# Constraint Satisfaction Problems over semilattice block Mal'tsev algebras

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#### **Abstract**

There are two well-known types of algorithms for solving CSPs: local propagation and generating a basis of the solution space. For several years the focus of the CSP research has been on 'hybrid' algorithms that somehow combine the two approaches. In this paper we present a new method of such hybridization that allows us to solve certain CSPs that has been out of reach for a quite a while.

We apply this method to CSPs parametrized by a universal algebra, an approach that has been very popular in the last decade or so. Specifically, we consider a fairly restricted class of algebras we will call semilattice block Mal'tsev. An algebra  $\mathbb{A}$  is called semilattice block Mal'tsev if it has a binary operation f, a ternary operation m, and a congruence  $\sigma$  such that the quotient  $\mathbb{A}/\sigma$  with operation f is a semilattice, f is a projection on every block of  $\sigma$ , and every block of  $\sigma$  is a Mal'tsev algebra with Mal'tsev operation m. This means that the domain in such a CSP is partitioned into blocks such that if the problem is considered on the quotient set  $\mathbb{A}/\sigma$ , it can be solved by a simple constraint propagation algorithm. On the other hand, if the problem is restricted on individual blocks, it can be solved by generating a basis of the solution space. We show that the two methods can be combined in a highly nontrivial way, and therefore the constraint satisfaction problem over a semilattice block Mal'tsev algebra is solvable in polynomial time.

#### 1 Introduction

In a Constraint Satisfaction Problem (CSP, for short) we need to decide whether or not a given set of constraints on values that can be assigned simultaneously to a given set of variables can be satisfied. While the general CSP is NP-complete, its versions restricted by specifying a constraint language, a set of allowed constraints, are sometimes solvable in polynomial time. For a constraint language  $\Gamma$  the corresponding restricted CSP is denoted  $\mathrm{CSP}(\Gamma)$  and called a nonuniform CSP. The study of the complexity of nonuniform CSPs has been initiated by Schaefer [31]. In

that paper Schaefer determined the complexity of  $CSP(\Gamma)$  for constraint languages on a 2-element set. The complexity of  $CSP(\Gamma)$  for constraint languages over finite sets has been attracting much attention since then. This research is guided by the Dichotomy Conjecture proposed by Feder and Vardi [18, 19] that states that every CSP of the form  $CSP(\Gamma)$  for a constraint language  $\Gamma$  on a finite set is either solvable in polynomial time or is NP-complete. The Dichotomy Conjecture has been restated and made more precise in different languages, see, e.g. [12, 29]. Also, several powerful approaches to the problem have been developed, through algebra, logic, and graph theory. So far the most successful method of studying the complexity of the CSP has been the algebraic approach introduced by Jeavons et al. [11, 12, 14, 23]. This approach relates the complexity of  $CSP(\Gamma)$  to the properties of a certain universal algebra  $\mathbb{A}_{\Gamma}$  associated with  $\Gamma$ . In particular it allows one to expand  $CSP(\Gamma)$  to the problem  $CSP(\mathbb{A}_{\Gamma})$  that depends only on the associated algebra, without changing its complexity. It therefore suffices to restrict ourselves to the study of the complexity of problems of the form CSP(A), where A is a finite universal algebra.

The dichotomy conjecture has been confirmed in a number of cases: for constraint languages on 2- and 3-element sets [7, 31] (a dichotomy result was also announced for languages over 4-, 5-, and 7-element sets [25, 32, 33]), for constraint languages containing all unary relations [1, 8, 9], and several others, see, e.g. [2, 3, 22]. One of the most remarkable phenomena discovered is that, generally, there are only two types of algorithms applicable to CSPs solvable in polynomial time. The first one has long been known to researchers in Artificial Intelligence as constraint propagation [17]. Algorithms of the other type resemble Gaussian elimination in the sense that they construct a small generating set of the set of all solutions [10, 22]. The scope of both types of algorithms is precisely known [2, 22].

General dichotomy results, however, cannot be proved using only algorithms of a single 'pure' type. In all such results, see, e.g. [1, 7, 8, 9] a certain mix of the two types of algorithms is needed. In some cases, for instance, [7] such a hybrid algorithm is somewhat ad hoc; in other cases, [1, 8, 9] it is based on intricate decompositions of the problem instance. It has become clear however that ad hoc hybridization and the decomposition techniques developed in the mentioned works are not sufficient. Therefore trying to identify new polynomial time solvable cases of the CSP through combining the two types of algorithms is the key to approaching the Dichotomy Conjecture. There have been several further attempts to design hybrid algorithms; however, most of them were not quite successful. In more successful cases such as [26, 27, 28, 30] the researchers tried to tackle somewhat limited cases, in which a combination of local consistency properties and Gaussian elimination type fragments is very explicit. To provide the context

for our results we explain those cases in details.

Suppose that a constraint language  $\Gamma$  is such that it is possible to partition its domain A into blocks with the property that the restriction of  $\mathrm{CSP}(\Gamma)$  on each block of the partition can be solved by an algorithm of one type; while if we collapse each block into a single element, the resulting quotient problem can be solved by an algorithm of another type. What can be said about  $\mathrm{CSP}(\Gamma)$  itself? For instance, consider constraint language  $\Gamma = \{R\}$  on  $A = \{0,1,2\}$  where the ternary relation R is given by (triples in R are written vertically)

If A is partitioned into  $B=\{0,1\}$  and  $C=\{2\}$ , then the restriction of R on the blocks B,C is one of the relations above separated by vertical lines (we can choose between B and C for different coordinate positions), and the corresponding CSP can be solved by Gaussian elimination. Indeed, the only nontrivial relation obtained this way is the first one, that is,  $R\cap B^3$ , and it is given by a linear equation x+y+z=0. The quotient relation R' then looks like

$$R' = \left(\begin{array}{cccc} B & C & C & C \\ B & B & C & B \\ B & B & B & C \end{array}\right),$$

and it follows from [31] that CSP(R') can be solved by a local propagation algorithm, as R' can be represented by a Horn clause. Solving  $CSP(\Gamma)$  is less easy, see, [7], and similar but more complicated cases have not been known to be polynomial time solvable until now.

To make constructions like the one above more precise we use the algebraic representation of nonuniform CSPs, in which a constraint language is replaced with its (universal) algebra of polymorphisms. This allows us to exploit structural properties of algebras to design a hybrid algorithm. So, starting from  $CSP(\Gamma)$ , where  $\Gamma$  is a constraint language on a set A, we first consider the corresponding algebra  $\mathbb{A}_{\Gamma}$  with base set A such  $CSP(\mathbb{A}_{\Gamma})$  is polynomial time reducible to  $CSP(\Gamma)$ . A partition of  $\mathbb{A}_{\Gamma}$  is given by a congruence of  $\mathbb{A}_{\Gamma}$ , that is, an invariant equivalence relation. Recall that due to the results of [12] the algebra  $\mathbb{A}_{\Gamma}$  can be assumed idempotent, this makes restrictions on congruence blocks possible. Now, suppose that an idempotent algebra  $\mathbb{A}$  is such that it has a congruence  $\sigma$  with the property that the CSP of its quotient  $\mathbb{A}/_{\sigma}$  can be solved by the small generating set algorithm, say, it is Mal'tsev, while for every  $\sigma$ -block  $\mathbb{B}$  (a subalgebra of  $\mathbb{A}$ ) the CSP over  $\mathbb{B}$  can be solved by a local propagation algorithm; or the other way round, see Figure 1. How can one solve the CSP over  $\mathbb{A}$  itself? Maroti in [27] considered the

first case, when  $\mathbb{A}/_{\sigma}$  can be solved by the small generating set algorithm. This case turns out to be easier because of the property of the  $\sigma$ -blocks we can exploit. Suppose for simplicity that every  $\sigma$ -block  $\mathbb{B}$  is a semilattice, as shown in Figure 1. Then every CSP instance on  $\mathbb{B}$  has some sort of a canonical solution that assigns the maximal element of the semilattice (that is element  $a \in \mathbb{B}$  such that ab = a for all  $b \in \mathbb{B}$ ) to every variable. It then can be shown that if we find a solution  $\varphi: V \to \mathbb{A}/_{\sigma}$  where V is the set of variables of the instance on  $\mathbb{A}/_{\sigma}$ , and then assign the maximal elements of the  $\sigma$ -block  $\varphi(v)$  to v, we obtain a solution of the original instance.

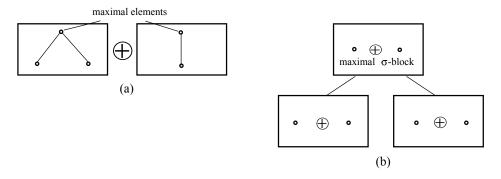


Figure 1: (a) Algebra  $\mathbb{A}$  such that  $\mathbb{A}/\sigma$  is Mal'tsev; (b) an SBM algebra. Rectangles represent  $\sigma$ -blocks, dots represent elements, lines show the semilattice structure, and  $\oplus$  represents a Mal'tsev operation acting on elements or  $\sigma$ -blocks.

The case when  $\mathbb{A}/_{\sigma}$  is a semilattice, while every  $\sigma$ -block is Mal'tsev is much more difficult. We will call such algebras *semilattice block Mal'tsev* algebras (SBM algebras, for short). More precisely, we consider idempotent algebras  $\mathbb{A}$  with the following property: There are a binary operation f and a ternary operation m, and a congruence  $\sigma$  of  $\mathbb{A}$  such that  $\mathbb{A}/_{\sigma}$  is a semilattice with a semilattice operation f, and every  $\sigma$ -block f is a Mal'tsev algebra with Mal'tsev operation f, and f is a projection. The main difficulty with this kind of algebras is that the only solution of a CSP over a semilattice we can reliably find is the canonical one assigning the maximal available element to each variable. Finding a second solution is already hard. On the other hand, if we restrict our instance only to the maximal  $\sigma$ -block  $\mathbb{B}$ , it may have no solution there, even though the original instance has a solution, which simply does not belong to the maximal block. If this is the case, it has been unclear for nearly 10 years how the domain can be reduced so that the maximal block is eliminated.

The problem has been resolved in some special cases. Firstly, Maroti in [28] showed that it suffices to consider SBM algebras of a certain restricted type. We

will use this result in this paper. Marcović and McKenzie suggested an algorithm that solves the CSP over SBM algebras  $\mathbb{A}$  when  $\mathbb{A}/_{\sigma}$  is a chain, that is,  $ab \in \{a,b\}$  for any  $a,b \in \mathbb{A}/_{\sigma}$ . In this case their algorithm is capable of eliminating the maximal block using the fact that if a semilattice is a chain, any of its subsets is a subalgebra. Finally, very recently Payne in [30] suggested an algorithm that works for a more general class of algebras than SBM, but algebras in this class have to satisfy an extra condition that in SBM algebras manifests itself as the existence of certain well behaving mappings between  $\sigma$ -blocks. In particular, this condition guarantees that the instance restricted to the maximal  $\sigma$ -block has a solution whenever the original problem has a solution.

In this paper we continue the effort started in [26, 28, 30] and present an algorithm that solves the CSP over an arbitrary SBM algebra.

#### **Theorem 1** *If* $\mathbb{A}$ *is a SBM algebra then* $CSP(\mathbb{A})$ *is solvable in polynomial time.*

The algorithm is based upon a new local consistency notion that we call block-minimality (although in our case it is necessarily not quite local, since it has to deal with Mal'tsev algebras). More specifically, our algorithm first separates the set V of variables of a CSP instance into overlapping subsets, coherent sets, and considers subproblems on these sets of variables. For block-minimality these subproblems have to be minimal, that is, every tuple from every constraint relation has to be a part of a solution. This can be achieved by solving the problem many times with additional constraints. However, this is not very straightforward, because coherent sets may contain all the variables from V. To overcome this problem we show that the subproblems restricted to coherent sets are either over a Mal'tsev domain and therefore can be solved efficiently, or they split up into a collection of disjoint instances, each of which has a strictly smaller domain. In the latter case we can recurse on these smaller instances. Finally, we prove that any block-minimal instance has a solution.

