{A} if and only if φ is pp-definable in \mathfrak{A} .

Proof. (Backwards.) Follows from Lemma 5.4.

(Forwards.) Since φ is preserved by the ω -polymorphisms of the "monster" elementary extension $\mathfrak{M} \succeq \mathfrak{A}$ constructed in Lemma 5.3, it follows from Theorem 5.5 that φ is ppdefinable on \mathfrak{M} . But this is a fortiori a pp-definition on \mathfrak{A} .

Corollary 5.8. Let T be an uncountably categorical fo-theory, and \mathfrak{A} a model of T of cardinality $\geq 2^{\omega}$. Then $\operatorname{Inv}(\operatorname{Pol}^{\omega}(\mathfrak{A})) \cap \langle \mathfrak{A} \rangle_{fo} = \langle \mathfrak{A} \rangle_{pp}$.

Proof. It is well-known that uncountable models of uncountably categorical theories are saturated in their own cardinality (Fact 1.2. in [32]). Hence, the statement follows from Theorem 5.5.

5.1. Tightness of Theorem 5.5

One might be interested in the following potential strengthenings of Theorem 5.5.

- 1. To derive the statement for arbitrary relations (not just for fo-definable relations).
- 2. To assume preservation under finitary polymorphisms (not infinitary polymorphisms).
- 3. To show the statement for arbitrary models of T (not just highly saturated structures).

The following proposition shows that each of these stronger assumptions is necessary.

Proposition 5.9.

- 1. There is a saturated structure \mathfrak{A}_{sat} of cardinality 2^{ω} such that $\operatorname{Inv}(\operatorname{Pol}^{\omega}(\mathfrak{A})) \neq \langle \mathfrak{A} \rangle_{pp}$.
- 2. There is a saturated structure \mathfrak{A}_{sat} of cardinality 2^{ω} such that $Inv(Pol(\mathfrak{A})) \cap \langle \mathfrak{A} \rangle_{fo} \neq \langle \mathfrak{A} \rangle_{pp}$.
- 3. There is a structure \mathfrak{A} such that $\operatorname{Inv}(\operatorname{Pol}^{\omega}(\mathfrak{A})) \cap \langle \mathfrak{A} \rangle_{\mathrm{fo}} \neq \langle \mathfrak{A} \rangle_{\mathrm{pp}}$.

Sketch proof.

- (1) Let $\mathfrak{A} := (\mathbb{Q}; +, 1, (u = v \lor x = y))$. Take \mathfrak{A}_{sat} to be the saturated elementary extension of \mathfrak{A} of cardinality 2^{ω} .
- (2) Let $\mathfrak{A} := (\mathbb{N}; U_i : i \in \omega)$, where $U_i := \mathbb{N} \setminus \{0, i\}$. Take \mathfrak{A}_{sat} to be the saturated elementary extension of \mathfrak{A} of cardinality 2^{ω} . A finite signature variant of this counterexample is provided in the appendix.
- (3) Take $\mathfrak{A} := (\mathbb{Q}; x = 1, x < 0, S_2(x, y)))$, where $S_2 := \{(x, y) : 2x < y, 0 < y \le 1\}$.

6. Applications

All polymorphisms are essentially unary. We will begin by demonstrating that the power of infinitary polymorphisms can be greatly limited. The forthcoming three lemmas are well-known for finite domains A (also wherever the connections Inv-Pol=pp and Inv-End=ep persist). They require a little care in the general, infinite case.

A function $f: A^{\alpha} \to A$ is essentially unary if there exists a $\beta < \alpha$ and $g: A \to A$ such that, for all $\overline{x} \in A^{\alpha}$, $f(\overline{x}) = g(x_{\beta})$. For $\overline{x}, \overline{w} \in D^{\alpha}$, and $X \subseteq \alpha$, let $\overline{x}[\overline{x}_X/\overline{w}_X]$ be the tuple \overline{x} with each entry x_{β} , where $\beta \in X$, substituted by w_{β} .

Lemma 6.1. A function $f : A^{\alpha} \to A$ is not essentially unary iff there exist two non-empty and disjoint $X, Y \subseteq \alpha$, such that both

- exist $\overline{x}, \overline{w}, \overline{w}' \in A^{\alpha}$ s.t. $f(\overline{x}[\overline{x}_X/\overline{w}_X]) \neq f(\overline{x}[\overline{x}_X/\overline{w}'_X])$, and
- exist $\overline{y}, \overline{z}, \overline{z}' \in A^{\alpha}$ s.t. $f(\overline{y}[\overline{y}_Y/\overline{z}_Y]) \neq f(\overline{y}[\overline{y}_Y/\overline{z}_Y])$.

Lemma 6.2. Let \mathfrak{A} be such that $(u = v \lor x = y) \in \langle \mathfrak{A} \rangle_{pp}$. Then all (finitary and infinitary) polymorphisms of \mathfrak{A} are essentially unary.

Lemma 6.3. Suppose \mathfrak{A} is such that $(x = y \lor u = v) \in \langle \mathfrak{A} \rangle_{pp}$. Then $\langle \mathfrak{A} \rangle_{pp} = \langle \mathfrak{A} \rangle_{ep}$.

Proposition 6.4. For all structures \mathfrak{A} , $\langle \mathfrak{A} \rangle_{pp} = \langle \mathfrak{A} \rangle_{ep}$ iff all ω -polymorphisms of all elementary extensions of \mathfrak{A} are essentially unary.

Proof. (Forwards.) If $\langle \mathfrak{A} \rangle_{\rm pp} = \langle \mathfrak{A} \rangle_{\rm ep}$ then $(u = v \lor x = y) \in \langle \mathfrak{A} \rangle_{\rm pp}$, and so $(u = v \lor x = y) \in \langle \mathfrak{A}' \rangle_{\rm pp}$ for all $\mathfrak{A}' \succeq \mathfrak{A}$. The result follows from Lemma 6.2.

(Backwards.) If all ω -polymorphisms of all elementary extensions of \mathfrak{A} are essentially unary, then in particular this is true of the "monster" elementary extension \mathfrak{M} built as in Lemma 5.3. It follows from Theorem 5.5 that $(u = v \lor x = y) \in \langle \mathfrak{M} \rangle_{\rm pp}$, which gives $\langle \mathfrak{M} \rangle_{\rm pp} = \langle \mathfrak{M} \rangle_{\rm ep}$ by Lemma 6.3. The result $\langle \mathfrak{A} \rangle_{\rm pp} = \langle \mathfrak{A} \rangle_{\rm ep}$ follows since $\mathfrak{A} \preceq \mathfrak{M}$.

We are able to prove that the stipulation of elementary extension in Proposition 6.4 is necessary, by exhibiting a structure whose ω -polymorphisms include only projections but for which $(x = y \lor u = v)$ is not pp-definable.

Lemma 6.5. The only ω -polymorphisms of $(\mathbb{Q}; +, 1, \neq)$ are projections.

It follows that $(x = y \lor u = v) \in \text{Inv}(\text{Pol}^{\omega}(\mathbb{Q}; +, 1, \neq))$, though $(x = y \lor u = v) \notin \langle (\mathbb{Q}; +, 1, \neq) \rangle_{\text{pp}}$ since if it were we could also derive $(x = y \lor u = v) \in \langle (\mathbb{R}; +, 1, \neq) \rangle_{\text{pp}}$ (since $(\mathbb{R}; +, 1, \neq)$ and $(\mathbb{Q}; +, 1, \neq)$ share the same theory). This would contradict Lemma 6.2

as $(\mathbb{R}; +, 1, \neq)$ has polymorphisms that are not essentially unary: indeed, there is an isomorphism between $(\mathbb{R}; +, 1)^2$ and $(\mathbb{R}; +, 1)$ (that we shall use again shortly), which gives a bijective homomorphism from $(\mathbb{R}; +, 1, \neq)^2$ to $(\mathbb{R}; +, 1, \neq)$.

The following definition comes from [5]. For a structure \mathfrak{A} and an ep-sentence φ , we generate the boolean sentence $F_{\mathfrak{A}}(\varphi)$ by removing all existential quantifiers and replacing each atom $R(x_1, \ldots, x_k)$, where $R^{\mathfrak{A}}$ is empty, with *false*, and replacing all other atoms with *true*. \mathfrak{A} is said to be *locally refutable* if for every ep-sentence φ , $\mathfrak{A} \models \varphi$ iff $F_{\mathfrak{A}}(\varphi)$ is true.

Proposition 6.6. Let \mathfrak{A} be a structure that is not locally refutable and for which all ω polymorphisms in all elementary extensions are essentially unary. Then $CSP(\mathfrak{A})$ is NPhard.

Proof. It is proved in [5] that the evaluation of ep-sentences on \mathfrak{A} is NP-hard. The result now follows from Corollary 6.4 (note that the recursive removal of disjunction induces a polynomial time reduction).

