# Quantifiers on languages and codensity monads* 

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#### Abstract

This paper contributes to the techniques of topo-algebraic recognition for languages beyond the regular setting as they relate to logic on words. In particular, we provide a general construction on recognisers corresponding to adding one layer of various kinds of quantifiers and prove a corresponding Reutenauer-type theorem. Our main tools are codensity monads and duality theory. Our construction hinges on a measure-theoretic characterisation of the profinite monad of the free $S$-semimodule monad for finite and commutative semirings $S$, which generalises our earlier insight that the Vietoris monad on Boolean spaces is the codensity monad of the finite powerset functor.


## 1 Introduction

It is well known that the combinatorial property of a language of being given by a star-free regular expression can be described both by algebraic and by logical means. Indeed, on the algebraic side, the star-free languages are exactly those languages whose syntactic monoids do not contain any non-trivial groups as subsemigroups. On the logical side, properties of words can be expressed in predicate logic by considering variables as positions in the word, relation symbols asserting that a position in a word has a certain letter of the alphabet, and possibly additional predicates on positions (known as numerical predicates). As shown by McNaughton and Papert in [16], the class of languages definable by first-order sentences over the numerical predicate $<$ consists precisely of the star-free ones.

The theory of formal languages abounds with such results showing the strong interplay between logic and algebra. For instance, Straubing, Thérien and Thomas introduced in [24] a class of additional quantifiers, the
 number of positions $x$ for which $\varphi(x)$ holds is congruent to $p$ modulo $q$ ). There it is shown for example that the languages definable using modular quantifiers of modulus $q$ are exactly the languages whose syntactic monoids are solvable groups of cardinality dividing a power of $q$.

Studying modular quantifiers is relevant for tackling open problems in Boolean circuit complexity, see for example [23] for a discussion. Since Boolean circuit classes contain non-regular languages, expanding the automata theoretic techniques beyond the regular setting is also relevant for addressing these problems.

A fundamental tool in studying the connection between algebra and logic in this setting is the availability of constructions on monoids which mirror the action of quantifiers. That is, given the syntactic monoid for a language with a free variable one wants to construct a monoid which recognises the quantified language. Constructions of this type abound, and are all versions of semidirect products, with the block product playing a central rôle as it allows one to construct recognisers for many different quantifiers [26].

[^0]The present article is an expanded and improved version of the publication [11], where the main results were first announced. Its purpose is to expand the techniques available for monoids and provide the topoalgebraic characterisation of adding one layer of various kinds of quantifiers, beyond the regular setting. A first step was made in [10], where a) we introduced a topological notion of recogniser, that will be motivated in the next subsection, and b) we gave a notion of unary Schützenberger product that corresponds, on the recogniser side, to adding one layer of the existential quantifier for arbitrary languages of words.

In Section 1.1 we provide a gentle introduction and motivate the duality-theoretic approach to language recognition. In Section 1.2 we present codensity monads, our tool of choice for systematically obtaining the relevant topological constructions, and we briefly discuss related work. Finally, in Section 1.3 we present the main contributions of this paper and provide an overview of the remainder of the paper.

### 1.1 Duality for language recognition

Stone duality plays an important rôle and has a long tradition in many areas of semantics, e.g. in domain theory and in modal logic. In [19] Pippenger made explicit the link between Stone duality and regular languages, by proving that the Boolean algebra of regular languages over a finite alphabet $A$ is the dual of the free profinite monoid on $A$. Yet, only recently, starting with the papers [8, 9], the deep connection between this field and formal language theory started to emerge. In these papers a new notion of language recognition, based on topological methods, was proposed for the setting of non-regular languages. Moreover, the scene was set for a new duality-theoretic understanding of the celebrated Eilenberg-Reiterman theorems, establishing a connection between varieties of languages, pseudo-varieties of finite algebras and profinite equations. This led to an active research area where categorical and duality-theoretic methods are used to encompass notions of language recognition for various automata models. See for example the monadic approach to language recognition put forward by Bojańczyk [5], or the series of papers on a category-theoretic approach to Eilenberg-Reiterman theory ([3] and references herein).

Let us illustrate the interplay between duality theory and the theory of regular languages by explaining the duality between the syntactic monoid of a regular language $L$ on a finite alphabet $A$, and the Boolean subalgebra $\mathcal{B} \hookrightarrow \mathcal{P}\left(A^{*}\right)$ generated by the quotients of $L$, i.e., by the sets

$$
w^{-1} L v^{-1}=\left\{u \in A^{*} \mid w u v \in L\right\}
$$

for $w, v \in A^{*}$. In this setting one makes use only of the finite duality between the category of finite Boolean algebras and the category of finite sets which, at the level of objects, asserts that each finite Boolean algebra is isomorphic to the powerset of its atoms.

Since the language $L$ is regular it has only finitely many quotients, say

$$
\left\{w_{1}^{-1} L v_{1}^{-1}, \ldots, w_{n}^{-1} L v_{n}^{-1}\right\}
$$

The finite Boolean algebra generated by this set has as atoms the non-empty subsets of $A^{*}$ of the form

$$
\bigcap_{i \in I} w_{i}^{-1} L v_{i}^{-1} \cap \bigcap_{j \in J}\left(w_{j}^{-1} L v_{j}^{-1}\right)^{c}
$$

for some partition $I \cup J$ of $\{1, \ldots, n\}$. We clearly see that such atoms are in one-to-one correspondence with the equivalence classes of the Myhill syntactic congruence $\sim_{L}$, and thus with the elements of the syntactic monoid $A^{*} / \sim_{L}$ of $L$.

However, the more interesting aspect of this approach is that one can also explain the monoid structure of $A^{*} / \sim_{L}$ and the syntactic morphism in duality-theoretic terms. For this, we have to recall first the duality between the category of sets and the category of complete atomic Boolean algebras. At the level of objects, every complete atomic Boolean algebra is isomorphic to the powerset of its atoms. So the dual of $A^{*}$ is $\mathcal{P}\left(A^{*}\right)$, but the duality also tells us that quotients on one side are turned into embeddings on the other. Thus, we have the duality between the following morphisms

$$
\mathcal{B} \longleftrightarrow \mathcal{P}\left(A^{*}\right) \quad \mid \quad A^{*} \longrightarrow A^{*} / \sim_{L} .
$$

Also, the left action of $A^{*}$ on itself given by appending a word $w$ on the left corresponds, on the dual side, to a left quotient operation (which is a right action):


Since the Boolean algebra $\mathcal{B}$ is closed under quotients, and thus we have commuting squares as the left one in diagram (1), by duality we obtain a left action of $A^{*}$ on $A^{*} / \sim_{L}$. By an analogous argument, one also obtains a right action of $A^{*}$ on $A^{*} / \sim_{L}$ and the two actions commute. It is a simple lemma, see [10], that since $A^{*} / \sim_{L}$ is a quotient of $A^{*}$ and it is equipped with commuting left and right $A^{*}$-actions (called in loc. cit. an $A^{*}$-biaction), then one can uniquely define a monoid multiplication on $A^{*} / \sim_{L}$ so that the quotient $A^{*} \rightarrow A^{*} / \sim_{L}$ is a monoid morphism.


This approach paves the way to a notion of recogniser and syntactic object pertinent for non-regular languages. In the case of a non-regular language $L$, the Boolean algebra $\mathcal{B}$ spanned by the quotients of $L$ is no longer finite, so the finite or discrete duality theorems we have employed previously are no longer applicable. Instead, we use the full power of Stone duality, which establishes the dual equivalence between the category of Boolean algebras and the category BStone of Boolean (Stone) spaces, that is, zero-dimensional compact Hausdorff spaces. In this setting, the dual of $\mathcal{P}\left(A^{*}\right)$ is the Stone-Čech compactification $\beta\left(A^{*}\right)$ of the discrete space $A^{*}$. The embedding of $\mathcal{B}$ into $\mathcal{P}\left(A^{*}\right)$ is turned by the duality theorem into a quotient of topological spaces as displayed below, where we denote the dual of $\mathcal{B}$ by $X$.

$$
\mathcal{B} \longleftrightarrow \mathcal{P}\left(A^{*}\right) \quad \mid \quad \beta\left(A^{*}\right) \longrightarrow X
$$

The syntactic monoid of the language $L$, now infinite, can be seen as a dense subset of $X$, and is indeed the image of the composite map $A^{*} \hookrightarrow \beta\left(A^{*}\right) \rightarrow X$ where the first arrow is the embedding of $A^{*}$ in its Stone-Cech compactification. We thus obtain a commuting diagram as follows.


Furthermore, one can show that the syntactic monoid acts (continuously) on $X$ both on the left and on the right, and these actions commute. This led us, in [10], to the definition of a Boolean space with an internal monoid (BiM) as a suitable notion for language recognition beyond the regular setting. We recall this (in fact a small variation of it) in Definition 2.1.

### 1.2 Profinite monads

Profinite methods have a long tradition in language theory, see for example [4]. To accommodate these tools in his monadic approach to language recognition, Bojańczyk [5] has recently introduced a construction transforming a monad $T$ on Set (the category of sets and functions) into a so-called profinite monad, again on the category of sets. The latter monad allowed him to study in this generic framework the profinite version of the objects modelled by $T$, such as profinite words, profinite countable chains and profinite trees.

A very much related construction of a profinite monad of $T$ was introduced in [2], this time as a monad on the category of Boolean spaces, obtained as a so-called codensity monad for a functor from the category of finitely carried $T$-algebras to Boolean spaces, that we describe in the next section.

The codensity monad is a standard construction in category theory, going back to the work of Kock in the 60 s. It is well known that any right adjoint functor $G$ induces a monad obtained by composition with its left adjoint, and this is exactly the codensity monad of $G$. In general, the codensity monad of a functor which is not necessarily right adjoint, provided it exists, is the best approximation to this phenomenon. For example the codensity monad of the forgetful functor $|-|:$ BStone $\rightarrow$ Set on Boolean spaces is the ultrafilter monad on Set obtained by composition with its left adjoint $\beta$ : Set $\rightarrow$ BStone. The same monad has yet another description as a codensity monad, this time for the inclusion of the category Set $_{f}$ of finite sets into Set, a fact proved in [12] and recently revisited in the elegant paper [14].

The starting point of the present paper is the observation that the unary Schützenberger product $(\diamond X, \diamond M)$ of a BiM $(X, M)$ from our paper [10] hinges, at a deeper level, on the fact that the Vietoris monad $\mathcal{V}$ on the category of Boolean spaces (which is heavily featured in that construction) is the profinite monad of the finite powerset monad $\mathcal{P}_{f}$ on Set. Recall that any Boolean space $X$ is the cofiltered (or inverse) limit of its finite quotients $X_{i}$. Then one can check that the Vietoris space $\mathcal{V} X$ can be obtained as the cofiltered limit of the finite sets $\mathcal{P}_{f} X_{i}$.

In order to find suitable recognisers for languages quantified by, e.g., modular existential quantifiers, we need a slightly different construction than $(\diamond X, \diamond M)$ of [10]. Specifically, we observe that the semantics of these quantifiers can be modelled, at least at the level of finite monoids, by the free $S$-semimodule monad $\mathcal{S}$, for a suitable choice of the semiring $S$. It should be noted that $\mathcal{P}_{f}$ is also an instance of the free $S$-semimodule monad, for the Boolean semiring 2. To obtain corresponding constructions at the level of Boolean spaces with internal monoids, one needs to understand the analogue of the Vietoris construction for the monad $\mathcal{S}$. And the obvious candidate, from a category-theoretic perspective, is the codensity monad of $\mathcal{S}$.

### 1.3 Contributions

This paper contributes to the connection between the topological approach to language recognition and logical formalisms beyond the setting of regular languages, and furthers, along the way, the study of profinite monads in formal language theory.

The main result of Section 3 allows one to extend finitary commutative Set-monads to the category of Boolean spaces with internal monoids. A particular instance of this result is presented in Section 4, where duality-theoretic insights are used to provide a concrete and useful description of the constructions involved in terms of measures. In Section 5 we develop a generic approach for mirroring operations on languages, such as modular quantifiers, associating to a $\operatorname{BiM}(X, M)$ a new $\operatorname{BiM}\left(\diamond_{S} X, \diamond_{S} M\right)$. Finally, Section 6 explains how these constructions are indeed canonical and provides a Reutenauer-type result characterising the Boolean algebra generated by the languages recognised by $\left(\nabla_{S} X,{\nabla_{S}} M\right)$.

## 2 Preliminaries

### 2.1 Logic on words

Fix an arbitrary finite set $A$, and write $A^{*}$ for the free monoid over $A$. A word over the alphabet $A$ (an $A$-word, for short) is an element $w \in A^{*}$. In the logical approach to language theory, the word $w$ is regarded as a (relational) structure on the set $\{1, \ldots,|w|\}$, where $|w|$ denotes the length of the word, equipped with a unary relation $P_{a}$ for each $a \in A$ which singles out the positions in the word where the letter $a$ appears. If $\varphi$ is a sentence (i.e., a formula in which every variable is in the scope of a quantifier) in a language interpretable over words, we denote by $L_{\varphi}$ the set of words satisfying $\varphi$.

Assume now $\varphi(x)$ is a formula with a free first-order variable $x$ (intuitively this means that $\varphi(x)$ can talk about positions in the word). In order to be able to interpret the free variable, we consider an extended alphabet $A \times 2$ which we think of as consisting of two copies of $A$, that is, we identify $A \times 2$ with the set $A \cup\left\{a^{\prime} \mid a \in A\right\}$, and we call the elements of the second copy of $A$ marked letters. Assuming $w=a_{1} \ldots a_{n}$ and $1 \leq i \leq|w|$, we write $w^{(i)}$ for the word $a_{1} \ldots a_{i-1} a_{i}^{\prime} a_{i+1} \ldots a_{n}$, i.e. for the word in $(A \times 2)^{*}$ having the same shape as $w$ but with the letter in position $i$ marked, and $w^{0}$ for the word $a_{1} \ldots a_{n}$ seen as a word in
$(A \times 2)^{*}$. Then we define $L_{\varphi(x)}$ as the set of all words in the alphabet $A \times 2$ with only one marked letter such that the underlying word in the alphabet $A$ satisfies $\varphi$ when the variable $x$ points at the marked position.

Now, given $L \subseteq(A \times 2)^{*}$, denote by $L_{\exists}$ the language consisting of those words $w=a_{1} \ldots a_{n}$ over $A$ such that there exists $1 \leq i \leq|w|$ with $a_{1} \ldots a_{i-1} a_{i}^{\prime} a_{i+1} \ldots a_{n} \in L$. Observe that $L=L_{\varphi(x)}$ entails $L_{\exists}=L_{\exists x . \varphi(x)}$, thus recovering the usual existential quantification.

Among the generalisations of the existential quantifier are the modular quantifiers. Consider the ring $\mathbb{Z}_{q}$ of integers modulo $q$, and pick $p \in \mathbb{Z}_{q}$. We say that an $A$-word $w$ satisfies the sentence $\exists_{p \text { mod } q x . \varphi(x) \text { if }}$ there exist $p$ modulo $q$ positions in $w$ for which the formula $\varphi(x)$ holds. Moreover, for an arbitrary language $L \subseteq(A \times 2)^{*}$, we define $L_{\exists_{p \bmod q}}$ as the set of $A$-words $w=a_{1} \ldots a_{n}$ such that the cardinality of the set

$$
\begin{equation*}
\left\{1 \leq i \leq|w| \mid a_{1} \ldots a_{i-1} a_{i}^{\prime} a_{i+1} \ldots a_{n} \in L\right\} \tag{2}
\end{equation*}
$$

is congruent to $p$ modulo $q$. Clearly, if the language $L$ is defined by the formula $\varphi(x)$, then $L_{\exists_{p \text { mod } q}}$ is defined by the formula $\exists_{p \bmod q} x . \varphi(x)$.

