Higher Semantics of Quantum Protocols

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***Abstract*—We propose a higher semantics for the description of quantum protocols, which deals with quantum and classical information in a unified way. Central to our approach is the modelling of classical data by information transfer to the environment, and the use of 2-category theory to formalize the resulting framework.**

This 2-categorical semantics has a graphical calculus, the diagrams of which correspond exactly to physically-implementable quantum procedures. Quantum teleportation in its most general sense is reformulated as the ability to remove correlations between a quantum system and its environment, and is represented by an elegant graphical identity. We use this new formalism to describe two new families of quantum protocols.

***Index Terms*—quantum computing, category theory, higher categories**

I. Introduction

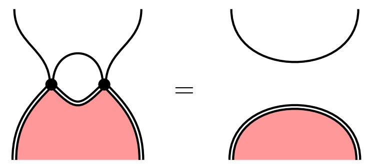
A. Overview and background

This work extends the pioneering contribution *Categorical Semantics of Quantum Protocols* [2] of Abramsky and Coecke, the first quantum information paper ever accepted for LiCS. The higher semantics described here adds expressiveness: composite expressions correspond exactly to physically-possible quantum informatic procedures, interference between difference measurement branches being prevented by the formalism. The powerful graphical calculus central to [2] is extended for our new semantics, and again plays a central organizing role.

A key insight of Abramsky and Coecke was that quantum information has a fundamentally *topological* nature. They also made clear the relevance of *monoidal category theory* as the appropriate formalism for making this precise [1, 2, 3, 4, 10]. The essential idea is that a quantum protocol gives rise to a flow of quantum information, which may not always be directed forwards in time, and that the overall effect of carrying out the protocol can in the simplest cases depend only on the *topology* of this flow. We take this topological perspective further by extending it to the flow of classical information. This is achieved mathematically by extending the monoidal category framework to *2-categorical* one.

The 2-category $\mathbf{2Hilb}$ in which our semantics is formulated was initially described by Baez [5], who has also emphasized, with collaborators, the great potential importance of higher-categorical methods for computer science, logic, topology and physics [6, 7]. This is part of a growing general understanding of the deep connections between all of these topics, with an important role played by higher category theory in providing the appropriate formal language to work with them. This background provided crucial motivation and perspective for the present work.

To give an example of our new scheme, we present the following diagrammatic equation which represents quantum teleportation (see Section IV-B for more detail):

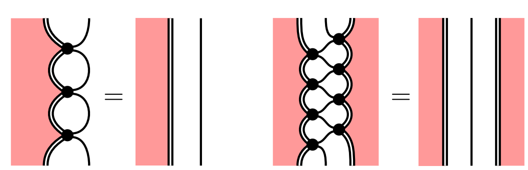


(1)

The left and right sides of this equation rigorously define particular composites of 2-cells. The shaded region represents the presence of a nontrivial *environment*, which records information about our quantum systems and leads to an effective quantum 'branching'. Once teleportation has been described in this form, it is a matter of pure mathematics to ascertain whether such a protocol can be physically implemented — that is, whether a solution to this equation can actually be found in $\mathbf{2Hilb}$. Since quantum teleportation is possible, we know that it can [8].

Our formalism ensures that any topological deformation of this equation will give a valid protocol, with the details of the physical implementation depending significantly on which deformation is chosen. Conventional quantum teleportation corresponds to one particular deformation.

We might well then consider: are there interesting generalizations of this equation? Do these also have solutions in $\mathbf{2Hilb}$? These are questions belonging to the realm of abstract algebra, and are of interest in their own right. The formalism of this paper ensures that any answers will be directly relevant for quantum information. We make a first step towards answering these questions by demonstrating that the following equations have solutions in $\mathbf{2Hilb}$:



(59,62)

The first of these is a variant of ordinary teleportation: while that has two stages, traditionally a measurement and a unitary correction, this new protocol has three. The second equation represents a certain interaction between a qubit and *two* regions of classical data, whose overall effect is for the qubit to be successfully transmitted.

These two equations represent only a small sample of the possible protocols which could be written down using this formalism. There is a large range of possible forms these could take, and the current state of the art in quantum information tells us very little about which of these forms give rise to implementable quantum protocols. We hope that further work on this question will give us a better understanding of the deep relationship between quantum information and topology.

B. Relationship to other work

There has been substantial activity in the past few years on the construction of categorical models encoding the interaction of quantum and classical data. One main body of work focuses on taking spaces of mixed states as the fundamental objects, and completely-positive maps as the appropriate notion of dynamical evolution. Selinger and others have described an abstract formalization of this perspective [11, 12, 20] in which classical and quantum data sit alongside each other, and Coecke and Perdrix have used this framework to characterize classical data as a quantum system equipped with a coupling to an environmental degree of freedom, which is then explicitly traced out [15]. Our model is different: classical information is always carried by a joint pure state, in the form of entanglement between the local system and its environment. The two approaches can be compared with the Churches of the Smaller and Larger Hilbert Space, to use phrases which find frequent informal use in the quantum information community.

On a technical level, our work builds on the results of Abramsky, Coecke and Pavlovic on categorical quantum mechanics [2, 4, 13]. Important tools developed as part of that programme continue to play a central role here, including classical structures and their modules.

C. Outline of paper

We begin in Section II by describing a simple model for the transfer of data from quantum systems to their environment. Section III then describes how this model gives rise to a 2-category **2Hilb** in a natural fashion, and introduces our graphical notation. Quantum teleportation is the focus of Section IV, and in Section V we describe two new families of quantum protocols.