The results of this paper can easily be made more general by removing some of the restrictions on the basic operations of SBM algebras. However, we hope that these results can be generalized well beyond SBM-like algebras and so we stop short of giving more general but also more technically involved proofs just restricting ourselves to demonstrating the general idea.

In Section 2 we recall the basic definitions on CSP and the algebraic approach. A somewhat simplified outline of the solution algorithm and block-minimality is given in Section 3. More advanced facts from algebra and a study of certain properties of SBM algebras are given in Section 4. In Section 5 we strengthen the results of [5] about the structure of relations over Mal'tsev algebras and extend them to

SBM algebras<sup>1</sup>. In Section 6 we extend these notions to CSP instances. Finally, in Section 7 we prove the main result and present a solution algorithm.

### 2 Preliminaries

#### 2.1 Multisorted Constraint Satisfaction Problem

By [n] we denote the set  $\{1,\ldots,n\}$ . Let  $A_1,\ldots,A_n$  be finite sets. Tuples from  $A_1\times\ldots\times A_n$  are denoted in boldface, say,  $\mathbf{a}$ , and their entries by  $\mathbf{a}[1],\ldots,\mathbf{a}[n]$ . A relation R over  $A_1,\ldots,A_n$  is a subset of  $A_1\times\cdots\times A_n$ . We refer to n as the arity of the tuple  $\mathbf{a}$  and the relation R. Let  $I=(i_1,\ldots,i_k)$  be an (ordered) multiset, a subset of [n]. Then let  $\mathrm{pr}_I\mathbf{a}=(\mathbf{a}[i_1],\ldots,\mathbf{a}[i_k])$  and  $\mathrm{pr}_IR=\{\mathrm{pr}_I\mathbf{a}\mid \mathbf{a}\in R\}$ . Relation R is said to be a subdirect product of  $A_1,\ldots,A_n$  if  $\mathrm{pr}_iR=A_i$  for  $i\in[n]$ . In some cases it will be convenient to consider tuples and relations whose entries are indexed by sets other than subsets of [n], most often those will be sets of variables. Then we either assume the index set is somehow ordered, or consider tuples as functions from the index set to the domain and relations as sets of such functions.

Let  $\mathcal{A}$  be a set of sets, in this paper  $\mathcal{A}$  is usually the set of universes of finite algebras derived from an SBM algebra; we clarify 'derived' later. An instance of a (Multisorted) Constraint Satisfaction Problem (CSP) over  $\mathcal{A}$  is given by  $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ , where V is a set of variables,  $\mathcal{A}$  is a collection of domains  $A_v \in \mathcal{A}$ , and  $\mathcal{C}$  is a set of constraints; every constraint  $\langle \mathbf{s}, R \rangle$  is a pair consisting of an ordered multiset  $\mathbf{s} = (v_1, \dots, v_k)$ , a subset of V, called the constraint scope and R, a relation over  $A_{v_1}, \dots, A_{v_k}$ , called the constraint relation.

#### 2.2 Algebraic structure of the CSP

For a detailed introduction to CSP and the algebraic approach to its structure the reader is referred to a very recent and very nice survey by Barto et al. [4]. Basics of universal algebra can be learned from the textbook [16] and monograph [21].

A (universal) algebra is a pair  $\mathbb{A}=(A;F)$ , where A is a set (always finite in this paper) called the *universe* of  $\mathbb{A}$ , and F is a set of basic operations, multi-ary operations on A. Algebras  $\mathbb{A}=(A,F^{\mathbb{A}})$  and  $\mathbb{B}=(B,F^{\mathbb{B}})$  are said to be similar if their basic operations are indexed by elements of the same set F in such a way that operations from  $F^{\mathbb{A}}$  and  $F^{\mathbb{B}}$  indexed by the same element have the same arity. Operations that can be obtained from the basic operations of  $\mathbb{A}$  or a class  $\mathfrak{A}$  of

<sup>&</sup>lt;sup>1</sup>Kearnes and Szendrei in [24] developed a technique based on so-called critical relations that resembles in certain aspects what can be achieved through coherent sets. However, [24] only concerns congruence modular algebras, and so cannot be used for SBM algebras.

similar algebras by means of compositions are said to be *term* operations of  $\mathbb{A}$  or, respectively,  $\mathfrak{A}$ .

The CSP is related to algebras through the notion of polymorphism. Let R be a relation on a set A and f is a k-ary operation on the same set. Operation f is said to be a *polymorphism* of R if for any  $\mathbf{a}_1,\ldots,\mathbf{a}_k\in R$  the tuple  $f(\mathbf{a}_1,\ldots,\mathbf{a}_k)$  also belongs to R. More generally, let R be a subset of  $A_1\times\cdots\times A_\ell$  and f is an operation symbol such that  $f^{\mathbb{A}_i}$  is a k-ary operation on  $A_i$  for  $i\in [\ell]$ . Then f is a polymorphism of R if for any  $\mathbf{a}_1,\ldots,\mathbf{a}_k\in R$  the tuple  $f(\mathbf{a}_1,\ldots,\mathbf{a}_k)$  belongs to R, where  $f(\mathbf{a}_1,\ldots,\mathbf{a}_k)=(f^{\mathbb{A}_1}(\mathbf{a}_1[1],\ldots,\mathbf{a}_k[1]),\ldots,f^{\mathbb{A}_\ell}(\mathbf{a}_1[\ell],\ldots,\mathbf{a}_k[\ell]))$ . Let  $\Gamma$  be a constraint language on a set A. Then  $\operatorname{Pol}(\Gamma)$  denotes the set of all operations f on f such that f is a polymorphism of every relation from f; also f and f a constraint language over f that is, a set of relations f denotes the set of sets and f a constraint language over f that is, a set of relations f denotes the set of all operations on sets from f such that f is a polymorphism of all relations from f. The corresponding set of algebras is denoted by f that is, for every f denotes f is a polymorphism of all relations from f. The corresponding set of algebras is denoted by f that is, for every f denotes f is a polymorphism of all relations from f. The corresponding set of algebras is denoted by f that is, for every f denotes f is a polymorphism of all relations from f. The set f contains algebra f is a polymorphism of that f is a polymorphism of the set f in f is a polymorphism of the set f is a polymorphism of that f is a polymorphism of the set f in f in

Any class of similar algebras also gives rise to a CSP. Let  $\mathfrak A$  be a class of similar finite algebras and  $\mathcal A$  the set of universes of algebras from  $\mathfrak A$ . Then  $\mathrm{CSP}(\mathfrak A)$  is the class of instances  $(V,\mathcal A,\mathcal C)$  of CSPs over  $\mathcal A$  such that every constraint relation R from  $\langle \mathbf s,R\rangle\in\mathcal C$ ,  $\mathbf s=(v_1,\ldots,v_k)$ , is a subalgebra of  $A_{v_1}\times\cdots\times A_{v_k}$ , where  $A_v$ ,  $v\in V$ , are viewed as algebras from  $\mathfrak A$ .

In this paper we will use two special types of operations.

**Example 2** A binary operation f on A is said to be *semilattice* if f(a,a) = a, f(a,b) = f(b,a), and f(f(a,b),c) = f(a,f(b,c)) for any  $a,b,c \in A$ . Similarly, f is a semilattice operation on a class  $\mathfrak A$  of similar algebras, if it is a term operation of that class and  $f^{\mathbb A}$  is a semilattice operation for every  $\mathbb A \in \mathfrak A$ . We will treat a semilattice operation as multiplication and denote it by  $\cdot$  or omit the sign altogether. A semilattice operation defines an order on its domain:  $a \leq b$  if and only if ab = b. This means that there is always the greatest element of such a semilattice order — the product of all the elements of A. We will denote this element by  $\max(\mathbb A)$ .

**Example 3** A ternary operation m is said to be Mal'tsev if it satisfies the equations m(a,b,b)=m(b,b,a)=a for any  $a,b\in A$ . A term operation m of a class  $\mathfrak A$  is Mal'tsev if  $m^{\mathbb A}$  is Mal'tsev for every  $\mathbb A\in \mathfrak A$ . An algebra with a Mal'tsev term operation is said to be Mal'tsev.

If  $\mathfrak A$  has a Mal'tsev term operation, the algorithm from [10] constructs a compact representation of the set of solutions of any instance from  $\mathrm{CSP}(\mathfrak A)$ , thus solving the problem in polynomial time.

A subalgebra of an algebra  $\mathbb{A}=(A,F)$  is a subset  $B\subseteq A$  equipped with the restrictions of operations from F on B and such that  $f(a_1,\ldots,a_k)\in B$  for every  $f\in F$  and  $a_1,\ldots,a_k\in B$ . An equivalence relation on A invariant with respect to the basic operations of  $\mathbb{A}$  is said to be a congruence of  $\mathbb{A}$ . If a,b are related by a congruence  $\alpha$ , we write  $a\stackrel{\alpha}{\equiv}b$ ; the  $\alpha$ -block containing a is denoted  $a^{\alpha}$ . The quotient algebra  $\mathbb{A}/\alpha$  has the universe  $A/\alpha$  and basic operations  $f^{\alpha}$ ,  $f\in F$ , such that for any  $a_1,\ldots,a_k\in A$  operation  $f^{\alpha}$  is given by  $f^{\alpha}(a_1^{\alpha},\ldots,a_k^{\alpha})=(f(a_1,\ldots,a_k))^{\alpha}$ . We will omit the superscript in  $f^{\alpha}$  whenever this does not lead to a confusion. Algebra  $\mathbb{A}$  is said to be idempotent if  $f(a,\ldots,a)=a$  for any  $f\in F$  and any  $f\in A$  and a subalgebra. In particular, every 1-element subset of f(a,a) is a subalgebra. Algebras f(a,a) with the same universe are called term equivalent if they have the same set of term operations. If f(a,a)=a is said to be a reduct of f(a,a).

Idempotent algebra  $\mathbb A$  is said to be *semilattice block Mal'tsev* if there are a binary term operation f and a ternary term operation m, and a congruence  $\sigma$  of  $\mathbb A$  such that  $\mathbb A/_\sigma$  is term equivalent to a semilattice with a semilattice operation f, operation m is a Mal'tsev operation on every  $\sigma$ -block B, and  $f|_B$  is a projection, that is,  $f|_B(x,y)=x$ .

#### 2.3 Partial solutions and local consistency

Let  $\mathcal{P}=(V,\mathcal{A},\mathcal{C})$  be a CSP instance Let  $W\subseteq V$ . By  $\mathcal{P}_W$  we denote the instance  $(W,\mathcal{A}^W,\mathcal{C}^W)$  defined as follows:  $A_v^W=A_v$  for each  $v\in W$ ; for every constraint  $C=\langle \mathbf{s},R\rangle,\,C\in\mathcal{C}$ , the set  $\mathcal{C}^W$  includes the constraint  $C^W=\langle \mathbf{s}',R'\rangle$ , where  $\mathbf{s}'=\mathbf{s}\cap W$  and  $R'=\operatorname{pr}_{\mathbf{s}'}R$ . A solution of  $\mathcal{P}_W$  is called a *partial solution* of  $\mathcal{P}$  on W. The set of all such solutions is denoted by  $\mathcal{S}_W$ . If  $W=\{v\}$  or  $W=\{u,v\}$ , we simplify notation to  $\mathcal{P}_v,\mathcal{S}_v$  and  $\mathcal{P}_{uv},\mathcal{S}_{uv}$ , respectively.