Finally, generalising the preceding situations, we can consider an arbitrary semiring ( $S,+, \cdot, 0_{S}, 1_{S}$ ) and an element $k \in S$. For $L \subseteq(A \times 2)^{*}$, an $A$-word $w=a_{1} \ldots a_{n}$ belongs to the quantified language, denoted by $\mathcal{Q}_{k}(L)$, provided that

$$
\underbrace{1_{S}+\cdots+1_{S}}_{m \text { times }}=k
$$

where $m$ is the cardinality of the set in (2).

### 2.2 Stone duality and the Vietoris hyperspace

Stone duality for Boolean algebras [22] establishes a categorical equivalence between the category of Boolean algebras and their homomorphisms, and the opposite of the category BStone of Boolean (Stone) spaces and continuous maps between them.

A Boolean space is a compact Hausdorff space that admits a basis of clopen (i.e., simultaneously closed and open) subsets. There is an obvious forgetful functor $|-|:$ BStone $\rightarrow$ Set. When clear from the context, we will omit writing $|-|$.

The dual of the Boolean space $X$ is the Boolean algebra $C l o p(X)$ of its clopen subsets, equipped with set-theoretic operations. Conversely, given a Boolean algebra $B$, the dual space $X$ may be taken either as the set of ultrafilters on $B$ (i.e., those proper filters $F$ satisfying $a \in F$ or $\neg a \in F$ for every $a \in B$ ) or as the Boolean algebra homomorphisms $h: B \rightarrow 2$ equipped with the topology generated by the sets

$$
\widehat{a}:=\{F \mid a \in F\} \cong\{h \mid h(a)=1\}, \text { for } a \in B
$$

An example of a Boolean space, central to our treatment, is the Stone-Čech compactification of an arbitrary set $K$. This is the dual space of the Boolean algebra $\mathcal{P} K$, and is denoted by $\beta K$. It is well known that the assignment $K \mapsto \beta K$ induces a functor $\beta$ : Set $\rightarrow$ BStone which is left adjoint to the forgetful functor $|-|$ : BStone $\rightarrow$ Set. Another functor, which played a key rôle in $[10]$ and will serve here as a leading example, is the Vietoris functor $\mathcal{V}: B S t o n e \rightarrow$ BStone. Given a Boolean space $X$, consider the collection $\mathcal{V} X$ of all closed subsets of $X$ equipped with the topology generated by the clopen subbasis

$$
\{\diamond V \mid V \in C l o p(X)\} \cup\left\{(\diamond V)^{c} \mid V \in C l o p(X)\right\}
$$

where $\diamond V:=\{K \in \mathcal{V} X \mid K \cap V \neq \emptyset\}$. The resulting space is called the Vietoris (hyper) space of $X$, and is again a Boolean space. Further, if $f: X \rightarrow Y$ is a morphism in BStone, then so is the direct image function $\mathcal{V} X \rightarrow \mathcal{V} Y, K \mapsto f[K]$. In fact, it is well known that this is the functor part of a monad $\mathcal{V}$ on BStone. The Vietoris hyperspace of an arbitrary topological space was first introduced by Vietoris [27]; for a complete account, including results stated here without proof, see [17].

### 2.3 Boolean spaces with internal monoids

In this section we give the definition of a Boolean space with an internal monoid, or BiM for short (see Definition 2.1 below), a topological recogniser well-suited for dealing with non-regular languages. In [10] a Boolean space with an internal monoid was defined as a pair $(X, M)$ consisting of a Boolean space $X$, a dense subspace $M$ equipped with a monoid structure, and a biaction (i.e., a pair of compatible left and right actions) of $M$ on $X$ with continuous components extending the obvious biaction of $M$ on itself. Here we use a small variation and simplification of this notion. Instead of imposing that the monoid is a dense subset of the space, we require a map from the monoid to the space with dense image.

In what follows, for a Boolean space $X$ we will denote by $[X, X]$ the set of continuous endofunctions on $X$, which comes with the obvious monoid multiplication o given by composition. Given a monoid $(M, \cdot)$, we will denote by $r: M \rightarrow M^{M}$ and $l: M \rightarrow M^{M}$ the two maps induced from the monoid multiplication via currying, which correspond to the obvious right, respectively left action of $M$ on itself.

Definition 2.1. A Boolean space with an internal monoid, or a BiM, is a tuple ( $X, M, h, \rho, \lambda$ ), where $X$ is a Boolean space, $M$ is a monoid, $h: M \rightarrow X, \lambda: M \rightarrow[X, X]$ and $\rho: M \rightarrow[X, X]$ are functions such that $h$ has a dense image and for all $m \in M$ the following diagrams commute in Set.


If no confusion arises we write $(X, M)$, or even just $X$, for the $\operatorname{BiM}(X, M, h, \rho, \lambda)$. A morphism between two BiMs $X$ and $X^{\prime}$ is a pair $(\tilde{\psi}, \psi)$ where $\tilde{\psi}: X \rightarrow X^{\prime}$ is a continuous map and $\psi: M \rightarrow M^{\prime}$ is a monoid morphism such that $\tilde{\psi} \circ h=h^{\prime} \circ \psi$. Note that since the image of $h$ is dense in $X$, given $\psi, \tilde{\psi}$ is uniquely determined if it exists. Accordingly, we will sometimes just write $\psi$ to designate the pair as well as each of its components. We denote the ensuing category of BiMs by BiM.

Remark 2.2. Notice that if $(X, M)$ is a BiM of the form $\left(\beta\left(A^{*}\right), A^{*}\right)$, and $\left(X^{\prime}, M^{\prime}\right)$ is any BiM , then every monoid morphism $\psi: A^{*} \rightarrow M^{\prime}$ yields a (unique) continuous extension $\tilde{\psi}: \beta\left(A^{*}\right) \rightarrow X^{\prime}$ making the pair $(\tilde{\psi}, \psi)$ into a BiM morphism. Thus BiM morphisms $\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow\left(X^{\prime}, M^{\prime}\right)$ are in one-to-one correspondence with monoid morphisms $A^{*} \rightarrow M^{\prime}$. For this reason we will often treat these two things as one and the same.

Remark 2.3. From Definition 2.1 it follows that $\rho$ and $\lambda$ induce in fact commuting right and left $M$-actions on $X$, so that $h$ is an $M$-biaction morphism. Indeed, since $h$ has a dense image in $X$ it follows that $\rho(m)$ and $\lambda(m)$ are the unique extensions on $X$ of $r(m)$, respectively $l(m)$. But the left and right actions of $M$ on itself commute, hence $\rho$ and $\lambda$ must enjoy the same properties. We also obtain that $(X, \operatorname{lm}(h))$ is a Boolean space with an internal monoid exactly as defined in [10].

Remark 2.4. An equivalent way of saying that the diagrams in (3) commute for all $m \in M$ is to say that the following diagrams commute in Set.


This will come handy in the proof of Theorem 3.4.
To conclude, we recall the associated notion of recognition. Under the bijection between subsets of a given set $K$ and clopens of its Stone-Čech compactification $\beta K$, we write $\widehat{L}$ for the clopen associated with the subset $L \in \mathcal{P} K$.

Definition 2.5. Let $A$ be a finite alphabet and $L \in \mathcal{P}\left(A^{*}\right)$. A morphism of BiMs $\psi:\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow(X, M)$ recognises the language $L$ if there is a clopen $C \subseteq X$ such that $\psi^{-1}(C)=\widehat{L}$. Moreover, we say that the $\operatorname{BiM}(X, M)$ recognises the language $L$ if there exists a BiM morphism $\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow(X, M)$ recognising $L$. Finally, if $\mathcal{B} \hookrightarrow \mathcal{P}\left(A^{*}\right)$ is a Boolean subalgebra, the $\operatorname{BiM}(X, M)$ is said to recognise $\mathcal{B}$ provided that it recognises each $L \in \mathcal{B}$.

Equivalently, a language $L \in \mathcal{P}\left(A^{*}\right)$ is recognised by the morphism of BiMs $\psi:\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow(X, M)$ when there exists a clopen $C \subseteq X$ such that $L=\psi^{-1}\left(h^{-1}(C)\right)$. The topology on $X$ specifies which subsets of $M$ can be used for recognition, namely the preimages under $h$ of the clopens in $X$. However, when $M$ is finite so is $X$. In fact, in this case $X$ has the same carrier set as $M$ and is equipped with the discrete topology, therefore in the regular setting we recover the usual notion of recognition.

### 2.4 Monads and algebras

We assume the reader is familiar with the basic notions of category theory, and especially with monads as a categorical approach to general algebra. Concerning the latter, we refer the reader to, e.g., [15, Chapter VI] or [6, Chapters 3-4].

Consider a monad $(T, \eta, \mu)$ on a category $\mathcal{C}$. Recall that an Eilenberg-Moore algebra for $T$ (or a $T$-algebra, for short) is a pair ( $X, h$ ) where $X$ is an object of $\mathcal{C}$ and $h: T X \rightarrow X$ is a morphism in $\mathcal{C}$ which behaves well with respect to the unit $\eta$ and the multiplication $\mu$ of the monad, that is, $h \circ \eta_{X}=\mathrm{id} \mathrm{A}_{X}$ and $h \circ T h=h \circ \mu_{X}$. A morphism of $T$-algebras $\left(X_{1}, h_{1}\right) \rightarrow\left(X_{2}, h_{2}\right)$ is a morphism $f: X_{1} \rightarrow X_{2}$ in $\mathcal{C}$ satisfying $f \circ h_{1}=h_{2} \circ T f$. Let $\mathcal{C}^{T}$ denote the category of $T$-algebras. When $T$ is a monad on the category Set of sets and functions, categories of the form Set $^{T}$ are, up to equivalence, precisely the varieties of (possibly infinite arity) algebras. This correspondence restricts to categories of Eilenberg-Moore algebras for finitary monads (i.e., monads preserving filtered colimits) and varieties of algebras in types consisting of finite arity operations. A $T$ algebra ( $X, h$ ) is said to be finitely carried (or sometimes just finite) provided $X$ is finite. We write $\operatorname{Set}_{f}^{T}$ for the full subcategory of Set ${ }^{T}$ on the finitely carried objects. The forgetful functor Set ${ }^{T} \rightarrow$ Set that sends $(X, h)$ to $X$ restricts to the finitely carried algebras, and gives rise to a functor $\operatorname{Set}_{f}^{T} \rightarrow \operatorname{Set}_{f}$.

In Section 5.2 we shall see how several logical quantifiers can be modelled by considering modules over a semiring and the appropriate profinite monad. Recall that a semiring is a tuple $(S,+, \cdot, 0,1)$ such that $(S,+, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, the operation $\cdot$ distributes over + , and $0 \cdot s=0=s \cdot 0$ for all $s \in S$. If no confusion arises, we will denote the semiring by $S$ only.

Example 2.6. A semiring $S$ induces a functor $\mathcal{S}$ : Set $\rightarrow$ Set which associates with a set $X$ the set of all functions $X \rightarrow S$ with finite support, that is

$$
\mathcal{S} X:=\{f: X \rightarrow S \mid f(x)=0 \text { for all but finitely many } x \in X\} .
$$

If $\psi: X \rightarrow Y$ is any function, define $\mathcal{S} \psi: \mathcal{S} X \rightarrow \mathcal{S} Y$ as $f \mapsto\left(y \mapsto \sum_{\psi(x)=y} f(x)\right)$. Any element $f \in \mathcal{S} X$ can be represented as a formal sum $\sum_{i=1}^{n} s_{i} x_{i}$, where $\left\{x_{1}, \ldots, x_{n}\right\}$ is the support of $f$ and $s_{i}=f\left(x_{i}\right)$ for each $i$. The functor $\mathcal{S}$ is part of a monad $(\mathcal{S}, \eta, \mu)$ on Set, called the free $S$-semimodule monad, whose unit is

$$
\eta_{X}: X \rightarrow \mathcal{S} X, \eta_{X}(x)\left(x^{\prime}\right)=1 \text { if } x^{\prime}=x \text { and } 0 \text { otherwise, }
$$

and whose multiplication is

$$
\mu_{X}: \mathcal{S}^{2} X \rightarrow \mathcal{S} X, \quad \sum_{i=1}^{n} s_{i} f_{i} \mapsto\left(x \mapsto \sum_{i=1}^{n} s_{i} f_{i}(x)\right)
$$

The category Set ${ }^{\mathcal{S}}$ is the category of modules over the semiring $S$. For example, if $S$ is the Boolean semiring 2 then $\mathcal{S}=\mathcal{P}_{f}$ (the finite powerset monad), whose Eilenberg-Moore algebras are join semilattices. If $S$ is $(\mathbb{N},+, \cdot, 0,1)$ or $(\mathbb{Z},+, \cdot, 0,1)$, then the algebras for the monad $\mathcal{S}$ are, respectively, Abelian monoids and Abelian groups.

### 2.5 Profinite monads

Throughout this subsection we fix a monad $T$ on Set. We begin by recalling the definition of the associated profinite monad $\widehat{T}$ on the category of Boolean spaces, following [2]. First we provide an intuitive idea of the construction, and then we give the formal definition. Given a Boolean space $X$, one considers all continuous maps $h_{i}: X \rightarrow Y_{i}$ where the $Y_{i}$ 's are finite sets equipped with Eilenberg-Moore algebra structures $\alpha_{i}: T Y_{i} \rightarrow Y_{i}$, as well as the algebra morphisms $u_{i j}: Y_{i} \rightarrow Y_{j}$ satisfying $u_{i j} \circ h_{i}=h_{j}$. Equipping the finite sets $Y_{i}$ with the discrete topology, one obtains a cofiltered diagram (or inverse limit system) $\mathcal{D}_{X}$ in BS tone, and $\widehat{T} X$ is the limit of this system. It turns out that $\widehat{T}$ is the underlying functor of a monad $(\widehat{T}, \widehat{\eta}, \widehat{\mu})$ on BStone, called the profinite monad associated with $T$. For example, it is not difficult to see how to obtain its unit $\widehat{\eta}_{X}$ from the universal property of the limit, as in the following diagram.


