Compositional Semantics for Relaxations of Differential Privacy

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We develop new abstractions for reasoning about relaxations of differential privacy: *Rényi differential privacy*, *zero-concentrated differential privacy*, and *truncated concentrated differential privacy*, which express different bounds on statistical divergences between two output probability distributions. In order to reason about such properties compositionally, we introduce *approximate span-lifting*, a novel construction extending the approximate relational lifting approaches previously developed for standard differential privacy to a more general class of divergences, and also to continuous distributions. As an application, we develop a program logic based on approximate span-liftings capable of proving relaxations of differential privacy and other statistical divergence properties.

CCS Concepts: • Software and its engineering  $\rightarrow$  General programming languages; • Social and professional topics  $\rightarrow$  History of programming languages;

#### **ACM Reference Format:**

Tetsuya Sato, Gilles Barthe, Marco Gaboardi, Justin Hsu, and Shin-ya Katsumata. 2018. Approximate Span Liftings: Compositional Semantics for Relaxations of Differential Privacy. *Proc. ACM Program. Lang.* 1, CONF, Article 1 (January 2018), 42 pages.

# **1 INTRODUCTION**

Differential privacy [Dwork et al. 2006] is a strong, statistical notion of data privacy that has attracted the attention of theoreticians and practitioners alike. One reason for its success is that differential privacy can often be proved *compositionally*, enabling easy construction of new private algorithms and making formal verification practical. By now, researchers have developed a wide variety of programming languages and program analysis tools to prove differential privacy [Albarghouthi and Hsu 2018; Barthe et al. 2015, 2013; Gaboardi et al. 2013; McSherry 2009; Reed and Pierce 2010; Winograd-Cort et al. 2017; Zhang and Kifer 2017] (Barthe et al. [2016c] provide a recent survey).

Seeking more refined composition properties, researchers have recently proposed new relaxations of differential privacy: *Rényi differential privacy* (RDP) [Mironov 2017], *zero-concentrated differential privacy* (zCDP) [Bun and Steinke 2016], and *truncated concentrated differential privacy* (tCDP) [Bun et al. 2018]. Roughly speaking, standard differential privacy requires a bound on the magnitude of a random variable measuring the privacy loss, while RDP, zCDP, and tCDP model finer bounds on the *moments* of this random variable. (Recall that the first moment of a random variable is its average value, and the second moment of a random variable is its variance.) These relaxations capture fine-grained aspects of the privacy loss, enabling more precise privacy analyses and allowing algorithms to add less random noise to achieve the same privacy level.

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Each of RDP, zCDP, and tCDP is defined in terms *Rényi divergences* [Renyi 1961], sophisticated distances on distributions originating from information theory. Inspiring our work, Barthe and Olmedo previously developed abstractions for reasoning about a family of divergences called *f*-divergences as part of their work on the program logic *f* pRHL [Barthe and Olmedo 2013; Olmedo 2014]. In particular, the semantic foundation of *f* pRHL is a 2-witness relational lifting for *f*-divergences, which tracks the *f*-divergence between relates pairs of distributions. However, this framework is not sufficient to establish about our target properties for two reasons. First, Rényi divergences are not *f*-divergences,<sup>1</sup> while zCDP and tCDP are properly described as *supremums* of *Rényi divergences*, rather than single divergences. As a result, these relaxations of differential privacy cannot be described in terms of *f*-divergences, nor captured in *f* pRHL. Accordingly, we develop new relational lifting supporting significantly more general divergences, allowing direct reasoning about RDP, zCDP, and tCDP.

A further challenge is that 2-witness relational liftings to date have only been proposed for discrete distributions, while many algorithms satisfying relaxations of differential privacy—indeed, the motivating examples of such algorithms—sample from continuous distributions, such as the Gaussian distribution. Handling these distributions requires a careful treatment of measure theory. Sato [2016] has previously considered a different semantic model for standard differential privacy over continuous distributions using *witness-free* relational lifting based on a categorical construction called *codensity lifting* [Katsumata and Sato 2015], but it is not clear how to handle more general divergences with this method.

To overcome these difficulties, we generalize 2-witness liftings in two directions. First, we replace the notion of f-divergence with a more general class of divergences, identifying the basic properties needed for compositional reasoning. Second, we generalize these liftings to about continuous probability measures. The main challenge is establishing a sequential composition principle—the continuous case introduces further measurability requirements for composition. Accordingly, we extend the structure of 2-witness liftings to a new notion called *approximate span-liftings*, which have the necessary data to ensure closure under sequential composition. Finally, we specialize our general model to Rényi divergence, divergences for zCDP, and divergences for tCDP, establishing categorical properties needed to build approximate span-liftings. As an extended application, we develop a relational program logic that can verify differential privacy, RDP, zCDP, and tCDP within a single logic for programs using discrete or continuous sampling, and interpret the logic via approximate span-liftings.

After motivating the various relaxations of differential privacy and presenting the key technical challenges (Section 2), and introducing mathematical preliminaries (Section 3), we present our main contributions.

- We identify a general class of divergences supporting basic properties composition properties, and we show that our class can model RDP, zCDP and tCDP (Section 4).
- We extend 2-witness relational liftings to the continuous case by introducing a novel notion of approximate span-lifting and showing how to translate composition properties of specific divergences to their corresponding approximate span-liftings (Section 5).
- We develop a program logic supporting four flavors of differential privacy-standard DP, RDP, zCDP, and tCDP-where programs may use both discrete and continuous random sampling, and show soundness (Section 6). We demonstrate our logic on three examples (Section 7).

We survey related work (Section 8) and then conclude with promising future directions (Section 9).

<sup>&</sup>lt;sup>1</sup>For instance, all *f*-divergences are jointly convex while Rényi divergences are only quasi-convex [Van Erven and Harremoës 2014].

#### 2 BACKGROUND: MOTIVATION AND TECHNICAL CHALLENGES

To better understand the key technical challenges, we first introduce relevant background on privacy, divergences, and existing relational verification techniques. For simplicity, in this section we consider probability distributions which have associated density functions.

# 2.1 Differential Privacy and its Relaxations

We first introduce differential privacy. A *randomized algorithm* is a measurable function  $\mathcal{A} \colon X \to \operatorname{Prob}(Y)$  from a set X of inputs to the set  $\operatorname{Prob}(Y)$  of *probability distributions* on a set Y of outputs.

Definition 2.1 (Differential Privacy (DP) [Dwork et al. 2006]). A randomized algorithm  $\mathcal{A}: X \to \operatorname{Prob}(Y)$  is  $(\varepsilon, \delta)$ -differentially private w.r.t an adjacency relation  $\Phi \subseteq X \times X$ , if for any pairs of inputs  $(x, x') \in \Phi$ , and any measurable subset  $S \subseteq Y$ , we have  $\Pr[\mathcal{A}(x) \in S] \leq e^{\varepsilon} \Pr[\mathcal{A}(x') \in S] + \delta$ .

Definition 2.2 (Rényi divergence [Renyi 1961]). Let  $\alpha > 1$ . The Rényi divergence of order  $\alpha$  between two probability distributions  $\mu_1$  and  $\mu_2$  on a measurable space X is defined by:

$$D_X^{\alpha}(\mu_1||\mu_2) \stackrel{\text{def}}{=} \frac{1}{\alpha - 1} \log \int_X \mu_2(x) \left(\frac{\mu_1(x)}{\mu_2(x)}\right)^{\alpha} dx. \tag{1}$$

Definition 2.3 (Rényi Differential Privacy (RDP) [Mironov 2017]). A randomized algorithm  $\mathcal{A}$  :  $X \to \operatorname{Prob}(Y)$  is  $(\alpha, \rho)$ -Rényi differentially private w.r.t an adjacency relation  $\Phi \subseteq X \times X$ , if for any pairs of inputs  $(x, x') \in \Phi$ , we have  $D_X^{\alpha}(\mathcal{A}(x)||\mathcal{A}(y)) \leq \rho$ .

Definition 2.4 (zero-Concentrated Differential Privacy (zCDP) [Bun and Steinke 2016]). A randomized algorithm  $\mathcal{A} : X \to \operatorname{Prob}(Y)$  is  $(\xi, \rho)$ -zero concentrated differentially private w.r.t an adjacency relation  $\Phi \subseteq X \times X$ , if for any pairs of inputs  $(x, x') \in \Phi$ , we have

$$\forall \alpha > 1. \ D_Y^{\alpha}(\mathcal{A}(x)||\mathcal{A}(x')) \le \xi + \alpha \rho.$$
<sup>(2)</sup>

Definition 2.5 (Truncated Concentrated Differential Privacy (tCDP) [Bun et al. 2018]). A randomized algorithm  $\mathcal{A} : X \to \operatorname{Prob}(Y)$  is  $(\rho, \omega)$ -truncated concentrated differentially private w.r.t an adjacency relation  $\Phi \subseteq X \times X$ , if for any input pairs  $(x, x') \in \Phi$ , we have

$$\forall 1 < \alpha < \omega. \ D_Y^{\alpha}(\mathcal{A}(x) || \mathcal{A}(x')) \le \alpha \rho.$$
(3)

While these notions may seem cryptic at first sight, they can all be understood as bounds on the *privacy loss*, defined for any two private inputs x, x' by

$$\mathcal{L}^{x \to x'}(y) = \frac{\Pr[\mathcal{A}(x) = y]}{\Pr[\mathcal{A}(x') = y]}$$

Intuitively, the privacy loss measures how much information is revealed when the output of a private algorithm is seen to be y. While output values with a high value of privacy loss are highly revealing—since they are far more likely to result from a private input x rather than a different private input x'—if these outputs are only seen with very small probability, then their influence can be discounted. Accordingly, the different privacy definitions bound different functions of the privacy loss function, evaluated at some output y drawn from the output distribution of the private algorithm. The following table summarizes these bounds.

Privacy notion of $\mathcal R$	Bound on privacy loss $\mathcal L$	
$(\varepsilon, \delta)$ -DP	$\Pr_{y \sim \mathcal{A}(x)}[\mathcal{L}^{x \to x'}(y) \le e^{\varepsilon}] \ge 1 - \delta$	
$(\alpha, \rho)$ -RDP	$\mathbb{E}_{y \sim \mathcal{A}(x)}[\mathcal{L}^{x \to x'}(y)^{\alpha}] \le e^{(\alpha - 1)\rho}$	
$(\xi, \rho)$ -zCDP	$\forall \alpha > 1. \mathbb{E}_{y \sim \mathcal{A}(x)} [\mathcal{L}^{x \to x'}(y)^{\alpha}] \le e^{(\alpha - 1)(\xi + \alpha \rho)}$	
$(\omega, \rho)$ -tCDP	$\forall 1 < \alpha < \omega. \mathbb{E}_{y \sim \mathcal{A}(x)} [\mathcal{L}^{x \to x'}(y)^{\alpha}] \le e^{(\alpha - 1)\alpha \rho}$	

In particular, DP bounds the maximum value of the privacy loss,  $(\alpha, \cdot)$ -RDP bounds the  $\alpha$ moment, zCDP bounds all moments, and  $(\cdot, \omega)$ -tCDP bounds the moments up to some cutoff  $\omega$ . Many conversions are known between these definitions; for instance, the relaxations of RDP, zCDP, and tCDP are known to sit between  $(\varepsilon, 0)$  and  $(\varepsilon, \delta)$ -differential privacy in terms of expressivity, up to some modification in the parameters. While this means that RDP, zCDP, and tCDP can sometimes be analyzed by reduction to standard differential privacy, converting between the different notions requires weakening the parameters and often the privacy analysis is simplest and most precise by working with RDP, zCDP, or tCDP directly. For further details, the interested reader can refer to the original papers [Bun and Steinke 2016; Mironov 2017].

A motivating example of a mechanism fitting these three definitions is the *Gaussian mechanism* and *Sinh Normal mechanism*, which add noise according to a Gaussian distribution and sinh-normal distribution over the real numbers respectively. The distributions are generated by continuous density functions.

# 2.2 2-witness Relational Liftings for *f*-divergences in Discrete Case

Barthe and Olmedo [2013] observed that standard differential privacy can be phrased in terms of a general class of divergences, called f-divergences.

Definition 2.6. A weight function is a convex function  $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$  continuous at  $0^2$ .