First-order definable CSPs. Recall $\varphi[\mathfrak{B}|\epsilon]$ to be the canonical query of \mathfrak{B} . CSP(\mathfrak{A}) is said to be *first-order definable* if there is a fo-sentence $\psi_{\mathfrak{A}}$ such that, for all finite \mathfrak{B} , $\mathfrak{A} \models \varphi[\mathfrak{B}|\epsilon]$ (i.e. $\varphi[\mathfrak{B}|\epsilon] \in \text{CSP}(\mathfrak{A})$) iff $\mathfrak{B} \models \psi_{\mathfrak{A}}$. The following definition comes from [21]. The *one-tolerant n-th power* ${}^{1}\mathfrak{A}^{n}$ of a τ -structure \mathfrak{A} is the τ -structure with domain A^{n} where a k-ary $R \in \tau$ denotes the relation consisting of all those k-tuples $((a_{1}^{1}, \ldots, a_{1}^{n}), \ldots, (a_{k}^{1}, \ldots, a_{k}^{n}))$ such that

$$|\{j: (a_1^j, \dots, a_k^j) \in R^{\mathfrak{A}}\}| \ge n-1.$$

For $n \geq 3$, an *n*-ary polymorphism f of \mathfrak{A} is called a *relational near-unanimity* polymorphism if f is a homomorphism from ${}^{1}\mathfrak{A}^{n}$ to \mathfrak{A} .

Theorem 6.7. Let \mathfrak{A} be a "monster" elementary extension (as constructed as in Lemma 5.3) on a finite signature. Then $CSP(\mathfrak{A})$ is first-order definable if and only if \mathfrak{A} has a relational near-unanimity polymorphism.

Corollary 6.8. Let \mathfrak{A} structure on a finite signature. Then $CSP(\mathfrak{A})$ is first-order definable if and only if \mathfrak{A} has an elementary extension which has a relational near-unanimity polymorphism.

Proof. By Lemma 5.3, \mathfrak{A} has a "monster" elementary extension \mathfrak{M} Since \mathfrak{M} and \mathfrak{A} satisfy the same primitive positive sentences, $CSP(\mathfrak{A})$ is first-order definable if and only if $CSP(\mathfrak{M})$ is. The statement follows immediately from Theorem 6.7.

Horn definability. We will briefly examine a class of structures for which we can give a neat algebraic condition as to whether a relation that is quantifier-free definable admits a quantifier-free Horn definition. Recalling known complexity results for fo-expansions of $(\mathbb{R}; +, 1)$ we will see that the presence of a certain polymorphism exactly delineates those fo-expansions for which the CSP is NP-complete for those which are in P. The following proposition is essentially from [3].

Proposition 6.9. Let \mathfrak{A} be a structure with a binary injective polymorphism e that is an embedding from \mathfrak{A}^2 into \mathfrak{A} . Then a relation R that is quantifier-free definable in the relations of \mathfrak{A} is preserved by e iff it admits a quantifier-free Horn definition in \mathfrak{A} .

We have already met an example of a structure with such an embedding: $(\mathbb{R}; +, 1)$.

Corollary 6.10. Let \mathfrak{B} be an fo-expansion of $(\mathbb{R}; +, 1)$ and let $e : (\mathbb{R}; +, 1)^2 \to (\mathbb{R}; +, 1)$ be an embedding. Then: if e is a polymorphism of \mathfrak{B} , then $CSP(\mathfrak{B})$ is in P; otherwise $CSP(\mathfrak{B})$ is NP-complete.

Proof. Note that $(\mathbb{R}; +, 1)$ admits quantifier elimination and so all fo-expansions may be specified as quantifier-free CNFs. It is proved in [6] that those that admit quantifier-free Horn definitions give a CSP that is in P while those that do not give CSPs that are NP-complete. The result follows from Proposition 6.9.

7. Concluding remarks and open problems

The results of this paper show that – at least in principle – the universal-algebraic approach can be applied to study the complexity of $\text{CSP}(\mathfrak{A})$ for arbitrary infinite-domain structures \mathfrak{A} . Among one of the first applications, we have presented a polymorphism-based characterization of those CSPs that are first-order definable.

A natural question is whether there are results from finite domain constraint satisfaction where there are principle obstacles for generalizations to infinite domains. We are not aware of any. However, candidates might arise from the following problems

- Is there an infinite structure \mathfrak{A} , epc in pp-Th(\mathfrak{A}), with a finite signature and a Mal'tsev polymorphism¹ such that $CSP(\mathfrak{A})$ is NP-hard?
- Is there an infinite structure \mathfrak{A} , epc in pp-Th(\mathfrak{A}), with a finite signature and a nearunanimity polymorphism² such that $CSP(\mathfrak{A})$ is NP-hard? The problem here is that it might be impossible to algorithmically establish k-consistency.

Further research questions are the following. Can we strengthen our preservation theorem (Theorem 5.5) to show, under the additional assumption that \mathfrak{A} is epc, that an for relation is pp-definable if and only if it is preserved by the *finitary* polymorphisms of \mathfrak{A} ? In particular, if \mathfrak{A} is saturated, are we forced to use infinitary polymorphisms even if the structure \mathfrak{A} is a core (see Proposition 3.6)? Finally, it would be very interesting to understand the polymorphisms of concrete and important CSPs from the literature; for example, the polymorphisms of the CSPs that were mentioned in the introduction. In particular, assuming that these CSPs are formulated with appropriate templates and that $P \neq NP$, our results (see Proposition 6.6) imply that essential ω -polymorphisms must exist.

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²That is, a polymorphism f of \mathfrak{A} that satisfies $f(x, x, \dots, x, y) = \dots = f(y, x, \dots, x) = x$ for all $x, y \in A$.

¹That is, a polymorphism f of \mathfrak{A} that satisfies f(x, x, y) = f(y, x, x) = y for all $x, y \in A$.

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Appendix

Lemma 2.1. Let \mathfrak{A} and \mathfrak{B} be τ -structures, where $|A| \leq |B|$ and \mathfrak{B} is pp-|A|-saturated. Suppose f is a mapping from $\{a_{\alpha} : \alpha < \mu\} \subseteq A$ ($\mu < |A|$) to B such that all pp-($\tau \cup \{c_{\alpha} : \alpha < \mu\}$)-sentences true on ($\mathfrak{A}; \langle a_{\alpha} \rangle_{\alpha < \mu}$) are true on ($\mathfrak{B}; \langle f(a_{\alpha}) \rangle_{\alpha < \mu}$). Then f can be extended to a homomorphism from \mathfrak{A} to \mathfrak{B} .

Proof. If \mathfrak{B} (and therefore \mathfrak{A}) is finite we may simply find witnesses in $(\mathfrak{B}; f(a_0), \ldots, f(a_{m-1}))$ for the existential variables of the canonical query of $(\mathfrak{A}; a_0, \ldots, a_{m-1})$, and we are done. Suppose that \mathfrak{B} is of infinite cardinality κ_B .

Suppose $\mu < |A| = \kappa_A \le \kappa_B$. Let $\langle a'_{\alpha} \rangle_{\alpha < \kappa_A}$ well-order \mathfrak{A} such that $\langle a'_{\alpha} \rangle_{\alpha < \mu} = \langle a_{\alpha} \rangle_{\alpha < \mu}$ (there is the implicit and harmless assumption that $\langle a_{\alpha} \rangle_{\alpha < \mu}$ contains no repetitions). Set $\langle b'_{\alpha} \rangle_{\alpha < \mu} := \langle f(a_{\alpha}) \rangle_{\alpha < \mu}$.

We will construct by transfinite recursion on β (up to κ_A) a sequence $\langle b'_{\alpha} \rangle_{\alpha < \beta}$ such that we maintain the inductive hypothesis

(*) all pp- $(\tau \cup \{c_{\alpha} : \alpha < \beta\})$ -sentences true on $(\mathfrak{A}; \langle a'_{\alpha} \rangle_{\alpha < \beta})$ are true on $(B; \langle b'_{\alpha} \rangle_{\alpha < \beta})$.

The result will clearly then follow by reading f as the map $\{a'_{\alpha} \mapsto b'_{\alpha}\}_{\alpha < \kappa_A}$.

(Base Case.) $\beta := \mu$. Follows from hypothesis of lemma.

(Inductive Step. Limit ordinals.) $\beta := \lambda$. Property (*) holds as a sentence can only mention a finite collection of constants, whose indices must all be less than some $\gamma < \lambda$.

(Inductive Step. Successor ordinals.) $\beta := \gamma^+ < \kappa_A$. Set

$$\Sigma := \{\varphi(x) : \varphi \text{ is a pp-}(\tau \cup \{c_{\alpha} : \alpha < \gamma\}) \text{-formula s.t. } (\mathfrak{A}; \langle a'_{\alpha} \rangle_{\alpha < \gamma}) \models \varphi(a'_{\gamma}) \}.$$

By (*), for every $\varphi \in \Sigma$, $(\mathfrak{B}; \langle b'_{\alpha} \rangle_{\alpha < \gamma}) \models \exists x. \varphi(x)$. By compactness, since Σ is closed under conjunction, we have that Σ is a pp-1-type of $(\mathfrak{B}; \langle b'_{\alpha} \rangle_{\alpha < \gamma})$. By pp-|A|-saturation of \mathfrak{B} it is realised by some element $b'_{\gamma} \in B$. By construction we maintain that all pp- $(\tau \cup \{c_{\alpha} : \alpha < \gamma^{+}\})$ -sentences true on $(\mathfrak{A}; \langle a'_{\alpha} \rangle_{\alpha < \gamma^{+}})$ are true on $(\mathfrak{B}; \langle b'_{\alpha} \rangle_{\alpha < \gamma^{+}})$.

Proposition 3.4. Let \mathfrak{A} be a countable structure of a set T of positively restricted \forall_2 sentences. Then there is a homomorphism from \mathfrak{A} to a countable epc structure \mathfrak{B} of T.