To give the formal definition of $\widehat{T}$, we introduce the functor $G$ : $\mathrm{Set}_{f}^{T} \rightarrow \mathrm{BS}$ tone obtained as the composition of the forgetful functor to $\operatorname{Set}_{f}$ with the embedding of $\operatorname{Set}_{f}$ into BStone :

$$
G: \operatorname{Set}_{f}^{T} \longrightarrow \operatorname{Set}_{f} \longrightarrow \text { BStone. }
$$

The shape of the diagram we constructed above for a Boolean space $X$ is the comma category $X \downarrow G$ whose objects are essentially the maps $h_{i}: X \rightarrow G\left(Y_{i}, \alpha_{i}\right)$, and whose arrows are the maps $u_{i j}$ as above. The diagram $\mathcal{D}_{X}$ is then formally given by the composition

$$
X \downarrow G \xrightarrow{\text { cod }} \operatorname{Set}_{f}^{T} \xrightarrow{G} \text { BStone }
$$

of the codomain functor $X \downarrow G \rightarrow \operatorname{Set}_{f}^{T}$, which maps $h_{i}: X \rightarrow G\left(Y_{i}, \alpha_{i}\right)$ to the algebra $\left(Y_{i}, \alpha_{i}\right)$, and $G: \operatorname{Set}_{f}^{T} \rightarrow$ BStone. Formally, for an arbitrary Boolean space $X$, we have $\widehat{T} X:=\lim \mathcal{D}_{X}$.

Notice that this is the pointwise limit computation of the right Kan extension of $G$ along itself (cf. [15, X.3]). That is, using standard category-theoretic notation, $\widehat{T}=\operatorname{Ran}_{G} G$. It is well known (see for example [14]) that the right Kan extension of a functor $G$ along itself, when it exists, is the functor part of a monad, called the codensity monad for $G$.

The universal property of the right Kan extension, along with the fact that the underlying-set functor
 transformation

$$
\begin{equation*}
\tau_{X}: T|X| \rightarrow|\widehat{T} X| \tag{4}
\end{equation*}
$$

which was also used in [2]. Here we give a presentation based on the limit computation of $\widehat{T} X$. Notice that the maps $\left|h_{i}\right|:|X| \rightarrow\left|Y_{i}\right|$ are functions into the carrier sets of the Eilenberg-Moore algebras $\alpha_{i}: T Y_{i} \rightarrow Y_{i}$ and thus, by the universal property of the free algebra $T|X|$, we can extend the maps $\left|h_{i}\right|$ to algebra morphisms $h_{i}^{\#}$ from $T|X|$ to $\left(Y_{i}, \alpha_{i}\right)$. The functions $h_{i}^{\#}$ form a cone for the diagram $|-| \circ \mathcal{D}_{X}$ in Set whose limit is $|\widehat{T} X|$, by virtue of the fact that the forgetful functor $|-|: B S t o n e \rightarrow$ Set preserves limits. By the universal property of the limit, this yields a unique map $\tau_{X}$ as in (4).

The natural transformation $\tau$ behaves well with respect to the units and multiplications of the monads $T$ and $\widehat{T}$, in the sense that the next two diagrams commute, see [1, Proposition B. 7$]$. Thus the pair $(|-|, \tau)$
is a monad morphism, or monad functor in the terminology of [25].


The fact that $(|-|, \tau)$ is a monad functor entails that the functor $|-|$ lifts to a functor $\widehat{-\mid}$ between the categories of Eilenberg-Moore algebras for the monads $\widehat{T}$ and $T$, as in the next diagram.


As a consequence we immediately obtain that the set $|\widehat{T} X|$ admits a $T$-algebra structure, a result also used in [1] for finite algebras. This structure is essentially the one obtained by applying the functor $\widehat{|-|}$ to the free $\widehat{T}$-algebra $\left(\widehat{T} X, \widehat{\mu}_{X}\right)$. In more detail,

Lemma 2.7. Given a Boolean space $X$, the composite map

$$
T|\widehat{T} X| \xrightarrow{\tau_{\widehat{T} X}}\left|\widehat{T}^{2} X\right| \xrightarrow{\left|\widehat{\mu}_{x}\right|}|\widehat{T} X|
$$

is a $T$-algebra structure on $|\widehat{T} X|$. Moreover, $\tau_{X}$ is a morphism of $T$-algebras from the free $T$-algebra on $|X|$ to $|\widehat{T} X|$ with the above structure.

Proof. This is a straightforward verification using the commutativity of the diagrams in (5).
While in some proofs it is essential to keep track of the forgetful functor, we will sometimes omit it in what follows and simply write $\tau_{X}: T X \rightarrow \widehat{T} X$. We recall a property of the natural transformation $\tau$ which will be crucial in the following.

Lemma 2.8. For every Boolean space $X$, the $\operatorname{map} \tau_{X}: T X \rightarrow \widehat{T} X$ has dense image. More generally, the composite

$$
T M \xrightarrow{T h} T X \xrightarrow{\tau_{X}} \widehat{T} X
$$

has dense image whenever $h: M \rightarrow X$ is a function with dense image.
Proof. For a proof of the fact that $\tau_{X}$ has dense image, see [20, Lemma 2.9]. An easy adaptation of the latter proof yields the second part of the statement.

Remark 2.9. Notice that, for an arbitrary monad $T$ on Set, the components of the natural transformation $\tau$ from (4) do not have to be injective. A counterexample is provided by the powerset monad $\mathcal{P}$ on Set. Indeed, both $\mathcal{P}$ and $\mathcal{P}_{f}$ generate the same profinite monad, namely the Vietoris monad on BStone. In the case of the monad $\mathcal{P}, \tau_{X}: \mathcal{P} X \rightarrow \mathcal{V} X$ sends a subset of the Boolean space $X$ to its closure, and this function is not injective in general. However, the components of $\tau$ are injective if $T$ is finitary and restricts to finite sets, e.g., if $T$ is the finite powerset monad on Set. For more details we refer the reader to [20, Section 2.2].

## 3 Extending Set-monads to BiMs

In this section we study liftings of monads from the category of sets to the category of BiMs. Let us fix, throughout the section, a monad $T$ on Set. In Section 2.5 we have seen that the profinite monad $\widehat{T}$ provides a canonical way of extending $T$ to Boolean spaces. On the other hand, in Section 3.1 we consider ways of lifting $T$ to the category of monoids. The combination of these two liftings, the topological and the monoid one, is considered in Section 3.2. In particular, in Theorem 3.4 we give sufficient conditions for $T$ to be extended in a canonical way to the category of BiMs , by combining the aforementioned liftings.

### 3.1 Lifting Set-monads to the category of monoids

It is well known that there are two "canonical" natural transformations of bifunctors

$$
\otimes, \otimes^{\prime}: T X \times T Y \rightarrow T(X \times Y)
$$

defined intuitively as follows. If we think of elements in $T X$ as terms $t\left(x_{1}, \ldots, x_{n}\right)$, then $t\left(x_{1}, \ldots, x_{n}\right) \otimes$ $s\left(y_{1}, \ldots, y_{m}\right)$ is defined as

$$
t\left(s\left(\left(x_{1}, y_{1}\right), \ldots,\left(x_{1}, y_{m}\right)\right), \ldots, s\left(\left(x_{n}, y_{1}\right), \ldots,\left(x_{n}, y_{m}\right)\right)\right)
$$

whereas $t\left(x_{1}, \ldots, x_{n}\right) \otimes^{\prime} s\left(y_{1}, \ldots, y_{m}\right)$ is defined as

$$
s\left(t\left(\left(x_{1}, y_{1}\right), \ldots,\left(x_{n}, y_{1}\right)\right), \ldots, t\left(\left(x_{1}, y_{m}\right), \ldots,\left(x_{n}, y_{m}\right)\right)\right)
$$

In general $\otimes$ and $\otimes^{\prime}$ do not coincide, and when they do the monad is called commutative, a notion due to Kock [13]. We give a formal definition in the case of the monad $T$. Every Set-monad has a unique strength, that is a natural transformation $\sigma_{X, Y}: X \times T Y \rightarrow T(X \times Y)$ such that the following diagrams commute.


This natural transformation can be explicitly described as follows. For any $x \in X$, write $f_{x}: Y \rightarrow X \times Y$ for the function sending $y$ to $(x, y)$. Then $\sigma_{X, Y}: X \times T Y \rightarrow T(X \times Y)$ sends a pair $(x, s)$ to the image of $s$ under $T f_{x}: T Y \rightarrow T(X \times Y)$. Associated with the strength $\sigma$, there is a costrength $\sigma_{X, Y}^{\prime}: T X \times Y \rightarrow T(X \times Y)$ defined as the composition

$$
T X \times Y \xrightarrow{\gamma_{T X, Y}} Y \times T X \xrightarrow{\sigma_{Y, X}} T(Y \times X) \xrightarrow{T \gamma_{Y, X}} T(X \times Y),
$$

where $\gamma_{X, Y}: X \times Y \rightarrow Y \times X$ is the function sending $(x, y)$ to $(y, x)$. The costrength $\sigma^{\prime}$ enjoys properties symmetric to those of the strength $\sigma$, expressed by the following commutative diagrams.


The monad $T$ is said to be commutative if, for any sets $X, Y$, the following square commutes.


Note that the commutativity of this diagram formalises the aforementioned idea that the natural transformations $\otimes$ and $\otimes^{\prime}$ coincide. Given a monoid $(M, \cdot, 1)$, one has two possibly different "canonical" ways of defining a binary operation on $T M$, obtained as either of the two composites

$$
\begin{equation*}
T M \times T M \underset{\otimes^{\prime}}{\stackrel{\otimes}{\leftrightarrows}} T(M \times M) \xrightarrow{T(\cdot)} T M \tag{10}
\end{equation*}
$$

If $e: 1 \rightarrow M$ denotes the map selecting the unit of the monoid, we can also define a map $1 \rightarrow T M$ obtained as the composite $T e \circ \eta_{1}$. That these data (with either of the two binary operations) give rise to monoid structures on $T M$ is a consequence of a more general result by Kock:
Theorem 3.1. If $T$ is a commutative Set-monad then $\otimes=\otimes^{\prime}$, and thus for every monoid $(M, \cdot, 1)$ the composition in (10) gives a monoid structure on TM. This yields a lifting of $T$ to a monad on the category of monoids and monoid homomorphisms.

Proof. This is a special case of [13, Theorem 2.1].

### 3.2 Combining the topological and monoid liftings

In Sections 2.5 and 3.1, respectively, we have seen that every Set-monad $T$ can be lifted to a monad $\widehat{T}$ on the category of Boolean spaces, and it can be lifted to a monad on the category of monoids provided it is commutative. In this section we show that, if $T$ is commutative and finitary, then the topological and monoid liftings can be combined to obtain a lifting of $T$ to the category of BiMs (see Theorem 3.4 below).

Let $T$ be a commutative Set-monad, $(X, M)$ a BiM and $h: M \rightarrow X$ the associated function with dense image. We would like to define a structure of BiM on the pair $(\widehat{T} X, T M)$. In particular, we should give a function $T M \rightarrow \widehat{T} X$ with dense image. To this aim, we define $\widehat{h}: T M \rightarrow \widehat{T} X$ as the composition

$$
\begin{equation*}
T M \xrightarrow{T h} T X \xrightarrow{\tau_{X}} \widehat{T} X \tag{11}
\end{equation*}
$$

By Lemma 2.8, this function has dense image. Further, since both $T h$ and $\tau_{X}$ are $T$-algebra morphisms, $\widehat{h}$ is also a $T$-algebra morphism.

Recall that, if $\alpha: T B \rightarrow B$ is a $T$-algebra and $A$ is any set, the set of functions $B^{A}$ carries an Eilenberg-Moore algebra structure for $T$, where the operations are defined pointwise. Further, whenever $\alpha_{i}: T B_{i} \rightarrow B_{i}$ for $i \in\{1,2\}$ are Eilenberg-Moore algebras for $T$ and $f: B_{1} \rightarrow B_{2}$ is an algebra morphism, then $\operatorname{Set}(A, f)=f \circ-: B_{1}^{A} \rightarrow B_{2}^{A}$ is a $T$-algebra morphism. We obtain at once the following fact.
Lemma 3.2. For any Set-monad $T$, the sets $T M^{T M}$ and $\widehat{T} X^{T M}$ carry $T$-algebra structures and the function

$$
\widehat{h} \circ-: T M^{T M} \rightarrow \widehat{T} X^{T M}
$$

is a T-algebra morphism.
Proof. With the notation of the previous paragraph, consider $A=T M, \alpha_{1}=\mu_{M}: T^{2} M \rightarrow T M, \alpha_{2}$ the $T$-algebra structure on $\widehat{T} X$ given as in Lemma 2.7, and $f=\widehat{h}$.

Thus, also the power algebra $\widehat{T} X^{\widehat{T} X}$ admits a $T$-algebra structure. Crucially, if the monad $T$ is finitary, the set $[\widehat{T} X, \widehat{T} X]$ of continuous endofunctions on $\widehat{T} X$ is a subalgebra of $\widehat{T} X^{\widehat{T} X}$. This is proved in the following proposition which will allow us to define, in the proof of Theorem 3.4, a biaction of $T M$ on $\widehat{T} X$.
Proposition 3.3. If $T$ is a finitary Set-monad, then $[\widehat{T} X, \widehat{T} X]$ is a subalgebra of the $T$-algebra $\widehat{T} X^{\widehat{T} X}$. With respect to this structure, the function

$$
-\circ \widehat{h}:[\widehat{T} X, \widehat{T} X] \rightarrow \widehat{T} X^{T M}
$$

is a $T$-algebra morphism.

Proof. It suffices to prove the first part of the statement, for then the function $-\circ \widehat{h}$ is a $T$-algebra morphism because it coincides with the following composition of $T$-algebra morphisms:

$$
[\widehat{T} X, \widehat{T} X] \longleftrightarrow \widehat{T} X^{\widehat{T} X} \xrightarrow{\widehat{T} X^{\widehat{h}}} \widehat{T} X^{T M}
$$

Recall from Section 2.5 that $\widehat{T} X$ is the cofiltered limit of finite sets $Y_{i}$ which carry $T$-algebra structures $\alpha_{i}: T Y_{i} \rightarrow Y_{i}$. We have the following isomorphisms in the category of sets:

$$
\begin{aligned}
{[\widehat{T} X, \widehat{T} X] } & \cong\left[\widehat{T} X, \lim _{i} Y_{i}\right] \\
& \cong \lim _{i}\left[\widehat{T} X, Y_{i}\right] \\
& \cong \lim _{i}\left[\lim _{j} Y_{j}, Y_{i}\right] \\
& \cong \lim _{i} \operatorname{colim}_{j}\left[Y_{j}, Y_{i}\right]
\end{aligned}
$$

where, for the last isomorphism, we have used the fact that the $Y_{i}$ are finite spaces, and consequently finitely copresentable. Moreover, notice that the colimit above is filtered.

The sets $Y_{i}$ carry $T$-algebra structures and so do the sets $\left[Y_{j}, Y_{i}\right] \cong Y_{i}^{Y_{j}}$ with respect to pointwise operations. Since $T$ is finitary, the forgetful functor Set ${ }^{T} \rightarrow$ Set preserves and reflects both filtered colimits and limits (see e.g. [6, Propositions 3.4.1-3.4.2]). Whence, $[\widehat{T} X, \widehat{T} X]$ carries a $T$-algebra structure. We claim that, with respect to this $T$-algebra structure, $[\widehat{T} X, \widehat{T} X]$ is a subalgebra of the power algebra $\widehat{T} X^{\widehat{T} X}$.