II. Modelling Information Transfer to the Environment

A. Introduction

Our model is built around a quantum system, its dynamics, its environment, and the interactions between these. When talking in general terms, we will use S and E to refer to our system and its environment respectively, and will also let these symbols stand for their Hilbert spaces of quantum states where appropriate.

The model does not describe arbitrary interactions between a system and its environment, but only those which cause information to be 'transferred' in a particularly straightforward way. Using the terminology of the *decoherence* programme, this is comparable to the situation when environmental interactions select out *robust states* of a quantum systems, with classical properties (see [19] for a survey.) This is a scenario in which the notion of 'classical data' extracted by the environment can most clearly be formalized, and is sufficient for modelling the projector-valued measurements that play a central role in quantum information theory.

B. Classical data types

A *classical data type* is a Hilbert space V equipped with *copying* and *deleting* maps

$$\delta \colon V \longrightarrow V \otimes V \tag{2}$$

$$\epsilon \colon V \longrightarrow \mathbb{C} \tag{3}$$

satisfying associativity, unit, and commutativity laws:

$$(\delta \otimes \mathrm{id}_V) \circ \delta = (\mathrm{id}_V \otimes \delta) \circ \delta \tag{4}$$

$$(\epsilon \otimes \mathrm{id}_V) \circ \delta = \mathrm{id}_V = (\mathrm{id}_V \otimes \epsilon) \circ \delta \tag{5}$$

$$\mathrm{swap}_{V,V} \circ \delta = \delta \tag{6}$$

Here we make use of the map

$$\mathrm{swap}_{V,V} \colon V \otimes V \longrightarrow V \otimes V,$$

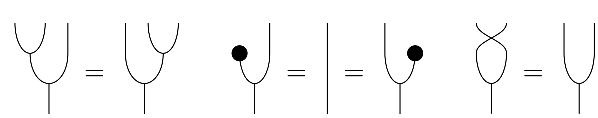
which we define by its action $|\phi\rangle \otimes |\psi\rangle \mapsto |\psi\rangle \otimes |\phi\rangle$ on product states. A common name for this structure is a *commutative coalgebra* (or *commutative comonoid*).

In our model, we assume that the environment E is given the structure of a classical data type. Physically, this is motivated by the idea that the environment stores certain kinds of information about our quantum system in a highly redundant manner, yielding these effective copying and deleting structures.

We will use a graphical notation to describe our algebraic operations [16, 21]. In this notation vertical lines represent Hilbert spaces, and vertices represent linear maps. Horizontal juxtaposition represents the tensor product operation, and vertical juxtaposition represents composition. Using this notation, our copying and deleting maps have the following form, which should be read from bottom to top:


(7)

Our axioms (4–6) are then depicted in the following way:

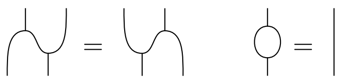

(8)

This graphical form of the axioms has the advantage of being more immediately comprehensible than the traditional algebraic presentation.

We will further require our classical data type to be a *special †-Frobenius algebra*, also called a *classical structure* by Coecke and Pavlovic [13]. The axioms for this involve the adjoints $\delta^\dagger : V \otimes V \longrightarrow V$ and $\epsilon^\dagger : \mathbb{C} \longrightarrow V$ of our copying and deleting maps, which are drawn in the following way:

(9)

The classical structure axioms then take the following form:



(10)

These axioms ensure that V is finite-dimensional, that δ acts by copying the elements of some orthonormal basis of V, and that ϵ acts by deleting them [14]:

$$\delta \colon |i\rangle \mapsto |i\rangle \otimes |i\rangle \tag{11}$$

$$\epsilon \colon |i\rangle \mapsto 1 \tag{12}$$

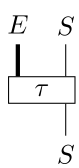
This is a significant restriction, but one which is appropriate for describing quantum informatic phenomena. It would be interesting to consider more general classical data types, especially for modelling *continuous* data; this would require dropping the classical structure axioms (10) above.

C. Environmental interactions

We now suppose that our system S and environment E interact, in such a way that information about the state of the system is transferred to the environment. To formalize this notion, we consider linear maps of the form

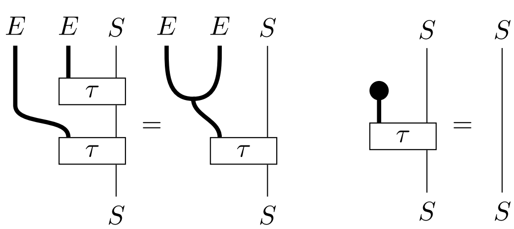
$$\tau \colon S \longrightarrow E \otimes S, \tag{13}$$

having the following graphical representation:



We use a thicker pen to draw the curve representing the environmental system, so it be easily distinguished.

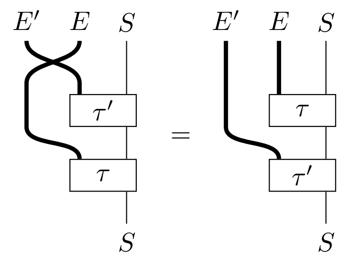
We now equip our environment E with the structure of a classical data type (E, δ, ϵ). We say that $\tau \colon S \longrightarrow E \otimes S$ is an *interaction map* if the following axioms hold:



(14,15)

If τ has these properties, then it can be thought of as transferring data of type (E, δ, ϵ) to the environment. This interpretation arises from the first axiom above: in words, extracting information twice from S is the same as extracting it once and copying the result. We also have a non-disturbance property, thanks to the second axiom: the action of τ can be undone by a linear map which acts solely on the environmental degrees of freedom. The axioms we have given here exactly match the conventional mathematical notion of a *comodule* for a comonoid.