Instance  $\mathcal{P}$  is called *minimal* if every tuple  $\mathbf{a} \in R$  for any constraint  $\langle \mathbf{s}, R \rangle \in \mathcal{C}$  can be extended to a solution of  $\mathcal{P}$ ; that is, there is  $\varphi \in \mathcal{S}$  such that  $\varphi(v) = \mathbf{a}[v]$  for  $v \in \mathbf{s}$ . Instance  $\mathcal{P}$  is called k-minimal if  $\mathcal{P}_W$  is minimal for all k-element  $W \subseteq V$ . For any fixed k every instance can be reduced to a k-minimal instance in polynomial time by a standard algorithm [13]: cycle over all k element subsets  $W \subseteq V$ , solve the problem  $\mathcal{P}_W$ , and for every constraint  $\langle \mathbf{s}, R \rangle$  exclude from R all tuples inconsistent with  $\mathcal{S}_W$ . If  $\mathcal{P} \in \mathrm{CSP}(\mathfrak{A})$  for some class  $\mathfrak{A}$  of similar algebras closed under subalgebras, the resulting problem also belongs to  $\mathrm{CSP}(\mathfrak{A})$ . In particular, from now on we will assume that all the instances we deal with are 1-minimal. For such problems we can also tighten the instance reducing the domains  $A_v, v \in V$ , to the sets  $\mathcal{S}_v$ . Every constraint relation will therefore be assumed to be a subdirect product of the respective domains. If  $\mathfrak{A}$  consists of idempotent

algebras, then any problem from  $\mathrm{CSP}(\mathfrak{A})$  can be reduced to a minimal one by solving polynomially many instances of  $\mathrm{CSP}(\mathfrak{A})$ . First of all, *constant relations*,  $R_a = \{(a)\}, \ a \in \mathbb{A} \in \mathfrak{A}, \ \text{are subalgebras of } \mathbb{A} \ \text{and therefore can be used in constraints. Then the algorithm proceeds as follows: cycle over all constraints <math>C = \langle \mathbf{s}, R \rangle \in \mathcal{C} \ \text{and all } \mathbf{a} \in R; \ \text{replace } C \ \text{with the collection of unary constraints} \ \langle (\mathbf{s}[i]), R_{\mathbf{a}[\mathbf{s}[i]]} \rangle; \ \text{solve the resulting instance } \mathcal{P}_{C,\mathbf{a}}; \ \text{remove a from } R \ \text{if } \mathcal{P}_{C,\mathbf{a}} \ \text{has no solutions.}$  However, this procedure obviously amounts to solving instances from  $\mathrm{CSP}(\mathfrak{A})$ , and therefore there is no guarantee this can be done in polynomial time.

**Example 4** If a class  $\mathfrak A$  of similar algebras has a semilattice term operation then  $\mathrm{CSP}(\mathfrak A)$  can be solved by establishing 1-minimality. More precisely, if  $\mathcal P=(V,\mathcal A,\mathcal C)$  is a 1-minimal instance from  $\mathrm{CSP}(\mathfrak A)$ , where  $A_v$  is the domain of  $v\in V$ , then the mapping  $\varphi(v)=\max(A_v)$  is a solution of  $\mathcal P$ .

#### 2.4 Congruences and polynomials

The set (lattice) of congruences of an algebra  $\mathbb{A}$  will be denoted by  $\mathsf{Con}(\mathbb{A})$ . So,  $\mathsf{Con}(\mathbb{A})$  is equipped with two binary operations of join,  $\vee$ , and meet,  $\wedge$ . The smallest congruence of  $\mathbb{A}$ , the equality relation, is denoted by  $\underline{0}_{\mathbb{A}}$ , and the greatest congruence, the total relation, is denoted by  $\underline{1}_{\mathbb{A}}$ . Let R be a subdirect product of  $\mathbb{A}_1,\ldots,\mathbb{A}_k$ , and  $\alpha_i\in\mathsf{Con}(\mathbb{A}_i)$ ,  $i\in[k]$ . Then by  $\overline{\alpha}_R$ , or simply  $\overline{\alpha}$  if R is clear from the context, we denote the congruence  $\alpha_1\times\cdots\times\alpha_k$  of R given by  $\mathbf{a}\stackrel{\overline{\alpha}}{\equiv}\mathbf{b}$  if and only if  $\mathbf{a}[i]\stackrel{\alpha_i}{\equiv}\mathbf{b}[i]$  for all  $i\in[k]$ . Also, if  $I=\{i_1,\ldots,i_\ell\}\subseteq[k]$  then by  $\overline{\alpha}_I$  we denote the congruence  $\alpha_{i_1}\times\cdots\times\alpha_{i_\ell}$  of  $\mathsf{pr}_IR$ .

Let  $\mathcal{P}=(V,\mathcal{A},\mathcal{C})$  be an instance of  $\mathrm{CSP}(\mathfrak{A})$  and  $\alpha_v$  a congruence of  $\mathbb{A}_v\in\mathfrak{A}$  for each  $v\in V$ . By  $\mathcal{P}_{\overline{\alpha}}$  we denote the instance  $(V,\mathcal{A}^{\overline{\alpha}},\mathcal{C}^{\overline{\alpha}})$ , in which  $\mathbb{A}_v^{\overline{\alpha}}=\mathbb{A}_v/\alpha_v$ , and a constraint  $\langle \mathbf{s},R'\rangle$ ,  $\mathbf{s}=(v_1,\ldots,v_k)$ , belongs to  $\mathcal{C}^{\overline{\alpha}}$  if and only if a constraint  $\langle \mathbf{s},R\rangle$ , where

$$R' = R/_{\overline{\alpha}} = \{\mathbf{a}^{\overline{\alpha}} = (\mathbf{a}[1]^{\alpha_{v_1}}, \dots, \mathbf{a}[k]^{\alpha_{v_k}}) \mid \mathbf{a} \in R\},$$

belongs to  $\mathcal{C}$ .

A pair of congruences  $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$  is said to be a *prime interval*, denoted  $\alpha \prec \beta$ , if  $\alpha \leq \beta$  and  $\alpha < \gamma < \beta$  for no congruence  $\gamma \in \mathsf{Con}(\mathbb{A})$ . Then  $\alpha \leq \beta$  means that  $\alpha \prec \beta$  or  $\alpha = \beta$ . For an operation f on  $\mathbb{A}$  we write  $f(\beta) \subseteq \alpha$  if, for any  $a, b \in \mathbb{A}$  with  $a \stackrel{\beta}{\equiv} b$ ,  $f(a) \stackrel{\alpha}{\equiv} f(b)$ .

Polynomials of  $\mathbb A$  are formed from term operations as follows. Let  $f(x_1,\dots,x_k,y_1,\dots,y_\ell)$  be a term operation of  $\mathbb A$  and  $a_1,\dots,a_\ell\in\mathbb A$ . Then the operation  $g(x_1,\dots,x_k)=f(x_1,\dots,x_k,a_1,\dots,a_\ell)$  is said to be a *polynomial* of  $\mathbb A$ . Note that although a polynomial does not have to be a polymorphism of

invariant relations of  $\mathbb{A}$ , unary polynomials and congruences of  $\mathbb{A}$  are in a special relationship: an equivalence relation  $\alpha$  is a congruence of  $\mathbb{A}$  if and only if it is preserved by every unary polynomial f, that is,  $f(\alpha) \subseteq \alpha$ . As usual, by an *idempotent unary polynomial* we mean a polynomial f(x) such that  $f \circ f = f$  or, equivalently, such that f(x) = x for any x from its range.

Let R be a subdirect product of  $\mathbb{A}_1,\ldots,\mathbb{A}_k$ . Similar to tuples from R, polynomials of R are also denoted in boldface, say,  $\mathbf{f}$ . A polynomial  $\mathbf{f}$  can be represented as  $\mathbf{f}(x_1,\ldots,x_k)=g(x_1,\ldots,x_k,\mathbf{a}^1,\ldots,\mathbf{a}^\ell)$  where g is a term operation of R and  $\mathbf{a}^1,\ldots,\mathbf{a}^l\in R$ . Then the polynomial  $g(x_1,\ldots,x_k,\mathbf{a}^1[i],\ldots,\mathbf{a}^\ell[i])$  of  $\mathbb{A}_i$  is denoted by  $f_i$ , and for  $I=\{i_1,\ldots,i_s\}\subseteq [n],$   $\mathbf{f}_I$  denotes the polynomial  $g(x_1,\ldots,x_k,\operatorname{pr}_I\mathbf{a}^1,\ldots,\operatorname{pr}_I\mathbf{a}^\ell)$  of  $\operatorname{pr}_IR$ . For any i, and any polynomial f of  $\mathbb{A}_i$ , there is a polynomial  $\mathbf{g}$  of R such that  $g_i=f$ . We shall call  $\mathbf{g}$  an extension of f to a polynomial of R. Finally, for  $I\subseteq [k]$ , and  $\mathbf{a}\in\prod_{i\in I}\mathbb{A}_i$  and  $\mathbf{b}\in\prod_{i\in [k]-I}\mathbb{A}_i$ ,  $(\mathbf{a},\mathbf{b})$  denotes the tuple  $\mathbf{c}$  such that  $\mathbf{c}[i]=\mathbf{a}[i]$  for  $i\in I$  and  $\mathbf{c}[i]=\mathbf{b}[i]$  if  $i\in [k]-I$ . To distinguish such concatenation of tuples from pairs of tuples, we will denote pairs of tuples by  $\langle \mathbf{a},\mathbf{b}\rangle$ .

The proposition below lists the main basic properties of relations over Mal'tsev algebras.

**Proposition 5 (Folklore)** *Let* R *be a subdirect product of Mal'tsev algebras*  $\mathbb{A}_1 \times \cdots \times \mathbb{A}_k$  *and*  $I \subseteq [k]$ . *Then the following properties hold* 

- (1) R is rectangular, that is if  $\mathbf{a}, \mathbf{b} \in \operatorname{pr}_I R, \mathbf{c}, \mathbf{d} \in \operatorname{pr}_{[k]-I} R$  and  $(\mathbf{a}, \mathbf{c}), (\mathbf{a}, \mathbf{d}), (\mathbf{b}, \mathbf{c}) \in R$ , then  $(\mathbf{b}, \mathbf{d}) \in R$ .
- (2) The relation  $\nu_I = \{ \langle \mathbf{a}, \mathbf{b} \rangle \in (\operatorname{pr}_I R)^2 \mid \text{there is } \mathbf{c} \in \operatorname{pr}_{[k]-I} R \text{ such that } (\mathbf{a}, \mathbf{c}), (\mathbf{b}, \mathbf{c}) \in R \} \text{ is a congruence of } \operatorname{pr}_I R.$

# 3 Outline of the algorithm

Our solution algorithm works by establishing some sort of minimality condition and repeatedly alternates two phases. The first phase is based on the results of Maroti [28] that allow us to reduce an instance over SBM algebras to one over SBM algebras with a *minimal* element. If  $\mathbb{A}$  is an SBM algebra then there is a congruence  $\sigma$  such that  $\mathbb{A}/\sigma$  is a semilattice. This means that  $\mathbb{A}/\sigma$  has a maximal or *absorbing* element a such that ax = xa = a for any  $x \in \mathbb{A}/\sigma$ . This element will be in the focus of our argument. We will also show with help of [28], Corollary 12, that it can always be assumed that  $\mathbb{A}/\sigma$  has a minimal or *neutral* element b such that bx = xb = x for any  $x \in \mathbb{A}/\sigma$ . In fact, one can assume an even stronger condition: that b is a 1-element  $\sigma$ -block.

For the second phase we introduce the *block-minimality* condition defined with the help of congruences and polynomials of an algebra. Let *R* be a subdirect prod-

uct of  $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$  and  $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_i)$ ,  $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_j)$  such that  $\alpha \prec \beta, \gamma \prec \delta$  for some  $i, j \in [n]$ . Interval  $(\alpha, \beta)$  can be separated from  $(\gamma, \delta)$  if there is a unary polynomial  $\mathbf{f}$  of R such that  $f_i(\beta) \not\subseteq \alpha$  while  $f_j(\delta) \subseteq \gamma$ . We are mostly interested in the situation when prime intervals cannot be separated.

Suppose that  $\mathcal{P}=(V,\mathcal{A},\mathcal{C})$  is a 3-minimal instance and the domain  $\mathbb{A}_v$  of  $v\in V$  is an SBM algebra and  $\sigma_v$  is such that  $\mathbb{A}_v/\sigma_v$  is a semilattice. Let  $\theta_v$  denote the congruence of  $\mathbb{A}_v$  such that the maximal element of  $\mathbb{A}_v/\sigma_v$  is one block of  $\theta_v$ , and all other  $\theta_v$ -blocks are singletons. We show, Lemma 9, that this is indeed a congruence. For every  $v\in V$  and  $\alpha,\beta\in \mathrm{Con}(\mathbb{A}_v)$  with  $\alpha\prec\beta\leq\theta_v$  let  $W_{v\alpha\beta}\subseteq V$  denote the set of variables w such that  $(\alpha,\beta)$  and  $(\gamma,\delta)$  for some  $\gamma,\delta\in\mathrm{Con}(\mathbb{A}_w)$  with  $\gamma\prec\delta\leq\theta_w$  cannot be separated from each other in the binary relation  $\mathcal{S}_{vw}$ . We call such sets of variables coherent sets. Instance  $\mathcal{P}$  is said to be block-minimal if for every  $v\in V$  and  $\alpha,\beta\in\mathrm{Con}(\mathbb{A}_v)$  with  $\alpha\prec\beta\leq\theta_v$  the problem  $\mathcal{P}_{W_{v\alpha\beta}}$  is minimal.