For each $x \in \widehat{T} X$, write $e v_{x}:[\widehat{T} X, \widehat{T} X] \rightarrow \widehat{T} X$ for the function sending $f$ to $f(x)$. Then the natural inclusion

$$
[\widehat{T} X, \widehat{T} X] \hookrightarrow \widehat{T} X^{\widehat{T} X}
$$

is a $T$-algebra morphism if, and only if, each $e v_{x}$ is a $T$-algebra morphism. Write $\left\{\pi_{i}: \widehat{T} X \rightarrow Y_{i} \mid i \in I\right\}$ for the cone of continuous functions defining $\widehat{T} X$ as the cofiltered limit of finite sets $Y_{i}$ which carry $T$-algebra structures. It is not difficult to see that each $\pi_{i}$ is a $T$-algebra morphism; for a proof, see [20, Proposition 2.10]. Therefore it suffices to show that each composition $\pi_{i} \circ e v_{x}:[\widehat{T} X, \widehat{T} X] \rightarrow Y_{i}$ is a $T$-algebra morphism. For any $j \in I$, denote by $\gamma_{j}: \widehat{T} X^{Y_{j}} \rightarrow Y_{i}$ the composite

$$
\widehat{T} X^{Y_{j}} \xrightarrow{\pi_{i} 0-} Y_{i}^{Y_{j}} \xrightarrow{e v_{\pi_{j}(x)}} Y_{i} .
$$

It follows by Lemma 3.2 that each $\gamma_{j}$ is a $T$-algebra morphism. Upon recalling that $[\widehat{T} X, \widehat{T} X] \cong \operatorname{colim}_{j}\left[Y_{j}, \widehat{T} X\right]$ in the category $\mathrm{Set}^{T}$, it is not difficult to see that $\pi_{i} \circ \mathrm{ev} v_{x}:[\widehat{T} X, \widehat{T} X] \rightarrow Y_{i}$ is the (unique) $T$-algebra morphism induced by the cocone $\left\{\gamma_{j}:\left[Y_{j}, \widehat{T} X\right] \rightarrow Y_{i} \mid j \in I\right\}$, thus concluding the proof.

Exploiting the previous observations we can prove the main result of this section, stating that every finitary commutative monad on the category of sets can be lifted to the category of BiMs.

Theorem 3.4. Any finitary commutative Set-monad $T$ can be extended to a monad on BiM mapping ( $X, M$ ) to $(\widehat{T} X, T M)$.

Proof. We first give the definition of the monad on an object ( $X, M, h, \rho, \lambda$ ). We will show that this is mapped to a $\operatorname{BiM}(\widehat{T} X, T M, \widehat{h}, \widehat{\rho}, \widehat{\lambda})$, where $\widehat{h}$ is as in equation (11), and $\widehat{\rho}$ and $\widehat{\lambda}$ are defined as follows. Recall that $[X, X]$ and $[\widehat{T} X, \widehat{T} X]$ denote the sets of continuous endofunctions on $X$ and $\widehat{T} X$, respectively. To define $\widehat{\rho}$, consider the composite of the following two maps, where $\widehat{T}_{X, X}$ is given by the application of the functor $\widehat{T}$ to a continuous function in $[X, X]$ :

$$
\begin{equation*}
M \xrightarrow{\rho}[X, X] \xrightarrow{\widehat{T}_{X, X}}[\widehat{T} X, \widehat{T} X] . \tag{12}
\end{equation*}
$$

By Proposition 3.3 we know that $[\widehat{T} X, \widehat{T} X]$ is a $T$-algebra, hence the map in (12) admits a unique extension to an algebra morphism $\widehat{\rho}: T M \rightarrow[\widehat{T} X, \widehat{T} X]$. The function $\widehat{\lambda}$ is defined similarly, as the unique $T$-algebra morphism extending $\widehat{T}_{X, X} \circ \lambda$.

In order to prove that $(\widehat{T} X, T M, \widehat{h}, \widehat{\rho}, \widehat{\lambda})$ is a BiM, it remains to prove that the functions $\widehat{h}, \widehat{\rho}$ and $\widehat{\lambda}$ make the diagrams in Definition 2.1 commute. Equivalently, by virtue of Remark 2.4, that the next square and the analogous one (with $\widehat{\rho}$ replaced by $\widehat{\lambda}$, and $\widehat{r}$ by $\widehat{l}$ ) commute,

$$
\begin{align*}
& {[\widehat{T} X, \widehat{T} X] \xrightarrow{-\circ \widehat{h}} \widehat{T} X^{T M}} \\
& \quad \widehat{\rho} \mid  \tag{13}\\
& \quad T M \xrightarrow{\widehat{h} \circ-} \\
& \quad T M^{T M}
\end{align*}
$$

where $\widehat{r}$ and $\widehat{l}$ denote the right and left action, respectively, of $T M$ on itself. To this end, notice that the following diagram commutes.


For the upper leftmost trapezoid, by definition of $\widehat{h}$, we must prove that for every $f \in[X, X]$ we have

$$
\tau_{X} \circ T f \circ T h=\tau_{X} \circ T(f \circ h)=\widehat{T} f \circ \widehat{h}=\widehat{T} f \circ \tau_{X} \circ T h
$$

In turn, this follows from the fact that $\tau_{X} \circ T f=\widehat{T} f \circ \tau_{X}$ by naturality of $\tau$. The lower rightmost trapezoid commutes by the very definition of $\widehat{h}$, whereas the inner square is a reformulation of the left commuting square in (3), cf. Remark 2.4.

We derive the commutativity of (13) using the universal property of the free $T$-algebra on $M$ and by observing that a) in the outer square in (14), the right vertical and the top horizontal arrows are morphisms of $T$-algebras by Lemma 3.2 and Proposition 3.3, respectively; b) the map $\widehat{\rho}$ was defined as the unique extension of $\widehat{T}_{X, X} \circ \rho$ to the free algebra $T M ; \mathrm{c}$ ) the map $\widehat{r}$ is the unique algebra morphism extending $T_{M, M} \circ r$ to $T M$. To settle item c), notice that it is equivalent to the commutativity of the following diagram,

where $\otimes$ denotes either of the two compositions in diagram (9), $\cdot: M \times M \rightarrow M$ is the monoid operation of $M$ and $\varepsilon(s, f)=T_{M, M}(f)(s)$ for every $(s, f) \in T M \times M^{M}$. Now, observe that the identity

$$
\begin{equation*}
\otimes \circ\left(\mathrm{id}_{T M} \times \eta_{M}\right)=\sigma_{M, M}^{\prime}, \tag{16}
\end{equation*}
$$

where $\sigma^{\prime}$ is the co-strength of $T$, holds provided the following two diagrams commute.


The triangle on the left commutes by the leftmost diagram in (7). To show that the other triangle commutes, since $\left(\mu_{M} \times \mathrm{id}_{M}\right) \circ\left(\eta_{T M} \times \mathrm{id}_{M}\right)=\mathrm{id}_{T M \times M}$, it suffices to show that the following diagram is commutative.


In turn, the top triangle commutes by the leftmost diagram in (8), while the lower square commutes by the rightmost diagram in (8). Therefore, by equation (16), the commutativity of diagram (15) is equivalent to the commutativity of the outer square below,

where $e v: M \times M^{M} \rightarrow M$ sends $(m, f) \in M \times M^{M}$ to $f(m)$. The upper leftmost triangle commutes by naturality of $\sigma^{\prime}$, while the rightmost triangle and the lower one are easily seen to be commutative. Hence, item c) above is satisfied and diagram (13) commutes, as was to be proved. Reasoning in a similar manner for the left action, one can see that $(\widehat{T} X, T M, \widehat{h}, \widehat{\rho}, \widehat{\lambda})$ is indeed a BiM.

It is now straightforward computations, using the commutativity of the monad $T$, to check that the assignment $(X, M) \mapsto(\widehat{T} X, T M)$ yields the functor part of a monad on the category of BiMs.

Remark 3.5. Assume that the monad $T$ is not commutative and we attempt to use in the proof of Theorem 3.4 the monoid multiplication on $T M$ given by $\otimes$. All is fine for the right action and indeed the right action $\widehat{r}$ of $T M$ on itself is the unique extension of $T_{M, M} \circ r$. However, this is not the case for the left action. Symmetrically, if we chose the multiplication of $T M$ stemming from $\otimes^{\prime}$, then the left action $\widehat{l}$ would be the extension of the map $T_{M, M} \circ l$, but this property would fail for the right action.

## 4 Extending the free semimodule monad to BiMs

In Theorem 3.4 we showed how to lift any finitary commutative monad on Set to a monad on BiM. The purpose of the present section is then twofold. On the one hand we provide an example of a family of

Set-monads to which this result applies, and on the other hand we give explicit descriptions of the various objects, maps and actions of the associated monads on BiM. This will be essential for our further work on recognisers.

Given a semiring $S$, recall from Example 2.6 the free $S$-semimodule monad $\mathcal{S}$ on Set. Notice that $\mathcal{S}$ is a commutative monad if and only if $S$ is a commutative semiring, i.e., the multiplication • is a commutative operation. Indeed, for a monoid $M$, the two monoid operations one can define on $\mathcal{S} M$ are given as follows. If $f, f^{\prime} \in \mathcal{S} M$, then one can define $f f^{\prime}(x)$ either by

$$
\sum_{m m^{\prime}=x} f(m) \cdot f^{\prime}\left(m^{\prime}\right) \quad \text { or } \quad \sum_{m^{\prime} m=x} f^{\prime}\left(m^{\prime}\right) \cdot f(m)
$$

and the two coincide precisely when the semiring is commutative. For this reason, for the rest of the paper we will only consider commutative semirings $S$. We also consider the associated Set-monad $\mathcal{S}$, along with the profinite monad $\widehat{\mathcal{S}}$ on BStone (cf. Section 2.5).

Throughout the section we fix an arbitrary finite and commutative semiring $S$. Let $X$ be a Boolean space, and denote by $B$ its dual algebra. Next, we provide a concrete description of the Boolean space $\widehat{\mathcal{S}} X$ in terms of measures on $X$. For more details and for the proofs of several facts mentioned in this section, the interested reader is referred to [20].

Definition 4.1. Let $X$ be a Boolean space and $B$ the dual algebra. An $S$-valued measure (or just a measure when the semiring is clear from the context) on $X$ is a function $\mu: B \rightarrow S$ which is finitely additive, that is

1. $\mu(0)=0$, and
2. $\mu(K \vee L)=\mu(K)+\mu(L)$ whenever $K, L \in B$ are disjoint.

We remark that in item 1 the first 0 is the bottom of the Boolean algebra, while the second 0 is in $S$. Also, one can express item 2 without reference to disjointness:

$$
2^{\prime} . \mu(K \vee L)+\mu(K \wedge L)=\mu(K)+\mu(L) \text { for all } K, L \in B
$$

Note that our notion of measure is not standard, as we only require finite additivity. Also, the measure is only defined on the clopens of the space $X$. Finally, it takes values in a (finite and commutative) semiring.

Notation 4.2. Let $X$ be a set and $f: X \rightarrow S$ a function. If $Y \subseteq X$ is a subset such that the sum $\sum_{x \in Y} f(x)$ exists in $S$, then we write

$$
\int_{Y} f:=\sum_{x \in Y} f(x) .
$$

If $B \subseteq \mathcal{P} X$, and $\int_{Y} f$ exists for each $Y \in B$, then $\int f: B \rightarrow S$ denotes the function taking $Y$ to $\int_{Y} f$.
Suppose $X$ is a Boolean space and $B$ is its dual algebra. The Boolean algebra $\widehat{B}$ dual to $\widehat{\mathcal{S}} X$ is the subalgebra of $\mathcal{P}(\mathcal{S} X)$ generated by the elements of the form

$$
[L, k]:=\left\{f \in \mathcal{S} X \mid \int_{L} f=k\right\}
$$

for $L \in B$ and $k \in S$. For a proof of this fact, see [20, Lemma 4.2]. Regarding the elements of $\widehat{\mathcal{S}} X$ as Boolean algebra homomorphisms $\varphi: \widehat{B} \rightarrow 2$, we can define a function

$$
\begin{equation*}
\widehat{\mathcal{S}} X \longrightarrow\{\mu: B \rightarrow S \mid \mu \text { is a measure on } X\}, \quad \varphi \mapsto \mu_{\varphi} \tag{17}
\end{equation*}
$$

where $\mu_{\varphi}$ is the measure sending $L \in B$ to the unique $k \in S$ such that $\varphi[L, k]=1$. In turn, the set of all measures on $X$ is equipped with a natural topology, generated by the sets of the form

$$
\begin{equation*}
\overline{[L, k]}=\{\mu: B \rightarrow S \mid \mu \text { is a measure on } X, \mu(L)=k\} \tag{18}
\end{equation*}
$$

for $L \in B$ and $k \in S$ (the notation is justified by Proposition 4.4 below). With respect to this topology, the space $\widehat{\mathcal{S}} X$ admits the following measure-theoretic characterisation.

Theorem 4.3. Let $S$ be a finite and commutative semiring. For any Boolean space $X$, the map in (17) yields a homeomorphism between $\widehat{\mathcal{S}} X$ and the space of $S$-valued measures on $X$.

Proof. See [20, Theorem 4.3].
The previous result allows for a concrete representation of the map $\tau_{X}$ in (4) which, in turn, yields the following concrete instantiation of Lemma 2.8 (cf. also Remark 2.9).

Proposition 4.4. If $X$ is a Boolean space, then the function

$$
\tau_{X}: \mathcal{S} X \rightarrow \widehat{\mathcal{S}} X, \quad f \mapsto \int f
$$

embeds $\mathcal{S} X$ in $\widehat{\mathcal{S}} X$ as a dense subspace. Moreover $\overline{[L, k]}$, as defined in (18), is the topological closure of $[L, k]$ whenever $L$ is a clopen of $X$, and $k \in S$.

Remark 4.5. Theorem 4.3 yields, in particular, a characterisation of the Vietoris space $\mathcal{V} X$, for $X$ a Boolean space, in terms of two-valued measures on $X$. In [20, Section 5] it is also shown that, provided the semiring is idempotent (hence, a semilattice), measures can be replaced by their densities, i.e., functions $X \rightarrow S$ which are continuous with respect to the down-set topology on $S$. Thus, we recover the classical representation of the Vietoris space of $X$ in terms of continuous functions from $X$ into the Sierpiński space.

As follows by the general results in Sections 2.5 and 3 , respectively, $\widehat{\mathcal{S}} X$ is a module over the semiring $S$ and it is a Boolean space with an internal monoid if $X$ is. Here we identify the concrete nature of this structure relative to the incarnation of $\widehat{\mathcal{S}} X$ as the space of measures on $X$. We state these as lemmas and, indeed, one can prove them directly. However, the results in this section are just special cases of the more general results in Sections 2.5 and 3.

Lemma 4.6. Let $X$ be a Boolean space and let $\mu, \nu \in \widehat{\mathcal{S}} X$. Then

$$
\mu+\nu: K \mapsto \mu(K)+\nu(K)
$$

is again a measure on $X$ and the ensuing binary operation on $\widehat{\mathcal{S}} X$ is continuous. Further, for any $k \in S$,

$$
k \mu: K \mapsto k \cdot \mu(K)
$$

is again a measure on $X$ and the ensuing unary operation on $\widehat{\mathcal{S}} X$ is continuous.
This accounts for the $S$-semimodule structure on $\widehat{\mathcal{S}} X$. Now assume that $X$ is not just a Boolean space, but a BiM. To improve readability, we assume $h: M \rightarrow X$ is injective and identify $M$ with its image. Firstly, we observe that $\mathcal{S} M$ sits as a dense subspace of $\widehat{\mathcal{S}} X$ by composing the map $\mathcal{S h}: \mathcal{S} M \rightarrow \mathcal{S} X$ with the integration map of Proposition 4.4. This is the concrete incarnation, in the case of the monad $\mathcal{S}$, of Lemma 2.8.