We will be interested in the more general situation where data is transferred to more than one type of environmental system. Suppose that we have two such interaction maps, $\tau \colon S \longrightarrow E \otimes S$ and $\tau' \colon S \longrightarrow E' \otimes S$, where both E and E' are equipped separately with the structure of a classical data type. Physically, we imagine that these environmental interactions are occurring in some 'uncontrollable' fashion, transferring data about the system S to both environments arbitrarily often. The only case in which the resulting information flow is well-defined will be when τ and τ' commute, in the following sense:



(16)

This commutativity condition can be readily extended to the situation of more than two interaction maps, in which case all pairs are required to commute.

An interesting example of an interaction map arises in the case that the quantum system is the *same* as the environment. In this case, we can choose the copying map $\delta \colon E \longrightarrow E \otimes E$ to itself be our interaction map.

Lemma II.1. *For a classical data type (E, δ, ϵ), the map $\delta : E \longrightarrow E \otimes E$ satisfies the axiom of an interaction map.*

It is reassuring that this is possible, as it emphasizes that despite the conceptual split between system and environment that lies at the heart of this framework, the environment can itself be treated as a system equipped with a canonical interaction.

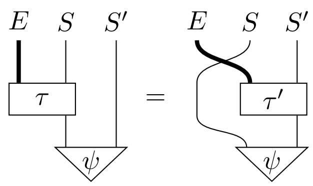
D. Simultaneous interactions

We now consider the possibility that two quantum systems S and S' interact simultaneously with the *same* environmental system E. In this situation, correlations are induced between states of the two systems, since if one measures the value of the environmental classical data type, one obtains a restriction on the joint state of $S \otimes S'$ to those states compatible with this value.

Suppose that our two environmental interaction maps for systems S and S' are

$$\tau \colon S \longrightarrow E \otimes S$$
$$\tau' \colon S' \longrightarrow E \otimes S'$$

We want to calculate those joint states of $S \otimes S'$ which are *consistent*, in the sense that they transfer the same classical data to the environment under the application of τ or τ'. We call this the tensor product of S and S' *with respect to the environment* E, written $S \otimes_E S'$. Suppose that $\psi \in S \otimes S'$ is such a joint state. Then we can express this consistency property by the following equation:



(17)

We wish to find the subspace of $S \otimes S'$ spanned by such states. Writing this subspace as $S \otimes_E S'$, we can construct it as the following equalizer:

$$S \otimes_E S' \xhookrightarrow{e} S \otimes S' \underset{(\mathrm{swap}_{E,S} \otimes \mathrm{id}_{S'}) \circ (\mathrm{id}_E \otimes \tau')}{\overset{\tau \otimes \mathrm{id}_{S'}}{\rightrightarrows}} E \otimes S \otimes S' \qquad (18)$$

We choose the embedding e to be an isometry. The state space $S \otimes_E S'$ can itself be given a canonical environmental interaction map, as the following composite:

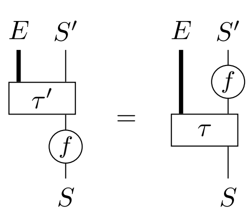
$$S \otimes_E S' \xrightarrow{e} S \otimes S' \xrightarrow{\tau \otimes \mathrm{id}_{S'}} E \otimes S \otimes S' \xrightarrow{\mathrm{id}_E \otimes e^\dagger} E \otimes (S \otimes_E S') \qquad (19)$$

Lemma II.2. *The composite* (19) *satisfies the axioms of an interaction map.*

E. Protected dynamics

The final layer of structure to add to our model is a notion of dynamics for our quantum systems. Since our systems are interacting with their environment in an uncontrollable fashion, the dynamics of our system will only be well-defined if it commutes with the interaction maps in an appropriate fashion.

Suppose $f : S \longrightarrow S'$ is the linear map representing dynamical evolution. We assume that our dynamics is *local*, and so S and S' interact with the same family of environments. We say that f is *protected from decoherence*, or simply *protected*, when the following equation holds:

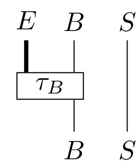


(20)

We require this to hold for each environment E with which the systems S and S' interact.

For a given system, precisely which dynamics are protected from decoherence will depend on the environmental interaction maps themselves. However, it is possible to set these up in such a way that *any* evolution of the system will be protected.

One way is to set the environment to be the trivial classical data type $(\mathbb{C}, 1, 1)$, and the interaction map as the canonical isomorphism of vector spaces $\tau \colon S \longrightarrow \mathbb{C} \otimes S$. A more realistic way to ensure arbitrary dynamics are 'protected' is to maintain a nontrivial environmental classical data type (E, δ, ϵ), but to choose the 'system' Hilbert space to be $B \otimes S$, where B is a 'buffer' system which interacts directly with the environment and 'protects' our true system S of interest. In this scenario, we require B to be equipped with its own interaction map $\tau_B \colon B \longrightarrow E \otimes B$. The interaction map $\tau_{B \otimes S} \colon (B \otimes S) \longrightarrow E \otimes (B \otimes S)$ is then defined as the tensor product of τ_B with the identity on S:



(21)

The following lemma establishes that this construction is valid.

Lemma II.3. *For an interaction map* $\tau_B : B \longrightarrow E \otimes B$*, then* $\tau_B \otimes \mathrm{id}_S : B \otimes S \longrightarrow E \otimes (B \otimes S)$ *is also an interaction map.*

We now see that this achieves our goal.