Proof. Set $\mathfrak{B}_0 = \mathfrak{A}$. Having constructed \mathfrak{B}_i , let $\{(\varphi_j, \bar{a}_j) \mid j < \omega\}$ be an enumeration of all pairs (φ, \bar{a}) where φ is existential-positive with free variables x_1, \ldots, x_n , and \bar{a} is an *n*-tuple from B_i . We construct a sequence $(\mathfrak{B}_i^0, f_i^0), (\mathfrak{B}_i^1, f_i^1), \ldots$ where \mathfrak{B}_i^j is a model of T and f_i^j is a homomorphism from \mathfrak{B}_i^j to \mathfrak{B}_i^{j+1} as follows.

Set $\mathfrak{B}_i^0 = \mathfrak{B}_i$. Let \bar{a}'_j be the image of \bar{a}_j in \mathfrak{B}_i^j under $f_i^{j-1} \circ \ldots \circ f_i^0$. If there is a model \mathfrak{A}' of T and a homomorphism $h: \mathfrak{B}_i^j \to \mathfrak{A}'$ such that $\mathfrak{A}' \models \varphi_j(h(\bar{a}'_j))$, set $\mathfrak{B}_i^{j+1} = \mathfrak{A}'$ and $f_i^j := h$, otherwise $\mathfrak{B}_i^{j+1} = \mathfrak{B}_i^j$ and f_i^j is the identity. Let \mathfrak{B}_{i+1} be $\lim_{j < \omega} \mathfrak{B}_i^j$ and let $f_i: \mathfrak{B}_i \to \mathfrak{B}_{i+1}$ be the homomorphism given by the limit of $\ldots \circ f_i^1 \circ f_i^0$.

By Proposition 3.3, $\mathfrak{B} = \lim_{i < \omega} \mathfrak{B}_i$ is a model of T. \mathfrak{B} is epc in T by construction, and the function h that is the limit of $\ldots \circ f_1 \circ f_0$ is a homomorphism from \mathfrak{A} to \mathfrak{B} .

Proposition 3.5. Let \mathfrak{A} be a countable epc model of a theory T. Each of the pp-types realised in \mathfrak{A} is a maximal pp-type of T.

Proof. Suppose $p(x_1, \ldots, x_n)$ is a pp-*m*-type, realised in \mathfrak{A} by (a_1, \ldots, a_n) , that is not maximal. Then there is a pp-formula $\varphi(x_1, \ldots, x_n)$ such that $\mathfrak{A} \not\models \varphi(a_1, \ldots, a_n)$ but $T \cup$

 $p(c_1, \ldots, c_n) \cup \{\varphi(c_1, \ldots, c_n)\} \text{ is consistent, with a model } (\mathfrak{B}; b_1, \ldots, b_n). \text{ Now, let } (\mathfrak{B}_{\mathrm{sat}}; b'_1, \ldots, b'_n) \text{ be an } \omega\text{-saturated model of Th}(\mathfrak{B}; b_1, \ldots, b_n) \text{ (such a model always exists, see Theorem 4.3.12 [22]). Clearly } (\mathfrak{B}_{\mathrm{sat}}; b'_1, \ldots, b'_n) \text{ is pp-}|A|\text{-saturated, and all pp-formulas true on } (\mathfrak{A}; a_1, \ldots, a_n) \text{ are true on } (\mathfrak{B}_{\mathrm{sat}}; b'_1, \ldots, b'_n). \text{ By Lemma 2.1, there is a homomorphism } h \text{ from } (\mathfrak{A}; a_1, \ldots, a_n) \text{ to } (\mathfrak{B}_{\mathrm{sat}}; b'_1, \ldots, b'_n). \text{ Now, since } \varphi(b'_1, \ldots, b'_n) \text{ holds on } \mathfrak{B}_{\mathrm{sat}} \text{ and } \mathfrak{A} \text{ is epc, we deduce the contradiction } \mathfrak{A} \models \varphi(a_1, \ldots, a_n).$

Proposition 3.6. If \mathfrak{A} is pp-saturated or finite, then \mathfrak{A} is a core iff \mathfrak{A} is epc for pp-Th(\mathfrak{A}).

Proof. (Backwards.) Suppose \mathfrak{A} is epc for pp-Th(\mathfrak{A}). Take a homomorphism $h : \mathfrak{A} \to \mathfrak{A}$. By epc, for a_1, \ldots, a_k in A, if $\mathfrak{A} \models R(h(a_1), \ldots, h(a_k))$ or $\mathfrak{A} \models h(a_1) = h(a_2)$, then $\mathfrak{A} \models R(a_1, \ldots, a_k)$ or $\mathfrak{A} \models a_1 = a_2$, respectively. It follows that h is an embedding.

(Forwards.) Suppose $\mathfrak{B} \models \text{pp-Th}(\mathfrak{A})$ and $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism. Suppose $\mathfrak{B} \models \varphi(h(\bar{a}))$, where $\varphi(\bar{x})$ is a pp-formula and \bar{a} is a tuple from A; we must prove that $\mathfrak{A} \models \varphi(\bar{a})$. First, we note that the image $h(\mathfrak{A}) \models \text{pp-Th}(\mathfrak{A})$. For the positive sentences of pp-Th(\mathfrak{A}), this follows from the homomorphism h; for the negative sentences of pp-Th(\mathfrak{A}) it follows from $\mathfrak{B} \models \text{pp-Th}(\mathfrak{A})$ together with $h(\mathfrak{A})$ being an induced substructure of \mathfrak{B} . Since \mathfrak{A} is pp-saturated, it follows from Lemma 2.1 that there is a homomorphism $g : h(\mathfrak{A}) \to \mathfrak{A}$. Therefore, we may derive that $\mathfrak{A} \models \varphi(g \circ h(\bar{a}))$. But, $g \circ h$ is an endomorphism of \mathfrak{A} , which must be an embedding since \mathfrak{A} is a core. The result $\mathfrak{A} \models \varphi(\bar{a})$ follows.

Theorem 4.1. For a complete pp- τ -theory T, the following are equivalent.

- (i) T has a finite or ω -categorical model.
- (*ii*) \sim_n^T has finite index for each *n*.
- (*iii*) T has finitely many maximal pp-n-types for each n.

We will prove this theorem in three stages. The first two provide little difficulty.

Proof of Theorem 4.1 (i) \Rightarrow (ii). For contradiction, suppose \sim_n^T has infinite index for some n, yet T has an ω -categorical model \mathfrak{A} such that (due to completeness) pp-Th(\mathfrak{A}) = T. Let φ_1 and φ_2 be two pp-formulas from different equivalence classes of \sim_n^T . Hence, there is a pp-formula φ_3 with free variables x_1, \ldots, x_n such that exactly one of the two formulas $\varphi_1 \wedge \varphi_3$ and $\varphi_2 \wedge \varphi_3$ is satisfiable relative to T. This shows that φ_1 and φ_2 define over \mathfrak{A} distinct relations. But we know that for ω -categorical structures there is only a finite number of first-order definable relations of arity n, and in particular only a finite number of inequivalent pp-definable relations of arity n; a contradiction.

Proof of Theorem 4.1 (ii) \Rightarrow (iii). We show that every maximal pp-n-type p is determined completely by the \sim_n^T equivalence classes of the pp-formulas contained in p. Since there are finitely many such classes, the result follows. Let p and q be maximal pp-n-types s.t. for every $\varphi_1 \in p$, exists $\varphi'_1 \in q$ s.t. $\varphi_1 \sim_n^T \varphi'_1$ and for every $\varphi_2 \in q$, exists $\varphi'_2 \in p$ s.t. $\varphi_2 \sim_n^T \varphi'_2$. We aim to prove that p = q. If not then there exists, w.l.o.g., $\psi \in p$ s.t. $\psi \notin q$. Clearly, $T \cup p \cup \psi$ is satisfiable, and, since q is maximal, $T \cup q \cup \psi$ is not satisfiable. By compactness $T \cup \{\theta_q, \psi\}$ is not satisfiable for some finite conjunction θ_q of formulas from q. Now, $\theta_q \in q$ by maximality and there exists by assumption $\theta'_q \in p$ s.t. $\theta_q \sim_n^T \theta'_q$. By definition of \sim_n^T we deduce $T \cup \{\theta'_q, \psi\}$ satisfiable iff $T \cup \{\theta_q, \psi\}$ satisfiable. Since the latter is not satisfiable, we deduce that neither is the former, which yields the contradiction that $T \cup p \cup \psi$ is not satisfiable. Proof of Theorem 4.1 (iii) \Rightarrow (i). Let the number of maximal pp-n-types, μ_n , be finite for all n. We will show that T has an ω -categorical model. We consider the signature τ' , which is the expansion of τ by μ_n relations of each arity n, corresponding to the maximal pp-ntypes of T. Any model of T has a canonical (unique) expansion as a τ' -structure (by the new relation symbols labelling tuples that attain their type). Consider the canonical τ' -expansion \mathfrak{A}' of a countable epc τ -model \mathfrak{A} of T, guaranteed to exist by Proposition 3.4. We will shortly prove that \mathfrak{A}' is homogeneous. From this it will follow that \mathfrak{A}' is ω -categorical (since there is only a finite number of inequivalent atomic formulas of each arity n), whereupon ω -categoricity is inherited by its τ -reduct \mathfrak{A} .