The result now follows from the following two statements. First, Proposition 20 claims that any instance  $\mathcal{P}$  over SBM algebras can be efficiently reduced to an equivalent block-minimal instance by solving polynomially many SBM instances over domains of smaller size. The second statement, Theorem 21, claims that any block-minimal SBM instance has a solution.

The key to the proof of Proposition 20 is Lemma 19 stating that every problem  $\mathcal{P}_{W_{v\alpha\beta}}$  is a disjoint union of problems over smaller domains, or its domains are Mal'tsev algebras. More precisely, in the first case there is k such that for every  $w \in W_{v\alpha\beta}$  the domain  $\mathbb{A}_w$  can be partitioned into a disjoint union  $\mathbb{A}_w^{(1)} \cup \cdots \cup \mathbb{A}_w^{(k)}$  in such a way that for any constraint  $\langle (v_1,\ldots,v_\ell),R\rangle$  of  $\mathcal{P}_{W_{v\alpha\beta}}$ , every tuple  $\mathbf{a} \in R$  belongs to  $\mathbb{A}_{v_1}^{(j)} \times \cdots \times \mathbb{A}_{v_k}^{(j)}$  for some  $j \in [k]$ . This property follows from the existence of a minimal element in every domain and the fact that certain prime intervals in congruence lattices of the domains of  $\mathcal{P}_{W_{v\alpha\beta}}$  cannot be separated from each other, Lemma 19. It means, of course, that it suffices to solve k problems  $\mathcal{P}_{W_{v\alpha\beta}}^{(j)}$  whose domains are  $\mathbb{A}_w^{(j)}$ .

We prove Theorem 21 by induction, showing that for every  $\overline{\beta} = (\beta_v)_{v \in V}$  with  $\beta_v \in \mathsf{Con}(\mathbb{A}_v)$  with  $\beta_v \leq \theta_v$  there is a collection of solutions  $\varphi_{v\alpha\beta}$  of  $\mathcal{P}_{W_{v\alpha\beta}}$  such that whenever  $u \in W_{v\alpha\beta} \cap W_{w\gamma\delta}$  we have  $\varphi_{v\alpha\beta}(u) \stackrel{\beta_u}{\equiv} \varphi_{w\gamma\delta}(u)$ . If every  $\beta_w$  equals  $\theta_w$  then such a collection exists because the maximal element of  $\mathbb{A}_w/\beta_w$  is a singleton, and we always can choose mappings  $\varphi_{v\alpha\beta}$  to be such that  $\varphi_{v\alpha\beta}(w)/\theta_v$  is the maximal element. On the other hand, if  $\beta_w$  is the equality relation for every  $w \in V$  then solutions  $\varphi_{v\alpha\beta}$  agree with each other and provide a solution of  $\mathcal{P}$ . Thus, showing that the existence of solutions  $\varphi_{v\alpha\beta}$  for some  $\overline{\beta}$  implies the existence of such solutions for smaller congruences  $\overline{\beta}'$  is the crux of our argument.

# 4 Semilattice block Mal'tsev algebras and minimal elements

#### 4.1 Minimal sets and polynomials

We will use several basic concepts of the tame congruence theory, [21].

An  $(\alpha, \beta)$ -minimal set is a minimal (under inclusion) set U such that  $U = f(\mathbb{A})$  for a unary polynomial of  $\mathbb{A}$  satisfying  $f(\beta) \not\subseteq \alpha$ . Sets B, C are said to be polynomially isomorphic in  $\mathbb{A}$  if there are unary polynomials f, g such that f(B) = C, g(C) = B, and  $f \circ g, g \circ f$  are identity mappings on C and B, respectively.

**Lemma 6 (Theorem 2.8, [21])** *Let*  $\alpha, \beta \in Con(\mathbb{A})$ ,  $\alpha \prec \beta$ . *Then the following hold.* 

- (1) Any  $(\alpha, \beta)$ -minimal sets U, V are polynomially isomorphic.
- (2) For any  $(\alpha, \beta)$ -minimal set U and any unary polynomial f, if  $f(\beta_U) \not\subseteq \alpha$  then f(U) is an  $(\alpha, \beta)$ -minimal set, U and f(U) are polynomially isomorphic, and f witnesses this fact.
- (3) For any  $(\alpha, \beta)$ -minimal set U there is a unary polynomial f such that  $f(\mathbb{A}) = U$ ,  $f(\beta) \not\subseteq \alpha$ , and f is idempotent, in particular, f is the identity mapping on U.
- (4) For any unary polynomial f such that  $f(\beta) \not\subseteq \alpha$  there is an  $(\alpha, \beta)$ -minimal set U such that f witnesses that U and f(U) are polynomially isomorphic.

Minimal sets of a Mal'tsev algebra form a particularly dense collection.

**Lemma 7 (Folklore)** *Let*  $\mathbb{A}$  *be a finite Mal'tsev algebra and*  $\alpha \prec \beta$  *for*  $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ . Then for any  $a, b \in \mathbb{A}$  with  $(a, b) \in \beta - \alpha$ , there is an  $(\alpha, \beta)$ -minimal set U such that  $a^{\alpha} \cap U \neq \emptyset$  and  $b^{\alpha} \cap U \neq \emptyset$ .

#### 4.2 Semilattice block Mal'tsev algebras

Since the fewer basic operations an algebra has, the richer the corresponding constraint language, we assume that the algebras we are dealing with have only two basic operations, just enough to guarantee the required properties. Therefore we assume that our semilattice block Mal'tsev algebras have only two basic operations: a binary operation  $\cdot$  that we will often omit, and a ternary operation m satisfying the conditions specified earlier. For elements  $a,b\in\mathbb{A}$  such that ab=ba=b we write  $a\leq b$ .

**Lemma 8** Let  $\mathbb{A}$  be an SBM algebra. By choosing a reduct of  $\mathbb{A}$  we may assume that

Operation  $\cdot$  satisfies the equation x(xy) = xy; and for any  $a, b \in \mathbb{A}$ ,  $a \le ab$ . Operation m can be chosen such that for any  $a, b, c \in \mathbb{A}$ ,  $m(a, b, c)^{\sigma_{\mathbb{A}}} = (abc)^{\sigma_{\mathbb{A}}}$ .

#### **Proof:** (1) Follows from Proposition 10 of [15].

(2) Consider the operation m'(x,y,z) = m(x,y,z)xyz. If B is a  $\sigma_{\mathbb{A}}$ -block, then, since ab = a for any  $a,b \in B$ , operation m' is Mal'tsev on B. Also, as  $\mathbb{A}/\sigma_{\mathbb{A}}$  is term equivalent to a semilattice,  $d = m(a,b,c)^{\sigma_{\mathbb{A}}}$  belongs to the subsemilattice of  $\mathbb{A}/\sigma_{\mathbb{A}}$  generated by  $a^{\sigma_{\mathbb{A}}}, b^{\sigma_{\mathbb{A}}}, c^{\sigma_{\mathbb{A}}}$ . Therefore  $m'(a,b,c)^{\sigma_{\mathbb{A}}} = d(abc)^{\sigma_{\mathbb{A}}} = (abc)^{\sigma_{\mathbb{A}}}$ , and we can choose m' for m.

Next we show some useful properties of SBM algebras. Let  $\mathbb{A}$  be an SBM algebra and  $\max(\mathbb{A})$  the maximal block of  $\sigma$ , that is,  $\max(\mathbb{A}) \cdot a \subseteq \max(\mathbb{A})$  for all  $a \in \mathbb{A}$ .

**Lemma 9** (1) The equivalence relation  $\theta_{\mathbb{A}}$  whose blocks are  $\max(\mathbb{A})$  and all the remaining elements form singleton blocks, is a congruence.

(2) Let R be a subdirect product of SBM algebras  $\mathbb{A}_1, \ldots, \mathbb{A}_n$  and the equivalence relation  $\theta_R$  is such that its blocks are  $\max(R) = R \cap (\max(\mathbb{A}_1) \times \cdots \times \max(\mathbb{A}_n))$ , and all the remaining elements form singleton blocks. Then  $\theta_R$  is a congruence.

**Proof:** (1) It suffices to observe that for any  $a \in \max(\mathbb{A})$  we have  $ax, xa, m(a, x, y), m(x, a, y), m(x, y, a) \in \max(\mathbb{A})$  for any x, y, and therefore all nonconstant polynomials of  $\mathbb{A}$  preserve  $\max(\mathbb{A})$ .

(2) is similar to (1).  $\Box$ 

When dealing with a relation over algebras  $\mathbb{A}_1, \dots, \mathbb{A}_n$  or a CSP with domains  $\mathbb{A}_v$  we will simplify the notation  $\theta_{\mathbb{A}_i}, \theta_{\mathbb{A}_v}$  to  $\theta_i, \theta_v$ .

**Lemma 10** Every  $(\alpha, \beta)$ -minimal set, for  $\alpha \prec \beta \leq \theta_{\mathbb{A}}$ , is a subset of  $\max(\mathbb{A})$ .

**Proof:** Let U be a  $(\alpha, \beta)$ -minimal set and f an idempotent polynomial with  $f(\mathbb{A}) = U$  and  $f(\beta) \not\subseteq \alpha$ . Since  $\beta \leq \theta_{\mathbb{A}}$ ,  $c, d \in U \cap \max(\mathbb{A})$  for some  $(c, d) \in \beta - \alpha$ , as otherwise we would have  $f(\beta) \subseteq \alpha$ . Take  $a \in \max(\mathbb{A})$  and set g(x) = f(x)a. For any  $b \in U \cap \max(\mathbb{A})$  we have g(b) = f(b)a = ba = b. Therefore  $g(\beta) \not\subseteq \alpha$  and  $g(\mathbb{A}) \subseteq \max(\mathbb{A})$ . Finally,  $f(\max(\mathbb{A})) \subseteq \max(\mathbb{A})$ , therefore  $f \circ g(\mathbb{A}) \subseteq U \cap \max(\mathbb{A})$  and  $f \circ g(x) = x$  for  $x \in U \cap \max(\mathbb{A})$ . As U is minimal,  $U = U \cap \max(\mathbb{A})$ .

#### 4.3 Maroti's reduction

In this section we describe a reduction introduced by Maroti in [28] that allows us to reduce CSPs over SBM algebras to CSPs over SBM algebras of a certain restricted type. More precisely, it allows us to assume that every domain A is

either a Mal'tsev algebra with m as a Mal'tsev operation, or it contains a *minimal* element a, that is, an element such that ab=ba=b for all  $b\in\mathbb{A}$ . Moreover, as is easily seen, such element is unique and forms a  $\sigma_{\mathbb{A}}$ -block, which is also the smallest element of the semilattice  $\mathbb{A}/\sigma_{\mathbb{A}}$ .

Let f be an idempotent unary polynomial of algebra  $\mathbb{A}$  and A the universe of  $\mathbb{A}$ . The *retract*  $f(\mathbb{A})$  of  $\mathbb{A}$  is the algebra with universe f(A), whose basic operations are of the form  $f \circ g$ , given by  $f \circ g(x_1, \ldots, x_n) = f(g(x_1, \ldots, x_n))$  for  $x_1, \ldots, x_n \in f(A)$ , where g is a basic operation of  $\mathbb{A}$ .