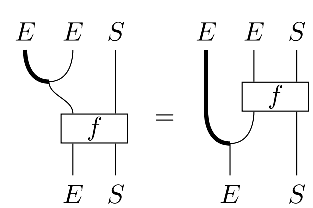
Lemma II.4. *For any linear map* $f : S \longrightarrow S'$*, the composite* $\mathrm{id}_B \otimes f : B \otimes S \longrightarrow B \otimes S'$ *is a protected linear map, with respect to the interaction maps described in Lemma II.3.*

This construction is a good model for a quantum system S placed inside a box B, which isolates it completely: while B interacts with the environment, S does not, and quantum evolution can take place without any information about S being transmitted to the environment.

F. Controlled operations

We saw in Lemma II.1 that the environment can itself be viewed as a system equipped with an interaction map. Using construction (21), we may therefore treat $E \otimes S$ itself as a system, for an arbitrary Hilbert space S, with an interaction map that copies the data stored in E. As we saw in Lemma II.4, any linear map $\mathrm{id}_E \otimes f$ for $f \colon S \longrightarrow S$ is protected with respect to this interaction.

However, our quantum system as a whole is now $E \otimes S$ in this new setting, and it is interesting to ask what the protected evolutions are of this composite system. Applying definition (20), they are maps $f \colon E \otimes S \longrightarrow E \otimes S$ satisfying the following equation:



(22)

For concreteness, we now suppose that the copying map associated to E acts by copying a basis of vectors $|i\rangle \in E$, where $i \in \{1, \ldots, \dim(E)\}$ acts as an index. Then

$$\delta \colon |i\rangle \mapsto |i\rangle \otimes |i\rangle$$
$$\epsilon \colon |i\rangle \mapsto 1,$$

and we can demonstrate that f is precisely series of *controlled operations*: operations on S which depend on the value stored in E.

Lemma II.5. *For a system with Hilbert space $E \otimes S$, classical data type (E, δ, ϵ) as above, and interaction map $\delta \otimes \mathrm{id}_S : E \otimes S \longrightarrow E \otimes E \otimes S$, the protected maps $f : E \otimes S \longrightarrow E \otimes S$ have the following form:*

$$f\colon |i\rangle \otimes |\sigma\rangle \mapsto |i\rangle \otimes f_i(|\sigma\rangle) \tag{23}$$

where $|\sigma\rangle$ is an arbitrary state of S, and $f_i\colon S \longrightarrow S$ is an indexed family of linear maps on S.

We see that protected evolutions of the composite system $E \otimes S$ are precisely *controlled operations*: operations on S which depend on the value stored in E.

III. 2-Categorical Quantum Information

A. Introduction

In this section we will describe how the model of environmental interaction described in Section II gives the ingredients for a monoidal 2-category **2Hilb**. We will see that the graphical calculus for this 2-category gives a concise and comprehensible notation for representing quantum systems, classical data, and performing measurements and controlled operations. For an introduction to 2-categories, see [17, 9].

B. Defining the 2-category

We begin the definition of **2Hilb** in the following way.

- **Objects** are classical structures, as defined by equations (8) and (10).
- **1-morphisms** $H\colon (E, \delta, \epsilon) \longrightarrow (E', \delta', \epsilon')$ are finite-dimensional Hilbert spaces H equipped with commuting interaction maps $\tau\colon H \longrightarrow E \otimes H$ and $\tau'\colon H \longrightarrow E' \otimes H$, as defined by equations (14), (15) and (16).
- **2-morphisms** $f\colon H \Longrightarrow J$ are protected linear maps, as defined by equation (20).
- **Composition of 1-morphisms** is defined as tensor product with respect to the common environment, as defined by equation (18):

$$\begin{aligned}(E_1, \delta_1, \epsilon_1) \xrightarrow{H} (E_2, \delta_2, \epsilon_2) \xrightarrow{J} (E_3, \delta_3, \epsilon_3) \\ := (E_1, \delta_1, \epsilon_1) \xrightarrow{H \otimes_{E_2} J} (E_3, \delta_3, \epsilon_3)\end{aligned} \tag{24}$$

- **Composition of 2-morphisms** is given by composition of protected linear maps.

This is not a complete definition of **2Hilb**. A significant missing component is the *associator*, a family of invertible 2-morphisms which accounts for the fact that given three composable 1-cells $E_1 \xrightarrow{H} E_2$, $E_2 \xrightarrow{J} E_3$ and $E_3 \xrightarrow{K} E_4$, the two possible compositions $K \circ (J \circ H)$ and $(K \circ J) \circ H$ are not necessarily equal, but only *isomorphic*.

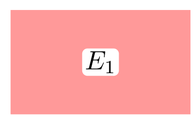
However, the 2-category we are constructing is closely related to a full subcategory of **Bimod**, a 2-category of wide use and great importance in categorical algebra. Each of the constructions we make in Section II corresponds more-or-less directly to 'reversals' of constructions used in the definition of that 2-category: classical data types correspond to algebras, commuting interaction maps correspond to bimodules, protected linear maps correspond to bimodule homomorphisms, and tensor product with respect to the environment corresponds to bimodule tensor product. As a result of this correspondence, the fact that our construction gives a well-defined 2-category follows immediately from the fact that **Bimod** does.

C. Graphical calculus

Here we summarize the standard graphical calculus for working with 2-categories. For a longer discussion, see [18, Section 2.2]. In fact, **2Hilb** is a *monoidal* 2-category, and the graphical calculus can be extended to cover this monoidal structure. While this is an important part of the structure, we will not need to make use of it in this paper.

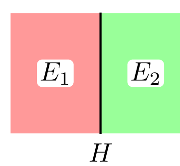
The graphical calculus for 2-categories is only a slight generalization of that for monoidal categories. Just as monoidal categories are 2-categories with one object, so the graphical calculus for monoidal categories is a one-object version of the graphical calculus for 2-categories.