It remains to prove that \mathfrak{A}' is homogeneous. A pp-formula $\varphi(\overline{x})$ is said to isolate a maximal pp-*n*-type $p(\overline{x})$ of T, if p is the only maximal pp-*n*-type of T of which φ is a member. If there is only a finite number of maximal pp-*n*-types of T, then it follows that each has an isolating formula. Let $f:(a_1,\ldots,a_m)\mapsto (b_1,\ldots,b_m)$ be a partial automorphism of \mathfrak{A}' (in the signature τ'). Let a' be an arbitrary element of A'. Consider the pp-*n*-types $p(x_1,\ldots,x_m)$ of (a_1,\ldots,a_m) and $q(x_1,\ldots,x_m,y)$ of (a_1,\ldots,a_m,a') in \mathfrak{A} . By Proposition 3.5, each of these types is maximal, and is isolated by the pp-formulas $\theta_p(x_1,\ldots,x_m)$ and $\theta_q(x_1,\ldots,x_m,y)$, respectively. Furthermore, the type of (b_1,\ldots,b_m) in \mathfrak{A} is p (as the partial automorphism of \mathfrak{A}' is in the signature τ'). But now, since $\exists y.\theta_q(x_1,\ldots,x_m,y) \in p$ (by maximality), we may deduce a b' s.t. $\mathfrak{A}' \models \theta_q(b_1,\ldots,b_m,b')$ and consequently $\mathfrak{A}' \models q(b_1,\ldots,b_m,b')$. It follows that $f': (a_1,\ldots,a_m,a') \mapsto (b_1,\ldots,b_m,b')$ is a partial automorphism of \mathfrak{A}' (in the signature τ'). A simple induction shows that we may extend to an automorphism of \mathfrak{A}' , and the result follows.

Lemma 5.2. If \mathfrak{M} is a saturated structure of cardinality $\kappa = \kappa^{\omega}$, then \mathfrak{M} has the homomorphism lifting property.

Proof. This follows immediately from Lemma 2.1, since $|M^{\omega}| \leq |M|$.

Lemma 5.3. For every τ -structure \mathfrak{A} there is a "monster" elementary extension $\mathfrak{M} \succeq \mathfrak{A}$ that is ω -saturated and has the homomorphism lifting property.

Proof. We will build \mathfrak{M} by transfinite induction, as the union of a chain of length \aleph_1 . Set $\mathfrak{M}_0 := \mathfrak{A}$. For successor ordinals γ^+ , we take an elementary extension $\mathfrak{M}_{\gamma^+} \succeq \mathfrak{M}_{\gamma}$ that is $|M_{\gamma}|^{\omega}$ -saturated (such always exists, see Theorem 4.3.12 [22]). For limit ordinals λ , set $\mathfrak{M}_{\lambda} := \bigcup_{\alpha < \lambda} \mathfrak{M}_{\alpha}$; finally, let $\mathfrak{M} := \mathfrak{M}_{\aleph_1}$.

 \mathfrak{M} is ω -saturated by construction. It remains to prove that \mathfrak{M} has the homomorphism lifting property. Consider the $b_1, \ldots, b_k \in M$ and $a_1, \ldots, a_k \in M^{\omega}$. The set of coordinates (of M) involved here,

$$A := \{b_1, \dots, b_k, a_1(1), a_1(2), \dots, a_2(1), a_2(2), \dots, \dots, a_k(1), a_k(2), \dots\},\$$

is of size $\leq \omega$. It follows that there is some $\mu < \aleph_1$ such that $A \subseteq M_{\mu}$ (this is why the chain used in the construction of \mathfrak{M} is of length \aleph_1).

Suppose we are given $a_1, \ldots, a_k \in \mathfrak{M}^{\omega}$ to $b_1, \ldots, b_k \in \mathfrak{M}$ such that all $\operatorname{pp}(\tau \cup \{c_1, \ldots, c_k\})$ sentences true in $(\mathfrak{M}^{\omega}; a_1, \ldots, a_k)$ are true in $(\mathfrak{M}; b_1, \ldots, b_k)$. Let f_{-1} be the partial map
from M^{ω} to M sending a_1, \ldots, a_k to b_1, \ldots, b_k . We first argue that all $\operatorname{pp}(\tau \cup \{c_1, \ldots, c_k\})$ sentences true in $(\mathfrak{M}_{\mu}^{\omega}; a_1, \ldots, a_k)$ are true in $(\mathfrak{M}_{\mu+1}; f_{-1}(a_1), \ldots, f_{-1}(a_k))$. Let φ be such

a sentence. Then:

$$\begin{aligned} (\mathfrak{M}_{\mu}{}^{\omega}; a_{1}, \ldots, a_{k}) &\models \varphi & \text{implies} \\ \text{for each } i & (\mathfrak{M}_{\mu}; a_{1}(i), \ldots, a_{k}(i)) \models \varphi & \text{implies} & [\text{since } \mathfrak{M}_{\mu} \preceq \mathfrak{M}] \\ \text{for each } i & (\mathfrak{M}; a_{1}(i), \ldots, a_{k}(i)) \models \varphi & \text{implies} \\ & (\mathfrak{M}^{\omega}; a_{1}, \ldots, a_{k}) \models \varphi & \text{implies} & [\text{by hypothesis}] \\ & (\mathfrak{M}; f_{-1}(a_{1}), \ldots, f_{-1}(a_{k})) \models \varphi & \text{implies} & [\text{since } \mathfrak{M} \succeq \mathfrak{M}_{\mu+1}] \\ & (\mathfrak{M}_{\mu+1}; f_{-1}(a_{1}), \ldots, f_{-1}(a_{k})) \models \varphi. \end{aligned}$$

It follows from Lemma 2.1 that there is a homomorphism $f_0: \mathfrak{M}_{\mu}^{\ \omega} \to \mathfrak{M}_{\mu+1}$ extending f_{-1} .

We will now proceed with a transfinite induction up to \aleph_1 . For successor ordinals, γ^+ , suppose that we have a homomorphism $f_{\gamma} : \mathfrak{M}_{\mu+\gamma}{}^{\omega} \to \mathfrak{M}_{\mu+\gamma^+}$. We will build a homomorphism $f_{\gamma^+} : \mathfrak{M}_{\mu+\gamma^+}{}^{\omega} \to \mathfrak{M}_{\mu+\gamma^{++}}$ extending f_{γ} . By the homomorphism f_{γ} , all pp- $(\tau \cup \{c_\alpha : \alpha < |M_{\mu+\gamma}|^{\omega}\})$ -sentences true in $(\mathfrak{M}_{\mu+\gamma}{}^{\omega}; \langle a_\alpha \rangle_{\alpha < |M_{\mu+\gamma}|^{\omega}})$ – where $\langle a_\alpha \rangle_{\alpha < |M_{\mu+\gamma}|^{\omega}}$ well-orders the elements of $\mathfrak{M}_{\mu+\gamma}{}^{\omega}$ – are true in $(\mathfrak{M}_{\mu+\gamma^+}; \langle f_{\gamma}(a_\alpha) \rangle_{\alpha < |M_{\mu+\gamma}|^{\omega}})$. It follows that all pp- $(\tau \cup \{c_\alpha : \alpha < |M_{\mu+\gamma}|^{\omega}\})$ -sentences true in $(\mathfrak{M}_{\mu+\gamma^+}; \langle a_\alpha \rangle_{\alpha < |M_{\mu+\gamma}|^{\omega}})$ are true in $(\mathfrak{M}_{\mu+\gamma^{++}}; \langle f(a_\alpha) \rangle_{\alpha < |M_{\mu+\gamma}|^{\omega}})$ – let φ be such a sentence, we give the derivation again:

	$(\mathfrak{M}_{\mu+\gamma^{+}}{}^{\omega};\langle a_{\alpha}\rangle_{\alpha< M_{\mu+\gamma} ^{\omega}})\models\varphi$	implies
for each \boldsymbol{i}	$(\mathfrak{M}_{\mu+\gamma^+};\langle a_{\alpha}(i)\rangle_{\alpha< M_{\mu+\gamma} ^{\omega}})\models\varphi$	implies
for each \boldsymbol{i}	$(\mathfrak{M}_{\mu+\gamma}; \langle a_{\alpha}(i) \rangle_{\alpha < M_{\mu+\gamma} ^{\omega}}) \models \varphi$	implies
	$(\mathfrak{M}_{\mu+\gamma}{}^{\omega};\langle a_{\alpha}\rangle_{\alpha< M_{\mu+\gamma} ^{\omega}})\models\varphi$	implies
	$(\mathfrak{M}_{\mu+\gamma^{+}};\langle f(a_{\alpha})\rangle_{\alpha< M_{\mu+\gamma} ^{\omega}})\models\varphi$	implies
	$(\mathfrak{M}_{\mu+\gamma^{++}};\langle f(a_{\alpha})\rangle_{\alpha< M_{\mu+\gamma} ^{\omega}})\models\varphi.$	

Now we can use Lemma 2.1 to derive some homomorphism $f_{\gamma^+} : \mathfrak{M}_{\mu+\gamma^+}^{\omega} \to \mathfrak{M}_{\mu+\gamma^{++}}$ extending f_{γ} . For limit ordinals λ , set $f_{\lambda} := \bigcup_{\alpha < \lambda} f_{\alpha}$.

Finally, we arrive at the homomorphism $f_{\aleph_1} : \mathfrak{M}^\omega \to \mathfrak{M}$, which has the desired property.

Lemma 5.4. For all structures \mathfrak{A} , $\langle \mathfrak{A} \rangle_{pp\infty} \subseteq Inv(Pol^{\infty}(\mathfrak{A}))$.