Definition 2.7 (*f*-divergence). For a weight function *f*, the *f*-divergence  $\Delta^f$  between two distributions  $\mu_1, \mu_2$  over a measurable space *X* is defined as

$$\Delta_X^f(\mu_1, \mu_2) = \int_X \mu_2(x) f\left(\frac{\mu_1(x)}{\mu_2(x)}\right) \, dx. \tag{4}$$

In particular, differential privacy can be modeled by the *f*-divergence  $\Delta^{DP(\varepsilon)}$  with weight function  $DP(\varepsilon)(t) = \max(0, 1 - e^{\varepsilon}t)$  [Barthe and Olmedo 2013; Olmedo 2014]. For any randomized algorithm  $\mathcal{A} : X \to Prob(Y)$  and adjacency relation  $\Phi \subseteq X \times X$ , we have

$$\mathcal{A} \text{ is } (\varepsilon, \delta) \text{-DP } \text{ iff } (\text{for all } (x, x') \in \Phi, \ \Delta_Y^{\mathsf{DP}(\varepsilon)}(\mathcal{A}(x), \mathcal{A}(x')) \leq \delta).$$

To verify *f*-divergence properties of probabilistic programs, Barthe and Olmedo introduced 2-witness relational lifting for *f*-divergences as a key abstraction. This construction lifts a relation  $R \subseteq X \times Y$  over discrete sets X, Y to a relation  $R^{\sharp(f,\delta)} \subseteq \text{Dist}(X) \times \text{Dist}(Y)$  over subprobability distributions:<sup>3</sup>

$$R^{\sharp(f,\delta)} = \left\{ (\mu_1, \mu_2) \mid \exists \mu_L, \mu_R \in \text{Dist}(R). \ \pi_1(\mu_L) = \mu_1, \ \pi_2(\mu_R) = \mu_2, \ \Delta_R^f(\mu_L, \mu_R) \le \delta \right\}.$$
(5)

Above,  $\pi_i(\mu)$  is the *i*-th marginal of  $\mu$ , that is,  $(\pi_1(\mu))(x) = \sum_{y \in Y} \mu(x, y)$  and  $(\pi_2(\mu))(y) = \sum_{x \in X} \mu(x, y)$ . The distributions  $\mu_L$  and  $\mu_R$  are called *witness distributions*, since to show that two distributions are related by a lifting, one must show the existence of two appropriate witnesses.

Barthe and Olmedo used these relational liftings as the foundation of their relational program logic f pRHL. These liftings have several attractive features. First, they reflect f-divergences:

$$\mathrm{Eq}_{X}^{\sharp(f,\,\delta)} = \{ (x,x) \mid x \in X \}^{\sharp(f,\,\delta)} = \left\{ (\mu_{1},\mu_{2}) \mid \Delta_{X}^{f}(\mu_{1},\mu_{2}) \le \delta \right\}.$$

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

<sup>&</sup>lt;sup>2</sup>As is conventional [Liese and Vajda 2006], we exclude the condition f(1) = 0 from the definition of weight function to support the exponential of Rényi divergence of order  $\alpha$ . We also assume  $0f(a/0) = \lim_{t\to 0+} tf(a/t)$  for a > 0 and 0f(0/0) = 0.

 $<sup>^{3}</sup>$ In order to reason about possibly non-terminating programs, they work with an extension of f-divergence to *subprobability distributions*.

So, they can be used to characterize differential privacy: a program  $\mathcal{A}: X \to \text{Dist}(Y)$  is  $(\varepsilon, \delta)$ differentially private w.r.t. an adjacency relation  $\Phi$ , if  $(\mathcal{A}(x), \mathcal{A}(x')) \in \text{Eq}_Y^{\sharp(\mathsf{DP}(\varepsilon), \delta)}$ , for every  $(x, x') \in \Phi$ . Second, 2-witness liftings satisfy various composition properties, enabling clean verification of probabilistic programs. However, this construction works only in the discrete case all subprobability distributions are over countable discrete sets—and the logic *f* pRHL cannot reason about programs that sample from continuous distributions, like the Gaussian distribution.

# 2.3 Challenge 1: Handling Richer Divergences

Much like standard differential privacy can be viewed in terms of f-divergences, we would like to view RDP, zCDP, and tCDP as bounds on more general divergences. A natural candidate for Rényi differential privacy is Rényi divergence  $D^{\alpha}$ , as in its original definition. Indeed, we have:

$$\mathcal{A}$$
 is  $(\alpha, \rho)$ -RDP iff (for all  $(x, x') \in \Phi$ ,  $D_Y^{\alpha}(\mathcal{A}(x)||\mathcal{A}(x')) \le \rho$ ).

However, the Rényi divergence  $D^{\alpha}(\mu_1||\mu_2)$  of order  $\alpha$  is not an f-divergence, and so it does not fit in the 2-witness lifting framework. Likewise, zCDP [Bun and Steinke 2016] and tCDP [Bun et al. 2018] can be defined via uniform bounds on families of Rényi divergence:

$$\Delta_X^{\text{zCDP}(\xi)}(\mu_1, \mu_2) = \sup_{1 < \alpha} \frac{1}{\alpha} \left( D_X^{\alpha}(\mu_1 || \mu_2) - \xi \right) \quad \text{for } 0 \le \xi,$$
(6)

$$\Delta_X^{\omega-\text{tCDP}}(\mu_1,\mu_2) = \sup_{1 < \alpha < \omega} \frac{1}{\alpha} \left( D_X^{\alpha}(\mu_1||\mu_2) \right) \quad \text{for } 1 < \omega, \tag{7}$$

000(4)

letting us reformulate zCDP and tCDP as

$$\mathcal{A} \text{ is } (\xi, \rho)\text{-zCDP iff (for all } (x, x') \in \Phi, \ \Delta_Y^{\text{zCDP}(\xi)}(\mathcal{A}(x), \mathcal{A}(x')) \leq \rho)$$
$$\mathcal{A} \text{ is } (\rho, \omega)\text{-tCDP iff (for all } (x, x') \in \Phi, \ \Delta_Y^{\omega-\text{tCDP}}(\mathcal{A}(x), \mathcal{A}(x')) \leq \rho).$$

These divergences are also not f-divergences. Furthermore, the RDP, zCDP and tCDP divergences may take negative values when applied to sub-probability distributions, which can arise from probabilistic computations that may not terminate with probability 1. Accordingly, we generalize the notion of divergence to go beyond f-divergences and also to handle sub-probability distributions. Starting from families of real valued functions from pairs of distributions, we introduce basic properties needed to give good composition properties for their corresponding liftings.

# 2.4 Challenge 2: Extending 2-witness Liftings to the Continuous Case

In order to support natural examples for RDP, zCDP, and tCDP, we need a framework supporting continuous distributions, such as Gaussian, Laplace, and sinh-normal distributions. Unfortunately, extending 2-witness relational liftings to the continuous case presents further technical challenges related to composition. The relational lifting  $(-)^{\sharp(DP(\varepsilon),\delta)}$  for standard differential privacy satisfies a sequential composition principle:

$$\begin{array}{c} (f,g)\colon R \to S^{\sharp(\mathsf{DP}(\varepsilon_1),\,\delta_1)} \text{ is a relation-preserving map.} \\ \\ (f^{\sharp},g^{\sharp})\colon R^{\sharp(\mathsf{DP}(\varepsilon_2),\,\delta_2)} \to S^{\sharp(\mathsf{DP}(\varepsilon_1+\varepsilon_2),\,\delta_1+\delta_2)} \text{ is a relation-preserving map.} \end{array}$$

Here,  $f^{\sharp}$  and  $g^{\sharp}$  are the Kleisli liftings of f and g with respect to the monad Dist of (discrete) subprobability distributions; this composition property gives 2-witness relational liftings a *graded monad* structure [Fujii et al. 2016; Katsumata 2014], highly useful for compositional reasoning. Since 2-witness lifting is defined through the existence of witness distributions, for any  $(d_1, d_2) \in R^{\sharp(\mathsf{DP}(\varepsilon_2), \delta_2)}$ , we then need witness distributions showing  $(f^{\sharp}(d_1), g^{\sharp}(d_2)) \in S^{\sharp(\mathsf{DP}(\varepsilon_1+\varepsilon_2), \delta_1+\delta_2)}$ . In the discrete case, these witnesses can be constructed in two steps:

(1) For any  $(x, y) \in R$ , there exist witnesses  $d'_L, d'_R \in \text{Dist}(S)$  proving  $(f(x), g(y)) \in S^{\sharp(\mathsf{DP}(\varepsilon_1), \delta_1)}$ . By applying the axiom of choice, we obtain a selection function

$$\langle l_1, l_2 \rangle \colon R \to \left\{ \left( d'_L, d'_R \right) \mid \Delta_S^{\mathsf{DP}(\varepsilon_1)}(d'_L, d'_R) \le \delta_1 \right\}$$

(2) For any witnesses  $d_L, d_R \in \text{Dist}(R)$  proving  $(d_1, d_2) \in R^{\sharp(\mathsf{DP}(\varepsilon_2), \delta_2)}, (l_1^{\sharp}(d_L), l_2^{\sharp}(d_R))$  is a pair of witness distributions proving  $(f^{\sharp}(d_1), g^{\sharp}(d_2)) \in S^{\sharp(\mathsf{DP}(\varepsilon_1 + \varepsilon_2), \delta_1 + \delta_2)}$  by composability of  $\Delta^{\mathsf{DP}(\varepsilon)}$ .

The first step is problematic to extend to the continuous case because the witness-selecting functions  $l_1$  and  $l_2$  obtained by the axiom of choice may not be measurable—the Kleisli extensions  $l_1^{\sharp}$  and  $l_2^{\sharp}$  in the second step may not be well-defined in the continuous case.

To resolve this difficulty, we introduce a novel notion of *approximate span-liftings*. The key idea is that morphisms between span-liftings carry a built-in measurable witness selection function, making it unnecessary to use the axiom of choice when proving sequential composition.

#### 3 MATHEMATICAL PRELIMINARIES

#### 3.1 Measure Theory

We briefly review some definitions from measure theory; readers should consult a textbook for more details [Rudin 1987]. Given a set *X*, a  $\sigma$ -algebra on *X* is a collection  $\Sigma$  of subsets of *X* including the empty set, closed under complements, countable unions, and countable intersections; a measurable space *X* is a set |X| with a  $\sigma$ -algebra  $\Sigma_X$ , called the measurable sets. A countable set *X* yields the *discrete* measurable space where all subsets are measurable:  $\Sigma_X = 2^X$ .

A map  $f: X \to Y$  between measurable spaces is *measurable* if  $f^{-1}(A) \in \Sigma_X$  for all  $A \in \Sigma_Y$ . Any subset S of measurable space X forms a *subspace* where the  $\sigma$ -algebra is given by  $\Sigma_S = \{A \cap S \mid A \in \Sigma_X\}$ .  $\Sigma_S$  is given as the coarsest one making the inclusion map  $S \hookrightarrow X$  measurable.

A measure on a measurable space is a map  $\mu: \Sigma_X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$  such that  $\mu(\emptyset) = 0$  and  $\mu(\bigcup_i X_i) = \sum_i \mu(X_i)$  for any countable family of disjoint measurable sets  $X_i$ . Measures with  $\mu(X) = 1$  are called *probability measures*, and measures with  $\mu(X) \leq 1$  are called *subprobability measures*.

For any pair of subprobability measures  $\mu_1$  on X and  $\mu_2$  on Y, the *product measure*  $\mu_1 \otimes \mu_2$  of  $\mu_1$ and  $\mu_2$  is the unique measure on  $X \times Y$  satisfying  $(\mu_1 \otimes \mu_2)(A \times B) = \mu_1(A) \cdot \mu_2(B)$ .

For any measurable space *X* and element  $x \in X$ , we write  $\mathbf{d}_x$  for the Dirac measure on *X* centered at *x*, defined as  $\mathbf{d}_x(A) = 1$  if  $x \in A$ , and  $\mathbf{d}_x(A) = 0$  otherwise.

Measurable spaces and measurable functions form a category Meas; this category has all limits and colimits, and finite products distribute over finite coproducts. We denote by Fin the full subcategory of Meas consisting of all finite discrete spaces.

## 3.2 The Sub-Giry Monad

The sub-Giry monad G is the subprobabilistic variant of the Giry monad [Giry 1982].