**Lemma 11** A retract of an SBM algebra through an idempotent polynomial is an SBM algebra.

**Proof:** Let f be an idempotent polynomial. Let  $g_1(x,y)=f(xy)$ ,  $m_1(x,y,z)=f(m(x,y,z))$  be the basic operations of the retract,  $\mathbb{A}_1=f(\mathbb{A})$ , and  $\sigma_1=\sigma_{\mathbb{A}|\mathbb{A}_1}$ . Firstly, note that  $\sigma_1$  is a congruence of  $\mathbb{A}_1$  and  $\mathbb{A}_1$  is an idempotent algebra. Since  $\mathbb{A}/\sigma_{\mathbb{A}}$  is term equivalent to a semilattice and any retract of a semilattice by a semilattice polynomial is a semilattice, so is  $\mathbb{A}_1/\sigma_1$ . Finally,

$$m_1(x, y, y) = f(m(x, y, y)) = f(x) = x$$
  
 $m_1(y, y, x) = f(m(y, y, x)) = f(x) = x$ ,

for any  $x, y \in \mathbb{A}_1$  with  $x \stackrel{\sigma_1}{\equiv} y$ .

The results of [28] imply the following. Let  $\mathfrak A$  be a class of similar finite algebras closed under subalgebras, and retracts via idempotent unary polynomials. Suppose that  $\mathfrak A$  has a term operation f satisfying the following conditions for some  $\mathbb B\in\mathfrak A$ :

- (1) f(x, f(x, y)) = f(x, y) for any  $x, y \in \mathbb{B}$ ;
- (2) for each  $a \in \mathbb{B}$  the mapping  $x \mapsto f(a, x)$  is not surjective;
- (3) the set C of  $a \in \mathbb{B}$  such that  $x \mapsto f(x, a)$  is surjective generates a proper subalgebra of  $\mathbb{B}$ .

Then  $CSP(\mathfrak{A})$  is polynomial time reducible to  $CSP(\mathfrak{A} - {\mathbb{B}})$ .

By Lemma 8 the operation  $\cdot$  of the class of SBM algebras from  $\mathfrak A$  satisfies condition (1). If the operation  $a \cdot x$  is surjective for some a, then  $a \leq x$  for all  $x \in \mathbb B$ . Therefore the only case when condition (2) is not satisfied is when  $\mathbb B$  has a minimal element. Finally, condition (3) is satisfied whenever  $\mathbb B$  is not a Mal'tsev algebra. Therefore, choosing  $\mathbb B$  to be a maximal (in terms of cardinality) algebra from  $\mathbb A$  satisfying conditions (1)–(3) we may only consider instances of  $\mathrm{CSP}(\mathbb A)$ , in which every domain has a minimal element or is a Mal'tsev algebra.

**Corollary 12** Every instance  $\mathcal{P} \in \mathrm{CSP}(\mathfrak{A})$  can be reduced in polynomial time to polynomially many instances over algebras each of which either is Mal'tsev or has a minimal element.

Throughout the rest of the paper  $\mathfrak A$  is a finite class of finite SBM algebras closed under taking subalgebras, quotient algebras, and retracts through unary idempotent polynomials.

# 5 Separating congruences

In this section we develop a method that will lead to some way to decompose CSPs over SBM algebras. First, we introduce and study the notion of separation of prime intervals. Let R be a subdirect product of  $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$  and  $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_i)$ ,  $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_j)$ , for some  $i, j \in [n]$ , such that  $\alpha \prec \beta, \gamma \prec \delta$ . Recall that interval  $(\alpha, \beta)$  can be separated from  $(\gamma, \delta)$  if there is a unary polynomial  $\mathbf{f}$  of R such that  $f_i(\beta) \not\subseteq \alpha$  while  $f_j(\delta) \subseteq \gamma$ . If  $\mathbf{f}$  satisfies this property we will also say that  $\mathbf{f}$  separates  $(\alpha, \beta)$  from  $(\gamma, \delta)$ . In the definition above it is possible that i = j or that n = 1; in this cases the argument in some proofs may be slightly different. To avoid such complications we will always assume that  $i \neq j$ , as the following lemma allows us to do.

**Lemma 13** Let Q be the binary equality relation on  $\mathbb{A}$ . Prime interval  $(\alpha, \beta)$ ,  $\alpha \prec \beta \leq \theta_{\mathbb{A}}$ , can be separated from  $(\gamma, \delta)$ ,  $\gamma \prec \delta \leq \theta_{\mathbb{A}}$ , as intervals in  $\mathsf{Con}(\mathbb{A})$  if and only if  $(\alpha, \beta)$  can be separated from  $(\gamma, \delta)$  in Q (as intervals in the congruence lattices of the factors of a binary relation).

**Proof:** Note that for any polynomial  $\mathbf{f}$  of Q its action on the first and second factors of Q is the same polynomial of  $\mathbb{A}$ . By definition  $\alpha \prec \beta$  can be separated from  $\gamma \prec \delta$  in  $\mathsf{Con}(\mathbb{A})$  if and only if there is a unary polynomial f of  $\mathbb{A}$ ,  $f(\beta) \not\subseteq \alpha$  while  $f(\delta) \subseteq \gamma$ . This condition can be expressed as follows: there is a unary polynomial  $\mathbf{f}$  of Q,  $f_1(\beta) \not\subseteq \alpha$  while  $f_2(\delta) \subseteq \gamma$ , which precisely means that  $(\alpha, \beta)$  can be separated from  $(\gamma, \delta)$  in Q

In Section 5.1 we study the sets of intervals that cannot be separated from each other. These sets will later give us some sort of decomposition of CSP instances. Collapsing polynomials introduced in Section 5.2 yeild one of the main ingredients of the solution algorithm. Section 5.3 provides a sufficient condition for separation of intervals and a related notion of decomposition, which is the second ingredient.

#### 5.1 Basic properties of separation

Let again R be a subdirect product of SBM algebras  $\mathbb{A}_1 \times \ldots \times \mathbb{A}_n$ ,  $i, j \in [n]$ , and  $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_i)$ ,  $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_j)$  with  $\alpha \prec \beta \leq \theta_i$ ,  $\gamma \prec \delta \leq \theta_j$ .

First, we show that separating polynomials can be chosen to satisfy certain simple conditions.

**Lemma 14** If  $(\alpha, \beta)$  can be separated from  $(\gamma, \delta)$  then there is a polynomial  $\mathbf{f}$  that separates  $(\alpha, \beta)$  from  $(\gamma, \delta)$  and such that  $f_{\ell}(\mathbb{A}_{\ell}) \subseteq \max(\mathbb{A}_{\ell})$  for every  $\ell \in [n]$ .

**Proof:** Let g separate  $(\alpha, \beta)$  from  $(\gamma, \delta)$ . Choose a tuple  $\mathbf{a} \in \max(R)$  and consider the polynomial  $\mathbf{f}(x) = \mathbf{g}(x) \cdot \mathbf{a}$ . As is easily seen,  $f_{\ell}(\mathbb{A}_{\ell}) \subseteq \max(\mathbb{A}_{\ell})$  for  $\ell \in [n]$ . Since  $g_j(\delta) \subseteq \gamma$ , we have  $f_j(\delta) \subseteq \gamma$ . Finally, take  $a, b \in \max(\mathbb{A}_i) \cap g_i(\mathbb{A}_i)$  with  $(a, b) \in \beta - \alpha$  and  $a', b' \in \max(\mathbb{A}_i)$  such that  $g_i(a') = a$ ,  $g_i(b') = b$ . By Lemma 6(4) and Lemma 10 such elements exist, because  $g_i(\beta) \not\subseteq \alpha$  and all the nontrivial (that is, different from an  $\alpha$ -block)  $\beta_i$ -blocks are inside  $\max(\mathbb{A}_i)$ . Then

$$f_i(a') = g_i(a')\mathbf{a}[i] = a\mathbf{a}[i] = a \neq b = b\mathbf{a}[i] = g_i(b')\mathbf{a}[i] = f_i(b').$$

From now on we assume that all polynomials separating intervals satisfy the conditions of Lemma 14.

**Lemma 15** If  $(\alpha, \beta)$  can be separated from  $(\gamma, \delta)$  then, for any  $(\alpha, \beta)$ -minimal set U, there is an idempotent unary polynomial g such that  $g_i(\mathbb{A}_i) = U$ , and g separates  $(\alpha, \beta)$  from  $(\gamma, \delta)$ .

**Proof:** Let  $\mathbf{f}$  separate  $(\alpha, \beta)$  from  $(\gamma, \delta)$ . Then by Lemma 6(4)  $f_i(\mathbb{A}_i)$  contains an  $(\alpha, \beta)$ -minimal set U', and there is an idempotent polynomial  $h_i$  with  $h_i(\mathbb{A}_i) = U'$ . The polynomial  $h_i$  can be extended to a polynomial  $\mathbf{h}$  of  $\mathbf{f}$ . Then  $\mathbf{f}' = \mathbf{h} \circ \mathbf{f}$  separates  $(\alpha, \beta)$  from  $(\gamma, \delta)$  and  $f_i'(\mathbb{A}_i) = U'$ .

By Lemma 6(2) there is an  $(\alpha, \beta)$ -minimal set U'' with  $f'_i(U'') = U'$  and an idempotent polynomial  $h'_i$  with  $h'_i(U') = U''$ . As above, the polynomial  $h'_i$  can be extended to a polynomial  $\mathbf{h}'$  of R. For a certain k,  $(\mathbf{f}' \circ \mathbf{h}')^k$  is idempotent, separates i from j, and  $(f'_i \circ h'_i)^k(\mathbb{A}_i) = U''$ . Now the lemma follows easily from Lemma 6(1).

Let  $\mathcal{I}_R$  be the set of triples  $(i, \alpha, \beta)$  such that  $i \in [n]$ ,  $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_i)$  and  $\alpha \prec \beta \leq \theta_i$ . The relation 'cannot be separated in R' on  $\mathcal{I}_R$  is clearly reflexive and transitive. Now, we prove it is also symmetric

**Lemma 16** If  $(\alpha, \beta)$  can be separated from  $(\gamma, \delta)$  then  $(\gamma, \delta)$  can be separated from  $(\alpha, \beta)$ .

**Proof:** Let  $U_1,\ldots,U_k$  be all the  $(\alpha,\beta)$ -minimal sets. By Lemma 15, for every  $U_\ell$ , there is an idempotent unary polynomial  $\mathbf{g}^{(\ell)}$  separating  $(\alpha,\beta)$  from  $(\gamma,\delta)$  and such that  $g_i^{(\ell)}(\mathbb{A}_i) = U_\ell$ . Take a  $\delta$ -block B that contains more than one  $\gamma$ -block, a tuple  $\mathbf{a} \in R$  such that  $\mathbf{a}[j] \in B$ , and set  $\mathbf{a}^{(\ell)} = \mathbf{g}^{(\ell)}(\mathbf{a})$ . By Lemmas 10 and 14  $\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(k)} \in \max(R)$  and  $U_1,\ldots,U_k \subseteq \max(\mathbb{A}_i)$ , and  $B \subseteq \max(\mathbb{A}_j)$ . The operation  $\mathbf{h}^{(\ell)}(x) = m(x,\mathbf{g}^{(\ell)}(x),\mathbf{a}^{(\ell)})$  satisfies the conditions

- $\bullet \ \ h_i^{(\ell)}(x) = m(x,g_i^{(\ell)}(x),\mathbf{a}^{(\ell)}[i]) = m(x,x,\mathbf{a}^{(\ell)}[i]) = \mathbf{a}^{(\ell)}[i] \ \text{for all} \ x \in U_\ell;$
- $\bullet \ \ h_j^{(\ell)}(x) = m(x,g_j^{(\ell)}(x),\mathbf{a}^{(\ell)}[j]) \stackrel{\alpha_j}{\equiv} m(x,\mathbf{a}^{(\ell)}[j],\mathbf{a}^{(\ell)}[j]) = x \text{ for all } x \in B;$
- $\mathbf{h}^{(\ell)}(R) \subseteq \max(R)$ .