Proof. We argue by induction on the term-complexity of the formula. Let $f: A^{\alpha} \to A$ be a polymorphism of \mathfrak{A} .

(Base Case.) $\varphi(\overline{v}) := R(\overline{v})$. Trivial.

(Inductive Step.) There are two subcases. In the following, suppose \overline{v} is an *m*-tuple. Let $\langle \overline{a}_{\beta} \rangle_{\beta < \alpha}$, be a sequence of *m*-tuples from \mathfrak{A} such that $\varphi(\overline{a}_{\beta})$, for all β .

(Existential Quantification.) $\varphi(\overline{v}) := \exists u.\psi(\overline{v}, u)$. Suppose we have $\varphi(\overline{a}_{\beta})$ for each $\beta < \alpha$. From each $\exists u.\psi(\overline{a}_{\beta}, u)$, derive the witness a'_{β} for u and use the inductive hypothesis (IH) to deduce that $\psi(f(\langle \overline{a}_{\beta} \rangle_{\beta < \alpha}), f(\langle a'_{\beta} \rangle_{\beta < \alpha}))$. It follows that $\exists u.\psi(f(\langle \overline{a}_{\beta} \rangle_{\beta < \alpha}), u)$ and we are able to deduce $\varphi(f(\langle \overline{a}_{\beta} \rangle_{\beta < \alpha}))$.

(Infinite Conjunctions.) $\varphi(\overline{v}) := \bigwedge_{\mu < \gamma} \psi_{\mu}(\overline{v})$. Suppose we have $\varphi(\overline{a}_{\beta})$ for each $\beta < \alpha$. Then for each $\mu < \gamma$ and $\beta < \alpha$ we have $\psi_{\mu}(\overline{a}_{\beta})$. By IH, we have each $\psi_{\mu}(f(\langle \overline{a}_{\beta} \rangle_{\beta < \alpha}))$. The result $\varphi(f(\langle \overline{a}_{\beta} \rangle_{\beta < \alpha}))$ follows. **Lemma 5.5.** Let \mathfrak{A} have the homomorphism lifting property. Then a fo-definable relation R is preserved by the ω -polymorphisms of \mathfrak{A} if and only if R is pp-definable in \mathfrak{A} , i.e.

$$\operatorname{Inv}(\operatorname{Pol}^{\omega}(\mathfrak{A})) \cap \langle \mathfrak{A} \rangle_{\operatorname{fo}} = \langle \mathfrak{A} \rangle_{\operatorname{pp}}$$

Proof. (Backwards.) That pp-formulas are preserved by ω -polymorphisms in any structure is a special case of Lemma 5.4.

(Forwards.) Suppose that R is a k-ary relation that is preserved by all ω -polymorphisms of \mathfrak{A} and that has a first-order definition φ in \mathfrak{A} . Let

$$\Psi := \{ \psi(x_1, \dots, x_k) : \psi \text{ is a pp-}\tau \text{-formula s.t. } \mathfrak{A} \models \varphi(x_1, \dots, x_k) \to \psi(x_1, \dots, x_k) \}.$$

We first show, for all $b_1, \ldots, b_k \in A$, that, if $\mathfrak{A} \models \Psi(b_1, \ldots, b_k)$, then $\mathfrak{A} \models \varphi(b_1, \ldots, b_k)$.

Take $b_1, \ldots, b_k \in A$ s.t. $\mathfrak{A} \models \psi(b_1, \ldots, b_k)$ for each $\psi \in \Psi$; if such elements do not exist there is nothing to show. Let U be the set of all pp- τ -formulas $\theta(x_1, \ldots, x_k)$ such that $\mathfrak{A} \models \neg \theta(b_1, \ldots, b_k)$. If U is empty then every pp- τ -formula is true on b_1, \ldots, b_k ; in particular $\mathfrak{A} \models \varphi(b_1, \ldots, b_k)$, and we are done. We may assume U to be countably infinite. We claim that for every $\theta \in U$ there exists a k-tuple $\overline{a}^{\theta} := (a_1^{\theta}, \ldots, a_k^{\theta})$ from A such that $\mathfrak{A} \models \neg \theta(a_1^{\theta}, \ldots, a_k^{\theta}) \land \varphi(a_1^{\theta}, \ldots, a_k^{\theta})$. Otherwise, $\mathfrak{A} \models \varphi(x_1, \ldots, x_k) \rightarrow \theta(x_1, \ldots, x_k)$, and we derive $\theta \in \Psi$ and the consequent contradiction $\mathfrak{A} \models \theta(b_1, \ldots, b_k)$.

Consider the k-tuple $\overline{a} := \prod_{\theta \in U} \overline{a}^{\theta}$ in \mathfrak{A}^{ω} . Observe that every pp- τ -formula $\chi(x_1, \ldots, x_k)$ s.t. $\mathfrak{A}^{\omega} \models \chi(\overline{a})$ is s.t. $\mathfrak{A} \models \chi(b_1, \ldots, b_k)$. To see this, suppose that $\mathfrak{A} \models \neg \chi(b_1, \ldots, b_k)$. Therefore $\chi \in U$, and by choice of \overline{a}^{χ} we have $\mathfrak{A} \models \neg \chi(\overline{a}^{\chi})$. But then $\mathfrak{A}^{\omega} \models \neg \chi(\overline{a})$.

Now, we have just shown that all pp- $(\tau \cup \{c_1, \ldots, c_k\})$ -sentences that hold on $(\mathfrak{A}^{\omega}; \overline{a})$ also hold on $(\mathfrak{A}; b_1, \ldots, b_k)$. Since \mathfrak{A} has the homomorphism lifting property, the existence of a homomorphism $f : (\mathfrak{A}^{\omega}; \overline{a}) \to (\mathfrak{A}; b_1, \ldots, b_k)$ follows from our definitions. But f is an ω -polymorphism of \mathfrak{A} , which preserves φ , and hence we derive $\mathfrak{A} \models \varphi(b_1, \ldots, b_k)$.

It remains to be shown that Ψ is equivalent on \mathfrak{A} to a single pp-formula. Note that $\Psi(c_1, \ldots, c_k) \cup \{\neg \varphi(c_1, \ldots, c_k)\} \cup \operatorname{Th}(\mathfrak{A})$ is unsatisfiable; for otherwise there is a $\mathfrak{B} \models \operatorname{Th}(\mathfrak{A})$ and $b'_1, \ldots, b'_k \in B$, s.t. $(\mathfrak{B}; b'_1, \ldots, b'_k) \models \Psi(c_1, \ldots, c_k)$ and $(\mathfrak{B}; b'_1, \ldots, b'_k) \models \neg \varphi(c_1, \ldots, c_k)$. In both Cases 1 and 2, \mathfrak{A} is ω -saturated, and this yields some $b''_1, \ldots, b''_k \in A$ s.t. $(\mathfrak{A}; b''_1, \ldots, b''_k) \models \neg \varphi(c_1, \ldots, c_k)$, which is a contradiction. By compactness of first-order logic there is a finite subset Ψ' of Ψ such that $\Psi'(c_1, \ldots, c_k) \cup \{\neg \varphi(c_1, \ldots, c_k)\} \cup \operatorname{Th}(\mathfrak{A})$ is unsatisfiable, i.e. $\Psi'(c_1, \ldots, c_k) \cup \operatorname{Th}(\mathfrak{A}) \models \varphi(c_1, \ldots, c_k)$. Set $\psi'(x_1, \ldots, x_k) := \bigwedge_{\psi \in \Psi'} \psi(x_1, \ldots, x_k)$, to derive $\operatorname{Th}(\mathfrak{A}) \models \psi'(x_1, \ldots, x_k) \to \varphi(x_1, \ldots, x_k)$.

Lemma 5.9.

- 1. There is a saturated structure \mathfrak{A} of cardinality 2^{ω} such that $\operatorname{Inv}(\operatorname{Pol}^{\omega}(\mathfrak{A})) \neq \langle \mathfrak{A} \rangle_{\operatorname{pp}}$.
- 2. There is a saturated structure \mathfrak{A} of cardinality 2^{ω} such that $\operatorname{Inv}(\operatorname{Pol}(\mathfrak{A})) \cap \langle \mathfrak{A} \rangle_{fo} \neq \langle \mathfrak{A} \rangle_{pp}$.
- 3. There is a structure \mathfrak{A} such that $\operatorname{Inv}(\operatorname{Pol}^{\omega}(\mathfrak{A})) \cap \langle \mathfrak{A} \rangle_{\mathrm{fo}} \neq \langle \mathfrak{A} \rangle_{\mathrm{pp}}$.

Necessity of intersection with FO. Let us consider the model $\mathfrak{A} =: (\mathbb{Q}; +, 1, (u = v \lor x = y))$. By Lemma 6.2, the infinitary polymorphisms of this structure are equivalent to its endomorphisms, and, in the presence of a fixed 1, it can easily be seen that its only endomorphism is the identity (indeed, there is a pp-definition of each of the rationals from 1 and +). It follows that all subsets of \mathbb{Q} are in $Inv(Pol^{\omega}(\mathfrak{A}))$, yet $\langle \mathfrak{A} \rangle_{pp}$ must be countable.

Of course, \mathfrak{A} is neither saturated nor of cardinality $\geq 2^{\omega}$. But the continuum of subsets of \mathbb{Q} will remain Inv-Pol^{ω} in a saturated model of Th(\mathfrak{A}) of such cardinality (a copy of \mathfrak{A} sits in all models of its theory). The existence of a saturated model of Th(\mathfrak{A}) of cardinality 2^{ω} follows from this theory's strong minimality (Fact 1.2. in [32]).