Objects E of our 2-category are represented by regions, which we distinguish by their shading:



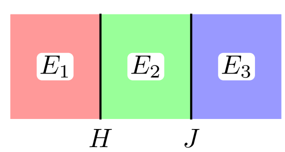
(25)

The 1-morphisms $H\colon E_1 \longrightarrow E_2$ of the 2-category are represented by vertical lines, with the domain of the 1-morphism on the left-hand side of the line and the codomain on the right-hand side:



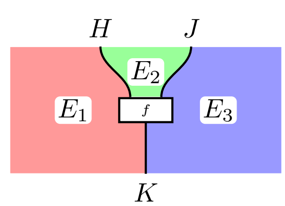
(26)

Composition of 1-morphisms is represented by placing the corresponding vertical lines side-by-side:



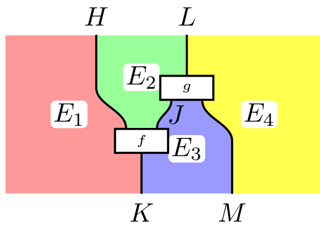
(27)

2-morphisms are represented by vertices:



(28)

In this case the morphism is $f\colon K \Longrightarrow H \otimes_{E_2} J$. Composition of 2-morphisms is represented by stacking vertically:



(29)

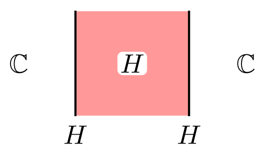
D. Quantum measurement

We now describe how quantum measurement works in our formalism. This was not a topic explicitly dealt with as a part of Section II, since it is most easily introduced with the help of the graphical calculus.

We begin by noting that, for any classical data type (H, δ, ϵ), the underlying Hilbert space H can be recovered in the following way. As described at the end of Section II-C, we can treat a classical data type itself as a quantum system. In our categorical model, this gives rise to 1-morphisms $H\colon (\mathbb{C}, 1, 1) \longrightarrow (H, \delta, \epsilon)$ and $H\colon (H, \delta, \epsilon) \longrightarrow (\mathbb{C}, 1, 1)$. We can then form the composite

$$H \otimes_H H \equiv (\mathbb{C}, 1, 1) \xrightarrow{H} (H, \delta, \epsilon) \xrightarrow{H} (\mathbb{C}, 1, 1), \qquad (30)$$

which has the following graphical representation:

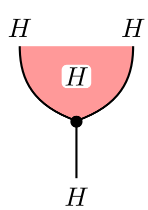


(31)

From now on we will usually drop the label $\mathbb{C}$ for the trivial classical data type, leaving such regions blank. In physical terms, we have two systems with Hilbert space H, each being transferring data to the classical data type (H, δ, ϵ), represented by the shaded region in the centre of the diagram.

Lemma III.1. *The composite* (30) *is isomorphic to H equipped with trivial interaction maps, and this isomorphism is a protected map.*

We have therefore have an isomorphism $H \simeq H \otimes_H H$, which is a 2-cell in our category. We represent it as follows:



(32)

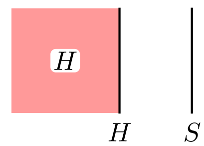
This 2-cell represents our measurement process.

The fact that this is an isomorphism is crucial to the interpretation of our formalism. There is no 'collapse of the wavefunction': rather, we introduce a nontrivial environmental system, to which our quantum systems transfer data via their interaction maps. In this case, those quantum systems are the two copies of H which form the left- and right-hand boundaries of the shaded region. This region is a 'future information cone', analogous to a future light cone in special relativity, in which the result of the measurement is available. The 'collapse' is replaced with the establishment of correlations between physical systems, in the spirit of the decoherence programme [19].

E. Controlled operations

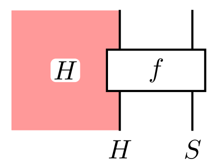
As described in Section II-F, we can use our formalism to describe controlled operations. We now see how that appears in our graphical notation.

A controlled operation is a protected map of type $f\colon H \otimes S \longrightarrow H \otimes S$, where we write H for the Hilbert space of the environment system, and where the system $H \otimes S$ carries the interaction map described by equation (21). This composite is represented in the graphical calculus as the following composite 1-cell:



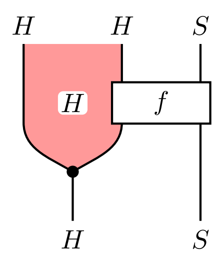
(33)

A controlled operation is therefore a 2-cell of the following form:



(34)

Putting this together with our graphical notation for quantum measurement, we can describe a scenario where we measure a quantum system H, and based on the result, perform an operation on another quantum system S:



(35)

The interplay between measurement and controlled operations is of great importance in our framework.

F. Graphical dictionary

In this section we give a complete graphical dictionary of all the 0-cells, 1-cells and 2-cells which are relevant in our framework, and the quantum computational phenomena to which they correspond.

For simplicity, we will only consider the case that the Hilbert space H of our quantum system is equal to that of our environment, which measures it in a nondegenerate way. We suppose the environment is equipped with the classical structure (H, δ, ϵ). For reasons of space, we omit the proofs that the 1-cells and 2-cells given here satisfy our axioms.