We are going to compose the polynomials  $\mathbf{h}^{(\ell)}$  such that the composition collapses  $\beta$ . To this end take a sequence  $1=\ell_1,\ell_2,\ldots$  such that  $U_{\ell_2}$  is a subset of the range of  $\overline{h}^{(1)}=h_i^{(\ell_1)}$ , and, for s>2,  $U_{\ell_s}$  is a subset of the range of  $\overline{h}^{(s-1)}=h_i^{(\ell_{s-1})}\circ\ldots\circ h_i^{(\ell_1)}$ . Since  $|\overline{h}^{(s)}(\mathbb{A}_i)|<|\overline{h}^{(s-1)}(\mathbb{A}_i)|$ , there is r such that  $|\overline{h}^{(r)}(\mathbb{A}_i)|$  contains no  $(\alpha,\beta)$ -minimal sets. Therefore, setting  $\mathbf{h}(x)=\mathbf{h}^{(\ell_r)}(\mathbf{h}^{(\ell_{r-1})}(\ldots\mathbf{h}^{(\ell_1)}(x)\ldots))$  we have that  $h_i$  collapses all the  $(\alpha,\beta)$ -minimal sets, and  $h_j$  acts identically on  $B/\alpha_j$ . Thus,  $\mathbf{h}$  separates  $(\gamma,\delta)$  from  $(\alpha,\beta)$ .

Lemma 16 together with the observation before it shows that the relation 'cannot be separated' is an equivalence relation on  $\mathcal{I}$ .

#### 5.2 Collapsing polynomials

Intuitively, a collapsing polynomial for some prime interval  $\alpha \prec \beta$  in an algebra or a subdirect product of algebras is a polynomial that collapses all prime intervals that can be separated from  $\alpha \prec \beta$  and only such prime intervals.

Let R be a subdirect product of SBM algebras  $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ , and  $(i, \alpha, \beta) \in \mathcal{I}_R$ . A unary idempotent polynomial  $\mathbf{f}$  of R is called  $(\alpha, \beta)$ -collapsing if the following conditions hold:

- (C1) for any  $(j, \gamma, \delta) \in \mathcal{I}_R$ , it holds  $f_j(\delta) \subseteq \gamma$ , unless  $(\alpha, \beta)$  and  $(\gamma, \delta)$  cannot be separated;
- (C2) for any  $(j, \gamma, \delta) \in \mathcal{I}_R$  such that  $(\alpha, \beta)$ ,  $(\gamma, \delta)$  cannot be separated, the set  $f_j(\mathbb{A}_j)$  is a  $(\gamma, \delta)$ -minimal set.

First, we show that  $(\alpha, \beta)$ -collapsing polynomials exist even if we impose some additional requirements.

**Lemma 17** Let R be a subdirect product of SBM algebras  $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$  and  $(i, \alpha, \beta) \in \mathcal{I}_R$ , and let  $\mathbf{a} \in R$  be such that  $\mathbf{a}[i]$  belongs to a  $\beta$ -block containing more than one  $\alpha$ -block and  $b \in \mathbb{A}_i$  with  $(\mathbf{a}[i], b) \in \beta - \alpha$ . Then there is an  $(\alpha, \beta)$ -collapsing polynomial  $\mathbf{f}$  of R such that  $\mathbf{f}(\mathbf{a}) = \mathbf{a}$  and  $f_i(b) \stackrel{\alpha}{\equiv} b$ .

**Proof:** First, we find an  $(\alpha, \beta)$ -collapsible polynomial. For every  $(j, \gamma, \delta) \in \mathcal{I}_R$  such that  $(\alpha, \beta)$  can be separated from  $(\gamma, \delta)$  there is an idempotent polynomial  $\mathbf{g}^{j\gamma\delta}$  such that  $g_j^{j\gamma\delta}(\delta) \subseteq \gamma$ , but  $g_i^{j\gamma\delta}(\beta) \not\subseteq \alpha$ . Moreover, we may assume by Lemma 15 that for every  $\mathbf{g}^{j\gamma\delta}$ ,  $g_i^{j\gamma\delta}(\mathbb{A}_i) = U$  for the same  $(\alpha, \beta)$ -minimal set U. Composing all such polynomials we obtain a polynomial  $\mathbf{h}$  such that  $h_i(\mathbb{A}_i) = U$ , and so  $h_i(\beta) \not\subseteq \alpha$ , and  $h_j(\delta) \subseteq \gamma$  for any  $j, \gamma, \delta$  as above. By iterating  $\mathbf{h}$  can be assumed idempotent. Choose  $\mathbf{h}$  to have the smallest image among unary idempotent polynomials such that  $h_i(\mathbb{A}_i)$  is an  $(\alpha, \beta)$ -minimal set and  $h_j(\delta) \subseteq \gamma$  for any  $(j, \gamma, \delta) \in \mathcal{I}_R$  such that  $(\alpha, \beta)$  can be separated from  $(\gamma, \delta)$ .

Suppose now that for some  $(j, \gamma, \delta) \in \mathcal{I}_R$  such that the interval  $(\alpha, \beta)$  cannot be separated from  $(\gamma, \delta)$  the set  $U' = h_j(\mathbb{A}_j)$  is not a  $(\gamma, \delta)$ -minimal set. Then, since  $h_j(\delta) \not\subseteq \gamma$ , the set U' contains a  $(\gamma, \delta)$ -minimal set U'' by Lemma 6(4). Let g be an idempotent polynomial of  $\mathbb{A}_j$  with  $g(\mathbb{A}_j) = U''$  and  $\mathbf{g}$  its extension to a polynomial of R. Then  $\mathbf{h}' = \mathbf{g} \circ \mathbf{h}$  satisfies the following conditions:

- $-h'_i(\mathbb{A}_j) = U''$  and  $h'_i(\delta) \not\subseteq \gamma$ ;
- $-h'_i(\beta) \not\subseteq \alpha$ , because  $(\alpha, \beta)$  cannot be separated from  $(\gamma, \delta)$ ;
- $-\left|\mathbf{h}'(R)\right|<\left|\mathbf{h}(R)\right|.$

Iterating  $\mathbf{h}'$  it can be assumed idempotent. Then the last property contradicts the choice of  $\mathbf{h}$ . Therefore  $\mathbf{h}$  is  $(\alpha, \beta)$ -collapsing.

Let  $\alpha_i = \alpha, \beta_i = \beta$ , and for  $j \in [n] - \{i\}$  let  $\alpha_j = \beta_j = \theta_j$ . It is not hard to see that  $\overline{\alpha} \leq \overline{\beta}$ . Indeed, suppose  $\eta \in \text{Con}(R)$  is such that  $\overline{\alpha} < \eta \leq \overline{\beta}$  and let i = n. Then there are  $(\mathbf{c}, c), (\mathbf{d}, d) \in R$  such that  $\langle (\mathbf{c}, c), (\mathbf{d}, d) \rangle \in \eta$  such that  $\langle (\mathbf{c}, d) \rangle \in \overline{\alpha}_{[n-1]}$  and  $\langle c, d \rangle \in \beta - \alpha$ . We show that for any  $\langle (\mathbf{c}', c'), (\mathbf{d}', d') \rangle \in \overline{\beta}$  we have  $\langle (\mathbf{c}', c'), (\mathbf{d}', d') \rangle \in \eta$ . In fact, by Proposition 5 it suffices to show that  $\langle (\mathbf{c}'', c''), (\mathbf{d}'', d'') \rangle \in \eta$  for some  $\mathbf{c}'', \mathbf{d}'' \in \text{pr}_{[n-1]}R'$  where  $R' = \max(R)$  and  $\langle \mathbf{c}'', \mathbf{d}'' \rangle \in \overline{\alpha}_{[n-1]}$ , and some  $c'', d'' \in \max(\mathbb{A}_n)$  with  $c'' \stackrel{\alpha}{\equiv} c', d'' \stackrel{\alpha}{\equiv} d'$ . Since R' is a Mal'tsev algebra by Lemma 7 applied to conguences  $\alpha \prec \beta$  there is a polynomial  $\mathbf{f}$  of R such that  $c'' = f_n(c) \stackrel{\alpha}{\equiv} c', d'' = f_n(d) \stackrel{\alpha}{\equiv} d'$  and  $\mathbf{f}(R) \subseteq R'$ . Let  $\mathbf{c}'' = \mathbf{f}_{[n-1]}(\mathbf{c}), \mathbf{d}'' = \mathbf{f}_{[n-1]}(\mathbf{d})$ . Then  $\langle (\mathbf{c}'', c''), (\mathbf{d}'', d'') \rangle \in \eta$ . Also, since  $\beta \neq \alpha$ , we have  $\overline{\alpha} \prec \overline{\beta}$ .

By Lemma 7 there is an  $(\alpha, \beta)$ -minimal set U such that  $\mathbf{a}[i]^{\alpha} \cap U, b^{\alpha} \cap U \neq \emptyset$ . Moreover, an  $(\alpha, \beta)$ -collapsing polynomial  $\mathbf{h}$  can be chosen such that  $h_i(\mathbb{A}_i) = U$ . Then set  $\mathbf{f}(x) = m(\mathbf{h}(x), \mathbf{h}(\mathbf{a}), \mathbf{a})$ . For the polynomial  $\mathbf{f}$  we have:

- $-\mathbf{f}(\mathbf{a}) = m(\mathbf{h}(\mathbf{a}), \mathbf{h}(\mathbf{a}), \mathbf{a}) = \mathbf{a};$
- $-c = f_i(b) = m(h_i(b), h_i(\mathbf{a}[i]), \mathbf{a}[i]) \stackrel{\alpha}{=} m(h_i(b), \mathbf{a}[i], \mathbf{a}[i]) = h_i(b) \stackrel{\alpha}{=} b$ , because, since  $\mathbf{h}$  is idempotent,  $h_i(\mathbf{a}[i]) \stackrel{\alpha}{=} \mathbf{a}[i]$  and  $h_i(b) \stackrel{\alpha}{=} b$ ;
- for any  $(j, \gamma, \delta) \in \mathcal{I}_R$  such that and  $(\alpha, \beta), (\gamma, \delta)$  can be separated,  $f_j(\delta) \subseteq \gamma$ .

By iterating  $\mathbf{f}$  we obtain an idempotent polynomial  $\mathbf{f}'$  that satisfies all the conditions above. Indeed, the first and third conditions are straightforward, while the second one follows from the equality  $f_i(c) \stackrel{\alpha}{\equiv} c$ . Finally, for any  $(j, \gamma, \delta) \in \mathcal{I}_R$  such that  $(\alpha, \beta), (\gamma, \delta)$  cannot be separated we have  $f'_j(\delta) \not\subseteq \gamma$ , because  $f'_i(\beta) \not\subseteq \alpha$ . Also,  $f'_i(\mathbb{A}_j)$  is a  $(\gamma, \delta)$ -minimal set, because  $h_j(\mathbb{A}_j)$  is a one.

Thus,  $\mathbf{f}'$  satisfies all the required conditions. The lemma is proved.

#### 5.3 Splits and alignments

In this section we present a sufficient condition for two prime intervals to be separated. As we shall see using this condition certain projections of a relation can be partitioned into a small number of subdirect products of smaller algebras.

Let R be a subdirect product of  $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ ,  $\alpha_i, \beta_i \in \mathsf{Con}(\mathbb{A}_i)$ ,  $i \in [n]$ , such that  $\alpha_i \prec \beta_i \leq \theta_{\mathbb{A}_i}$ . An element  $a \in \mathbb{A}_i$ ,  $i \in [n]$ , is called  $\alpha_i \beta_i$ -split if there is a  $\beta_i$ -block B and  $b, c \in B$  such that  $ab \not\equiv ac$ . Note that no element from  $\max(\mathbb{A}_i)$  is  $\alpha_i \beta_i$ -split, while the minimal element is  $\alpha_i \beta_i$ -split. We say that  $i, j \in [n]$  are not  $\overline{\alpha} \overline{\beta}$ -aligned if there is  $\mathbf{a} \in R$  such that  $\mathbf{a}[i]$  is not  $\alpha_i \beta_i$ -split and  $\mathbf{a}[j]$  is  $\alpha_j \beta_j$ -split, or the other way round.

**Lemma 18** If i, j are not  $\overline{\alpha}\overline{\beta}$ -aligned then  $(\alpha_i, \beta_i)$  can be separated from  $(\alpha_j, \beta_j)$ .