Necessity of infinitary polymorphisms. Let $\{U_i : i \in \omega\}$ be a set of unary relations. Consider the model $\mathfrak{A} := (\mathbb{N}; U_i : i \in \omega)$, involving a countable set of unary relations, defined by $U_i := \mathbb{N} \setminus \{0, i\}$. Diagrammatically,

	U_1	U_2	U_3	• • •
0	×	×	×	• • •
1	×			• • •
2		×		• • •
3			×	• • •
:	:	:	:	

Consider the first-order definable unary relation $P(v) := U_1(v) \vee U_2(v)$, i.e. $P := \mathbb{N} \setminus \{0\}$. It is straightforward to verify that P is closed under the finitary polymorphisms of \mathfrak{A} and is not pp-definable over \mathfrak{A} . Note that P is not preserved under the infinitary polymorphism $f : \mathbb{N}^{\omega} \to \mathbb{N}$ of \mathfrak{A} defined by $f(\overline{w}) = 0$, if \overline{w} contains all elements of $\mathbb{N} \setminus \{0\}$, and $f(\overline{w}) = w_0$ (the first element of the sequence \overline{w}), otherwise. Again, these properties will remain if we move to a saturated model \mathfrak{A}_{sat} of cardinality 2^{ω} (such a model will simply be \mathfrak{A} augmented with a continuum of elements for which all of the relations $\{U_i : i \in \omega\}$ hold).

We now detail a finite signature variant of the above structure that also serves as a suitable (counter)example. Consider the signature $\langle E, R \rangle$ involving two binary relations, edge and red edge. Let the structure \mathfrak{A} contain

• a directed ω -*E*-path: i.e., vertices $\{[0, i] : i < \omega\}$ and *E*-edges $\{([0, i], [0, i + 1]) : i < \omega\}$.

and for each $j < \omega$:

• a directed ω -*E*-path with overlaid undirected *R*-path omitting only the *j*th edge: i.e. vertices $\{[j,i] : i < \omega\}$ with *E*-edges $\{([j,i], [j,i+1]) : i < \omega\}$ and *R*-edges $\{([j,i], [j,i+1]), ([j,i+1], [j,i]) : i < \omega, i+1 \neq j\}$.

Consider the first-order definable unary relation $P(v) := \exists x, y. R(v, x) \lor (E(v, x) \land R(x, y))$. It is not hard to verify that P is preserved by the finitary polymorphisms of \mathfrak{A} , but is not pp-definable over \mathfrak{A} (as it is not preserved by the ω -polymorphisms of \mathfrak{A}). These properties transfer to the saturated elementary extension \mathfrak{A}_{sat} of cardinality 2^{ω} .

Necessity of highly saturated structures. Consider the structure $\mathfrak{A} := (\mathbb{Q}; x = 1, x < 0, S_2(x, y))$, where $S_2 := \{(x, y) : 2x < y, 0 < y \leq 1\}$. Now, $x \leq 0$ is clearly first-order definable in \mathfrak{A} . It is also in $Inv(Pol^{\omega}(\mathfrak{A}))$, being definable by the following infinite conjunction of pp-formulas in one free variable (see Lemma 5.4).

$$\bigwedge_{i\in\omega} \exists z \ \exists y_1,\ldots,y_i. \ S_2(x,y_1) \land S_2(y_1,y_2) \land \ldots \land S_2(y_i,z) \land z = 1.$$

We will now argue that it is not pp-definable.

Lemma. Let $\overline{x} := (x_1, \ldots, x_k)$ and suppose that $\varphi(\overline{x}) \in \langle \mathfrak{A} \rangle_{\text{pp}}$. If $\mathfrak{A} \models \varphi(\overline{a})$ and $a_{\lambda_1}, \ldots, a_{\lambda_j}$ list exactly the elements of \overline{a} that are 0, then there exists $\epsilon > 0$ such that, for all $\epsilon \ge \delta \ge 0$, $\mathfrak{A} \models \varphi(\overline{a}[a_{\lambda_1}/\delta, \ldots, a_{\lambda_j}/\delta])$.

Proof. By induction on the term complexity of φ .

(Base Cases.) φ is an atom. The statement is trivially true if $\varphi(x) := x = 1, x < 0$ or x = x. Suppose $\varphi(x_1, x_2) := S_2(x_1, x_2)$; if $S_2(a_1, a_2)$, then only a_1 could be zero. Set $\epsilon := a_2/2.$

(Inductive Step.) There are two subcases.

 $\varphi(\overline{x}) := \psi_1(\overline{x}) \wedge \psi_2(\overline{x})$. There exist respective witnesses ϵ_1 and ϵ_2 for $\psi_1(\overline{a})$ and $\psi_2(\overline{a})$: we may set $\epsilon := \min\{\epsilon_1, \epsilon_2\}$ as the witness for $\varphi(\overline{a})$.

 $\varphi(\overline{x}) := \exists y.\psi(y,\overline{x})$. If $\varphi(\overline{a})$ holds, then we may choose a b s.t. $\psi(b,\overline{a})$. By inductive hypothesis, there exists an appropriate ϵ for $\psi(b, \overline{a})$ and this may also be used for $\varphi(\overline{a})$.

That $x \leq 0$ is not pp-definable is a trivial consequence of the lemma, for suppose it were defined by $\varphi(x)$. Since $\varphi(0)$ holds, we may derive the contradiction that $\varphi(\epsilon)$ holds for some $\epsilon > 0$. Note that the first part of the inductive step in the previous lemma would fail for infinite conjunctions. Finally, suppose \mathfrak{A}_{sat} were a saturated model of $Th(\mathfrak{A})$ of cardinality $\geq 2^{\omega}$. While we have $\langle \mathfrak{A} \rangle_{\rm fo} \cap \operatorname{Inv}(\operatorname{Pol}^{\omega}(\mathfrak{A})) \neq \langle \mathfrak{A} \rangle_{\rm pp}$, we must have $\langle \mathfrak{A}_{\rm sat} \rangle_{\rm fo} \cap \operatorname{Inv}(\operatorname{Pol}^{\omega}(\mathfrak{A}_{\rm sat})) =$ $\langle \mathfrak{A}_{\mathrm{sat}} \rangle_{\mathrm{pp}}$. We note that $x \leq 0$ is not $\mathrm{Inv}(\mathrm{Pol}^{\omega}(\mathfrak{A}_{\mathrm{sat}}))$.

Lemma 6.1. A function $f: A^{\alpha} \to A$ is not essentially unary iff there exist two non-empty and disjoint $X, Y \subseteq \alpha$, such that both

- exist $\overline{x}, \overline{w}, \overline{w}' \in A^{\alpha}$ s.t. $f(\overline{x}[\overline{x}_X/\overline{w}_X]) \neq f(\overline{x}[\overline{x}_X/\overline{w}'_X])$, and exist $\overline{y}, \overline{z}, \overline{z}' \in A^{\alpha}$ s.t. $f(\overline{y}[\overline{y}_Y/\overline{z}_Y]) \neq f(\overline{y}[\overline{y}_Y/\overline{z}'_Y])$.

Proof. We will benefit from the following local definition. A set $Z \subseteq \alpha$ is termed good if the following holds: for all $\overline{x}, \overline{w}, \overline{w}' \in A^{\alpha}$ we have $f(\overline{x}[\overline{x}_Z/\overline{w}_Z]) = f(\overline{x}[\overline{x}_Z/\overline{w}_Z])$. If Z is not good, then we term it bad. Note that good sets are closed under union; i.e., if X and Yare both good, then so is $X \cup Y$. The contrapositive of the lemma is the assertion that f is essentially unary iff, for any two non-empty and disjoint $X, Y \subseteq \alpha$, at least one of X and Y is good.

(Backwards.) By contraposition. If f is essentially unary, then let β and g be s.t. $f(\overline{x}) = q(x_{\beta})$. Now, take any two non-empty and disjoint $X, Y \subseteq \alpha$. At least one does not contain β , and it must be a good set.

(Forwards.) By contraposition. Assume that, for any two non-empty and disjoint $X, Y \subseteq \alpha$, at least one of X and Y is good. If there are no bad subsets of α , i.e. f is constant, then clearly f is essentially unary. Assume the existence of some bad set. We will derive the existence of a bad set of cardinality 1; for otherwise let Z be a minimal bad set (under the total lexicographical order on the 0-1 characteristic sequence of length α) of cardinality greater than 1. Let Z_1 and Z_2 be a non-trivial partition of Z. At least one of Z_1 and Z_2 must be good, by assumption. Hence the other must be bad (as good sets are closed under union, and $Z := Z_1 \cup Z_2$ is bad), contradicting minimality of Z. Let $Z = \{\beta\}$ be a minimal bad set. Set

$$g(x_{\beta}) := f(x_{\beta}^{\alpha}) = f(x_{\beta}, x_{\beta}, \ldots),$$

i.e. each variable $x_{\gamma}, \gamma \leq \alpha$, is substituted by x_{β} (of course the choice of x_{β} as the variable here is not important). That $f(\overline{x}) = g(x_{\beta})$ now follows from $\alpha \setminus \{\beta\}$ being a good set.

Lemma 6.2. Let \mathfrak{A} be such that $(u = v \lor x = y) \in \langle \mathfrak{A} \rangle_{pp}$. Then all (finitary and infinitary) polymorphisms of \mathfrak{A} are essentially unary.