*Definition 3.1.* The sub-Giry monad  $(\mathcal{G}, \eta, (-)^{\sharp})$  over **Meas** is defined as follows:

- For any  $X \in$  Meas, the measurable space GX is the set of subprobability measures (measures whose mass is equal or less than 1) on X equipped with the coarsest  $\sigma$ -algebra induced by the evaluation functions  $ev_A : GX \to [0, 1]$  defined by  $v \mapsto v(A)$  ( $A \in \Sigma_X$ ).
- For each  $f: X \to Y$  in Meas,  $Gf: GX \to GY$  is defined by  $(Gf)(\mu) = \mu(f^{-1}(-))$ .
- The unit  $\eta$  is defined by the Dirac distributions  $\eta_X(x) = \mathbf{d}_x$ .
- The Kleisli extension  $f^{\sharp}: \mathcal{G}X \to \mathcal{G}Y$  of  $f: X \to \mathcal{G}Y$  is given by for any  $\mu \in \mathcal{G}X$  and  $A \in \Sigma_Y$ ,  $f^{\sharp}(\mu)(A) = \int_X f(x)(A) d\mu(x)$ .

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

The sub-Giry monad satisfies useful properties for interpreting probabilistic programs. It is commutative and strong with respect to the Cartesian products of **Meas**, where the double strength  $dst_{X,Y}: \mathcal{G}(X) \times \mathcal{G}(Y) \Rightarrow \mathcal{G}(X \times Y)$  is given by the product measures  $dst_{X,Y}(v_1, v_2) = v_1 \otimes v_2$ . The double strength is used to define semantics for composition and to interpret typing contexts. Additionally, the sub-Giry monad provides a structure to interpret loops. Namely, we can introduce an  $\omega$ CPO<sub>⊥</sub> structure over measurable functions of type  $X \to \mathcal{G}(Y)$  with the following order:<sup>4</sup>

$$f \sqsubseteq g \iff \forall x \in X, B \in \Sigma_Y. f(x)(B) \le g(x)(B) \quad (f, g: X \to \mathcal{G}(Y) \text{ in Meas}).$$

#### 3.3 Graded Monads

A graded monad [Fujii et al. 2016; Katsumata 2014] is a monad refined by indices from a monoid. Let  $A = (A, \cdot, 1_A, \leq)$  be a preordered monoid. An *A*-graded monad on a category  $\mathbb{C}$  consists of

- a family  $\{T_e\}_{e \in M}$  of endofunctors  $T_e$  on  $\mathbb{C}$ ,
- a morphism  $\eta_X \colon X \to T_{1_A}X$  for  $X \in \mathbb{C}$  (unit),
- a morphism  $(-)^{e_1 \not\equiv e_2} \colon \mathbb{C}(X, T_{e_2}Y) \to \mathbb{C}(T_{e_1}X, T_{e_1e_2}Y)$  for  $X, Y \in \mathbb{C}$  and  $e_1, e_2 \in A$  (Kleisli lifting),
- a family  $\{\sqsubseteq^{e_1, e_2}\}_{e_1 \leq e_2}$  of natural transformations  $\sqsubseteq^{e_1, e_2} \colon T_{e_1} \Rightarrow T_{e_2}$  (inclusion)

satisfying the following compatibility condition: for any  $f: X \to T_{e_1}Y$  and  $g: Y \to T_{e_2}Z$ ,

$$\sqsubseteq_{Z}^{(e_{2}e_{1}),(e_{2}e_{3})} \circ f^{e_{2}\sharp e_{1}} = (\sqsubseteq_{Y}^{e_{1},e_{2}} \circ f)^{e_{2}\sharp e_{3}}, \quad f^{e_{3}\sharp e_{1}} \circ \sqsubseteq_{X}^{e_{2},e_{3}} = \sqsubseteq_{Y}^{(e_{2}e_{1}),(e_{3}e_{1})} \circ f^{e_{2}\sharp e_{1}},$$
$$f^{1\sharp e_{1}} \circ \eta_{X} = f, \quad \eta_{X}^{1\sharp e} = \operatorname{id}_{T_{e}X}, \quad (g^{e_{1}\sharp e_{2}} \circ f)^{e_{0}\sharp e_{1}e_{2}} = g^{e_{0}e_{1}\sharp e_{2}} \circ f^{e_{0}\sharp e_{1}}.$$

A typical way of constructing a graded monad is by refining a plain monad with indices. An *A*-graded lifting of a monad  $(T, \eta^T, (-)^{\sharp})$  on  $\mathbb{D}$  along a functor  $U : \mathbb{C} \to \mathbb{D}$  is an *A*-graded monad  $\{T_e\}_{e \in A}$  on  $\mathbb{C}$  satisfying  $U \circ T_e = T \circ U$ ,  $U(f^{e_2 \sharp e_1}) = (Uf)^{\sharp}$ ,  $U(\eta_D) = \eta_{UD}^T$ , and  $U(\sqsubseteq_D^{e_1, e_2}) = \operatorname{id}_{TUD}$ . The functor U erases the grading of  $T_e$ , yielding the original (plain) monad T.

### 3.4 The Category of Spans on Measurable Spaces

To extend the relational lifting approach to the continuous setting, we work with the category of *spans*, whose objects generalize relations by taking arbitrary functions in place of projections. Morphisms between spans will encode the information needed to ensure good compositional behavior.

Definition 3.2. The category Span(Meas) of spans in Meas consists of:

- Objects  $(X, Y, \Phi, \rho_1, \rho_2)$  given by span  $X \xleftarrow{\rho_1} \Phi \xrightarrow{\rho_1} Y$  in Meas.
- Morphisms  $(X, Y, \Phi, \rho_1, \rho_2) \rightarrow (Z, W, \Psi, \rho'_1, \rho'_2)$  given by triples (h, k, l) of morphisms  $h: X \rightarrow Z, k: Y \rightarrow W$ , and  $l: \Phi \rightarrow \Psi$  in **Meas** satisfying  $h \circ \rho_1 = \rho'_1 \circ l$  and  $k \circ \rho_2 = \rho'_2 \circ l$ .

For simplicity, we often denote a **Span**(**Meas**)-object ( $X, Y, \Phi, \rho_1, \rho_2$ ) by  $\Phi$ . The category **Span**(**Meas**) has several useful properties. First, the category has binary products:

$$(X, Y, \Phi, \rho_1, \rho_2) \dot{\times} (Z, W, \Psi, \rho_1', \rho_2') = (X \times Z, Y \times W, \Phi \times \Psi, \rho_1 \times \rho_1', \rho_2 \times \rho_2').$$

We will frequently use two notions of pairing on functions. Let  $f_1: X \to Y$ ,  $f_2: X \to W$ , we have  $\langle f_1, f_2 \rangle: X \to Y \times W$  and  $f_1 \times f_2: X \times X \to Y \times W$ . As functions,  $\langle f_1, f_2 \rangle$  takes a single input x and returns a pair  $(f_1(x), f_2(x))$ . On the other hand,  $f_1 \times f_2$  take a pair of inputs (x, y) and returns  $(f_1(x), f_2(y))$ .

<sup>&</sup>lt;sup>4</sup>This ordering gives an  $\omega$ CPO<sub>1</sub>-enrichment of the Kleisli category Meas<sub>G</sub>, which is equivalent to the partial additivity of stochastic relations [Panangaden 1999].

The category Span(Meas) also has coproducts:

$$(X, Y, \Phi, \rho_1, \rho_2) \dotplus (X', Y', \Phi', \rho_1', \rho_2') = (X + X', Y + Y', \Phi + \Phi', \rho_1 + \rho_1', \rho_2 + \rho_2')$$

Standard binary relations can be interpreted as spans. For  $X, Y \in$  **Meas**, any binary relation  $\Phi \subseteq |X| \times |Y|$  determines a span  $X \xleftarrow{\pi_1} \Phi \xrightarrow{\pi_2} Y$  in **Meas**, where  $\pi_1$  and  $\pi_2$  are projections, and  $\Phi$  is regarded as a subspace of  $X \times Y$ .

Finally, relation-preserving maps can be interpreted as morphisms of spans. Consider two binary relations  $\Phi \subseteq |X| \times |Y|$  and  $\Psi \subseteq |Z| \times |W|$ , and suppose that they are interpreted as spans  $(X, Y, \Phi, \pi_1, \pi_2)$  and  $(Z, W, \Psi, \pi_1, \pi_2)$  as above. If  $f: X \to Z$  and  $g: Y \to W$  in **Meas** satisfy  $(f(x), g(y)) \in \Psi$  for any  $(x, y) \in \Phi$ , then we have the following morphism

 $(f, g, f \times g|_{\Phi}): (X, Y, \Phi, \pi_1, \pi_2) \to (Z, W, \Psi, \pi_1, \pi_2)$  in Span(Meas)

where  $f \times g|_{\Phi}$  is the restriction of  $f \times g$  on  $\Phi$  (we often write just  $f \times g$ ). These features are crucial to interpret probabilistic program logics, as we will see in Section 6.

# 4 GENERAL STATISTICAL DIVERGENCES

Now that we have covered the preliminaries, our goal is to build a suitable graded monad on Span(Meas)—this will be our abstraction for relational reasoning about divergences. We proceed in two stages. In this section, we introduce a general class of *divergences*, real-valued functions on two measures over the same space. Then, we identify important composition properties inspired from analogous properties of *f*-divergences [Barthe and Olmedo 2013; Liese and Vajda 2006]. We will leverage these properties to give a graded monad structure on Span(Meas) capturing these divergences in the next section. We write  $\mathbb{R}$  for the set  $\mathbb{R} \cup \{-\infty, +\infty\}$  of extended reals. We regard both  $\mathbb{R}$  and  $\mathbb{R}_{\geq 0}$  as partially ordered additive monoids. For the former one, the addition is extended by  $\infty + (-\infty) = -\infty$ .

Definition 4.1. A divergence is a family  $\Delta = {\Delta_X}_{X \in Meas}$  of functions

$$\Delta_X \colon |\mathcal{G}X| \times |\mathcal{G}X| \to \mathbb{R}$$

To describe composition of divergences, it is useful to work with indexed families of divergences; often, two divergences can be combined to give a new divergence with different indices. For instance, the notion of zCDP can be characterized by the family  $\{\Delta^{zCDP(\xi)}\}_{0 \le \xi}$  of divergences  $\Delta^{zCDP(\xi)}$  introduced in Section 2 (Equation 6). For this reason, we introduce the notion of graded families of divergences.

Definition 4.2. Let  $(A, \cdot, 1_A, \leq)$  be a preordered monoid. An *A*-graded family of divergences is a family  $\Delta = {\Delta^{\alpha}}_{\alpha \in A}$  such that

$$\alpha \leq \beta \implies (\forall X \in \mathbf{Meas}. \ \forall \mu_1, \mu_2 \in \mathcal{G}X. \ \Delta_X^\beta(\mu_1, \mu_2) \leq \Delta_X^\alpha(\mu_1, \mu_2)).$$

Note that the preorder on the grading is contravariant. We will regard a divergence  $\Delta$  as a singleton-graded family { $\Delta$ }.

#### 4.1 Basic Properties of Divergences

We define basic properties of graded families of divergences for given  $(A, \cdot, 1_A, \leq)$ .

Definition 4.3. An A-graded family  $\Delta = {\Delta^{\alpha}}_{\alpha \in A}$  of divergences is: **reflexive:** if  $\Delta^{\alpha}_{X}(\mu, \mu) \leq 0$ . **functorial:** if  $\Delta^{\alpha}_{Y}(\mathcal{G}k(\mu_{1}), \mathcal{G}k(\mu_{2})) \leq \Delta^{\alpha}_{X}(\mu_{1}, \mu_{2})$  for any  $k: X \to Y$ . **substitutive:** if  $\Delta^{\alpha}_{Y}(f^{\sharp}\mu_{1}, f^{\sharp}\mu_{2}) \leq \Delta^{\alpha}_{Y}(\mu_{1}, \mu_{2})$  for any  $f: X \to \mathcal{G}Y$ .

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

additive: if  $\Delta_{X \times Y}^{\alpha \cdot \beta}(\mu_1 \otimes \mu_3, \mu_2 \otimes \mu_4) \leq \Delta_X^{\alpha}(\mu_1, \mu_2) + \Delta_Y^{\beta}(\mu_3, \mu_4)$ . continuous: if  $\Delta_X^{\alpha}(\mu_1, \mu_2) = \sup \left\{ \Delta_I^{\alpha}(\mathcal{G}k(\mu_1), \mathcal{G}k(\mu_2)) \mid I \in \operatorname{Fin}, k \colon X \to I \right\}$ . composable: if  $\Delta_Y^{\alpha \cdot \beta}(f^{\sharp}\mu_1, g^{\sharp}\mu_2) \leq \Delta_X^{\alpha}(\mu_1, \mu_2) + \sup_{x \in X} \Delta_Y^{\beta}(f(x), g(x))$  for any  $f, g \colon X \to \mathcal{G}Y$ .

All functions are assumed to be measurable.