Lemma 4.7. Let $(X, M)$ be a Boolean space with an internal monoid. Then

$$
\mathcal{S} M \rightarrow \widehat{\mathcal{S}} X, \quad f \mapsto \int f
$$

is the map $\widehat{h}$ from (11) transporting $\mathcal{S} M$ into a dense subspace of $\widehat{\mathcal{S}} X$.
We remark that, since we assumed $h$ is injective, so is the map $\widehat{h}$ in the previous lemma (cf. Remark 2.9). Now, to exhibit the BiM structure of $\widehat{\mathcal{S}} X$, we start by identifying the actions of $M$ on $\widehat{\mathcal{S}} X$.

Lemma 4.8. Let $(X, M)$ be a Boolean space with an internal monoid. Further, let $\mu \in \widehat{\mathcal{S}} X$ and $m \in M$. Then

$$
m \mu: K \mapsto \mu\left(m^{-1} K\right)
$$

where $m^{-1} K=\{x \in X \mid m x \in K\}$ whenever $K \subseteq X$ is clopen, is again a measure on $X$. This defines a left action of $M$ on $\widehat{\mathcal{S}} X$ with continuous components. Similarly,

$$
\mu m: K \mapsto \mu\left(K m^{-1}\right)
$$

defines a right action of $M$ on $\widehat{\mathcal{S}} X$ with continuous components, and these actions are compatible in the sense that $(m \mu) n=m(\mu n)$.

Using the $S$-semimodule structure of $\widehat{\mathcal{S}} X$ (see Lemma 2.7), along with the biaction of $M$ on $\widehat{\mathcal{S}} X$ provided by the previous lemma, it is easy to obtain the biaction of $\mathcal{S} M$ on $\widehat{\mathcal{S}} X$. The following can be regarded as the specific incarnation of Theorem 3.4.

Proposition 4.9. Let $(X, M)$ be a Boolean space with an internal monoid. The map

$$
\mathcal{S} M \times \widehat{\mathcal{S}} X \rightarrow \widehat{\mathcal{S}} X,(f, \mu) \mapsto f \mu:=\sum_{m \in M} f(m) \cdot m \mu
$$

is a left action of $\mathcal{S} M$ on $\widehat{\mathcal{S}} X$ with continuous components. A right action with continuous components may be defined similarly. These two actions are compatible and provide the BiM structure on $(\widehat{\mathcal{S}} X, \mathcal{S} M)$.

Finally, we consider a restriction of the above action of $\mathcal{S} M$ on $\widehat{\mathcal{S}} X$ which we will need for the construction of the space $\diamond_{S} X$ in Section 5 . This is given by precomposing with the unit of the monad $\widehat{\mathcal{S}}$ :

$$
\widehat{\eta}_{X}: X \rightarrow \widehat{\mathcal{S}} X, x \mapsto \mu_{x}
$$

where $\mu_{x}(K)=1$ if $x \in K$, and $\mu_{x}(K)=0$ otherwise. That is, $\mu_{x}=\int \chi_{x}$ where $\chi_{x}$ is the characteristic function of $\{x\}$ into $S$. It is immediate that this map embeds $X$ as a (closed) subspace of $\widehat{\mathcal{S}} X$. Thus we obtain an "action"

$$
\mathcal{S} M \times X \rightarrow \widehat{\mathcal{S}} X,(f, x) \mapsto f \mu_{x} .
$$

Next we observe that this "action" factors through the map $\mathcal{S} X \rightarrow \widehat{\mathcal{S}} X$ defined in Proposition 4.4.
Lemma 4.10. Let $(X, M)$ be a Boolean space with an internal monoid. Consider the map

$$
\mathcal{S} M \times X \rightarrow \mathcal{S} X, \quad(f, x) \mapsto f x
$$

where $f x(y):=\sum_{m x=y} f(m)$. Then we have

$$
f \mu_{x}=\int f x
$$

Furthermore, for each $f \in \mathcal{S} M$, the assignment $x \mapsto \int f x$ is continuous.

## 5 Recognisers for operations given by $S$-valued transductions

In this section we will see how we can use the extension of a Set-monad $T$ to BiMs obtained in Section 3 to generate recognisers for languages obtained by applying an operation modelled by the monad $T$.

It is by now a standard result in the theory of formal languages that many operations on languages can be modelled using transductions, i.e., maps of the form $M \rightarrow \mathcal{P} N$ for two monoids $M$ and $N$, see [18]. The starting point of this work is the observation that the existential quantifier can also be modelled as a transduction, as we will see in Section 5.2. Furthermore, modular quantifiers $\exists_{p \bmod q}$ of modulus $q$ fit into the same pattern. The only difference is that instead of using transductions of the form $M \rightarrow \mathcal{P} N$ one needs to replace the powerset $\mathcal{P} N$ with the free $\mathbb{Z}_{q}$-semimodule over $N$. More generally, we are interested in operations that can be modelled as maps $M \rightarrow \mathcal{S} N$ with $\mathcal{S}$ denoting as before the free $\mathcal{S}$-semimodule monad. In category theory these maps are known as Kleisli maps for $\mathcal{S}$, the morphisms in the so-called Kleisli category of $\mathcal{S}$.

We start Section 5.1 by briefly recalling the definition of the Kleisli maps for a monad. Then we present the blueprint of our approach, using an additional assumption on the $T$-Kleisli map under consideration (namely that it is a monoid morphism), and in Section 5.2 we instantiate $T$ to the free $S$-semimodule monads for commutative semirings $S$ and we adapt the general theory developed previously.

### 5.1 Recognising operations modelled by a monad $T$

Consider a monad $(T, \eta, \mu)$ on a category $\mathcal{C}$. The Kleisli category $\operatorname{KI}(T)$ for $T$ is equivalent to the category of free $T$-algebras and has played a crucial rôle in program semantics for modelling functions with side-effects. Formally, the objects of $\mathrm{KI}(T)$ are the objects in the underlying category $\mathcal{C}$ and a morphism $X \rightarrow Y$ in $\mathrm{KI}(T)$ (called a $T$-Kleisli map) is a morphism $X \rightarrow T Y$ in $\mathcal{C}$. One can think of an object $X$ in $\mathrm{KI}(T)$ as the generator of the free algebra $T X$. Notice that morphisms $X \rightarrow Y$ in $\mathrm{KI}(T)$ are in one-to-one correspondence with $T$-algebra morphisms $T X \rightarrow T Y$ between the corresponding free algebras.

Hereafter we assume $T$ is an arbitrary commutative and finitary monad on Set, and let $A, B$ be finite sets. We start by observing that a Kleisli map $R: A^{*} \rightarrow T\left(B^{*}\right)$ could be used to transform languages in the alphabet $B$ into languages in the alphabet $A$. Assume that $L=\phi^{-1}(P)$ for some monoid morphism $\phi: B^{*} \rightarrow M$ and some $P \subseteq M$. We consider the function

$$
A^{*} \xrightarrow{R} T\left(B^{*}\right) \xrightarrow{T \phi} T M .
$$

Since $T$ is a commutative monad, we know that it lifts to the category of monoids and thus we can see $T \phi$ as a monoid morphism. If $R$ is also a monoid morphism, and we will assume this only in this subsection, then so is $T \phi \circ R$, and it could be used for language recognition in the standard way. Assuming that we have a way of turning the recognising sets in $M$ into recognising sets in $T M$, i.e., that we have a predicate transformer $\mathcal{P} M \rightarrow \mathcal{P} T M$ mapping $P$ to $\widetilde{P}$, we obtain a language $\widetilde{L}$ in $A^{*}$ as the preimage of $\widetilde{P}$ under the morphism $T \phi \circ R$.
Remark 5.1. In the running example of the next subsection we will need maps $R$ that are not monoid morphisms, and in that setting we will have to use a matrix representation of the transduction instead. Nevertheless, the techniques used in the next subsection can be seen as an adaptation of the theory developed here for the case when $R$ is indeed a monoid morphism.

In this work we go beyond regular languages, so we are interested in languages recognised by a BiM morphism as follows:


We recall that to improve readability, and since $\tilde{\phi}$ is uniquely determined by its restriction to $B^{*}$, we sometimes denote such a morphism of BiMs simply by $\phi$, instead of $(\tilde{\phi}, \phi)$.

By Theorem 3.4, we know that $(\widehat{T} X, T M)$ is a BiM, and in what follows we use it for recognising $A$ languages by constructing another BiM morphism $\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow(\widehat{T} X, T M)$ as in Lemma 5.2 below. To this end, we need a way of lifting the Kleisli map $R: A^{*} \rightarrow T\left(B^{*}\right)$ to a Kleisli map for the monad $\widehat{T}$. This can be done in a natural way using a natural transformation

$$
\tau^{\#}: \beta T \rightarrow \widehat{T} \beta
$$

obtained from the natural transformation $\tau_{X}: T|X| \rightarrow|\widehat{T} X|$ defined in (4) using the unit $\iota$ and the counit $\varepsilon$ of the adjunction $\beta \dashv|-|$. Explicitly, $\tau^{\#}$ is obtained as the composite

$$
\begin{equation*}
\beta T \xrightarrow{\beta T \iota} \beta T|-|\beta \xrightarrow{\beta \tau \beta} \beta|-| \widehat{T} \beta \xrightarrow{\varepsilon \widehat{T} \beta} \widehat{T} \beta . \tag{20}
\end{equation*}
$$

(This is a rather standard construction in category theory, see for example [25, Theorem 9]). In down-to-earth terms, the component of $\tau^{\#}$ at the set $Y$ is the free extension of the composite

$$
T Y \xrightarrow{T \iota_{Y}} T|\beta Y| \xrightarrow{\tau_{\beta Y}}|\widehat{T} \beta Y| .
$$

It follows that the natural transformation $\tau^{\#}: \beta T \rightarrow \widehat{T} \beta$ also behaves well with respect to the units and multiplications of the monads. That is, in the terminology of [25], the pair ( $\beta, \tau^{\#}$ ) is a monad opfunctor. This in turn implies that $\beta$ can be lifted to a functor $\widehat{\beta}$ between the Kleisli categories, making the next square commute, where the vertical functors are the free functors from the base to the Kleisli categories.


The functor $\widehat{\beta}$ maps the Kleisli map $R: A^{*} \rightarrow T\left(B^{*}\right)$ to the Kleisli map $\widehat{R}: \beta\left(A^{*}\right) \rightarrow \widehat{T} \beta\left(B^{*}\right)$ given by the composite

$$
\begin{equation*}
\widehat{R}: \beta\left(A^{*}\right) \xrightarrow{\beta R} \beta T\left(B^{*}\right) \xrightarrow{\tau^{\#}} \widehat{T} \beta\left(B^{*}\right) . \tag{21}
\end{equation*}
$$

Lemma 5.2. Assume $R: A^{*} \rightarrow T\left(B^{*}\right)$ is a monoid morphism. If the pair $(\tilde{\phi}, \phi)$ from (19) is a morphism of $\mathrm{BiM} s$, then so is the pair $(\widehat{T} \tilde{\phi} \circ \widehat{R}, T \phi \circ R)$ described in the next diagram.


Proof. In the statement of the lemma we have omitted writing the forgetful functor $|-|$ on the top line of the diagram. We will need it nevertheless in the proof. Using the definition of $\widehat{R}$, we need to show that the next diagram commutes:


The two rectangles in the diagram above commute by naturality of $\iota$, respectively $\tau$, and the bottom right rhombus commutes because $\phi$ is a morphism of BiMs . To prove that the middle trapezoid is commutative, we just have to recall how the transformation $\tau^{\#}$ is defined, see (20). In a 2-categorical terminology, this is a simple exercise involving the mates $\tau$ and $\tau^{\#}$ :


The squares commute by the naturality of $\iota$, whilst the triangle commutes because $|-|\varepsilon \circ \iota|-|=$ id.

### 5.2 Recognising quantified languages via $S$-transductions

Here we show how to construct BiMs recognising quantified languages. We point out that the content of this subsection could be easily adapted to arbitrary Kleisli maps for the monads of the form $\widehat{\mathcal{S}}$, for commutative
semirings $S$. We start with a language $L$ in the extended alphabet $(A \times 2)^{*}$ recognised by a BiM morphism as in the following diagram.


In other words, there exists a clopen $C$ in $X$ such that $L=\phi^{-1}(C \cap M)$. The aim of this subsection is to construct recognisers for the quantified languages $L_{\exists}$ and $L_{\exists_{p \text { mod } q}}$, as defined in Section 2.1. To this end, using the formal sum notation in the definition of the monad $\mathcal{S}$, we consider the map $R: A^{*} \rightarrow \mathcal{S}\left((A \times 2)^{*}\right)$ defined by

$$
w \mapsto \sum_{i=1}^{|w|} 1_{S} \cdot w^{(i)} .
$$

If $S$ is the Boolean semiring 2 , then $R$ simply associates with each word $w$ the set of all words in $(A \times 2)^{*}$ with the same shape as $w$ and with exactly one marked letter. The framework developed in the previous subsection does not immediately apply, since $R$ is not a monoid morphism. So the first step we have to take is to obtain a monoid morphism from $R$, which will then be used to construct BiM recognisers for quantified languages.

Upon viewing $R$ as an $S$-transduction (see [21]), we observe that it is realised by the rational $S$-transducer $\mathcal{T}_{R}$ pictured in Figure 1, in which we have drawn transition maps only for a generic letter $a \in A$.


Figure 1: The $S$-transducer $\mathcal{T}_{R}$ realising $R$. All the transitions have weights $1_{S}$, and thus the transducer outputs value $1_{S}$ for all pairs of the form $\left(w, w^{(i)}\right)$, with $w \in A^{*}$ and $1 \leq i \leq|w|$.

This transducer provides the following representation of $R$ in terms of a monoid morphism

$$
\begin{equation*}
\mathrm{R}: A^{*} \rightarrow \mathcal{M}_{2}\left(\mathcal{S}\left((A \times 2)^{*}\right)\right), \tag{22}
\end{equation*}
$$

where $\mathcal{M}_{n}\left(\mathcal{S}\left((A \times 2)^{*}\right)\right)$ denotes the set of $n \times n$-matrices over the semiring $\mathcal{S}\left((A \times 2)^{*}\right)$. For a word $w \in A^{*}$, the matrix $\mathrm{R}(w)$ has at position $(i, j)$ the formal sum of output words obtained from the transducer $\mathcal{T}_{R}$ by going from state $i$ to state $j$ while reading input word $w$. That is, R is given by

$$
w \mapsto\left(\begin{array}{cc}
1_{S} \cdot w^{0} & \sum_{i} 1_{S} \cdot w^{(i)} \\
0_{S} & 1_{S} \cdot w^{0}
\end{array}\right) .
$$

The next two examples provide the motivation for considering the particular transduction $R$ in the first place.
Example 5.3. Assume $S$ is the Boolean semiring 2, thus $\mathcal{S}=\mathcal{P}_{f}$, and $R(w)=\left\{w^{(i)}|1 \leq i \leq|w|\}\right.$. The language $L_{\exists} \subseteq A^{*}$ is recognised by the following composite monoid morphism, that will be denoted by $\phi_{\exists}$.

$$
A^{*} \xrightarrow{\mathrm{R}} \mathcal{M}_{2}\left(\mathcal{P}_{f}\left((A \times 2)^{*}\right)\right) \xrightarrow{\mathcal{M}_{2}\left(\mathcal{P}_{f} \phi\right)} \mathcal{M}_{2}\left(\mathcal{P}_{f} M\right)
$$

Indeed, if $L=\phi^{-1}(P)$ for some $P \subseteq M$ then $L_{\exists}=\phi_{\exists}^{-1}(\widetilde{P})$, where $\widetilde{P}$ is the set of matrices in $\mathcal{M}_{2}\left(\mathcal{P}_{f} M\right)$ such that the finite set in position $(1,2)$ intersects $P$.