0-cells:

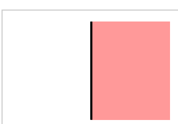
Region coupled to trivial environment $(\mathbb{C}, \mathrm{id}_{\mathbb{C}}, \mathrm{id}_{\mathbb{C}})$ (36)

Region coupled to nontrivial environment (H, δ, ϵ) (37)

1-cells: For a 1-cell $H : E_1 \longrightarrow E_2$, we write $[H; \tau_1; \tau_2]$ to denote the Hilbert space along with its environmental interaction maps $\tau_1 : H \longrightarrow E_1 \otimes H$ and $\tau_2 : H \longrightarrow H \otimes E_2$, in the manner of Section II-C. In the pictures below, the sources and targets of each 1-cell can be determined by the 0-cells at the left-hand and right-hand edges.

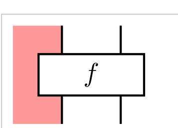
The system $[H; \mathrm{id}_H; \mathrm{id}_H]$ (38)

The system $[H; \delta; \mathrm{id}_H]$ (39)

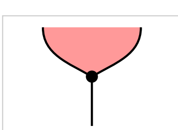


The system $[H; \mathrm{id}_H; \delta]$ (40)

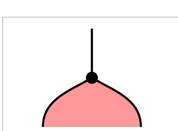
2-cells: For each of our component 2-cells, we give the source and target 1-cells explicitly. It can be checked that each linear map given satisfies the protectedness axiom (20), and so gives rise to a valid 2-cell in **2Hilb**. Most of these can be proven from the classical data type axioms (4–6), but some also rely on the stronger †-Frobenius axioms.



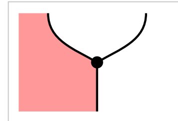
Perform a controlled operation
$[H \otimes S; \delta \otimes \mathrm{id}_S; \mathrm{id}_{H \otimes S}]$ (41)
$\xrightarrow{f} [H \otimes S; \delta \otimes \mathrm{id}_S; \mathrm{id}_{H \otimes S}]$



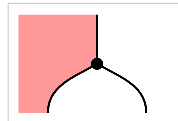
Perform a measurement
$[H; \mathrm{id}_H; \mathrm{id}_H] \xrightarrow[\simeq]{\mathrm{id}_H}$ (42)
$[H \otimes_H H = H; \mathrm{id}_H; \mathrm{id}_H]$



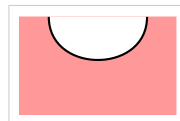
Undo classical correlations
$[H \otimes_H H = H; \mathrm{id}_H; \mathrm{id}_H]$ (43)
$\xrightarrow[\simeq]{\mathrm{id}_H} [H; \mathrm{id}_H; \mathrm{id}_H]$



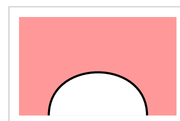
Extract a copy of the classical data
$[H; \delta; \mathrm{id}_H] \xrightarrow{\delta}$ (44)
$[H \otimes H, \delta \otimes \mathrm{id}_H; \mathrm{id}_{H \otimes H}]$



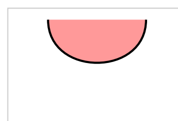
Compare quantum data with classical data
$[H \otimes H; \delta \otimes \mathrm{id}_H; \mathrm{id}_{H \otimes H}]$ (45)
$\xrightarrow{\delta^\dagger} [H, \delta; \mathrm{id}_H]$



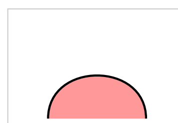
Copy classical data
$[H; \delta; \delta] \xrightarrow{\delta}$ (46)
$[H \otimes H; \delta \otimes \mathrm{id}_H; \mathrm{id}_H \otimes \delta]$



Compare classical data
$[H \otimes H; \delta \otimes \mathrm{id}_H; \mathrm{id}_H \otimes \delta]$ (47)
$\xrightarrow{\delta^\dagger} [H; \delta; \delta]$



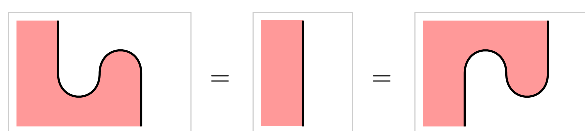
Create uniform classical data
$[\mathbb{C}; \mathrm{id}_{\mathbb{C}}; \mathrm{id}_{\mathbb{C}}] \xrightarrow{\epsilon^\dagger}$ (48)
$[H \otimes_H H = H; \mathrm{id}_H; \mathrm{id}_H]$

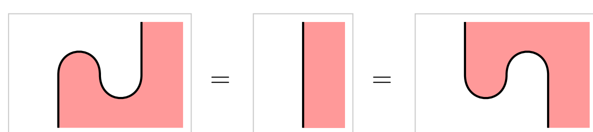


Forget classical data
$[H \otimes_H H = H; \mathrm{id}_H; \mathrm{id}_H]$ (49)
$\xrightarrow{\epsilon} [\mathbb{C}; \mathrm{id}_{\mathbb{C}}; \mathrm{id}_{\mathbb{C}}]$

G. Compactness

Just as *compactness* was a crucial feature of Abramsky's and Coecke's original work on categorical semantics of quantum protocols [2], the same is true here. In our setting, each 1-cell has an *ambidextrous adjoint* [18]. Given the graphical dictionary above, we can represent this by the following family of graphical equations:


(50)

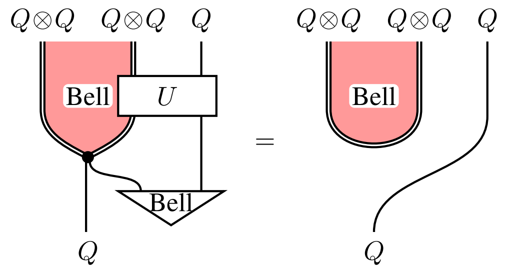

(51)

These axioms allow us to topologically deform our identities, leading to protocols with different physical interpretations.