These properties are inspired by properties from the literature on f-divergences and differential privacy. For instance, substitutivity is the generalization of the usual notion of *data-processing inequality* for f-divergences [Pardo and Vajda 1997, Chapter 2], while functoriality is the special case where the data-processing function is deterministic. These two properties are also known in the differential privacy literature as *resilience to post-processing* [Dwork and Roth 2013, Proposition 2.1], in the randomized and deterministic case. Composability corresponds to composition in differential privacy, which states that we can adaptively compose two differentially private mechanisms. Additivity corresponds to a simple instance of composition where the second mechanism does not depend on the result of the first. Continuity is the generalization of the continuity of f-divergences [Pardo and Vajda 1997, Theorem 16], which approximates divergences of continuous distributions by divergences of discrete distributions.

Reflexivity and composability are key properties to give a structure of graded monad. Intuitively, reflexivity gives a unit, and composability gives a (graded) Kleisli lifting. We also need additivity to give a *strength* of the graded monad, allowing a lifting on real-valued distributions—often available from known results in probability theory—to be converted into a lifting on distributions over larger spaces (e.g., program memories). In some ways, composability is the key property: reflexivity is usually immediate, and additivity is a consequence.

THEOREM 4.4. An A-graded family  $\Delta$  is additive if it is continuous and composable.

Although these properties have been studied before in the discrete case, there are subtleties when passing to our continuous ones. For example, in the case of discrete distributions, additivity is an instance of composability [Barthe and Olmedo 2013, Proposition 4]. In the case of continuous distributions, this may no longer hold. However, one can recover additivity from composability by using a continuity property.

To prove composability, it is often easier to establish two other properties of families of divergences first: approximability and finite-composability. These properties describe divergences that are well-behaved with respect to discretization, in order to smoothly extend properties in the discrete case to the continuous case.

*Definition 4.5.* An *A*-graded family  $\Delta = {\Delta^{\alpha}}_{\alpha \in A}$  of divergences is:

**approximable:** if for any  $X \in$  **Meas** and  $I \in$  **Fin**,  $f, g: X \to GI$ , and  $\mu_1, \mu_2 \in GX$ , there are  $J_n \in$  **Fin** and  $m_n^*: X \to J_n$  and  $m_n: J_n \to X$  in **Meas** such that

$$\Delta_I^{\alpha}(f^{\sharp}(\mu_1), g^{\sharp}(\mu_2)) = \lim_{n \to \infty} \Delta_I^{\alpha}((f \circ m_n \circ m_n^*)^{\sharp}(\mu_1), (g \circ m_n \circ m_n^*)^{\sharp}(\mu_2)).$$

**finite-composable:** if for any  $I, J \in Fin, f, g: I \rightarrow GJ$ , and  $d_1, d_2 \in GI$ ,

$$\Delta_J^{\alpha \cdot \beta}(f^{\sharp}d_1, g^{\sharp}d_2) \le \Delta_I^{\alpha}(d_1, d_2) + \sup_{i \in I} \Delta_J^{\beta}(f(i), g(i)).$$

The function  $m_n^*$  in the definition of the approximability of  $\Delta$  discretizes points in X to  $J_n$ , and  $m_n$  reconstructs points in X from  $J_n$ . Finite-composability of  $\Delta$  means the composability of  $\Delta$  in the discrete case.

These properties allow us to extend composability of divergences in the discrete case, witnessed by finite-composability, to the continuous case. Finite-composability is often known for standard 1:10 Tetsuya Sato, Gilles Barthe, Marco Gaboardi, Justin Hsu, and Shin-ya Katsumata

divergences, or can be established by direct calculations. If  $\Delta$  is approximable and continuous, finite-composability implies composability. Formally, we have the following theorem.

THEOREM 4.6. A continuous approximable A-graded family  $\Delta$  is composable if finite-composable.

# 4.2 Basic Properties of *f*-divergences

To discuss basic properties of divergences for DP, RDP, zCDP, and tCDP, we begin with basic properties of *f*-divergences since DP can be formulated by a graded family  $\Delta^{DP} = \{\Delta^{DP(\varepsilon)}\}_{0 \le \varepsilon}$  of *f*-divergences, and Rényi divergences are logarithms of *f*-divergences. An *f*-divergence  $\Delta^f$  of subprobability measures is defined in the same way as *f*-divergence of probability measures (4). The *f*-divergences are not necessarily positive for subprobability measures, though they are positive for proper probability measures. We can extend the continuity of *f*-divergences [Liese and Vajda 2006, Theorem 16] to support subprobability measures.

THEOREM 4.7 (CF. LIESE AND VAJDA [2006, THEOREM 16]). For any weight function f, the f-divergence  $\Delta^f$  is continuous:<sup>5</sup> for any subprobability measures  $\mu_1, \mu_2 \in \mathcal{G}X$  on X, we have

$$\Delta_X^f(\mu_1,\mu_2) = \sup\left\{\sum_{i=0}^n \mu_2(A_i) f\left(\frac{\mu_1(A_i)}{\mu_2(A_i)}\right) \mid \{A_i\}_{i=0}^n \text{ is a measurable finite partition of } X\right\}.$$

As we have seen, DP can be formulated by the  $\mathbb{R}_{\geq 0}$ -graded family  $\Delta^{DP} = {\Delta^{DP(\varepsilon)}}_{0 \leq \varepsilon}$  of *f*-divergences, while the Rényi divergences supporting RDP, zCDP, and tCDP are logarithms of *f*-divergences. Before proving basic properties of divergences for DP, RDP, zCDP, and tCDP, we first need two important basic properties of *f*-divergences, continuity and approximability, and we show that finite-composability of *f*-divergences are extended to (proper) composability.

THEOREM 4.8. The *f*-divergence  $\Delta^f$  is approximable for any weight function *f*.

Therefore, any finite-composable family of f-divergences is composable.

THEOREM 4.9. An A-graded family  $\Delta = {\Delta^{f_{\alpha}}}_{\alpha \in A}$  of the  $f_{\alpha}$ -divergences is composable if it is finite-composable.

We remark here that any composable family of f-divergences is also additive by applying Theorem 4.4, since f-divergences are always continuous (Theorem 4.7).

# 4.3 Properties of Divergences for DP, RDP, zCDP, and tCDP

As we have seen, DP can be formulated by the  $\mathbb{R}_{\geq 0}$ -graded family  $\Delta^{DP}$  of f-divergences. By Theorem 4.4 and 4.9 and Barthe and Olmedo [2013, Theorem 1], we obtain the basic properties of the divergences  $\Delta^{DP}$  for DP as follows:

THEOREM 4.10 (CF. BARTHE AND OLMEDO [2013, THEOREM 1]). The  $\mathbb{R}_{\geq 0}$ -graded family  $\Delta^{DP} = \{\Delta^{DP(\varepsilon)}\}_{0 \leq \varepsilon}$  is reflexive, continuous, approximable, composable, and additive.

Similarly, we can obtain basic properties for RDP, zCDP, and tCDP. First, by Theorem 4.7 and Theorem 4.8, the exponential  $\exp(D^{\alpha})$  of Rényi divergence of order  $\alpha$  is continuous and approximable because is exactly the *f*-divergence with weight function  $t \mapsto \exp(\alpha/(1-\alpha))t^{\alpha}$ .

Since the logarithm function is monotone and continuous except at 0, Rényi divergence is continuous and approximable too. Reflexivity and finite-composability of Rényi divergences follow by direct calculations. Theorem 4.9 yields:

<sup>&</sup>lt;sup>5</sup>Note that a measurable finite partition  $\{A_i\}_{i=0}^n$  on X is equivalent to a measurable function  $k: X \to I$  where  $I = \{0, 1, \ldots, n\}$ .

THEOREM 4.11. For any  $\alpha > 1$ , the Rényi divergence  $D^{\alpha}$  of order  $\alpha$  is reflexive, continuous, approximable, composable, and additive (as a singleton-graded family).

We extend the following properties of Rényi divergences which give the transitive laws of RDP and zCDP to support subprobability measures. (An known analogous law for tCDP is not known.)

PROPOSITION 4.1 (CF. VAN ERVEN AND HARREMOËS [2014, THEOREM 3]). We have

$$1 < \alpha \leq \beta \implies D_X^{\alpha}(\mu_1 || \mu_2) \leq D_X^{p}(\mu_1 || \mu_2).$$

PROPOSITION 4.2 (CF. LANGLOIS ET AL. [2014, LEMMA 4.1]). For any  $\alpha > 1$ ,  $\mu_1, \mu_2, \mu_3 \in GX$ , and p, q > 1 satisfying  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$D_X^{\alpha}(\mu_1||\mu_3) \le \frac{p\alpha - 1}{p(\alpha - 1)} D_X^{p\alpha}(\mu_1||\mu_2) + D_X^{\frac{q}{p}(p\alpha - 1)}(\mu_1||\mu_2).$$

As we have seen in Section 2.4, we can define divergences for zCDP and tCDP by Equation (6) and Equation (7). Explicitly, we introduce the divergences for zCDP and tCDP by  $\Delta^{zCDP(\xi,\rho)} = \sup_{1 < \alpha} \frac{1}{\alpha} (D^{\alpha} - \xi)$  and  $\Delta^{\omega-tCDP(\rho)} = \sup_{1 < \alpha < \omega} \frac{1}{\alpha} D^{\alpha}$  respectively. Since two supremums are commutative (sup<sub>x</sub> sup<sub>y</sub>  $A(x, y) = \sup_y \sup_x A(x, y)$ ) in general, the following basic properties of the graded family of zCDP and the divergence of tCDP are obtained from Theorem 4.11.

THEOREM 4.12. The  $\mathbb{R}_{\geq 0}$ -graded family  $\Delta^{zCDP} = {\Delta^{zCDP(\xi)}}_{0 \leq \xi}$  for zCDP is reflexive, continuous, composable, and additive.

THEOREM 4.13. For each  $1 < \omega$ , the divergence  $\Delta^{\omega-tCDP}$  for  $\omega$ -tCDP is reflexive, continuous, composable, and additive.

Note that we may not have approximability, but the family is still composable. These results also hold for subprobability measures where Rényi divergence and divergences for zCDP and tCDP are defined in a way similar to Equation (1) and Equation (2) respectively.

## 5 APPROXIMATE SPAN-LIFTING

4/4

We are now ready to combine graded divergences with spans, leading to our new relational liftings. Given an *A*-graded family  $\Delta = {\Delta^{\alpha}}_{\alpha \in A}$  of divergences, we introduce a graded monad on **Span(Meas)** called the *approximate span-lifting*  $(-)^{\sharp(\Delta, \alpha, \delta)}$  for the family  $\Delta$ , where  $\alpha \in A$  and  $\delta \in \mathbb{R}$ . We first define its action on objects.

Definition 5.1. We define the span-constructor  $(-)^{\sharp(\Delta, \alpha, \delta)}$  as follows: for any  $(X, Y, \Phi, \rho_1, \rho_2)$  in **Span(Meas**), we define the **Span(Meas**)-object

$$(X, Y, \Phi, \rho_1, \rho_2)^{\mathfrak{p}(\Delta, \alpha, \delta)} = (\mathcal{G}X, \mathcal{G}Y, W(\Phi, \Delta, \alpha, \delta), \mathcal{G}\rho_1 \circ \pi_1, \mathcal{G}\rho_1 \circ \pi_2)$$
  
where  $W(\Phi, \Delta, \alpha, \delta) = \left\{ (v_1, v_2) \in \mathcal{G}\Phi \times \mathcal{G}\Phi \mid \Delta_{\Phi}^{\alpha}(v_1, v_2) \le \delta \right\}.$ 

We view  $W(\Phi, \Delta, \alpha, \delta)$  as a subspace of the measurable space  $\mathcal{G}\Phi \times \mathcal{G}\Phi$ .

Intuitively,  $(X, Y, \Phi, \rho_1, \rho_2)^{\sharp(\Delta, \alpha, \delta)}$  relates subprobability measures with  $\Delta^{\alpha}$ -distance at most  $\delta$ . The set  $W(\Phi, \Delta, \alpha, \delta)$  contains all possible witness distributions, and  $\pi_1$  and  $\pi_2$  are canonical projections from  $W(\Phi, \Delta, \alpha, \delta)$  to  $\mathcal{G}\Phi$ . As a special case, the approximate span-lifting  $(-)^{\sharp(\Delta, \alpha, \delta)}$  recovers the divergence  $\Delta^{\alpha}$  by applying the equality relation  $(X, X, \mathsf{Eq}_X, \pi_1, \pi_1)^{\sharp(\Delta, \alpha, \delta)}$ .