Example 5.4. Assume $S$ is the semiring $\mathbb{Z}_{q}$. The language $L_{\exists_{p \bmod q}} \subseteq A^{*}$ is recognised by the following composite monoid morphism, that will be denoted by $\phi_{\exists_{p \bmod q}}$.

$$
A^{*} \xrightarrow{\mathrm{R}} \mathcal{M}_{2}\left(\mathcal{S}\left((A \times 2)^{*}\right)\right) \xrightarrow{\mathcal{M}_{2}(\mathcal{S} \phi)} \mathcal{M}_{2}(\mathcal{S} M)
$$

Indeed, if $L=\phi^{-1}(P)$ with $P \subseteq M$ then $L_{\exists_{p \bmod q}}=\phi_{\exists_{p \bmod q}}^{-1}(\widetilde{P})$, where $\widetilde{P}$ is the set of matrices in $\mathcal{M}_{2}(\mathcal{S} M)$ such that the finitely supported function $f: \mathbb{Z}_{q} \rightarrow M$ in position $(1,2)$ has the property that $\int_{P} f=p$ in $\mathbb{Z}_{q}$.

In view of Theorem 3.4, we know that whenever $(X, M)$ is a BiM, then so is $(\hat{\mathcal{S}} X, S M)$ with the actions of the internal monoid as in Proposition 4.9. Using this fact, we can prove the following lemma.

Lemma 5.5. If $(X, M)$ is a BiM, then so is

$$
\left(\mathcal{M}_{n}(\widehat{\mathcal{S}} X), \mathcal{M}_{n}(\mathcal{S} M)\right)
$$

for any integer $n \geq 1$.
Proof. The set $\mathcal{M}_{n}(\widehat{\mathcal{S}} X)$ is a Boolean space with respect to the product topology of $n \times n$ copies of $\widehat{\mathcal{S}} X$. The statement then follows easily upon defining the actions of the monoid $\mathcal{M}_{n}(\mathcal{S} M)$ on $\mathcal{M}_{n}(\widehat{\mathcal{S}} X)$ by using the actions of $\mathcal{S} M$ on $\widehat{\mathcal{S}} X$ via matrix multiplication, and the $S$-semimodule structure of $\widehat{S} X$. For example, the left action of $\left(f_{i j}\right)_{i, j} \in \mathcal{M}_{n}(\mathcal{S} M)$ on $\left(\mu_{i j}\right)_{i, j} \in \mathcal{M}_{n}(\widehat{\mathcal{S}} X)$ yields a matrix of measures in $\widehat{\mathcal{S}} X$ having at position $(i, j)$ the measure $\sum_{k=1}^{n} f_{i k} \mu_{k j}$.

We next prove a result which entails that the monoid morphisms $\phi_{\exists}$ and $\phi_{\exists_{p \bmod q}}$ constructed in Examples 5.3 and 5.4 can be extended to BiM morphisms recognising $L_{\exists}$ and $L_{\exists_{p \bmod q}}$, respectively.
Lemma 5.6. If the pair $(\tilde{\phi}, \phi)$ from (19) is a morphism of BiM s and $\mathrm{R}: A^{*} \rightarrow \mathcal{M}_{n}\left(\mathcal{S}\left(B^{*}\right)\right)$ is a monoid morphism, then the pair $\left(\mathcal{M}_{n}(\widehat{\mathcal{S}} \tilde{\phi}) \circ \widehat{\mathrm{R}}, \mathcal{M}_{n}(\mathcal{S} \phi) \circ \mathrm{R}\right)$ described in the next diagram is a BiM morphism,

where $\widehat{\mathrm{R}}$ is the unique continuous extension of the following composite map:

$$
A^{*} \xrightarrow{\mathrm{R}} \mathcal{M}_{n}\left(\mathcal{S}\left(B^{*}\right)\right) \xrightarrow{\mathcal{M}_{n}(\iota)} \mathcal{M}_{n}\left(\beta \mathcal{S}\left(B^{*}\right)\right) \xrightarrow{\mathcal{M}_{n}\left(\tau^{\#}\right)} \mathcal{M}_{n}\left(\widehat{\mathcal{S}} \beta\left(B^{*}\right)\right)
$$

Proof. This follows essentially by Lemma 5.2 by setting $T=\mathcal{S}$, along with the functoriality of $\mathcal{M}_{n}(-)$. Note that the aforementioned lemma applies to this setting because R is a monoid morphism.

If we apply the previous lemma to the monoid morphism R in equation (22) we obtain the BiM $\left(\mathcal{M}_{2}(\widehat{\mathcal{S}} X), \mathcal{M}_{2}(\mathcal{S} M)\right)$ which, when instantiated with the appropriate semiring $S$, recognises the quantified languages $L_{\exists}$ and $L_{\exists_{p \bmod q}}$.

For instance, suppose the semiring $S$ is $\mathbb{Z}_{q}$. If $L$ is recognised by a clopen $C \subseteq X$ then, upon recalling from (18) that subbasic clopens of $\widehat{\mathcal{S}} X$ are of the form $\overline{[K, k]}$ for $K$ a clopen of $X$ and $k \in S$, one can easily prove that the quantified language $L_{\exists_{p \bmod q}}$ is recognised by the clopen subset of $\mathcal{M}_{2}(\widehat{\mathcal{S}} X)$ given by the product $\widehat{\mathcal{S}} X \times \overline{[C, p]} \times \widehat{\mathcal{S}} X \times \widehat{\mathcal{S}} X$, where the elements of the clopen $\overline{[C, p]}$ should appear in position $(1,2)$ in the matrix view of the space.

However, notice that the image of the morphism $\mathcal{M}_{2}(\mathcal{S} \tilde{\phi}) \circ \widehat{\mathrm{R}}$ is contained in the subspace of $\mathcal{M}_{2}(\widehat{\mathcal{S}} X)$ which can be represented by the matrix

$$
\left(\begin{array}{cc}
X & \widehat{\mathcal{S}} X \\
0 & X
\end{array}\right)
$$

As a consequence, we can use for the same recognition purpose a smaller BiM, through which the morphism $\mathcal{M}_{2}(\mathcal{S} \tilde{\phi}) \circ \widehat{\mathrm{R}}$ factors. We denote this BiM morphism by

$$
\diamond_{S} \phi:\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow\left(\diamond_{S} X, \diamond_{S} M\right),
$$

where

$$
\diamond_{S} X:=\widehat{\mathcal{S}} X \times X \text { and } \diamond_{S} M:=\mathcal{S} M \times M
$$

with monoid structure and biactions defined essentially by identifying the products above with upper triangular matrices, and then using the matrix multiplication and the concrete description of several monoid actions from Lemmas 4.8 and 4.10. Using the notations described in these lemmas, the left action of $\nabla_{S} M$ on $\nabla_{S} X$ can be described by

$$
\left(\begin{array}{cc}
m & f \\
0 & m
\end{array}\right)\left(\begin{array}{cc}
x & \mu \\
0 & x
\end{array}\right)=\left(\begin{array}{cc}
m x & m \mu+\int f x \\
0 & m x
\end{array}\right)
$$

where $(f, m) \in \nabla_{S} M$ and $(\mu, x) \in \nabla_{S} X$. Recall from Section 2.1 that the language $\mathcal{Q}_{k}(L)$ in the alphabet $A$ is obtained by quantifying the language $L \subseteq(A \times 2)^{*}$ with respect to the quantifier associated with a semiring $S$ and an element $k \in S$. We summarise the preceding observations in the following theorem.

Theorem 5.7. Let $S$ be a commutative semiring, and $k \in S$. Suppose a language $L \subseteq(A \times 2)^{*}$ is recognised by the BiM morphism $\phi:\left(\beta\left((A \times 2)^{*}\right),(A \times 2)^{*}\right) \rightarrow(X, M)$. Then the quantified language $\mathcal{Q}_{k}(L) \subseteq A^{*}$ is recognised by the BiM morphism $\diamond_{S} \phi:\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow\left(\diamond_{S} X, \diamond_{S} M\right)$.

As an immediate consequence, taking $S=2$ the Boolean semiring and $k=1$, we recover the result in [10, Proposition 13] on existential quantification:

Corollary 5.8. Consider a formula $\varphi(x)$ with a free first-order variable $x$. If the language $L_{\varphi(x)} \subseteq(A \times 2)^{*}$ is recognised by the BiM morphism $\phi:\left(\beta\left((A \times 2)^{*}\right),(A \times 2)^{*}\right) \rightarrow(X, M)$, then the existentially quantified language $L_{\exists x . \varphi(x)} \subseteq A^{*}$ is recognised by the BiM morphism $\diamond_{2} \phi:\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow\left(\mathcal{V} X \times X, \mathcal{P}_{f} M \times M\right)$.

## 6 Duality-theoretic account of the construction

Let $S$ be a finite and commutative semiring, and $(X, M)$ a BiM. As earlier, we denote by $B$ the dual algebra of $X$. Further, let $\phi:\left(\beta\left((A \times 2)^{*}\right),(A \times 2)^{*}\right) \rightarrow(X, M)$ be a BiM morphism. We denote by $\mathcal{B}$ the preimage under $\phi$ of $B$. That is, $\mathcal{B}$ is the Boolean algebra, closed under quotients in $\mathcal{P}\left((A \times 2)^{*}\right)$, of languages recognised by the BiM morphism $\phi$.

In Section 5.2 we introduced the map $\nabla_{S} \phi$ as a recogniser for the quantified languages obtained from the languages in $\mathcal{B}$. Here we prove, by duality, that $\nabla_{S} \phi$ is in fact the minimal possible BiM recogniser for these quantified languages. This will allow us to get a Reutenauer-type theorem for $\nabla_{S} X$, see Theorem 6.12. The idea is the following. On the language side, we are interested in the Boolean algebra generated by the languages of the form $\mathcal{Q}_{k}(L)$, for $k \in S$ and $L \in \mathcal{B}$. This coincides with the Boolean algebra $\mathcal{Q B}$ obtained as the preimage of $\widehat{B}$, the Boolean algebra of clopens of $\widehat{\mathcal{S}} X$, under the composite

$$
\phi_{Q}: A^{*} \longrightarrow \mathcal{S}\left((A \times 2)^{*}\right) \longrightarrow \mathcal{S} M \longrightarrow \widehat{\mathcal{S}} X
$$

where the first map sends $w \in A^{*}$ to $\sum_{i=1}^{|w|} 1_{S} \cdot w^{(i)}$, the second one is $\mathcal{S} \phi: \mathcal{S}\left((A \times 2)^{*}\right) \rightarrow \mathcal{S} M$, and the third one is the integration map (cf. Lemma 4.7). Indeed, suppose $\phi^{-1}(K)=L \in \mathcal{B}$ for $K$ a clopen of $X$. Then, for every $k \in S$,

$$
\begin{aligned}
\phi_{Q}^{-1}(\overline{[K, k]}) & =\left\{w \in A^{*} \mid \int_{K} \sum_{i=1}^{|w|} 1_{S} \cdot \phi\left(w^{(i)}\right)=k\right\} \\
& =\left\{w \in A^{*} \mid w \in \mathcal{Q}_{k}\left(\phi^{-1}(K)\right)\right\}=\mathcal{Q}_{k}(L) .
\end{aligned}
$$

The Boolean algebra $\mathcal{Q B}$ is not closed under quotients. Since we want a BiM recogniser, and not just a "Boolean space recogniser", we want to recognise the Boolean algebra closed under quotients generated by $\mathcal{Q B}$. Furthermore, from the viewpoint of logic we are adding one layer of quantifiers. Thus, by inductive hypothesis, it makes sense to include also the languages of the form $L_{0}=\left\{w \in A^{*} \mid w^{0} \in L\right\}$, for $L \in \mathcal{B}$. These are the languages in the alphabet $A$ which are recognised by $\phi$ upon composing with the embedding

$$
()^{0}: A^{*} \rightarrow \beta\left((A \times 2)^{*}\right), w \mapsto w^{0}
$$

Let $\mathcal{B}_{0}$ be the Boolean algebra that is the preimage of $\mathcal{B}$ under ( $)^{0}$. We thus want a BiM recogniser for $\mathcal{B}^{\prime}$, the closure under quotients of $\left\langle\mathcal{Q B} \cup \mathcal{B}_{0}>_{B A}\right.$. We show that:

1. The Boolean algebra $\left\langle\mathcal{Q B} \cup \mathcal{B}_{0}>_{B A}\right.$ is already closed under quotients, whence $\mathcal{B}^{\prime}=<\mathcal{Q B} \cup \mathcal{B}_{0}>_{B A}$.
2. This allows us to see $\mathcal{B}^{\prime}$ as a quotient of the coproduct of $\mathcal{Q B}$ and $\mathcal{B}_{0}$, hence also of $\widehat{B}$ and $B$. By describing the quotienting operations on these subalgebras we can define a compatible quotienting operation on the coproduct, which makes the natural map $\widehat{B}+B \rightarrow \mathcal{P}\left(A^{*}\right)$ a homomorphism of Boolean algebras with biactions.
3. Finally, dualising the quotienting operation on $\widehat{B}+B$ we get the actions of $\diamond_{S} M$ on $\diamond_{S} X$ given by matrix multiplication in Section 5.2. Further, we recover $\diamond_{S} \phi$ as dual to the homomorphism $\widehat{B}+B \rightarrow \mathcal{P}\left(A^{*}\right)$.

To improve readability, throughout this section we omit reference to the semiring $S$, and write $\diamond \phi, \diamond X, \diamond M$ instead of $\nabla_{S} \phi, \nabla_{S} X, \diamond_{S} M$.

### 6.1 The space $\diamond X$ by duality

Recall from (21) the Kleisli map $\widehat{R}$, and consider the continuous map

$$
\phi_{Q}: \beta\left(A^{*}\right) \xrightarrow{\widehat{R}} \widehat{\mathcal{S}} \beta\left((A \times 2)^{*}\right) \xrightarrow{\widehat{\mathcal{S}}_{\phi}} \widehat{\mathcal{S}} X
$$

which is given for $w \in A^{*}$ by

$$
\phi_{Q}(w):=\int f_{w}
$$

where

$$
f_{w}:=\sum_{1 \leq i \leq|w|} 1_{S} \cdot \phi\left(w^{(i)}\right)
$$

For any $k \in S$ and $L \in \mathcal{B}$, the clopen in $\beta\left(A^{*}\right)$ corresponding to $\mathcal{Q}_{k}(L)$ is $\phi_{Q}{ }^{-1}(\overline{[K, k]})$, where $K \subseteq X$ is the clopen in $X$ recognising $L$ via $\phi$ and $\overline{[K, k]}$ is as in equation (18). By Theorem 4.3, the clopens of $\widehat{\mathcal{S}} X$ are generated by the sets of the form $\overline{[K, k]}$ with $k \in S$ and $K \subseteq X$ clopen, thus we have:

Proposition 6.1. The Boolean algebra $\mathcal{Q B}$ of those languages over $A$ which are inverse images of clopens under $\phi_{Q}$ is generated by the quantified languages $\mathcal{Q}_{k}(L)$, for $k \in S$ and $L \in \mathcal{B}$.