IV. Results on Quantum Teleportation

A. Conventional teleportation

As a first application of our new formalism, we begin by describing quantum teleportation of a single qubit [8] in our framework. The protocol is defined by the following graphical identity, where $Q = \mathbb{C}^2$ is the state space of a qubit:



(52)

On the left-hand side of the equation, a state is prepared, a measurement is performed, and a controlled operation is then executed. The right-hand diagram is much simpler: the original qubit exits at the top-right corner without interacting, and the classical region is disconnected. To assert that the protocol works as desired is precisely to claim that these composites are equal 2-cells in $\mathbf{2Hilb}$.

We now examine the components of the left-hand diagram in more detail. Executing the quantum teleportation protocol begins with a single qubit, and the preparation of a Bell state. At this stage the systems are isolated from any environmental interaction, and so the adjoining regions are unshaded to indicate the trivial data type $(\mathbb{C}, 1, 1)$. The next stage requires a Bell-basis measurement on a 2-qubit system, comprising the initial qubit and one half of the Bell state which has been prepared. We represent this using a classical data type $(Q \otimes Q, \delta_{\text{Bell}}, \epsilon_{\text{Bell}})$, referred to simply as 'Bell', where δ_{Bell} and ϵ_{Bell} copy and delete the elements of the Bell basis:

$$|\text{Bell}_1\rangle = \tfrac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) \qquad |\text{Bell}_2\rangle = \tfrac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big)$$
$$|\text{Bell}_3\rangle = \tfrac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big) \qquad |\text{Bell}_4\rangle = \tfrac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)$$

Since these are entangled states, the corresponding measurement vertex cannot be decomposed. The final stage of the protocol involves performing a controlled unitary operation on the remaining qubit. In our scheme, this is represented by a linear map $U \colon Q \otimes Q \otimes Q \longrightarrow Q \otimes Q \otimes Q$ which is protected with respect to the appropriate interaction map, which is defined as follows:

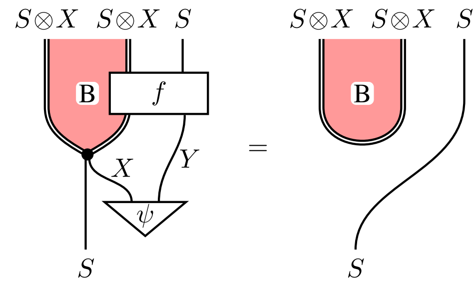
$$\begin{aligned} U = {} & |\text{Bell}_1\rangle\langle\text{Bell}_1| \otimes \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) + |\text{Bell}_2\rangle\langle\text{Bell}_2| \otimes \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) \\ & + |\text{Bell}_3\rangle\langle\text{Bell}_3| \otimes \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right) + |\text{Bell}_4\rangle\langle\text{Bell}_4| \otimes \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right) \end{aligned} \tag{53}$$

It is a short calculation to demonstrate that this satisfies the protectedness axiom.

The right-hand diagram represents a scenario in which the qubit is fully preserved, and classical data is created with a uniform distribution over the Bell basis states. The topological disconnection between the qubit and the classical region indicates that the quantum and classical data is uncorrelated.

B. Generalized teleportation

We now focus on the following question: given a basis of a multi-partite Hilbert space $S \otimes A$, can a measurement in this basis form part of a generalized teleportation protocol for the system S, where A is an ancilla system? We suppose that our experiment takes the following form, generalizing that of the standard teleportation protocol:

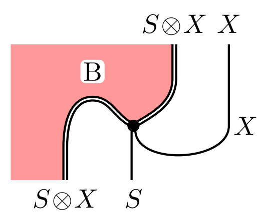


(54)

Here S represents the quantum system whose state is to be teleported, and X and Y are ancilla systems which are prepared in a joint state $|\psi\rangle \in X \otimes Y$. The morphism f represents a controlled operation. We write 'B' for the classical data type on $S \otimes X$ which copies elements of our chosen basis.

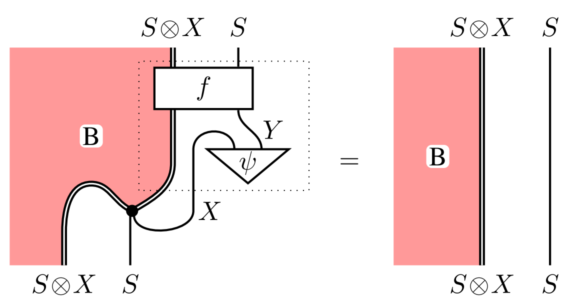
We now prove the following theorem. Recall a *retraction* for a 2-cell is a left inverse under vertical composition.

Theorem IV.1. *A nondegenerate bipartite measurement in a basis* B *can form part of a generalized teleportation protocol of the form* (54) *iff the following composite 2-cell has a retraction:*



(55)

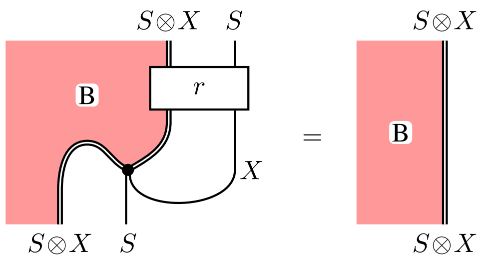
Proof: Suppose the protocol (54) can be carried out. Then we use compactness to deform it in the following way:



(56)

All unknown quantities have been moved into the same part of the diagram, and surrounded by a dotted box to emphasize that they can be treated as a single composite 2-cell. It is clear that the contents of this box gives a retraction for the composite described in the hypothesis of the theorem.