THEOREM 5.2. For any A-graded family  $\Delta$ ,  $\alpha \in A$ , and  $\delta \in \mathbb{R}$ , we have

$$(X, X, X, \operatorname{id}_X, \operatorname{id}_X)^{\sharp(\Delta, \alpha, \delta)} = (\mathcal{G}X, \mathcal{G}X, \left\{ (\mu_1, \mu_2) \mid \Delta_X^{\alpha}(\mu_1, \mu_2) \le \delta \right\}, \pi_1, \pi_2).$$

Here,  $(X, X, X, id_X, id_X)$  is isomorphic to the equality relation  $(X, X, Eq_X, \pi_1|_{Eq_X}, \pi_1|_{Eq_X})$ .

Next, we give approximate span-liftings the structure of a graded monad with double strength. We consider the important case where  $\Delta$  is a reflexive, composable, and additive *A*-graded family of divergences; in some cases, we can recover more limited versions of approximate span-liftings by dropping or weakening these properties.

THEOREM 5.3. If an A-graded family  $\Delta$  is reflexive, composable, and additive, then the approximate span-lifting  $(-)^{\sharp(\Delta,\alpha,\delta)}$  form an  $A \times \mathbb{R}$ -graded monad with double strength. Namely, there are maps

**Functor:** For any morphism (h, k, l):  $(X, Y, \Phi, \rho_1, \rho_2) \rightarrow (Z, W, \Psi, \rho'_1, \rho'_2)$  in the category Span(Meas) and any  $(\alpha, \delta) \in A \times \overline{\mathbb{R}}$ ,

 $(\mathcal{G}h, \mathcal{G}k, \mathcal{G}l \times \mathcal{G}l) \colon (X, Y, \Phi, \rho_1, \rho_2)^{\sharp(\Delta, \alpha, \delta)} \to (Z, W, \Psi, \rho_1', \rho_2')^{\sharp(\Delta, \alpha, \delta)}.$ 

**Unit:** For any morphism  $(X, Y, \Phi, \rho_1, \rho_2)$  in Span(Meas),

 $(\eta_X, \eta_Y, \langle \eta_\Phi, \eta_\Phi \rangle) \colon (X, Y, \Phi, \rho_1, \rho_2) \to (X, Y, \Phi, \rho_1, \rho_2)^{\sharp(\Delta, 1_A, 0)}$ 

**Kleisli lifting:** For any morphism (h, k, l):  $(X, Y, \Phi, \rho_1, \rho_2) \rightarrow (Z, W, \Psi, \rho'_1, \rho'_2)^{\sharp(\Delta, \alpha, \delta)}$  in Span(Meas) and  $(\beta, \gamma) \in A \times \overline{\mathbb{R}}$ ,

$$(h^{\sharp}, k^{\sharp}, (\pi_1 \circ l)^{\sharp} \times (\pi_2 \circ l)^{\sharp}) \colon (X, Y, \Phi, \rho_1, \rho_2)^{\sharp(\Delta, \beta, \gamma)} \to (Z, W, \Psi, \rho'_1, \rho'_2)^{\sharp(\Delta, \alpha\beta, \delta+\gamma)}$$

**Inclusions:** For any  $(X, Y, \Phi, \rho_1, \rho_2)$  in Span(Meas), and any  $\alpha \leq \beta$  and  $\delta \leq \gamma$ ,

 $(\mathrm{id}_{GX},\mathrm{id}_{GY},\mathrm{id}_{G\Phi}\times\mathrm{id}_{G\Phi})\colon (X,Y,\Phi,\rho_1,\rho_2)^{\sharp(\Delta,\alpha,\delta)} \to (X,Y,\Phi,\rho_1,\rho_2)^{\sharp(\Delta,\beta,\gamma)}.$ 

**Double strength:** For any  $(X, Y, \Phi, \rho_1, \rho_2)$  and  $(Z, W, \Psi, \rho'_1, \rho'_2)$  in Span(Meas), and parameters

 $(\alpha, \delta)$  and  $(\beta, \gamma)$  in  $A \times \overline{\mathbb{R}}$ , by letting  $\theta_i = \mathsf{dst}_{\Phi, \Psi} \circ (\pi_i \times \pi_i)$  where  $i = 1, 2, \beta$ 

 $(\mathsf{dst}_{X,Z},\mathsf{dst}_{Y,W},\langle\theta_1,\theta_2\rangle)$ 

 $: (X, Y, \Phi, \rho_1, \rho_2)^{\sharp(\Delta, \alpha, \delta)} \times (Z, W, \Psi, \rho_1', \rho_2')^{\sharp(\Delta, \beta, \gamma)} \to (\Phi \times \Psi)^{\sharp(\Delta, \alpha\beta, \delta+\gamma)}.$ 

PROOF SKETCH. Checking of the axioms of graded monad is straightforward since all structures are inherited from the sub-Giry monad  $\mathcal{G}$ . It suffices to prove the well-definedness of the above maps. For example, we check the well-definedness of the Kleisli lifting of a morphism  $(h, k, l): (X, Y, \Phi, \rho_1, \rho_2) \rightarrow (Z, W, \Psi, \rho'_1, \rho'_2)^{\sharp(\Delta, \alpha, \delta)}$  in **Span(Meas**). To prove this, we first show that the third component  $(\pi_1 \circ l)^{\sharp} \times (\pi_2 \circ l)^{\sharp}$  of the Kleisli lifting forms a measurable function from  $W(\Phi, \Delta, \beta, \gamma)$  to  $W(\Psi, \Delta, \alpha\beta, \delta + \gamma)$  by using the composability of  $\Delta$  where measurability is obvious since  $W(\Phi, \Delta, \beta, \gamma)$  and  $W(\Psi, \Delta, \alpha\beta, \delta + \gamma)$  are the subspaces of  $\mathcal{G}\Phi \times \mathcal{G}\Phi$  and  $\mathcal{G}\Psi \times \mathcal{G}\Psi$ . Next, we show  $\mathcal{G}\rho'_1 \circ \pi_1 \circ ((\pi_1 \circ l)^{\sharp} \times (\pi_2 \circ l)^{\sharp}) = h^{\sharp} \circ \rho_1$  and  $\mathcal{G}\rho'_2 \circ \pi_2 \circ ((\pi_1 \circ l)^{\sharp} \times (\pi_2 \circ l)^{\sharp}) = k^{\sharp} \circ \rho_2$ , but this is given from the assumption  $\rho'_1 \circ l = h \circ \rho_1$  and  $\rho'_2 \circ l = k \circ \rho_2$ .

Similary, the well-definedness of functor part and unit are proved by using the composability and reflexivity of  $\Delta$ ; the inclusion is obtained from the definition of *A*-graded family of divergences; the double strength is obtained from the additivity of  $\Delta$ .

#### 5.1 Remark: Adaptive Compositions

Many composition theorems of differential privacy are based on the notion of *k*-fold adaptive composition [Winograd-Cort et al. 2017, Definition 2.3] and [Dwork et al. 2010, Section A]. Roughly speaking, for *k* programs  $q_1, \ldots, q_k$  their *k*-fold adaptive composition  $q_1 \triangleright q_2 \triangleright \cdots \triangleright q_k$  calculates in the following way:

- (1) The first program  $q_1$  takes an input x in X, and returns an output  $y_1$  in  $Y_1$ .
- (2) The second program  $q_2$  takes an input  $x \in X$  and the output  $y_1 \in Y_1$  of the previous program  $q_1$ , and returns an output  $y_2 \in Y_2$ .

<sup>• • •</sup> 

(*k*) The *k*-th program  $q_k$  takes an input  $x \in X$  and the outputs  $y_1, \ldots, y_{k-1}$  of previous programs  $q_1, \ldots, q_{k-1}$ , and returns an output  $y_k \in Y_k$ .

We observe that our definition of composability of divergences covers the *standard* composability with respect to *k*-fold adaptive composition.<sup>6</sup> For example, adaptive composition of two randomized programs can be formulated categorically as follows: let  $f: X \to GY$  and  $f: Y \times X \to GX$  be two randomized programs. The adaptive composition  $f \triangleright g: X \to G(Y \times Z)$  is defined by

 $f \triangleright g = (\operatorname{st}_{Y,Z} \circ (\operatorname{id}_Y \times g) \circ \alpha_{Y,Y,Z})^{\sharp} \circ \operatorname{st}'_{Y \times Y,X} \circ (\mathcal{G}\operatorname{copy}_Y \times \operatorname{id}_X) \circ (f \times \operatorname{id}_X) \circ \operatorname{copy}_X.$ 

Here, st'\_{Y\times Y,X} is the costrength  $\mathcal{G}(Y\times Y)\times X \to \mathcal{G}((Y\times Y)\times X)$ ; copy<sub>X</sub> is the diagonal map  $X \to X\times X$ ( $x \mapsto (x, x)$ ) on X;  $\alpha_{Y,Y,Z}$  is the associativity ( $Y \times Y$ )  $\times Z \to Y \times (Y \times Z)$  of cartesian product of **Meas**. We show that the composability of  $\Delta$  is stronger than the adaptive composability. Suppose that  $\Delta$  reflexive, continuous and composable. Since  $(-)^{\sharp(\Delta,\alpha,\delta)}$  is a graded span-lifting with a double strength, the adaptive composition of the following two morphisms  $(f_1, f_2, f_3) \colon \Phi \to \Psi^{\sharp(\Delta,\alpha,\delta)}$  and  $(g_1, g_2, g_3) \colon \Psi \times \Phi \to \Omega^{\sharp(\Delta,\beta,\gamma)}$  of spans is given by  $(f_1 \triangleright g_1, f_2 \triangleright g_2, l) \colon \Phi \to (\Psi \times \Omega)^{\sharp(\Delta,\alpha\beta,\delta+\gamma)}$  (we omit details of l).

# 5.2 Approximate Span-liftings for DP, RDP, and zCDP

Finally, we build approximate span-liftings for DP, RDP, zCDP, and tCDP by combining Theorems 4.10, 4.11, 4.12, and 4.13 with the construction of categorical structures of approximate span-liftings (Theorem 5.3).

THEOREM 5.4 (APPROXIMATE SPAN-LIFTING FOR DP, RDP, zCDP, TCDP). The following approximate span-liftings are graded liftings with a double strength of  $\mathcal{G} \times \mathcal{G}$  along U: Span(Meas)  $\rightarrow$  Meas×Meas.

Privacy	(Graded family of )Divergence	Approximate span-lifting	Grading Monoid
DP	$\Delta^{DP} = \{\Delta^{DP(\varepsilon)}\}_{0 \le \varepsilon}$	$\{(-)^{\sharp(\Delta^{\mathrm{DP}},\varepsilon,\delta)}\}_{0\leq\varepsilon,0\leq\delta}$	$\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$
RDP	$D^{\alpha}$ (Rényi divergence; see (1))	$\{(-)^{\sharp(D^{\alpha},*,\rho)}\}_{*\in\{*\},\rho\in\overline{\mathbb{R}}}$	$\overline{\mathbb{R}}$
zCDP	$\Delta^{zCDP} = \{\Delta^{zCDP(\xi)}\}_{0 \le \xi}  (see \ (6))$	$\{(-)^{\sharp(\Delta^{zCDP},\xi,\rho)}\}_{0\leq\xi,\rho\in\overline{\mathbb{R}}}$	$\mathbb{R}_{\geq 0}  imes \overline{\mathbb{R}}$
tCDP	$\Delta^{\omega-\text{tCDP}} = \{\Delta^{\omega-\text{tCDP}}\}  (see (7))$	$\{(-)^{\sharp(\Delta^{\omega-tCDP},*,\rho)}\}_{*\in\{*\},\rho\in\overline{\mathbb{R}}}$	$\overline{\mathbb{R}}$

### 6 CASE STUDY: THE PROGRAM LOGIC SPAN-APRHL

The previous section showed that the RDP, zCDP, and tCDP relaxations of differential privacy can be captured by relational liftings with the same categorical properties enjoyed by relational liftings for standard differential privacy. As a result, we can use these liftings to give the semantic foundation for formal verification of these relaxations. To demonstrate a concrete application, we design a program logic span-apRHL that can prove DP, RDP, zCDP, and tCDP for randomized algorithms, supporting both discrete and continuous random samplings.

<sup>&</sup>lt;sup>6</sup>For differential privacy, there are *advanced composition* theorems such as Dwork et al. [2010, Theorem 3.3], Dwork and Roth [2013, Theorem 3.20], which give stronger privacy guarantees.