Note that $\mathcal{Q B}$, as defined in the previous proposition, is not closed under quotients. This is the reason we had to make an adjustment between Sections 5.1 and 5.2 above.

We denote by $\mathcal{B}_{0}$ the Boolean algebra of languages closed under quotients which is recognised by the BiM morphism

$$
\phi_{0}:\left(\beta\left(A^{*}\right), A^{*}\right) \xrightarrow{()^{0}}\left(\beta\left((A \times 2)^{*}\right),(A \times 2)^{*}\right) \xrightarrow{\phi}(X, M) .
$$

In other words, $\mathcal{B}_{0}$ consists of the languages of the form $L_{0}:=\phi_{0}{ }^{-1}(K)$, obtained as the preimage under ()$^{0}$ of languages $L=\phi^{-1}(K)$ in $\mathcal{B}$. Taking the product map, it now follows that

$$
\diamond \phi=\phi_{Q} \times \phi_{0}: \beta\left(A^{*}\right) \rightarrow \widehat{\mathcal{S}} X \times X
$$

viewed just as a map of Boolean spaces, "recognises" the Boolean algebra generated by $\mathcal{Q B} \cup \mathcal{B}_{0}$, in the sense that the elements of the latter Boolean algebra are exactly those of the form $\diamond \phi^{-1}(C)$ for some clopen $C \subseteq \widehat{\mathcal{S}} X \times X$. However, since $\mathcal{Q B}$ is not closed under quotients, a priori, neither is $\left\langle\mathcal{Q B} \cup \mathcal{B}_{0}\right\rangle_{B A}$.

The Boolean algebra $\mathcal{B}^{\prime}$ that we are interested in is the closure under quotients of $\left\langle\mathcal{Q B} \cup \mathcal{B}_{0}\right\rangle_{B A}$. The important observation is that $\left\langle\mathcal{Q B} \cup \mathcal{B}_{0}>_{B A}\right.$ is already closed under the quotient operations, thus explaining why $\widehat{\mathcal{S}} X \times X$, along with the above product map, is the right recogniser space-wise.

Proposition 6.2. The Boolean algebra generated by $\mathcal{Q B} \cup \mathcal{B}_{0}$ is closed under quotients. That is,

$$
\mathcal{B}^{\prime}=<\mathcal{Q}_{k}(L), L_{0} \mid L \in \mathcal{B} \text { and } k \in S>_{B A}
$$

Proof. Since $\mathcal{B}_{0}$ is closed under quotients, it suffices to consider the quotienting of languages of the form $\mathcal{Q}_{k}(L)=\phi_{Q}{ }^{-1}(\overline{[K, k]})$ where $K \subseteq X$ is the clopen recognising $L$ via $\phi$. For $u \in A^{*}$ we have

$$
\begin{aligned}
u^{-1} \mathcal{Q}_{k}(L) & =\left\{w \in A^{*} \mid u w \in \mathcal{Q}_{k}(L)\right\} \\
& =\left\{w \in A^{*} \mid \int f_{u w} \in \overline{[K, k]}\right\}
\end{aligned}
$$

Since the free variable in the word $u w$ either occurs in $u$ or in $w$,

$$
f_{u w}=\phi\left(u^{0}\right) f_{w}+f_{u} \phi\left(w^{0}\right)
$$

Further, since $\int\left(\phi\left(u^{0}\right) f_{w}+f_{u} \phi\left(w^{0}\right)\right)=\int \phi\left(u^{0}\right) f_{w}+\int f_{u} \phi\left(w^{0}\right)$, we have

$$
\begin{aligned}
u^{-1} \mathcal{Q}_{k}(L) & =\left\{w \in A^{*} \mid \int \phi\left(u^{0}\right) f_{w}+\int f_{u} \phi\left(w^{0}\right) \in \overline{[K, k]}\right\} \\
& =\bigcup_{k_{1}+k_{2}=k}\left\{w \in A^{*} \mid \int \phi\left(u^{0}\right) f_{w} \in \overline{\left[K, k_{1}\right]} \text { and } \int f_{u} \phi\left(w^{0}\right) \in \overline{\left[K, k_{2}\right]}\right\} .
\end{aligned}
$$

Now,

$$
\begin{equation*}
\int \phi\left(u^{0}\right) f_{w} \in \overline{\left[K, k_{1}\right]} \Longleftrightarrow \int f_{w} \in \overline{\left[\phi\left(u^{0}\right)^{-1} K, k_{1}\right]} \tag{23}
\end{equation*}
$$

which in turn is equivalent to $w \in \mathcal{Q}_{k_{1}}\left(\left(u^{0}\right)^{-1} L\right)$, which is an element of $\mathcal{Q B}$. We now proceed with the second condition. We have $\int f_{u} \phi\left(w^{0}\right) \in \overline{\left[K, k_{2}\right]}$ if, and only if, there is a set

$$
I \subseteq \operatorname{Sup}\left(f_{u}\right):=\left\{m \in M \mid f_{u}(m) \neq 0\right\}
$$

with

- $\int_{I} f_{u}=k_{2}$;
- $m \phi\left(w^{0}\right) \in K$ for each $m \in I$;
- $m \phi\left(w^{0}\right) \notin K$ for each $m \notin I$.

Observe that $m \phi\left(w^{0}\right) \in K$ if, and only if, $w \in \phi_{0}{ }^{-1}\left(m^{-1} K\right)$. Thus

$$
\left\{w \in A^{*} \mid \int f_{u} \phi\left(w^{0}\right) \in \overline{\left[K, k_{2}\right]}\right\}
$$

is equal to

$$
\begin{equation*}
\bigcup_{\substack{I \subseteq \operatorname{Sup}\left(f_{u}\right) \\ \int_{I} f_{u}=k_{2}}}\left(\left[\bigcap_{m \in I} \phi_{0}^{-1}\left(m^{-1} K\right)\right] \cap\left[\bigcap_{m \in I^{c}} \phi_{0}^{-1}\left(m^{-1} K^{c}\right)\right]\right) \tag{24}
\end{equation*}
$$

which is in $\mathcal{B}_{0}$.

Corollary 6.3. The dual space of $\mathcal{B}^{\prime}$ is a closed subspace of $\widehat{\mathcal{S}} X \times X$. In particular, $\mathcal{B}^{\prime}$ is recognised as a Boolean algebra by $\widehat{\mathcal{S}} X \times X$.

Proof. By the previous proposition, $\mathcal{B}^{\prime}=\left\langle\mathcal{Q B} \cup \mathcal{B}_{0}>_{B A}\right.$. But $\mathcal{B}_{0}$ is exactly the preimage of the dual of $X$ under $\phi_{0}$, and $\mathcal{Q B}$ is exactly the preimage of the dual of $\widehat{\mathcal{S}} X$ under $\phi_{Q}$. Thus $\mathcal{B}^{\prime}$ is exactly the preimage of the dual of $\widehat{\mathcal{S}} X \times X$ under $\diamond \phi$, and therefore $\mathcal{B}^{\prime}$ is recognised as a Boolean algebra by $\widehat{\mathcal{S}} X \times X$.

Now, factoring the map $\diamond \phi$, we obtain a closed subspace $Y$ of $\widehat{\mathcal{S}} X \times X$ so that

$$
\diamond \phi: \beta\left(A^{*}\right) \rightarrow Y \hookrightarrow \widehat{\mathcal{S}} X \times X
$$

Since the dual of the quotient map $\beta\left(A^{*}\right) \rightarrow Y$ is an embedding whose image is $\mathcal{B}^{\prime}$, the dual of $Y$ is isomorphic to $\mathcal{B}^{\prime}$.

### 6.2 The internal monoid structure of $\diamond X$ by duality

In Section 5.2 the monoid operation of $\diamond M=\mathcal{S} M \times M$ and the actions of $\diamond M$ on $\diamond X$ were given in terms of matrix multiplication. This multiplication was introduced in an ad hoc manner. Here we show that these actions (and, in particular, the monoid operation) need not be guessed, as they can be derived by duality. In fact, they are the appropriate actions on $\Delta X$ for making $\diamond \phi$ a BiM morphism. For this purpose, we consider the homomorphism dual to $\diamond \phi$ :

$$
\varphi: \widehat{B}+B \rightarrow \mathcal{P}\left(A^{*}\right), \overline{[K, k]} \mapsto \phi_{Q}^{-1}(\overline{[K, k]}), K \mapsto \phi_{0}^{-1}(K)
$$

We already know, by Proposition 6.2, that the image of $\varphi$ is closed under quotients. The point is, in fact, that Proposition 6.2 tells us that we can define a biaction of $\diamond M$ on $\widehat{B}+B$ so that $\varphi$ becomes a homomorphism of Boolean algebras with biactions. Thus, for each $(f, m) \in \diamond M$, we want to define a "left quotient" by $(f, m)$ (that is, the component at $(f, m)$ of a right action) on $\widehat{B}+B$ (and a "right quotient", which is a left action) so that $\varphi$ becomes a homomorphism of Boolean algebras with biactions.

The monoid morphism from $A^{*}$ to $\diamond M$ is given by sending the internal monoid element $u \in A^{*}$ to the internal monoid element $\left(f_{u}, \phi\left(u^{0}\right)\right) \in \mathcal{S} M \times M$, where $f_{u}$ is defined as at the beginning of Section 6.1. Now, the component at $(f, m)$ of a "left quotient" operation on $\widehat{B}+B$ is a homomorphism

$$
\Lambda(f, m): \widehat{B}+B \rightarrow \widehat{B}+B
$$

Given the nature of coproducts, such a homomorphism is determined by its components $\Lambda_{1}(f, m): \widehat{B} \rightarrow \widehat{B}+B$ and $\Lambda_{2}(f, m): B \rightarrow \widehat{B}+B$. Our goal then, is to show that:

- the computation of quotient operations in the image of $\varphi$ combined with wanting $\varphi$ to be a morphism of Boolean algebras with biactions, dictates what $\Lambda_{1}(f, m)$ and $\Lambda_{2}(f, m)$ must be;
- the left action of $\diamond M$ on $\diamond X$ dual to $\Lambda$ coincides with the one defined in Section 5.2.

The symmetric facts for the right action are similar and thus we only consider the left action. Also, note that we will not prove directly that the $\Lambda(f, m)$ 's that we define are components of a right action on a Boolean algebra, as this will follow from the second bullet point above.

So, we want to define the action such that $\varphi$ becomes a homomorphism sending the action of $\left(f_{u}, \phi\left(u^{0}\right)\right)$ to the action of the quotient operation $u^{-1}()$ on $\mathcal{P}\left(A^{*}\right)$. The computations in the proof of Proposition 6.2 tell us the components of $u^{-1} \phi_{Q}{ }^{-1}(\overline{[K, k]})$ in $\mathcal{Q B}$ and in $\mathcal{B}_{0}$, respectively. Since $\mathcal{Q B}$ and $\mathcal{B}_{0}$ are precisely the images under $\varphi$ of $\widehat{B}$ and $B$, respectively, the computation tells us how to define $\Lambda_{1}\left(f_{u}, \phi\left(u^{0}\right)\right)$ using components $\Lambda_{11}(f, m): \widehat{B} \rightarrow \widehat{B}$ and $\Lambda_{12}(f, m): \widehat{B} \rightarrow B$.

By the computation in (23), we have that the component $\Lambda_{11}(f, m): \widehat{B} \rightarrow \widehat{B}$ depends only on the second coordinate of the pair $\left(f_{u}, \phi\left(u^{0}\right)\right)$ and it sends $\overline{[K, k]}$ to $\overline{\left[\left(\phi\left(u^{0}\right)\right)^{-1} K, k\right]}$. Stating it for an arbitrary element $(f, m) \in \mathcal{S} M \times M$, we have

$$
\Lambda_{11}(f, m): \widehat{B} \rightarrow \widehat{B}, \overline{[K, k]} \mapsto \overline{\left[m^{-1} K, k\right]} .
$$

Similarly, the computation in (24), stated for an arbitrary element $(f, m) \in \mathcal{S} M \times M$, yields $\Lambda_{12}(f, m): \widehat{B} \rightarrow$ $B$ given by

$$
\begin{equation*}
\overline{[K, k]} \mapsto \bigcup_{\substack{I \subseteq \operatorname{Sup}(f) \\ \int_{I} f=k}}\left(\left[\bigcap_{n \in I} n^{-1} K\right] \cap\left[\bigcap_{n \in I^{c}} n^{-1} K^{c}\right]\right) \tag{25}
\end{equation*}
$$

The above observations imply that
Proposition 6.4. The map $\varphi: \widehat{B}+B \rightarrow \mathcal{P}\left(A^{*}\right)$ is a homomorphism of Boolean algebras with biactions when the left quotient operation $\Lambda(f, m)$ of $\widehat{B}+B$ is defined on $\widehat{B}$ by

$$
\Lambda_{1}(f, m): \overline{[K, k]} \mapsto \bigvee_{k_{1}+k_{2}=k}\left(\Lambda_{11}\left(\overline{\left[K, k_{1}\right]}\right) \wedge \Lambda_{12}\left(\overline{\left[K, k_{2}\right]}\right)\right)
$$

and on $B$ by $\Lambda_{2}(f, m): K \mapsto m^{-1} K$.
It is now an easy verification that the maps $\Lambda_{11}(f, m)$ and $\Lambda_{12}(f, m)$ are dual to the summands of the first component of the action of $(f, m)$ on $\diamond X$, and that $\Lambda_{1}(f, m)$ and $\Lambda_{2}(f, m)$ are dual, respectively, to

$$
\lambda_{1}(f, m): \widehat{\mathcal{S}} X \times X \rightarrow \widehat{\mathcal{S}} X, \quad(\mu, x) \mapsto m \mu+\int f x
$$

and

$$
\lambda_{2}(f, m): \widehat{\mathcal{S}} X \times X \rightarrow X, \quad(\mu, x) \mapsto m x
$$

Lemma 6.5. The homomorphism $\Lambda_{11}(f, m): \widehat{B} \rightarrow \widehat{B}$ given by $\overline{[K, k]} \mapsto \overline{\left[m^{-1} K, k\right]}$ is dual to the continuous function $\lambda_{11}(f, m): \widehat{\mathcal{S}} X \rightarrow \widehat{\mathcal{S}} X$ given by $\mu \mapsto m \mu$, where

$$
m \mu: B \rightarrow S, K \mapsto \mu\left(m^{-1} K\right)
$$

Proof. The function $\lambda_{11}(f, m)$ is dual to $\Lambda_{11}(f, m)$ if, and only if, for all $\mu \in \widehat{\mathcal{S}} X$ and all $\overline{[K, k]} \in \widehat{B}$ we have

$$
\lambda_{11}(f, m) \mu \in \overline{[K, k]} \quad \Longleftrightarrow \quad \mu \in \Lambda_{11}(f, m) \overline{[K, k]}
$$