Proof. Let $P_4 := (u = v \lor x = y) \in \langle \mathfrak{A} \rangle_{\text{pp}}$. It follows from Lemma 5.4 that P_4 must be preserved by the polymorphisms of \mathfrak{A} . Suppose for contradiction that \mathfrak{A} has a polymorphism $f: M^{\alpha} \to M$ that is not essentially unary. From Lemma 6.1, we deduce non-empty and disjoint $X, Y \subseteq \alpha$, s.t. there exist $\overline{x}, \overline{w}, \overline{w}' \in A^{\alpha}$ with $f(\overline{x}[\overline{x}_X/\overline{w}_X]) \neq f(\overline{x}[\overline{x}_X/\overline{w}_X])$ and $\overline{y}, \overline{z}, \overline{z}' \in A^{\alpha}$ with $f(\overline{y}[\overline{y}_Y/\overline{z}_Y]) \neq f(\overline{y}[\overline{y}_Y/\overline{z}_Y])$. But, for each $\beta \in \alpha$,

 $P_4(\overline{x}[\overline{x}_X/\overline{w}_X]_\beta, \overline{x}[\overline{x}_X/\overline{w}_X']_\beta, \overline{y}[\overline{y}_Y/\overline{z}_Y]_\beta, \overline{y}[\overline{y}_Y/\overline{z}_Y']_\beta)$

holds, by disjointness of X and Y, while

$$P_4(f(\overline{x}[\overline{x}_X/\overline{w}_X]), f(\overline{x}[\overline{x}_X/\overline{w}_X']), f(\overline{y}[\overline{y}_Y/\overline{z}_Y]), f(\overline{y}[\overline{y}_Y/\overline{z}_Y']))$$

does not.

Lemma 6.3. Suppose \mathfrak{A} is such that $(x = y \lor u = v) \in \langle \mathfrak{A} \rangle_{pp}$. Then $\langle \mathfrak{A} \rangle_{pp} = \langle \mathfrak{A} \rangle_{ep}$.

Proof. The proof will be by simulation of the binary \vee . Take $\varphi \in \langle \mathfrak{A} \rangle_{ep}$ in prenex form; we will recursively remove disjunctions of the form

$$\psi_1(x_1,\ldots,x_n,y_1,\ldots,y_p)\vee\psi_2(x_1,\ldots,x_n,z_1,\ldots,z_q).$$

We may assume that each of ψ_1 and ψ_2 is alone satisfiable, for otherwise their disjunction is logically equivalent to just one of them. We will introduce new variables $x'_1, \ldots, x'_n, y'_1, \ldots, y'_p$ and $x''_1, \ldots, x''_n, z'_1, \ldots, z'_q$. Note that it follows from [7] that there is a $\theta \in \langle (A; x = y \lor u = v) \rangle_{\text{pp}}$ such that $\theta \equiv$

$$(x'_1 = x_1 \land \ldots \land x'_k = x_k \land y'_1 = y_1 \land \ldots \land y'_p = y_p) \lor (x''_1 = x_1 \land \ldots \land x''_k = x_k \land z'_1 = z_1 \land \ldots \land z'_q = z_q).$$

The disjunct $\psi_1 \vee \psi_2$ should be replaced with the following, in which the existential quantifiers should be read as all coming before the conjunction.

$$\exists x_1', \dots, x_n', y_1', \dots, y_p'. \quad \psi_1(x_1', \dots, x_n', y_1', \dots, y_p') \land \psi_2(x_1'', \dots, x_n'', z_1', \dots, z_q') \land \exists x_1'', \dots, x_n'', z_1', \dots, z_p'. \quad \theta(x_1', \dots, x_n', y_1', \dots, y_p', x_1'', \dots, x_n'', z_1', \dots, z_q')$$

Lemma 6.5. The only ω -polymorphisms of $(\mathbb{Q}; +, 1, \neq)$ are projections.

Proof. We give the proof for polymorphisms of arity ω , but the argument works just as well for any infinite or finite arity. A function $f: D^{\omega} \to D$ is *idempotent* if $f(d, d, \ldots) = d$, for all $d \in D$. It is *conservative* if it further satisfies $f(d_1, d_2, \ldots) \in \{d_1, d_2, \ldots\}$, for all $d_1, d_2, \ldots \in$ D. Let $f: \mathbb{Q}^{\omega} \to \mathbb{Q}$ be a polymorphism of $(\mathbb{Q}; +, 1, \neq)$. It is clear that f is idempotent as the only endomorphism of $(\mathbb{Q}; +, 1)$ is the identity. Further, by preservation of \neq , it is easy to see that f must be conservative. Consider $\{0, 1\}^{\omega}$ with the total lexicographical ordering indduced by 0 < 1. Choose some minimal $\langle z_\lambda \rangle_{\lambda < \omega} \in \{0, 1\}^{\omega}$ s.t. $f(\langle z_\lambda \rangle_{\lambda < \omega}) = 1$ (since $f(1, 1, \ldots) = 1$, such a $\langle z_\lambda \rangle_{\lambda < \omega}$ exists). If $\langle z_\lambda \rangle_{\lambda < \omega}$ had more than one index that is a 1, then there would exist $\langle z'_\lambda \rangle_{\lambda < \omega}$ and $\langle z''_\lambda \rangle_{\lambda < \omega}$ s.t. $\langle z'_\lambda \rangle_{\lambda < \omega}, \langle z''_\lambda \rangle_{\lambda < \omega} = \langle z_\lambda \rangle_{\lambda < \omega}$ and $\langle z'_\lambda \rangle_{\lambda < \omega} + \langle z''_\lambda \rangle_{\lambda < \omega} = \langle z_\lambda \rangle_{\lambda < \omega}$ and so, by preservation of +, one of $\langle z'_\lambda \rangle_{\lambda < \omega}, \langle z''_\lambda \rangle_{\lambda < \omega} = 1$, contradicting minimality of $\langle z_\lambda \rangle_{\lambda < \omega}$. So, for some $i, \langle z_\lambda \rangle_{\lambda < \omega}$ is of the form

$$(0,\ldots,0,\overbrace{1}^{i \text{th position}},0,\ldots)$$

By preservation of +, it follows, for each $q \in \mathbb{Q}$, that $f(q \cdot \langle z_{\lambda} \rangle_{\lambda < \omega}) = q$.

Firstly, we consider $\langle x_{\lambda} \rangle_{\lambda < \omega} \in \mathbb{Q}^{\omega}$ s.t. $q \notin \{x_{\lambda} : \lambda < \omega\} \neq \mathbb{Q}$. If $f(\langle x_{\lambda} \rangle_{\lambda < \omega}) = p \neq x_i$, then, by preservation of +,

$$f(\langle x_{\lambda} \rangle_{\lambda < \omega} + (q - p) \langle z_{\lambda} \rangle_{\lambda < \omega}) = f(\langle x_{\lambda} \rangle_{\lambda < \omega}) + (q - p) f(\langle z_{\lambda} \rangle_{\lambda < \omega}) = q$$

But this violates conservativity of f as q does not appear in $\langle x_{\lambda} \rangle_{\lambda < \omega} + (q-p) \langle z_{\lambda} \rangle_{\lambda < \omega}$ (since $p \neq x_i$).

Finally, we take an arbitrary $\langle x_{\lambda} \rangle_{\lambda < \omega} \in \mathbb{Q}^{\omega}$. Consider the set $\Lambda := \{\lambda : x_{\lambda} = 1\}$ and $\langle x'_{\lambda} \rangle_{\lambda < \omega}$ and $\langle x''_{\lambda} \rangle_{\lambda < \omega}$ obtained according to $x'_{\lambda} = x_{\lambda}$, if $\lambda \notin \Lambda$, and = 0 otherwise; and $x''_{\lambda} = 1$, if $\lambda \in \Lambda$, and = 0 otherwise. Clearly $\langle x_{\lambda} \rangle_{\lambda < \omega} = \langle x'_{\lambda} \rangle_{\lambda < \omega} + \langle x''_{\lambda} \rangle_{\lambda < \omega}$, and $\langle x'_{\lambda} \rangle_{\lambda < \omega}$ and $\langle x''_{\lambda} \rangle_{\lambda < \omega}$ satisfy the condition of the previous paragraph, i.e. that neither $\{x'_{\lambda} : \lambda < \omega\}$ nor $\{x''_{\lambda} : \lambda < \omega\}$ is \mathbb{Q} . The result follows by preservation of +.

Theorem 6.7. Let \mathfrak{A} be a "monster" elementary extension with a finite signature. Then $CSP(\mathfrak{A})$ is first-order definable if and only if \mathfrak{A} has a relational near-unanimity polymorphism.

Proof. In fact we need both a local lemma and some local definitions.

Local lemma. Let \mathfrak{M} be a "monster" extension. If all finite substructures \mathfrak{C} of ${}^{1}\mathfrak{M}^{n+1}$ map homomorphically to \mathfrak{M} , then ${}^{1}\mathfrak{M}^{n+1}$ maps homomorphically to \mathfrak{M} .