**Proof:** It suffices to consider the case n=2, i=1, j=2. Let  $(a,b) \in R$  be such that a is  $\alpha_i\beta_i$ -split, while b is not  $\alpha_j\beta_j$ -split. Let also  $(c,d) \in R' = \max(R)$ . Consider operation  $\mathbf{f}((x_1,x_2)) = (a,b) \cdot ((x_1,x_2) \cdot (c,d))$ . We claim that  $f_1(\beta_1) \not\subseteq \alpha_1$  while  $f_2(\beta_2) \subseteq \alpha_2$ .

First, observe that all the values of the operation  $g((x_1,x_2))=(x_1,x_2)\cdot (c,d)$  belong to  $\max(R)$ , and  $g((x_1,x_2))=(x_1,x_2)$  for any  $(x_1,x_2)\in\max(R)$ . Then, for any  $\beta_2$ -block  $B_2$  and any  $a',b'\in B_2$  we have  $f_2(a')=b(a'd)\stackrel{\alpha_2}{\equiv}b(b'd)=f_2(b')$ , as b is not  $\alpha_2\beta_2$ -split. Thus  $f_2(\beta_2)\subseteq\alpha_2$ . On the other hand, since a is  $\alpha_1\beta_1$ -split, there is a  $\beta_1$ -block  $B_1$  and  $a'',b''\in B_1$  such that  $f_1(a'')=a(a''c)=aa''\stackrel{\alpha_1}{\equiv}ab''=a(b''c)=f_1(b'')$ . The second and the second last equalities hold because, as  $\beta_1\subseteq\theta_1$  and  $B_1$  is a nontrivial  $\beta_1$ -block, we have  $B_1\subseteq\max(\mathbb{A}_1)$ . Therefore  $f_1(\beta_1)\not\subseteq\alpha_1$ .

#### **6** From relations to instances

Here we apply the results of the previous section to CSP instances. In particular, we introduce coherent sets of an instance and show that if an instance has solutions on every coherent set, which are consistent in some weak sense, then the entire instance has a solution.

Let  $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$  be a 3-minimal instance of  $\mathrm{CSP}(\mathfrak{A})$ . We assume that the domain  $\mathbb{A}_v$  of each variable  $v \in V$  is the set of solutions  $\mathcal{S}_v$ , and so the constraint relations are subdirect products of the domains.

Since separation of prime intervals depends only on binary projections of a relation, it can be defined for 3-minimal instances as well. More precisely, let  $\mathcal{I}_{\mathcal{P}}$  (or just  $\mathcal{I}$  if  $\mathcal{P}$  is clear from the context) be the set of all triples  $(v, \alpha, \beta)$ , where  $v \in V$ ,  $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$  are such that  $\alpha \prec \beta \leq \theta_v$ . Let  $(v, \alpha, \beta), (w, \gamma, \delta) \in \mathcal{I}$ ; we say that  $(\alpha, \beta)$  cannot separated from  $(\gamma, \delta)$  if this is the case for  $\mathcal{S}_{vw}$ . Due to 3-minimality — we can consider sets of solutions on 3 variables — this relation is transitive. It is also reflexive and symmetric by Lemma 16.

Next we define two partitions of a CSP instance  $\mathcal{P}$ . The first one, link partition allows us to reduce solving subinstances of  $\mathcal{P}$  to instances over smaller domains. The second one provides a sufficient condition to have a link partition and is defined through alignment properties.

Let again  $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$  be a 3-minimal instance of  $\mathrm{CSP}(\mathfrak{A})$ . Partitions  $A_{v1} \cup \ldots \cup A_{vk_v} = \mathbb{A}_v$  for  $v \in V$  are called a *link partition* if the following condition holds:

• For any  $v,w\in V$ ,  $k_v=k_w$ , and there is a bijection  $\varphi_{vw}:[k_v]\to [k_w]$  such that for any  $(a,b)\in \mathcal{S}_{vw}$  and any  $j\in [k_v]$ ,  $a\in A_{vj}$  if and only if  $b\in A_{w\varphi_{vw}(j)}$ .

Observe that, since  $\mathcal{P}$  is 3-minimal, the mappings  $\varphi_{vw}$  are consistent, that is, for any  $u, v, w \in V$  it holds that  $\varphi_{vw} \circ \varphi_{uv} = \varphi_{uw}$ . Without loss of generality we will assume that  $\varphi_{vw}$  is an identity mapping.

As is easily seen the partition  $A_{v1} \cup ... \cup A_{vk_v} = \mathbb{A}_v$  defines a congruence of  $\mathbb{A}_v$ . In particular, each of  $A_{vi}$  is a subalgebra of  $\mathbb{A}_v$ .

Let  $\alpha_v, \beta_v \in \mathsf{Con}(\mathbb{A}_v)$  for  $v \in V$  be such that  $\alpha_v \prec \beta_v \leq \theta_v$ . Variables  $v, w \in V$  are  $\overline{\alpha}\overline{\beta}$ -aligned if they are  $\overline{\alpha}\overline{\beta}$ -aligned in  $\mathcal{S}_{vw}$ . In the following lemma we assume that every domain  $\mathbb{A}_v$  of  $\mathcal{P}$  either has a minimal element, or  $\sigma_{\mathbb{A}_v}$  is the full congruence, and so  $\mathbb{A}_v$  is a Mal'tsev algebra.

**Lemma 19** (1) If variables  $v, w \in V$  of an instance  $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$  are  $\overline{\alpha}\overline{\beta}$ -aligned and  $\mathbb{A}_v$  has a minimal element then  $\mathbb{A}_w$  also has a minimal element. (2) If every domain of an instance  $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$  has a minimal element and any **Proof:** For every  $v \in V$  let  $L_v$  denote the set of  $\alpha_v \beta_v$ -split elements of  $\mathbb{A}_v$  and let  $N_v$  denote the set of  $\alpha_v \beta_v$ -non-split elements. As we observed before Lemma 18, both sets are nonempty if  $\mathbb{A}_v$  has a minimal element, and  $L_v = \emptyset$  if  $\mathbb{A}_v$  is a Mal'tsev algebra.

- (1) If  $\mathbb{A}_w$  is a Mal'tsev algebra then v, w cannot be  $\overline{\alpha}\overline{\beta}$ -aligned since  $L_w = \emptyset$ , while  $L_v, N_v \neq \emptyset$ , and  $S_{vw}$  is a subdirect product.
- (2) For any  $v, w \in V$  and any pair  $(a, b) \in \mathcal{S}_{vw}$ ,  $a \in L_v$  if and only if  $b \in L_w$ . Therefore  $\mathcal{S}_{vw}$  is link-partitioned, as well as R for any constraint  $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ .  $\square$

# 7 The algorithm

In the first part of this section we introduce the property of block-minimality, the key property of CSP instances for our algorithm. We also prove that block-minimality can be efficiently established. Then in the second part we show that block-minimality is sufficient for the existence of a solution, Theorem 21, which is the main result of this section, and provides a polynomial time algorithm for CSPs over SBM algebras.

#### 7.1 Block-minimality

Let  $\mathcal{P}=(V,\mathcal{A},\mathcal{C})$  be a 3-minimal instance such that for every its domain  $\mathbb{A}_v$  either  $\sigma_{\mathbb{A}_v}$  is the full congruence, and so  $\mathbb{A}_v$  is a Mal'tsev algebra with Mal'tsev operation m, or  $\mathbb{A}_v$  has a minimal element.

Recall that  $\mathcal{I}_{\mathcal{P}}$  or just  $\mathcal{I}$  denotes the set of all triples  $(v, \alpha, \beta)$ , where  $v \in V$ ,  $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$  are such that  $\alpha \prec \beta \leq \theta_v$ . For a triple  $(v, \alpha, \beta) \in \mathcal{I}$  by  $\mathcal{I}(v, \alpha, \beta)$  we denote the set of all triples  $(w, \gamma, \delta) \in \mathcal{I}$  such that  $(\alpha, \beta)$  cannot be separated from  $(\gamma, \delta)$ . Also, by  $W_{v\alpha\beta}$  we denote the set  $\{w \mid (w, \gamma, \delta) \in \mathcal{I}(v, \alpha, \beta)\}$ . Sets of the form  $W_{v\alpha\beta}$  are called *coherent sets*.

Instance  $\mathcal{P}$  is said to be *block-minimal* if for any  $(v, \alpha, \beta) \in \mathcal{I}$  the instance  $\mathcal{P}_{W_{v\alpha\beta}}$  is minimal.

In the next section we prove, Theorem 21, that every block-minimal instance has a solution. To show that Theorem 21 gives rise to a polynomial-time algorithm for  $\mathrm{CSP}(\mathfrak{A})$  we need to show how block-minimality can be established. We prove that establishing block-minimality can be reduced to solving polynomially many smaller instances of  $\mathrm{CSP}(\mathfrak{A})$ .

**Proposition 20** Transforming an instance  $\mathcal{P} = (V, \mathcal{A}, \mathcal{C}) \in \mathrm{CSP}(\mathfrak{A})$  to a block-minimal instance can be reduced to solving polynomially many instances  $\mathcal{P}' =$ 

 $(V', \mathcal{A}', \mathcal{C}') \in \mathrm{CSP}(\mathfrak{A})$  such that  $V' \subseteq V$  and either  $\mathbb{A}'_v$  is a Mal'tsev algebra for all  $v \in V'$ , or  $|\mathbb{A}'_v| < |\mathbb{A}_v|$  for all  $v \in V'$ .

Since the cardinalities of algebras in  $\mathfrak A$  are bounded, the depth of recursion when establishing block-minimality is also bounded. Therefore, together with Theorem 21 this proposition gives a polynomial time algorithm for  $\mathrm{CSP}(\mathfrak A)$ .

**Proof:** Using the standard propagation algorithm and Maroti's reduction (Section 4.3) we may assume that  $\mathcal{P}$  is 3-minimal and every  $\mathbb{A}_v$  is either Mal'tsev or has a minimal element. Take  $(v,\alpha,\beta)\in\mathcal{I}$  as in the definition of blockminimality. We need to show how to make problems  $\mathcal{P}_{Wv\alpha\beta}$  minimal. If every  $\mathbb{A}_w$  for  $w\in W_{v\alpha\beta}$  is Mal'tsev,  $\mathcal{P}_{Wv\alpha\beta}$  can be made minimal using the algorithm from [10]. If  $\mathbb{A}_w$  has a minimal element for some  $w\in W_{v\alpha\beta}$  then set  $\alpha_v=\alpha,\beta_v=\beta$ , and for each  $w\in W_{v\alpha\beta}$  choose  $\alpha_w,\beta_w$  in such a way that  $(w,\alpha_w,\beta_w)\in\mathcal{I}(v,\alpha,\beta)$ . Then by Lemma 19 and 18  $\mathcal{P}_{Wv\alpha\beta}$  is link partitioned, that is, it is a disjoint union of instances  $\mathcal{P}_1\cup\cdots\cup\mathcal{P}_m$ , where  $\mathcal{P}_i=(W_{v\alpha\beta},\mathcal{A}^i,\mathcal{C}^i)$  are such that  $\mathbb{A}_w=\mathbb{A}_w^1\cup\cdots\cup\mathbb{A}_w^m$  is a disjoint union. We then transform them to minimal instances separately.

If at any stage there is a tuple from a constraint relation that does not extend to a solution of a certain subinstance, we tighten the original problem  $\mathcal{P}$  and start all over again. Observing that the set tuples from a constraint relation that can be extended to a solution of the subinstance is a subalgebra, the resulting instance belongs to  $CSP(\mathfrak{A})$  as well.

#### 7.2 Block-minimality and solutions of the CSP

We now prove that block-minimality is a sufficient condition to have a solution.

**Theorem 21** Every block-minimal instance  $\mathcal{P} \in \mathrm{CSP}(\mathfrak{A})$  with nonempty constraint relations has a solution.