But $\lambda_{11}(f, m) \mu=m \mu$, so

$$
\begin{aligned}
\lambda_{11}(f, m) \mu \in \overline{[K, k]} & \Longleftrightarrow m \mu \in \overline{[K, k]} \\
& \Longleftrightarrow m \mu(K)=k \\
& \Longleftrightarrow \mu\left(m^{-1} K\right)=k \\
& \Longleftrightarrow \mu \in \overline{\left[m^{-1} K, k\right]}=\Lambda_{11}(f, m)
\end{aligned}
$$

as was to be proved.
Lemma 6.6. The homomorphism $\Lambda_{12}(f, m): \widehat{B} \rightarrow B$ given as in (25) is dual to the continuous function $\lambda_{12}(f, m): X \rightarrow \widehat{\mathcal{S}} X$ given by $x \mapsto \int f x$, where

$$
\int f x: B \rightarrow S, K \mapsto \int_{K x}-f_{1} .
$$

Proof. Let $x \in X$ and $[K, k] \in \widehat{B}$. Then

$$
\begin{aligned}
\int f x \in[K, k] & \Longleftrightarrow \int_{K} f x=k \\
& \Longleftrightarrow \int_{K x} f=k \\
& \Longleftrightarrow \sum_{x \in n^{-1} K} f(n)=k
\end{aligned}
$$

and the latter is true if, and only if, there exists $I \subseteq \operatorname{Sup}(f)$ with $\int_{I} f=k$ and $x \in n^{-1} K$ for each $n \in I$ and $x \notin n^{-1} K$ for each $n \notin I$. That is,

$$
\int f x \in[K, k] \quad \Longleftrightarrow \quad x \in \Lambda_{12}(f, m)[K, k] .
$$

Therefore, the homomorphism $\Lambda_{12}(f, m)$ is dual to the continuous map $\lambda_{12}(f, m)$.
Lemma 6.7. The homomorphism $\Lambda_{1}(f, m): \widehat{B} \rightarrow \widehat{B}+B$ given as in Proposition 6.4 is dual to the continuous function $\lambda_{1}(f, m): \widehat{\mathcal{S}} X \times X \rightarrow \widehat{\mathcal{S}} X$ given by $(\mu, x) \mapsto m \mu+\int f x$.

Proof. Let $(\mu, x) \in \widehat{\mathcal{S}} X \times X$ and $[K, k] \in \widehat{B}$. Then

$$
\begin{aligned}
\lambda_{1}(f, m)(\mu, x) \in[K, k] & \Longleftrightarrow \lambda_{11}(f, m) \mu+\lambda_{12}(f, m) x \in[K, k] \\
& \Longleftrightarrow \exists k_{1}, k_{2}\left(k_{1}+k_{2}=k, \lambda_{11}(f, m) \mu \in\left[K, k_{1}\right], \text { and } \lambda_{12}(f, m) x \in\left[K, k_{2}\right]\right) \\
& \Longleftrightarrow \exists k_{1}, k_{2}\left(k_{1}+k_{2}=k, \mu \in \Lambda_{11}(f, m)\left[K, k_{1}\right], \text { and } x \in \Lambda_{12}(f, m)\left[K, k_{2}\right]\right) \\
& \Longleftrightarrow(\mu, x) \in \Lambda_{1}[K, k],
\end{aligned}
$$

as was to be shown.
It is straightforward that $\Lambda_{2}(f, m): K \mapsto m^{-1} K$ is dual to $\lambda_{2}(f, m): x \mapsto m x$. Since the continuous map $\lambda_{1}(f, m) \times \lambda_{2}(f, m): \diamond X \rightarrow \diamond X$ coincides with the component at $(f, m)$ of the left action of $\diamond M$ on $\diamond X$ defined in Section 5.2 through matrix multiplication, we conclude that
Corollary 6.8. The dual of the left quotienting operation $\Lambda$ on $\widehat{B}+B$ defined in Proposition 6.4 is the left action of $\diamond M$ on $\diamond X$ defined in Section 5.2.

A similar result holds of course for the right action, and the monoid operation of $\diamond M$ can be recovered by restricting the actions on $\diamond X$ to $\diamond M$. As a consequence, we have

Theorem 6.9. Let $\phi:\left(\beta\left((A \times 2)^{*}\right),(A \times 2)^{*}\right) \rightarrow(X, M)$ be a BiM morphism. Then the homomorphism of Boolean algebras with biactions

$$
\varphi: \widehat{B}+B \rightarrow \mathcal{P}\left(A^{*}\right), \quad \overline{[K, k]} \mapsto \phi_{Q}^{-1}(\overline{[K, k]}), \quad K \mapsto \phi_{0}^{-1}(K)
$$

obtained by equipping $\widehat{B}+B$ with the biaction of $\diamond M$ as indicated in Proposition 6.4, is dual to the BiM morphism

$$
\diamond \phi:\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow(\diamond X, \diamond M)
$$

derived in Section 5.2.

### 6.3 A Reutenauer theorem for $\diamond X$

In this last subsection we prove a Reutenauer-type result (see Theorem 6.12 below) characterising the Boolean algebra closed under quotients generated by all languages recognised by the space $\diamond X$ through length preserving morphisms.

Definition 6.10. We call a BiM morphism $\psi:\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow(\diamond X, \diamond M)$ length preserving provided, for each $a \in A$, we have that

$$
\pi_{1} \circ \psi(a): M \rightarrow S
$$

is the characteristic function $\chi_{m_{a}}$ for some single $m_{a} \in M$. That is, $\pi_{1} \circ \psi(a)\left(m_{a}\right)=1$ and $\pi_{1} \circ \psi(a)(m)=0$ for all $m \in M$ with $m \neq m_{a}$.

Recall that, given any BiM morphism $\phi:\left(\beta\left((A \times 2)^{*}\right),(A \times 2)^{*}\right) \rightarrow(X, M)$, we obtain a BiM morphism

$$
\diamond \phi:\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow(\diamond X, \diamond M), \quad w \mapsto\left(\int f_{w}, \phi\left(w^{0}\right)\right)
$$

Upon defining $f_{a}:=\pi_{1} \circ \diamond \phi(a)$, we have $f_{a}=\chi_{m_{a}}$ where $m_{a}=\phi(a, 1)$. Hence, $\diamond \phi$ is length preserving. It is now a matter of a straightforward computation to prove the following proposition.

Proposition 6.11. Let $X$ be a BiM. Every length preserving BiM morphism $\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow(\diamond X, \diamond M)$ is of the form $\diamond \phi$ for some BiM morphism $\phi:\left(\beta\left((A \times 2)^{*}\right),(A \times 2)^{*}\right) \rightarrow(X, M)$.
Proof. Consider an arbitrary length preserving BiM morphism $\psi:\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow(\diamond X, \diamond M)$. We define $\phi:\left(\beta\left((A \times 2)^{*}\right),(A \times 2)^{*}\right) \rightarrow(X, M)$ by

$$
\begin{aligned}
\phi:(A \times 2)^{*} & \rightarrow M \\
(a, 0) & \mapsto \pi_{2} \circ \psi(a) \\
(a, 1) & \mapsto m_{a}
\end{aligned}
$$

where $m_{a} \in M$ is such that $\pi_{1} \circ \psi(a)=\chi_{m_{a}}$. The universal property of the Stone-Čech compactification guarantees that $\phi$ is a BiM morphism with the topological component $\tilde{\phi}=\beta \phi$. It now suffices to show that $\psi(a)=\diamond \phi(a)$ for each $a \in A$ :

$$
\diamond \phi(a)=\left(f_{a}, \phi_{0}(a)\right)=\left(\chi_{\phi(a, 1)}, \phi(a, 0)\right)=\left(\chi_{m_{a}}, \pi_{2} \circ \psi(a)\right)=\left(\pi_{1} \circ \psi(a), \pi_{2} \circ \psi(a)\right)=\psi(a)
$$

Theorem 6.12. Let $X$ be a BiM, and $A$ a finite alphabet. The Boolean subalgebra closed under quotients of $\mathcal{P}\left(A^{*}\right)$ generated by all languages over $A$ which are recognised by a length preserving BiM morphism into $\diamond X$ is generated as a Boolean algebra by the languages over $A$ recognised by $X$, and the languages $\mathcal{Q}_{k}(L)$ for $L$ a language over $A \times 2$ recognised by $X$.

Proof. Let us denote by $\mathcal{B}^{\prime \prime}$ the Boolean algebra generated by the languages over $A$ recognised by $X$, and the languages $\mathcal{Q}_{k}(L)$ for $L$ a language over $A \times 2$ recognised by $X$.

If $L^{\prime} \in \mathcal{P}\left(A^{*}\right)$ is recognised by a length preserving BiM morphism $\psi:\left(\beta\left(A^{*}\right), A^{*}\right) \rightarrow(\diamond X, \diamond M)$, then by Proposition 6.11 there is a BiM morphism $\phi:\left(\beta\left((A \times 2)^{*}\right),(A \times 2)^{*}\right) \rightarrow(X, M)$ such that $\rangle \phi=\psi$. That is, $L^{\prime}$ lies in the Boolean algebra called $\mathcal{B}^{\prime}$ in the beginning of this section. Since $\mathcal{B}^{\prime} \subseteq \mathcal{B}^{\prime \prime}$ by Proposition 6.2, we have $L^{\prime} \in \mathcal{B}^{\prime \prime}$.

For the other inclusion, if $L$ is a language over $A \times 2$ recognised by $X$, then $\mathcal{Q}_{k}(L)$ is recognised by $\diamond X$ through a length preserving morphism in view of Theorem 5.7. Finally, suppose $L$ is a language over $A$ recognised by $\eta: \beta\left(A^{*}\right) \rightarrow X$ through the clopen $K$. Consider any function $\phi: \beta\left((A \times 2)^{*}\right) \rightarrow M$ satisfying $\phi(a, 0)=\eta(a)$ for each $a \in A$. Then $L=\diamond \phi^{-1}(\widehat{\mathcal{S}} X \times K)$, showing that $L$ is recognised by $\diamond X$ through a length preserving morphism.

## 7 Conclusion

In this paper we provide a general construction for recognisers which captures the action of quantifier-like operations on languages of words, drawing heavily on a combination of general categorical tools and more concrete duality theoretic ones.

This paper is a stepping stone in a long-term research programme aimed at finding meaningful ultrafilter equations that characterise logically defined classes of non-regular languages. The next step is to understand the effect on equations of the constructions introduced here.

The generic development of Section 5 allows this work to be extended to encompass a wider range of operations on languages modelled by rational transducers which, by the Kleene-Schützenberger theorem (see, e.g., [21]), admit a matrix representation. Also, the duality-theoretic account in Section 6 leads to a Reutenauer-type characterisation theorem, akin to the one in [10]. It would also be interesting to understand a common framework for our contributions and the recent work [7].

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## References

[1] J. Adámek, L.-T. Chen, S. Milius, and H. Urbat. Profinite monads, profinite equations and Reiterman's theorem. CoRR, abs/1511.02147, 2015.
[2] J. Adámek, L.-T. Chen, S. Milius, and H. Urbat. Profinite monads, profinite equations, and Reiterman's theorem. In FoSSaCS, volume 9634 of Lecture Notes in Computer Science, pages 531-547. Springer, 2016.
[3] J. Adámek, S. Milius, R. Myers, and H. Urbat. Varieties of languages in a category. In LICS, pages 414-425. IEEE Computer Society, 2015.
[4] J. Almeida and P. Weil. Profinite categories and semidirect products. Journal of Pure and Applied Algebra, 123(1):1-50, 1998.
[5] M. Bojańczyk. Recognisable languages over monads. In DLT, volume 9168 of Lecture Notes in Computer Science, pages 1-13. Springer, 2015.
[6] F. Borceux. Handbook of categorical algebra 2, Categories and Structures. Encyclopedia of mathematics and its applications. Cambridge University Press, Cambridge, New York, 1994.
[7] L.-T. Chen and H. Urbat. Schützenberger products in a category. In DLT, volume 9840 of Lecture Notes in Computer Science, pages 89-101. Springer, 2016.
[8] M. Gehrke, S. Grigorieff, and J.-É. Pin. Duality and equational theory of regular languages. In ICALP (2), volume 5126 of Lecture Notes in Computer Science, pages 246-257. Springer, 2008.
[9] M. Gehrke, S. Grigorieff, and J.-É. Pin. A topological approach to recognition. In ICALP (2), volume 6199 of Lecture Notes in Computer Science, pages 151-162. Springer, 2010.
[10] M. Gehrke, D. Petrişan, and L. Reggio. The Schützenberger product for syntactic spaces. In ICALP, volume 55 of LIPIcs, pages 112:1-112:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.
[11] M. Gehrke, D. Petrişan, and L. Reggio. Quantifiers on languages and codensity monads. In 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, pages 1-12, 2017.
[12] J. F. Kennison and D. Gildenhuys. Equational completion, model induced triples and pro-objects. Journal of Pure and Applied Algebra, 1(4):317-346, 1971.
[13] A. Kock. Monads on symmetric monoidal closed categories. Journal of The Australian Mathematical Society, 21:1-10, 1970.
[14] T. Leinster. Codensity and the ultrafilter monad. Theory and Applications of Categories, pages 332-370, 2013.
[15] S. Mac Lane. Categories for the working mathematician, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1998.
[16] R. McNaughton and S. Papert. Counter-free automata. The M.I.T. Press, Cambridge, Mass.-London, 1971. With an appendix by William Henneman, M.I.T. Research Monograph, No. 65.
[17] E. Michael. Topologies on spaces of subsets. Trans. Amer. Math. Soc., 71:152-182, 1951.
[18] J.-É. Pin and J. Sakarovitch. Some operations and transductions that preserve rationality. In Proceedings of the 6th GI-Conference on Theoretical Computer Science, pages 277-288. Springer-Verlag, 1982.
[19] N. Pippenger. Regular languages and stone duality. Theory of Computing Systems, 30(2):121-134, 1997.
[20] L. Reggio. Codensity, profiniteness and algebras of semiring-valued measures. Journal of Pure and Applied Algebra, 2019, in press.
[21] J. Sakarovitch. Elements of Automata Theory. Cambridge University Press, New York, NY, USA, 2009.
[22] M. H. Stone. The theory of representations for Boolean algebras. Trans. Amer. Math. Soc., 40(1):37-111, 1936.
[23] H. Straubing and D. Thérien. Modular quantifiers. In Logic and Automata, volume 2 of Texts in Logic and Games, pages 613-628. Amsterdam University Press, 2008.
[24] H. Straubing, D. Thérien, and W. Thomas. Regular languages defined with generalized quantifiers. Information and Computation, 118(2):289-301, 1995.
[25] R. Street. The formal theory of monads. Journal of Pure and Applied Algebra, 2(2):149-168, 1972.
[26] P. Tesson and D. Thérien. Logic meets algebra: the case of regular languages. Logical Methods in Computer Science, 3(1), 2007.
[27] L. Vietoris. Bereiche zweiter Ordnung. Monatsh. Math. Phys., 32(1):258-280, 1922.


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