Proof of local lemma. We note, for structures \mathfrak{A} and \mathfrak{B} , that if \mathfrak{A} and \mathfrak{B} are elementarily equivalent, then so are ${}^{1}\mathfrak{A}^{n+1}$ and ${}^{1}\mathfrak{B}^{n+1}$ (as ${}^{1}\mathfrak{A}^{n+1}$ is fo-definable in \mathfrak{A}). The assumption of the lemma may be restated as that all pp- τ -sentences true in ${}^{1}\mathfrak{M}^{n+1}$ are true in \mathfrak{M} . Let \mathfrak{M}_{0} be the structure from which the "monster" was originally built. We deduce that all pp- τ -sentences true in ${}^{1}\mathfrak{M}_{0}^{n+1}$ are true in \mathfrak{M}_{1} , and, using Lemma 2.1, the consequent homomorphism $f_{0}: {}^{1}\mathfrak{M}_{0}^{n+1} \to \mathfrak{M}_{1}$. Now we undertake the transfinite induction as before. In fact, the successor steps are simpler: as when all pp- $\tau \cup \{c_{\alpha}: \alpha < |M_{\gamma}|^{n+1}\}$)-sentences true in $({}^{1}\mathfrak{M}_{\gamma}{}^{n+1}; \langle a_{\alpha} \rangle_{\alpha < |M_{\gamma}|^{n+1}})$ are true in $(\mathfrak{M}_{\gamma^{+}}; \langle a_{\alpha} \rangle_{\alpha < |M_{\gamma}|^{n+1}})$, it follows immediately, by the properties of elementary extension, that all pp- $\tau \cup \{c_{\alpha}: \alpha < |M_{\gamma}|^{n+1}\}$)-sentences true in $({}^{1}\mathfrak{M}_{\gamma^{+}}{}^{n+1}; \langle a_{\alpha} \rangle_{\alpha < |M_{\gamma}|^{n+1}})$ are true in $(\mathfrak{M}_{\gamma^{+}+1}; \langle a_{\alpha} \rangle_{\alpha < |M_{\gamma}|^{n+1}})$. Limit ordinal steps proceed as before. Note that we do not actually need the transfinite induction up to \aleph_{1} here – though it does no harm – an induction up to ω would have sufficed. \Box

A finite τ -structure \mathfrak{C} is an obstruction for the τ -structure \mathfrak{A} if there is no homomorphism from \mathfrak{C} to \mathfrak{A} . A family \mathfrak{F} of obstructions for \mathfrak{A} is called a *complete set of obstructions* if for every τ -structure \mathfrak{B} that does not admit a homomorphism to \mathfrak{A} there exists some $\mathfrak{C} \in \mathfrak{F}$ which admits a homomorphism to \mathfrak{B} . The structure \mathfrak{A} is said to have *finite duality* if it admits a finite complete set of obstructions. An obstruction \mathfrak{C} for \mathfrak{A} is called *critical* if every proper (not necessarily induced) substructure of \mathfrak{C} admits a homomorphism to \mathfrak{A} . For any set A, let pr_k^n denote the projection map from A^n to A which maps any tuple to its k-th coordinate. We claim the following (essentially from [21]:

Claim. If there exists an (n+1)-ary relational near-unanimity polymorphism of \mathfrak{A} then the critical obstructions of \mathfrak{A} have at most n hyperedges. If \mathfrak{A} is a "monster" extension, the converse holds as well.

Proof of Claim. (Forwards.) By contraposition. Let \mathfrak{C} be a critical obstruction of \mathfrak{A} with m distinct hyperedges $e_1, \ldots, e_m, m > n$. Then for $k \in \{1, \ldots, m\}$, the τ -structure \mathfrak{C}_k obtained from \mathfrak{C} by removing e_k (without changing the domain) admits a homomorphism h_k to \mathfrak{A} . By definition of ${}^{1}\mathfrak{A}^m$, the map $h = (h_1, \ldots, h_m)$ is a homomorphism from \mathfrak{C} to

 ${}^{1}\mathfrak{A}^{m}$. Therefore there is no homomorphism from ${}^{1}\mathfrak{A}^{m}$ to \mathfrak{A} , and in particular none from ${}^{1}\mathfrak{A}^{n+1}$ to \mathfrak{A} .

(Backwards.) Conversely, suppose that \mathfrak{A} is a "monster" extension, and that there is no homomorphism from ${}^{\mathfrak{A}}\mathfrak{A}^{n+1}$ to \mathfrak{A} . It follows from our local lemma that there exists a finite substructure \mathfrak{C} of ${}^{\mathfrak{A}}\mathfrak{A}^{n+1}$ which has no homomorphism to \mathfrak{A} . Hence, \mathfrak{C} is an obstruction of \mathfrak{A} which admits a homomorphism h to ${}^{\mathfrak{A}}\mathfrak{A}^{n+1}$. Let \mathfrak{C}' be a (not necessarily induced) substructure of \mathfrak{C} that is critical (such a \mathfrak{C}' always exists). For every $k \in \{1, \ldots, n+1\}$ there exists a hyperedge e_k of \mathfrak{C}' which is not preserved by $\mathrm{pr}_k^{n+1} \circ h$, since $\mathrm{pr}_k^{n+1} \circ h$ is not a homomorphism from \mathfrak{C} to \mathfrak{A} . By definition of ${}^{\mathfrak{A}}\mathfrak{A}^{n+1}$, e_k is respected by $\mathrm{pr}_j^{n+1} \circ h$ for every $j \neq k$, and thus $e_j \neq e_k$ for $j \neq k$. Therefore \mathfrak{C} has at least n+1 hyperedges. \Box

Proof of Theorem 6.7. (Forwards.) Suppose first that $\text{CSP}(\mathfrak{A})$ is first-order definable. Since $\text{CSP}(\mathfrak{A})$ is a class of finite structures that is closed under inverse homomorphisms, by the dual version of Rossman's Theorem, [28], there is a universal first-order τ -sentence φ that holds on a finite structure \mathfrak{B} if and only if \mathfrak{B} homomorphically maps to \mathfrak{A} . Bringing φ into prenex negation normal form, it is straightforward to read from φ a finite complete set \mathcal{F} of obstructions to \mathfrak{A} . Let m be the maximal number of hyperedges in the obstructions from \mathcal{F} . By the claim above, since \mathfrak{A} is a "monster", there is a homomorphism from ${}^{1}\mathfrak{A}^{m+1}$ to \mathfrak{A} . This is by definition a relational near-unanimity polymorphism of \mathfrak{A} .

(Backwards.) Now suppose that for some n the structure ${}^{1}\mathfrak{A}^{n+1}$ admits a homomorphism to \mathfrak{A} . By the claim above the critical obstructions of \mathfrak{A} have at most n hyperedges. Since our signature is finite and relational, this implies that there are finitely many critical obstructions to \mathfrak{A} . This implies that the set of all critical obstructions is a finite obstruction set for \mathfrak{A} . It is now straightforward to write down a (universal) first-order definition of $\mathrm{CSP}(\mathfrak{A})$.

Proposition 6.9. Let \mathfrak{A} be a structure with a binary injective polymorphism e that is an embedding from \mathfrak{A}^2 into \mathfrak{A} . Then a relation R that is quantifier-free definable in the relations of \mathfrak{A} is preserved by e iff it admits a quantifier-free Horn definition in \mathfrak{A} .

Proof. In this proof $\overline{x}_1 \dots, \overline{x}_k$ should be read as variable subtuples of the variable tuple \overline{x} . Likewise with the element subtuples $\overline{a}_1 \dots, \overline{a}_k$ of \overline{a} .

(Forwards.) Let \mathcal{F} be a Horn definition of R. Suppose \overline{a} and $\overline{a}' \in R^{\mathfrak{A}}$. It suffices to demonstrate the preservation of each clause in \mathcal{F} of the form $(R_1(\overline{x}_1 \wedge \ldots \wedge R_l(\overline{x}_l)) \rightarrow R_{l+1}(\overline{x}_{l+1}), \text{ for } R_1, \ldots, R_{l+1} \in \mathfrak{A}.$

If the fomer clauses are true, there are two cases. Either some antecedent $R_i(\overline{a}_i)$ or $R_i(\overline{a}'_i)$ is false, in which case $R_i(e(\overline{a}_i, \overline{a}'_i))$ is false, and the latter clause is true. Or, if all antecedents in both former clauses are true, then both $R_{l+1}(\overline{a}_{l+1})$ and $R_{l+1}(\overline{a}'_{l+1})$ are true, so it follows that $R_{l+1}(e(\overline{a}_{l+1}, \overline{a}'_{l+1}))$ is true, and and the latter clause is true.

(Backwards.) Consider a CNF definition \mathcal{F} of R in \mathfrak{A} that is irreducible in the sense that it has no redundant literals in its clauses (or, indeed, redundant clauses). Because it can not be Horn, there exists a clause $R_1(\overline{x}_1) \vee R_2(\overline{x}_2) \vee S_3(\overline{x}_3) \vee \ldots \vee S_l(\overline{x}_l)$, with R_1, R_2 positive literals S_3, \ldots, S_l positive or negative literals, with $\overline{a}, \overline{a}' \in R^{\mathfrak{A}}$ s.t.

$$R_1(\overline{a}_1) \wedge \neg R_2(\overline{a}_2) \wedge \neg S_3(\overline{a}_3) \wedge \ldots \wedge \neg S_l(\overline{a}_l) \\ \neg R_1(\overline{a}_1) \wedge R_2(\overline{a}_2) \wedge \neg S_3(\overline{a}_3) \wedge \ldots \wedge \neg S_l(\overline{a}_l)$$

Consider the tuple $e(\overline{a}, \overline{a}')$. Clearly it will fail to satisfy the clause.