**Proof:** Let  $\mathcal{P}=(V,\mathcal{A},\mathcal{C})$  be a 3-minimal and block-minimal instance from  $\mathrm{CSP}(\mathfrak{A})$ , and such that every domain  $\mathbb{A}_v$  is either a Mal'tsev algebra or has a minimal element. We make use of the following construction. Let  $\gamma_v \in \mathrm{Con}(\mathbb{A}_v)$ ,  $\gamma_v \leq \theta_v$  for  $v \in V$ . A collection of mappings  $\mathcal{M} = \{\varphi_{v\alpha\beta} \mid (v,\alpha,\beta) \in \mathcal{I}\}$  is called an  $\overline{\gamma}$ -ensemble for  $\mathcal{P}$  if

- (1) for every  $(v, \alpha, \beta) \in \mathcal{I}$  the mapping  $\varphi_{v\alpha\beta}$  is a solution of  $\mathcal{P}_{W_{v\alpha\beta}}$ ; and
- (2) for every  $(v, \alpha, \beta), (w, \gamma, \delta) \in \mathcal{I}$ , and any  $u \in W_{v\alpha\beta} \cap W_{w\gamma\delta}$ , it holds  $\varphi_{v\alpha\beta}(u) \stackrel{\gamma_u}{\equiv} \varphi_{w\gamma\delta}(u)$ ;

(3) for any  $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$  the tuple  $\mathbf{a}$  where  $\mathbf{a}[u] = \varphi_{v\alpha\beta}/\gamma_v$  for  $u \in \mathbf{s}$  and any  $(v, \alpha, \beta) \in \mathcal{I}$  with  $u \in W_{v\alpha\beta}$ , belongs to  $R/\overline{\gamma_e}$ .

We prove that for any  $\gamma_v \in \mathsf{Con}(\mathbb{A}_v)$ ,  $\gamma_v \leq \theta_v$  for  $v \in V$  the instance  $\mathcal{P}$  has a  $\overline{\gamma}$ -ensemble.

If  $\gamma_v = \theta_v$  for each  $v \in V$  then any collection of solutions  $\varphi_{v\alpha\beta}$  of  $\mathcal{P}_{W_{v\alpha\beta}}$  such that  $\varphi_{v\alpha\beta}(u) \in \max(\mathbb{A}_u)$  for all  $(v,\alpha,\beta) \in \mathcal{I}$ , and  $u \in W_{v\alpha\beta}$ , satisfies the conditions of a  $\overline{\gamma}$ -ensemble. Moreover by the block-minimality of  $\mathcal{P}$  such solutions exist.

If  $\gamma_v = \underline{0}_v$  for  $v \in V$  then for any  $(v, \alpha, \beta), (w, \gamma, \delta) \in \mathcal{I}$  condition (2) implies  $\varphi_{v\alpha\beta}(u) = \varphi_{w\gamma\delta}(u)$  for  $u \in W_{v\alpha\beta} \cap W_{w\gamma\delta}$ . Let us denote this value by  $\psi(u)$ . Then condition (3) implies that  $\psi$  is a solution of  $\mathcal{P}$ .

Finally, the inductive step follows from Lemma 22.

**Lemma 22** Let  $\mathcal{P} = (V, \mathcal{A}, \mathcal{C}) \in \mathrm{CSP}(\mathfrak{A})$  be a 3-minimal and block-minimal instance such that every  $\mathbb{A}_v$ ,  $v \in V$ , either is Mal'tsev or has a minimal element. Let  $v \in V$  and  $\beta_w, \gamma_w \in \mathrm{Con}(\mathbb{A}_w)$ ,  $w \in V$ , be such that  $\beta_w \preceq \gamma_w \leq \theta_w$ ,  $\beta_v \prec \gamma_v$  and  $\beta_w = \gamma_w$  for  $w \neq v$ . If there is a  $\overline{\gamma}$ -ensemble for  $\mathcal{P}$  then there is a  $\overline{\beta}$ -ensemble for  $\mathcal{P}$ .

**Proof:** Let  $\mathcal{M}=\{\varphi_{w\gamma\delta}\mid (w,\gamma,\delta)\in\mathcal{I}\}$  be a  $\overline{\gamma}$ -ensemble and  $\xi(u)=\varphi_{w\gamma\delta}(u)^{\gamma_u}$  for  $u\in W_{w\gamma\delta}$ . By condition (2) for  $\overline{\gamma}$ -ensembles this definition is consistent. If  $\xi(v)$  is a  $\gamma_v$ -block that is equal to an  $\beta_v$ -block, then  $\mathcal{M}$  is also a  $\overline{\beta}$ -ensemble, and there is nothing to prove.

Otherwise let B be the  $\beta_v$ -block containing  $\varphi_{v\alpha\beta}(v)$ . We show that for every  $(w,\gamma,\delta)\in\mathcal{I}$  with  $v\in W_{w\gamma\delta}$  a solution  $\varphi'_{w\gamma\delta}$  can be found such that  $\varphi'_{w\gamma\delta}(v)\in B$  and  $\varphi'_{w\gamma\delta}(u)\stackrel{\gamma_u}{\equiv}\varphi_{w\gamma\delta}(u)$ . Then, setting  $\varphi'_{w\gamma\delta}=\varphi_{w\gamma\delta}$  for  $(w,\gamma,\delta)\in\mathcal{I}$  such that  $v\not\in W_{w\gamma\delta}$  and  $\mathcal{M}'=\{\varphi'_{w\gamma\delta}\mid (w,\gamma,\delta)\in\mathcal{I}\}$  we conclude that  $\mathcal{M}'$  is a  $\overline{\beta}$ -ensemble.

Let  $(w, \gamma, \delta) \in \mathcal{I}$  be such that  $v \in W_{w\gamma\delta}$ , and let  $W = W_{v\alpha\beta}$ ,  $U = W_{w\gamma\delta}$ ,  $\varphi = \varphi_{v\alpha\beta}|_{W\cap U}$ ,  $\psi = \varphi_{w\gamma\delta}$ . Note that in this notation  $\mathcal{S}_W$ ,  $\mathcal{S}_U$ , and  $\mathcal{S}_{W\cap U}$  are the sets of solutions of  $\mathcal{P}_{W_{v\alpha\beta}}$ ,  $\mathcal{P}_{W_{w\gamma\delta}}$ , and  $\mathcal{P}_{W_{v\alpha\beta}\cap W_{w\gamma\delta}}$ . It will often be convenient for us to treat these sets as relations rather than sets of solutions of a CSP. Then  $\operatorname{pr}_{W\cap U}\mathcal{S}_W$ ,  $\operatorname{pr}_{W\cap U}\mathcal{S}_U\subseteq\mathcal{S}_{W\cap U}$ , and so  $\varphi$ ,  $\operatorname{pr}_{W\cap U}\psi\in\mathcal{S}_{W\cap U}$ .

Let  $\mathbf{f}$  be a  $(\beta_v, \gamma_v)$ -collapsing polynomial of  $\mathcal{S}_U$ . By Lemma 17 it can be selected such that  $\psi \in \mathbf{f}(\mathcal{S}_U)$  and  $B \cap f_v(\mathbb{A}_v) \neq \emptyset$ . Let  $\pi = \mathbf{f}_{W \cap U}(\varphi)$ . We show that the mapping  $\varphi'$  on U given by  $\varphi'(u) = \pi(u)$  for  $u \in W \cap U$ , and  $\varphi'(u) = \psi(u)$  for  $u \in U - W$  is a solution from  $\mathcal{S}_U$ . Since  $\varphi(v) \in B$  and  $B \cap f_v(\mathbb{A}_v) \neq \emptyset$ , that is,  $f_v(B) \subseteq B$  as  $\mathbf{f}$  is idempotent, we have  $\pi(v) = f_v(\varphi(v)) \in B$ . Also, as for

every  $u \in (W \cap U) - \{v\}$ , we have

$$\varphi'(u) = \pi(u) = f_u(\varphi(u)) \stackrel{\beta_u}{\equiv} f_u(\psi(u)) = \psi(u).$$

Therefore,  $\varphi'$  satisfies condition (2) of  $\overline{\beta}$ -ensembles for w, j.

Now we prove that  $\varphi'$  is a solution from  $\mathcal{S}_U$ . Let  $C = \langle \mathbf{s}, R \rangle$  be a constraint from  $\mathcal{P}_U$ ,  $W' = \mathbf{s} \cap W$  and  $\mathbf{a} = \mathrm{pr}_{W'} \varphi$ . Then, since  $\varphi$  is a solution from  $\mathcal{S}_{W \cap U}$ , there is  $\mathbf{b} \in R$  with  $\mathbf{a} = \mathrm{pr}_{W'} \mathbf{b}$ . Let  $\mathbf{c} = \mathbf{f}_{\mathbf{s}}(\mathbf{b})$ , clearly,  $\mathbf{c} \in R$ . For the tuple  $\mathbf{c}$  we have:

$$-\mathbf{c}[u] = f_u(\mathbf{a}[u]) = f_u(\varphi(u)) = \varphi'(u) \text{ for } u \in W';$$

 $-\mathbf{c}[u] = f_u(\mathbf{b}[u]) = \psi(u)$  for  $u \in \mathbf{s} - W'$ , because in this case  $f_u(\theta_u) \subseteq \underline{0}_u$ , and therefore, as  $f_u(\psi(u)) = \psi(u)$ , we have  $f_u(\max(\mathbb{A}_u)) = \{\psi(u)\}$ .

Thus,  $\mathbf{c} = \mathrm{pr}_{\mathbf{s}} \varphi'$ , and thus  $\varphi'$  is a solution from  $\mathcal{S}_{W \cap U}$ .

So far we have defined mappings  $\varphi'_{w\gamma\delta}$ , proved that they are solutions of the respective subinstances, that is, condition (1), and that they are consistent modulo  $\overline{\beta}$ , that is, condition (2). It remains to verify condition (3). Let  $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$  and  $\xi(u) = \varphi_{w\gamma\delta}(u)^{\beta_u}, \xi'(u) = \varphi'_{w\gamma\delta}(u)^{\beta_u}$  for  $u \in V$  and any  $(w, \gamma, \delta) \in \mathcal{I}$ , such that  $u \in W_{w\gamma\delta}$ . We need to show that  $\operatorname{pr}_{\mathbf{s}}\xi' \in R' = R/\overline{\beta}_{\epsilon}$ .

We use a simplified version of the argument above. Let  $W'=W\cap \mathbf{s}$ . If  $v\not\in \mathbf{s}$ , the result follows from condition (3) for  $\overline{\gamma}$ . Suppose  $v\in W'$  and let  $\mathbf{f}$  be a  $(\beta_v,\gamma_v)$ -collapsing polynomial of R'. Also, let  $\mathbf{a}=\mathrm{pr}_{\mathbf{s}}\xi,\,\mathbf{b}'=\mathrm{pr}_{W\cap\mathbf{s}}\varphi/\overline{\beta}_{W\cap\mathbf{s}}$ , where  $\varphi=\varphi_{v\alpha\beta}$  as before, and  $\mathbf{b}\in R'$  such that  $\mathbf{b}'=\mathrm{pr}_{W\cap\mathbf{s}}\mathbf{b}$ . By Lemma 17  $\mathbf{f}$  can be selected such that  $\mathbf{a}\in \mathbf{f}(R')$  and  $\mathbf{b}[v]\in f_v(\mathbb{A}_v/\beta_v)$ . Let  $\mathbf{c}=\mathbf{f}_{W\cap U}(\mathbf{b})$ . We have

$$-\mathbf{c}[v] = \mathbf{b}'[v];$$

$$-\mathbf{c}[u] = f_u(\mathbf{b}'[u]) = f_u'(\mathbf{a}[u]) = \mathbf{a}[u] \text{ for } u \in W' - \{u\}, \text{ as } \varphi(u) \in \xi(u) = \xi'(u);$$

$$-\mathbf{c}[u] = f_u(\mathbf{b}[u]) = f_u(\mathbf{a}[u]) = \mathbf{a}[u]$$
 for  $u \in \mathbf{s} - W'$ , as in this case  $f_u(\theta_u) \subseteq \beta_u$ , and therefore, since  $f_u(\mathbf{a}[u]) = \mathbf{a}[u]$ , we have  $f_u(\max(\mathbb{A}_u/\beta_u)) = \{\mathbf{b}[u]\}$ .

Therefore  $\mathbf{c} \in R'$ , and as  $\mathbf{c} = \operatorname{pr}_{\mathbf{a}} \xi'$ , the result follows.

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