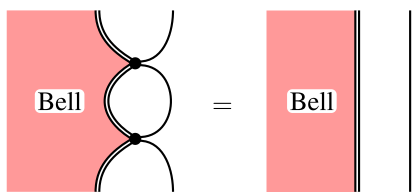
We now suppose that the composite of the hypothesis has a retraction r. This is exactly the following statement:



(57)

We see that this has the form of a generalized teleportation protocol, in the sense of (54), for a particular choice of Y and ψ, which establishes the theorem. ∎

For any particular basis B, the composite (55) can be constructed, and it is then a matter of calculation to see whether it has a retraction. In fact, for a Bell basis measurement, the retraction of the composite (55) has a particularly elegant form, and the algebraic essence of the conventional teleportation protocol can be presented in the following minimal way:



(58)

The methodology of the proof of Theorem IV.1 allows us to establish the following result.

Theorem IV.2. *If a nondegenerate bipartite measurement can form part of a teleportation procedure, then Bell-type shared entanglement is sufficient.*

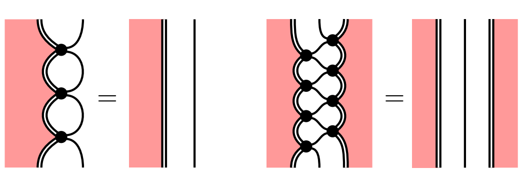
Proof: By the previous theorem, if the measurement can form part of a teleportation procedure, then the composite (55) has a retraction r. This gives equation (57), which is a generalized teleportation protocol with Bell-type shared entanglement, as required. ∎

V. More General Protocols

A. Introduction

Our graphical calculus allows us to write down general quantum informatic procedures in a precise way. Quite easily, schemes can be written down which do not correspond to anything described in the literature, and it is in general a difficult problem to decide whether these procedures can actually be performed.

For reasons of space, we cannot go into great depth here. We restrict to consideration of the following two schemes:

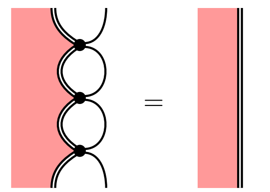


(59,62)

Remarkably, these two protocols are enabled by the *same* two-qubit measurement basis, which is constructed geometrically via a tetrahedron inscribed within the Bloch sphere. Giving this basis allows these protocols to be implemented in principle.

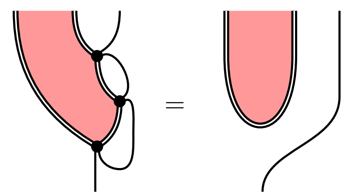
B. Three-stage teleportation

We have described how the essence of the ordinary teleportation protocol can be reduced to the equation (58), in its most minimal form. The two vertices of that diagram refer to the two *stages* of conventional teleportation: usually, these comprise measurement and correction steps. We now consider *three-stage teleportation*, defined by the following equation:



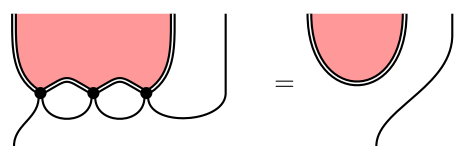
(59)

We omit the label on the shaded region representing the measurement basis. Different topological deformations of the identity above cause these stages to manifest themselves in different ways, as according to the graphical library given in Section III-F. Here is possible deformation:



(60)

In this form, the three stages manifest as a single measurement and two controlled operations. Another deformation gives a different view:

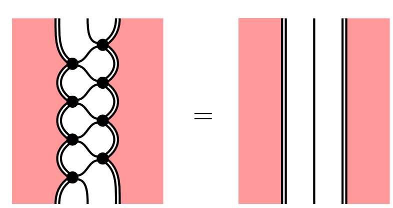


(61)

The left-hand side of this equation now represents a protocol involving 7 identical quantum systems, of which pairs $(2,3)$, $(4,5)$ and $(6,7)$ are prepared in Bell states. Identical measurements are then performed on qubit pairs $(1,2)$, $(3,4)$ and $(5,6)$. The results of these measurements are then compared, and we postselect for the case where all measurement outcomes are the same.

C. Interlaced teleportation

Interlaced teleportation is represented by the following identity:



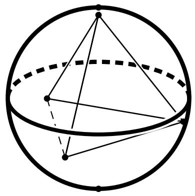
(62)

A key novel aspect of this protocol is that it involves *two* regions of classical data, which interact in a nontrivial way.

It is possible that this is the first quantum protocol to be proposed with this property. The design of this protocol is rather intricate, but it is the simplest protocol we are aware of with this property for which an implementation can be found. Unfortunately, we lack the space here to give more details.

D. Implementation

Describing the abstract form of the protocol is a different matter to establishing whether it can actually be implemented. For that, we begin by considering a regular tetrahedron inscribed within a Bloch sphere, in any position:



(63)

Each point on the Bloch sphere corresponds to a rank-1 projector on $\mathbb{C}^2$, and so the vertices of any such tetrahedron give rise to four projectors $P_i \colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ with $i \in \{1, 2, 3, 4\}$. From each projector we build a unitary operator $U_i := (\mathrm{id} - P_i) + \omega P_i$, where ω is a third root of unity. These projectors form the elements of our measurement basis.

Lemma V.1. *The set of unitaries U_i form an orthogonal basis.*

We can use this basis to form a 2-qubit measurement vertex, using the canonical isomorphism $\mathcal{B}(\mathbb{C}^2) \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$ given by the computational basis.

Theorem V.2. *The basis U_i provides a solution to the three-stage teleportation equation* (59).

Theorem V.3. *The basis U_i provides a solution to the interlaced teleportation equation* (62).

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