## 6.1 The Language pWHILE

We take a standard, first-order language pWHILE, augmenting the usual imperative commands with a random sampling statement (we omit the grammar of expressions which is largely standard).

$$\begin{aligned} \tau &::= \text{bool} \mid \text{int} \mid \text{real} \mid \tau^{d} \ (d \in \mathbb{N}) \mid \dots & \text{(basic types)} \\ e &::= x \mid b \in \mathbb{B} \mid n \in \mathbb{Z} \mid r \in \mathbb{R} \mid e_{1} \oplus e_{2} \mid e_{1} \bowtie e_{2} \mid e_{1}[e_{2}] \mid \dots & \text{(expressions)} \\ & \oplus &::= + \mid - \mid * \mid / \mid \min \mid \max \mid \wedge \mid \vee \quad \bowtie ::= \leq \mid \geq \mid = \mid \neq \mid < \mid > \\ v &::= \text{Dirac}(e) \mid \text{Bern}(e) \mid \text{Lap}(e_{1}, e_{2}) \mid \text{Gauss}(e_{1}, e_{2}) \mid \dots & \text{(probabilistic expression)} \\ c &::= \text{skip} \mid x \xleftarrow{\$} v \mid c_{1}; c_{2} \mid \text{if } e \text{ then } c_{1} \text{ else } c_{2} \mid \text{while } e \text{ do } c \end{aligned}$$

Here, *b*, *n*, and *r* are constants;  $\tau$  is a *value type*; *x* is a *variable*; *e* is an *expression*; *v* is a *probabilistic expression*; Dirac, Bern, Lap, and Gauss represent the Dirac, Bernoulli, Laplace, and the Gaussian distributions, respectively; *c* is a *command/program*. We will use the following shorthands:  $x \leftarrow e \stackrel{\text{def}}{=} x \stackrel{\$}{\leftarrow} \text{Dirac}(e)$  and if *b* then  $c \stackrel{\text{def}}{=} i f b$  then *c* else skip. We consider programs that are well typed. The type system is largely standard, with three kinds of judgments:  $\Gamma \vdash^t e: \tau, \Gamma \vdash^p v: \tau$ ,

and  $\Gamma \vdash c$  for expressions, distributions and programs, respectively. For details, see Appendix.

# 6.2 Relational Assertions

Our assertion logic uses formulas of the form

$$\Phi, \Psi ::= \mathcal{E} \mid \Phi \land \Psi \mid \Phi \lor \Psi \mid \neg \Phi$$

where  $\mathcal{E}$  represents basic relational expressions, namely:

$$\mathcal{E} ::= e_1 \langle 1 \rangle \bowtie e_2 \langle 2 \rangle \mid (e_1 \langle 1 \rangle \oplus_1 e_2 \langle 2 \rangle) \bowtie (e_3 \langle 1 \rangle \oplus_2 e_4 \langle 2 \rangle).$$

As usual in relational logics, we use the tags  $\langle 1 \rangle$  and  $\langle 2 \rangle$  to distinguish expressions evaluated in the first and second memory, respectively. For simplicity, we consider only the relations given in the above syntax, the language can be easily extended with other constructions. In the following we will use some syntactic sugar for constant  $k: (e\langle 1 \rangle \bowtie k) \stackrel{\text{def}}{=} (e \bowtie k) \langle 1 \rangle = \text{true} \langle 2 \rangle$ , and  $(e\langle 2 \rangle \bowtie$  $k) \stackrel{\text{def}}{=} \text{true} \langle 1 \rangle = (e \bowtie k) \langle 2 \rangle$ . We consider only relation expression  $\Phi$  that are well-formed in a context  $\Gamma$ , and we denote this by the judgment  $\Gamma \vdash^R \Phi$ . Rules for deriving this kind of judgments are standard, and postponed to Appendix.

Since we use span-liftings instead of relational liftings, we interpret relational assertions as spans, that is, as **Span(Meas)**-objects. This can be done by first interpreting assertions  $\Gamma \vdash^R \Phi$  as binary relations  $\llbracket \Phi \rrbracket \subseteq \llbracket \Gamma \rrbracket \times \llbracket \Gamma \rrbracket$ , and then converting to spans ( $\llbracket \Gamma \rrbracket, \llbracket \Gamma \rrbracket, \pi_1, \pi_2$ ). We describe the semantics of relation assertions in the next section.

We will also use implications of relations  $\Gamma \vdash^{I} \Phi \implies \Psi$ , which is defined when  $\Gamma \vdash^{R} \Phi$  and  $\Gamma \vdash^{R} \Psi$ , and the implication  $\Phi \implies \Psi$  forms a tautology under the typing context  $\Gamma$ . For example, we have the following inclusion, where  $\Gamma \vdash^{t} x$ : real:

$$\Gamma \vdash^{I} ((x\langle 1 \rangle \leq x\langle 2 \rangle) \land (x\langle 1 \rangle \geq x\langle 2 \rangle)) \implies (x\langle 1 \rangle = x\langle 2 \rangle).$$

# 6.3 Relational Program Logic Judgments, Axioms and Rules

In span-apRHL we can prove three kinds of judgments corresponding to differential privacy, RDP, zCDP, and tCDP. For well-typed commands  $\Gamma \vdash c_1$  and  $\Gamma \vdash c_2$  and assertions  $\Gamma \vdash^R \Phi$  and  $\Gamma \vdash^R \Psi$ , we

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

$$\begin{split} \Gamma \vdash x_{1} \leftarrow e_{1} \sim^{\Lambda}_{1_{A},0} x_{2} \leftarrow e_{2} \colon \Phi\{e_{1}\langle 1 \rangle, e_{2}\langle 2 \rangle / x_{1}\langle 1 \rangle, x_{2}\langle 2 \rangle\} \implies \Phi \quad [\text{assn}] \\ \\ \frac{\Gamma \vdash c_{1} \sim^{\Lambda}_{\alpha,\delta} c_{1}' \colon \Phi \implies \Phi' \quad \Gamma \vdash c_{2} \sim^{\Lambda}_{\beta,\gamma} c_{2}' \colon \Phi' \implies \Psi}{\Gamma \vdash c_{1}; c_{2} \sim^{\Lambda}_{\alpha\beta,\delta+\gamma} c_{1}'; c_{2}' \colon \Phi \implies \Psi} \quad [\text{seq}] \\ \\ \frac{\Gamma \vdash^{I} \Phi' \implies \Phi \ \Gamma \vdash^{I} \Psi \implies \Psi' \ \Gamma \vdash c_{1} \sim^{\Lambda}_{\alpha,\delta} c_{2} \colon \Phi \implies \Psi \ \alpha \leq \beta \ \delta \leq \gamma}{\Gamma \vdash c_{1} \sim^{\Lambda}_{\beta,\gamma} c_{2} \colon \Phi' \implies \Psi'} \quad [\text{weak}] \end{split}$$

Fig. 1. Selection of span-apRHL basic rules.

define judgments:

$$\begin{split} & \Gamma \vdash c_1 \sim_{\epsilon,\delta}^{\mathsf{DP}} c_2 \colon \Phi \implies \Psi \quad (\epsilon,\delta) \text{-differential privacy (DP)} \\ & \Gamma \vdash c_1 \sim_{\rho}^{\alpha-\mathsf{RDP}} c_2 \colon \Phi \implies \Psi \quad (\alpha,\rho) \text{-Rényi differential privacy (RDP)} \\ & \Gamma \vdash c_1 \sim_{\xi,\rho}^{\mathsf{zCDP}} c_2 \colon \Phi \implies \Psi \quad (\xi,\rho) \text{-zero-concentrated differential privacy (zCDP)} \\ & \Gamma \vdash c_1 \sim_{\rho}^{\omega-\text{tCDP}} c_2 \colon \Phi \implies \Psi \quad (\omega,\rho) \text{-truncated-concentrated differential privacy (tCDP)} \end{split}$$

We divide the proof rules of span-apRHL in four classes: basic rules (Figure 1), rules for basic mechanisms (Figure 2), rules for reasoning about transitivity (Figure 3), and rules for conversions (Figure 4). The basic rules can be used to reason about either differential privacy, RDP, zCDP, and tCDP. We describe the basic rules in a parametric way by considering  $\{\sim_{\alpha,\delta}^{\Delta}\}_{\alpha \in A, 0 \leq \delta}$  to stand for one of the families  $\{\sim_{\epsilon,\delta}^{DP}\}_{0 \leq \epsilon, 0 \leq \delta}, \{\sim_{\rho}^{\alpha-RDP}\}_{* \in \{*\}, 0 \leq \rho}, \{\sim_{\xi,\rho}^{zCDP}\}_{0 \leq \xi, 0 \leq \rho}, \text{and } \{\sim_{\rho}^{\omega-tCDP}\}_{0 \leq \rho}\}$ . We give a selection of the proof rules in Figure 1; the rest of the rules are standard and we defer them to the appendix. Here, we comment briefly on the rules. The [assn] rule for assignment is mostly standard, the only non-standard aspect is that depending on which notion of privacy we want to use, we need to select the corresponding unit  $1_A$ . The rule [seq] is the sequential composition of commands and takes the same form no matter which family of divergence we consider. The rule [weak] is our version of the usual consequence rule, where additionally we can weaken also the privacy parameters for each of the privacy definitions.

In Figure 2, we show some rules for the basic mechanisms that we support: Bernoulli, Laplace, and Gauss. We give several of them to show the difference, in terms of the parameters, for the same mechanism, that we have in the different logics. All of them are supported in the continuous case. We show only DP rules for Bernoulli and Laplace mechanisms, and postpone other Bernoulli and Laplace mechanism rules to the Appendix.

In Figure 3, we show rules for transitivity in span-apRHL. Transitivity is important because it allows one to reason about group privacy [Dwork and Roth 2013]. The different flavors of the logic have different numeric parameters for these rules, reflecting the slight differences in group privacy [Bun and Steinke 2016; Dwork and Roth 2013; Mironov 2017]. Finally, Figure 4 gives rules for converting between judgments for different flavors of differential privacy. In some of them we have a loss in the parameters, in others there is no loss. These rules correspond to the different conversion theorems for the different logics [Bun and Steinke 2016; Mironov 2017]. Notice that most of these rules require lossless programs because they have been formulated in terms of distributions, rather than subdistributions.

$$\begin{split} \Gamma \vdash x_1 & \stackrel{\$}{\leftarrow} \operatorname{Bern}(e_1) \sim^{\operatorname{DP}}_{\log\max(p, 1-p) - \log\min(p, 1-p), 0} x_2 & \stackrel{\$}{\leftarrow} \operatorname{Bern}(e_2): \\ & ((e_1 \langle 1 \rangle = p) \land (1 - e_1 \langle 1 \rangle = e_2 \langle 2 \rangle) \implies (x_1 \langle 1 \rangle = x_2 \langle 2 \rangle) \end{split} \tag{DP-Bern}$$

$$\Gamma \vdash x_1 \stackrel{\$}{\leftarrow} \operatorname{Bern}(e_1) \sim_{0,0}^{\operatorname{DP}} x_2 \stackrel{\$}{\leftarrow} \operatorname{Bern}(e_2):$$

$$(e_1\langle 1 \rangle = e_2\langle 2 \rangle) \implies (x_1\langle 1 \rangle = x_2\langle 2 \rangle)$$
[DP-Bern-Eq]

$$\Gamma \vdash x_1 \xleftarrow{} \mathsf{Lap}(e_1, \lambda) \sim_{r/\lambda, 0}^{\mathsf{DP}} x_2 \xleftarrow{} \mathsf{Lap}(e_2, \lambda):$$

$$(|e_1\langle 1 \rangle - e_2\langle 2 \rangle| \le r) \implies (x_1\langle 1 \rangle = x_2\langle 2 \rangle)$$
[DP-Lap]

$$\Gamma \vdash x_1 \xleftarrow{\$} \mathsf{Gauss}(e_1, \sigma^2) \sim_{\alpha r^2/2\sigma^2}^{\alpha - \mathsf{RDP}} x_2 \xleftarrow{\$} \mathsf{Gauss}(e_2, \sigma^2) (|e_1\langle 1 \rangle - e_2\langle 2 \rangle| \le r) \implies (x_1\langle 1 \rangle = x_2\langle 2 \rangle)$$
[RDP-G]

$$\Gamma \vdash x_1 \stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_1, \sigma^2) \sim_{0, r^2/2\sigma^2}^{\mathsf{zCDP}} x_2 \stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_2, \sigma^2) \\ (|e_1\langle 1 \rangle - e_2\langle 2 \rangle| \le r) \implies (x_1\langle 1 \rangle = x_2\langle 2 \rangle)$$
[zCDP-G]

$$\Gamma \vdash x_1 \stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_1, \sigma^2) \sim_{0, r^2/2\sigma^2}^{\mathsf{tCDP}} x_2 \stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_2, \sigma^2) \\ (|e_1\langle 1 \rangle - e_2\langle 2 \rangle| \le r) \implies (x_1\langle 1 \rangle = x_2\langle 2 \rangle)$$
 [tCDP-G]

$$\frac{\exists c > \frac{1+\sqrt{3}}{2}. \left(2\log(0.66/\delta) \le c^2\right) \land \left(\frac{cr}{\varepsilon} \le \sigma\right)}{\left(\frac{cr}{\varepsilon} \le \sigma\right)}$$
[DP-G]

$$\begin{split} \Gamma \vdash x_{1} &\stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_{1}, \sigma^{2}) \sim^{\mathsf{DP}}_{e, \delta} x_{2} \stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_{2}, \sigma^{2}): \\ & (|e_{1}\langle 1 \rangle - e_{2}\langle 2 \rangle| \leq r) \implies (x_{1}\langle 1 \rangle = x_{2}\langle 2 \rangle) \\ & 1 < 1/\sqrt{\rho} \leq A/\delta \\ \hline \Gamma \vdash x_{1} \stackrel{\$}{\leftarrow} e_{1} + A \cdot \operatorname{arsinh}\left(\frac{1}{A}\mathsf{Gauss}(0, \delta^{2}/2\rho)\right) \end{split}$$
 [tCDP-SinhG]

$$\sim_{16\rho}^{A/8\delta-\text{tCDP}} x_2 \stackrel{\$}{\leftarrow} e_2 + A \cdot \operatorname{arsinh}\left(\frac{1}{A}\operatorname{Gauss}(0, \delta^2/2\rho)\right):$$
$$(|e_1\langle 1\rangle - e_2\langle 2\rangle| \le \delta) \implies (x_1\langle 1\rangle = x_2\langle 2\rangle)$$

# Fig. 2. Rules for basic mechanisms for DP, RDP, zCDP, and tCDP in span-apRHL.

$$\frac{\Gamma + c_{1} \sim_{\epsilon_{1},\delta_{1}}^{DP} c_{2}: \Phi \implies x_{1}\langle 1 \rangle = x_{2}\langle 2 \rangle \quad \Gamma + c_{2} \sim_{\epsilon_{2},\delta_{2}}^{DP} c_{3}: \Psi \implies x_{2}\langle 1 \rangle = x_{3}\langle 2 \rangle}{\Gamma + c_{1} \sim_{\rho_{1}}^{DP} c_{2}: \Phi \implies x_{1}\langle 1 \rangle = x_{2}\langle 2 \rangle} \qquad [DP-Trans]$$

$$\frac{\Gamma + c_{1} \sim_{\rho_{1}}^{p\alpha-RDP} c_{2}: \Phi \implies x_{1}\langle 1 \rangle = x_{2}\langle 2 \rangle}{\Gamma + c_{2} \sim_{\rho_{2}}^{q(p\alpha-1)/p-RDP} c_{3}: \Psi \implies x_{2}\langle 1 \rangle = x_{3}\langle 2 \rangle \quad \frac{1}{p} + \frac{1}{q} = 1 \quad 1 
$$\frac{\Gamma + c_{1} \sim_{\xi,\rho}^{\alpha-RDP} c_{3}: \Psi \implies x_{2}\langle 1 \rangle = x_{3}\langle 2 \rangle \quad \frac{1}{p} + \frac{1}{q} = 1 \quad 1 
$$\frac{\Gamma + c_{1} \sim_{\xi,\rho}^{zCDP} c_{3}: \Psi \implies x_{2}\langle 1 \rangle = x_{3}\langle 2 \rangle \quad k \in \mathbb{N} \quad 1 < k}{\Gamma + c_{1} \sim_{\xi,\rho}^{zCDP} c_{3}: \Phi \oplus \Psi \implies x_{1}\langle 1 \rangle = x_{3}\langle 2 \rangle} \qquad [zCDP-Trans]$$$$$$

#### Fig. 3. Span-apRHL transitivity rules for group privacy

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

$$\frac{\Gamma + c_{1} \sim_{\epsilon, 0}^{\mathsf{DP}} c_{2} : \Phi \implies \Psi \quad c_{1}, c_{2} : \text{lossless}}{\Gamma + c_{1} \sim_{\epsilon, 0}^{\mathsf{2CDP}} c_{2} : \Phi \implies \Psi \quad c_{1}, c_{2} : \text{lossless}} \quad [D/z]$$

$$\frac{\Gamma + c_{1} \sim_{\epsilon, 0}^{\mathsf{2CDP}} c_{2} : \Phi \implies \Psi}{\forall \alpha > 1. \ \Gamma + c_{1} \sim_{\rho}^{\alpha - \mathsf{RDP}} c_{2} : \Phi \implies \Psi} \quad [z/R]$$

$$\frac{\Gamma + c_{1} \sim_{\xi, \rho}^{\mathsf{2CDP}} c_{2} : \Phi \implies \Psi \quad c_{1}, c_{2} : \text{lossless} \quad 0 < \delta < 1}{\Gamma + c_{1} \sim_{\rho}^{\mathsf{DP}} c_{2} : \Phi \implies \Psi} \quad [z/D]$$

$$\frac{\Gamma + c_{1} \sim_{\rho}^{\mathsf{DP}} c_{2} : \Phi \implies \Psi \quad c_{1}, c_{2} : \text{lossless} \quad 0 < \delta < 1}{\Gamma + c_{1} \sim_{\rho}^{\mathsf{DP}} c_{2} : \Phi \implies \Psi} \quad [z/D]$$

$$\frac{\Gamma + c_{1} \sim_{\rho}^{\mathsf{DP}} c_{2} : \Phi \implies \Psi, c_{1}, c_{2} : \text{lossless}, \beta = \min(\omega, 1 + \sqrt{\log(1/\delta)/\rho}), 0 < \delta < 1}{\Gamma + c_{1} \sim_{\rho}^{\mathsf{DP}} c_{1} : \delta : c_{2} : \Phi \implies \Psi} \quad [t/D]$$

Fig. 4. Rules for conversions between DP, RDP and zCDP in span-apRHL.

#### 6.4 Denotational Semantics of pWHILE

To prove the soundness of span-apRHL we interpret pWHILE in **Meas** using the sub-Giry monad  $\mathcal{G}$ . Most of the definitions are standard. The value types are interpreted as expected. To give a semantics to expressions, distribution expressions, and commands, we interpret their associated typing/well-formedness judgments in some context  $\Gamma$ , which is interpreted as usual as a product. We interpret an expression judgment  $\Gamma \vdash^t e: \tau$  as a measurable function  $\llbracket \Gamma \vdash^t e: \tau \rrbracket$ .  $\llbracket \Gamma \rrbracket \to \llbracket \tau \rrbracket$ , for instance, the variable case  $\Gamma \vdash^t x: \tau$  is interpreted as the projection  $\pi_x: \llbracket \Gamma \rrbracket \to \llbracket \tau \rrbracket$ . Note that all operators  $\oplus$  and comparisons  $\bowtie$  are interpreted to measurable functions  $\oplus: \llbracket \tau \rrbracket \to \llbracket \tau \rrbracket$  and  $\bowtie: \llbracket \tau \rrbracket \times \llbracket \tau \rrbracket \to \llbracket bool \rrbracket$  respectively. Likewise, we interpret a distribution expression judgment  $\Gamma \vdash^p v: \tau$  as a measurable function expression judgment  $\Gamma \vdash^p v: \tau$  as a measurable function expression judgment  $\Gamma \vdash^p v: \tau$  as a measurable function expression  $\exists \tau \rrbracket \to \llbracket \tau \rrbracket$  and  $\bowtie: \llbracket \tau \rrbracket \to \llbracket bool \rrbracket$  respectively. Likewise, we interpret a distribution expression judgment  $\Gamma \vdash^p v: \tau$  as a measurable function  $\llbracket \Gamma \vdash^r e_1: real \rrbracket, \llbracket \Gamma \vdash \to \mathcal{G} \llbracket \tau \rrbracket$ ; for instance, the Gaussian expression  $\Gamma \vdash^p v: \tau \rrbracket: \llbracket \Gamma \rrbracket \to \mathcal{G} \llbracket \tau \rrbracket$ ; for instance, the Gaussian expression  $\Gamma \vdash^p v: \tau \rrbracket$ :  $\llbracket \Gamma \amalg \to \mathcal{G} \llbracket \tau \rrbracket$ ; for instance, the Gaussian expression  $\Gamma \vdash^p Gauss(e_1, e_2): real is interpreted as a Gaussian distribution. <math>\mathcal{N}(\llbracket \Gamma \vdash^r e_1: real \rrbracket, \llbracket \Gamma \vdash \to \mathcal{G} \llbracket \Gamma \rrbracket$  defined inductively as

$$\begin{split} \llbracket \Gamma \vdash x \xleftarrow{\$} \nu \rrbracket &= \mathcal{G}(\operatorname{rw}\langle \Gamma \mid x \colon \tau \rangle) \circ \operatorname{st}_{\llbracket \Gamma \rrbracket, \llbracket \tau \rrbracket} \circ \langle \operatorname{id}_{\llbracket \Gamma \rrbracket}, \llbracket \nu \rrbracket \rangle, \qquad \llbracket \Gamma \vdash c_1; c_2 \rrbracket = \llbracket \Gamma \vdash c_2 \rrbracket^{\sharp} \circ \llbracket \Gamma \vdash c_1 \rrbracket, \\ \llbracket \Gamma \vdash \operatorname{skip} \rrbracket &= \eta_{\llbracket \Gamma \rrbracket} \qquad \llbracket \Gamma \vdash \operatorname{if} b \text{ then } c_1 \text{ else } c_2 \rrbracket = [\llbracket \Gamma \vdash c_1 \rrbracket, \llbracket \Gamma \vdash c_2 \rrbracket] \circ \operatorname{br}\langle \Gamma \rangle \circ \langle \llbracket \Gamma \vdash b \rrbracket, \operatorname{id}_{\llbracket \Gamma \rrbracket} \rangle \end{split}$$

Here,  $\operatorname{rw}(\Gamma \mid x: \tau) \colon \llbracket \Gamma \rrbracket \times \llbracket x: \tau \rrbracket \to \llbracket \Gamma \rrbracket (x: \tau \in \Gamma)$  is an overwriting operation of memory  $((a_1, \ldots, a_k, \ldots, a_n), b_k) \mapsto (a_1, \ldots, b_k, \ldots, a_n)$ , which is given from the Cartesian products in **Meas**. The function  $\operatorname{br}(\Gamma) : 2 \times \llbracket \Gamma \rrbracket \to \llbracket \Gamma \rrbracket + \llbracket \Gamma \rrbracket$  comes from the canonical isomorphism  $2 \times \llbracket \Gamma \rrbracket \cong \llbracket \Gamma \rrbracket + \llbracket \Gamma \rrbracket$  given from the distributivity of **Meas**.

To interpret loops, we introduce the dummy "abort" command  $\Gamma \vdash \text{null}$  that is interpreted by the null/zero measure  $[\Gamma \vdash \text{null}] = 0$ , and the following commands corresponding to the finite

unrollings of the loop:

$$[\text{while } b \text{ do } c]_n = \begin{cases} \text{if } b \text{ then null else skip,} & \text{if } n = 0\\ \text{if } b \text{ then } c; [\text{while } b \text{ do } c]_k, & \text{if } n = k + 1 \end{cases}$$

We then interpret loops as:  $[\Gamma \vdash while b \text{ do } c] = \sup_{n \in \mathbb{N}} [\Gamma \vdash [while e \text{ do } c]_n]$ . This is well-defined, since the family  $\{[\Gamma \vdash [while e \text{ do } c]_n]\}_{n \in \mathbb{N}}$  is an  $\omega$ -chain with respect to the  $\omega CPO_{\perp}$ -enrichment  $\sqsubseteq$  of **Meas**<sub> $\mathcal{G}$ </sub>.

# 6.5 Semantics of Relations

Since we use span-liftings instead of relational liftings, we need to interpret relation expressions to spans, that is, **Span(Meas)**-objects. We proceed in two steps: first interpreting expressions as binary relations, and then converting relations to spans. In the first step, we interpret a relation expression  $\Gamma \vdash^R \Phi$  as a binary relation over  $\llbracket \Gamma \rrbracket$ :

$$\begin{split} \left( \Gamma \vdash^{R} e_{1}\langle 1 \rangle \bowtie e_{2}\langle 2 \rangle \right) \\ &= \left\{ \left( m_{1}, m_{2} \right) \in \left[ \Gamma \right] \times \left[ \Gamma \right] \mid \left[ \Gamma \vdash^{t} e_{1} : \tau \right] (m_{1}) \bowtie \left[ \Gamma \vdash^{t} e_{2} : \tau \right] (m_{2}) \right\} \\ \left( \Gamma \vdash^{R} \left( e_{1}\langle 1 \rangle \otimes_{1} e_{2}\langle 2 \rangle \right) \bowtie \left( e_{3}\langle 1 \rangle \otimes_{2} e_{4}\langle 2 \rangle \right) \right) \\ &= \left\{ \left( m_{1}, m_{2} \right) \in \left[ \Gamma \right] \times \left[ \Gamma \right] \mid \left[ \Gamma \vdash^{t} e_{1} : \tau \right] (m_{1}) \otimes_{1} \left[ \Gamma \vdash^{t} e_{2} : \tau \right] (m_{2}) \\ & \bowtie \left[ \Gamma \vdash^{t} e_{3} : \tau \right] (m_{1}) \otimes_{2} \left[ \Gamma \vdash^{t} e_{4} : \tau \right] (m_{2}) \right\} \end{split}$$

We interpret the connectives in the expected way:

$$(\Gamma \vdash^{R} \Phi \land \Psi) = (\Gamma \vdash^{R} \Phi) \cap (\Gamma \vdash^{R} \Psi) \qquad (\Gamma \vdash^{R} \Phi \lor \Psi) = (\Gamma \vdash^{R} \Phi) \cup (\Gamma \vdash^{R} \Psi)$$
$$(\Gamma \vdash^{R} \neg \Phi) = (\llbracket \Gamma \rrbracket \times \llbracket \Gamma \rrbracket) \setminus (\Gamma \vdash^{R} \Phi)$$

Then, we can convert the binary relation  $(\Gamma \vdash^R \Phi) \subseteq [\Gamma] \times [\Gamma]$  to the span

$$\llbracket \Gamma \vdash^{R} \Phi \rrbracket = (\llbracket \Gamma \rrbracket, \llbracket \Gamma \rrbracket, \llbracket \Gamma \rrbracket, \langle \Gamma \vdash^{R} \Phi \rangle, \pi_{1}|_{\langle \Gamma \vdash^{R} \Phi \rangle}, \pi_{2}|_{\langle \Gamma \vdash^{R} \Phi \rangle}).$$

We interpret the implication  $\Gamma \vdash^{I} \Phi \implies \Psi$  by the following morphism in **Span**(**Meas**):

$$\llbracket \Gamma \vdash^{I} \Phi \implies \Psi \rrbracket = (\mathrm{id}_{\llbracket \Gamma \rrbracket}, \mathrm{id}_{\llbracket \Gamma \rrbracket}, (\mathrm{id}_{\llbracket \Gamma \rrbracket} \times \mathrm{id}_{\llbracket \Gamma \rrbracket})|_{(\Gamma \vdash^{R} \Phi)}) \colon \llbracket \Gamma \vdash^{R} \Phi \rrbracket \rightarrow \llbracket \Gamma \vdash^{R} \Psi \rrbracket$$

## 6.6 Validity of Judgments

We say a judgment  $\Gamma \vdash c_1 \sim^{\Delta}_{\alpha,\delta} c_2 \colon \Phi \implies \Psi$  is valid if there exists a measurable function  $l \colon [\![\Gamma \vdash^R \Phi]\!] \to W([\![\Gamma \vdash^R \Psi]\!], \Delta, \alpha, \delta)$  (we call it a *witness function*) such that

$$(\llbracket\Gamma \vdash c_1\rrbracket, \llbracket\Gamma \vdash c_2\rrbracket, l) \colon \llbracket\Gamma \vdash^R \Phi\rrbracket \to \llbracket\Gamma \vdash^R \Psi\rrbracket^{\sharp(\Delta, \alpha, \delta)}$$

is a morphism in Span(Meas). Concretely, we define the validity in span-apRHL as follows:

$$\begin{split} \Gamma &\models c_1 \sim_{\varepsilon,\delta}^{\mathsf{DP}} c_2 \colon \Phi \implies \Psi \text{ iff } \exists l. \left( [\![\Gamma \vdash c_1]\!], [\![\Gamma \vdash c_2]\!], l \right) \colon [\![\Gamma \vdash^R \Phi]\!] \rightarrow [\![\Gamma \vdash^R \Psi]\!]^{\sharp(\Delta^{\mathsf{DP}}, \varepsilon, \delta)}, \\ \Gamma &\models c_1 \sim_{\rho}^{\alpha - \mathsf{RDP}} c_2 \colon \Phi \implies \Psi \text{ iff } \exists l. \left( [\![\Gamma \vdash c_1]\!], [\![\Gamma \vdash c_2]\!], l \right) \colon [\![\Gamma \vdash^R \Phi]\!] \rightarrow [\![\Gamma \vdash^R \Psi]\!]^{\sharp(D^{\alpha}, *, \rho)}, \\ \Gamma &\models c_1 \sim_{\xi,\rho}^{\mathsf{2CDP}} c_2 \colon \Phi \implies \Psi \text{ iff } \exists l. \left( [\![\Gamma \vdash c_1]\!], [\![\Gamma \vdash c_2]\!], l \right) \colon [\![\Gamma \vdash^R \Phi]\!] \rightarrow [\![\Gamma \vdash^R \Psi]\!]^{\sharp(\Delta^{\mathsf{2CDP}}, \xi, \rho)} \\ \Gamma &\models c_1 \sim_{\rho}^{\omega - \mathsf{tCDP}} c_2 \colon \Phi \implies \Psi \text{ iff } \exists l. \left( [\![\Gamma \vdash c_1]\!], [\![\Gamma \vdash c_2]\!], l \right) \colon [\![\Gamma \vdash^R \Phi]\!] \rightarrow [\![\Gamma \vdash^R \Psi]\!]^{\sharp(\Delta^{\omega - \mathsf{tCDP}}, *, \rho)}. \end{split}$$

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

#### 6.7 Soundness

THEOREM 6.1. If  $\Gamma \vdash c_1 \sim^{\Delta}_{\alpha, \delta} c_2 \colon \Phi \implies \Psi$  is derivable in span-apRHL, then it is valid.

PROOF SKETCH. The soundness of the basic rules is derived from the unit, graded Kleisli liftings, and inclusions of the graded span-lifting  $\{(-)^{\sharp(\Delta, \alpha, \delta)}\}_{\alpha, \delta}$  given in Section 5. We focus here on the soundness of the [seq] rule. Since the judgments  $\Gamma \vdash c_1 \sim^{\Delta}_{\alpha, \delta} c'_1 \colon \Phi \implies \Phi'$  and  $\Gamma \vdash c_2 \sim^{\Delta}_{\alpha\beta, \delta+\gamma} c'_2 \colon \Phi' \implies \Psi$  are valid, for some witness functions  $l_1$  and  $l_2$  we have

$$(\llbracket \Gamma \vdash c_1 \rrbracket, \llbracket \Gamma \vdash c'_1 \rrbracket, l_1) \colon \llbracket \Phi \rrbracket \to \llbracket \Phi' \rrbracket^{\sharp(\Delta, \alpha, \delta)}, (\llbracket \Gamma \vdash c_2 \rrbracket, \llbracket \Gamma \vdash c'_2 \rrbracket, l_2) \colon \llbracket \Phi' \rrbracket \to \llbracket \Psi \rrbracket^{\sharp(\Delta, \beta, \gamma)}.$$

By taking the graded Kleisli extension of the second morphism  $(\llbracket \Gamma \vdash c_2 \rrbracket, \llbracket \Gamma \vdash c'_2 \rrbracket, l_2)$ , for some witness function  $l_3$  given by the construction in Theorem 5.3 (Kleisli lifting), we have the following morphism in the category **Span**(**Meas**):

$$(\llbracket \Gamma \vdash c_2 \rrbracket^{\sharp}, \llbracket \Gamma \vdash c'_2 \rrbracket^{\sharp}, l_3) \colon \llbracket \Phi' \rrbracket^{\sharp(\Delta, \alpha, \delta)} \to \llbracket \Psi \rrbracket^{\sharp(\Delta, \alpha\beta, \delta+\gamma)}.$$

Composing them, we conclude the validity of  $\Gamma \vdash c_1; c_2 \sim^{\Delta}_{\alpha\beta,\delta+\gamma} c'_1; c'_2: \Phi \implies \Psi.$ 

The soundness of the mechanism rules are proved by interpreting known results of mechanisms for DP, RDP, zCDP, and tCDP to span-liftings. For example, the soundness of [RDP-G] proved by interpreting the Rényi differential privacy of Gaussian mechanism to span-liftings. First, the function  $f = \mathcal{N}(-, \sigma^2) \colon \mathbb{R} \to \mathcal{G}\mathbb{R}$  describing a Gaussian mechanism is measurable. From the previous result Mironov [2017, Proposition 7] of Rényi differential privacy of the Gaussian mechanism, the measurable function f satisfies the following implication:

$$|x - y| \le r \implies D^{\alpha}(f(x)||f(y)) \le \alpha r^2/2\sigma^2$$

This implies that we have the below morphism in the category **Span**(**Meas**):

$$(f, f, (f \times f)|_{\Phi}): \{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid |x - y| \le r \} \to \mathrm{Eq}_{\mathbb{R}}^{\#(D^{\alpha}, *, \alpha r^{2}/2\sigma^{2})}.$$

From this, by straightforward calculations, we obtain the soundness of [RDP-G].

Note that we need to give *measurable* functions l selecting witness distributions when proving these rules—in the discrete case, these functions can be obtained by the axiom of choice. In the case of [RDP-G], we could give the witness  $l = f \times f$  directly.

Similary, the soundness of the rest of mechanism rules follows from the following previous results on DP, RDP, zCDP and tCDP: Mironov [2017, Propositions 6], Dwork et al. [2006, Proposition 1], Sato [2016, Lemma 4.2] (an enhancement of Dwork and Roth [2013, Theorem 3.22]), and Bun et al. [2018, Theorem 19], and the soundness of transitive rules follows from: Olmedo [2014, Lemma 4.2(iii)], Bun and Steinke [2016, Proposition 27] and Langlois et al. [2014, Lemma 4.1]. The soundness of the conversion rules follows by applying the comparison theorems of divergences Bun and Steinke [2016, Proposition 4], Mironov [2017, Proposition 3], Bun and Steinke [2016, Lemma 8.2, 3.5], Bun et al. [2018, Lemma 8] to the following inclusion between the approximate span-liftings:

$$(\Delta^1_{\alpha} \le \delta \implies \Delta^2_{\beta} \le \gamma) \implies ((\mathrm{id}, \mathrm{id}) \colon (\Phi)^{\sharp(\Delta^1, \alpha, \delta)} \to (\Phi)^{\sharp(\Delta^2, \beta, \gamma)} \text{ in } \mathrm{Span}(\mathrm{Meas})).$$

#### 7 VERIFICATION EXAMPLES

We show how we can use the span-pRHL program logic to verify concrete programs. We stress an important point here, since the guarantees provided by RDP, zCDP, and tCDP can all be converted in guarantees about ( $\epsilon$ ,  $\delta$ )-differential privacy, one could just use the latter for analyze all the examples we will show. The interest however in performing as much reasoning as possible using these

relaxations is that one can achieve better values of the parameters. This will become particularly evident in the last example.

# 7.1 One-way Marginals

As a warm up, we begin with the following classic example of a one-way marginal algorithm with additive noise.

Algorithm 1 A mechanism estimates the attribute means

1: procedure AttMean(n: int,  $\rho$ : real (const.), x: bool<sup>n</sup> (dataset), i: int, y, z, w: real) 2:  $i \leftarrow 0; y \leftarrow 0;$ 3: while i < n do 4:  $y \leftarrow y + x[i]; i \leftarrow i + 1;$ 5:  $z \leftarrow y/n;$ 6:  $w \leftarrow Gauss(z, 1/2n^2\rho);$ 

We first show the Rényi-differential privacy of AttMean. We set a typing context  $\Gamma$  of AttMean by  $x: bool^n$  (dataset), i: int, and y, z, w: real. We show the following judgment:

$$\Gamma \vdash \text{AttMean} \sim_{\alpha \rho}^{\text{RDP}} \text{AttMean}: \operatorname{adj}(x\langle 1 \rangle, x\langle 2 \rangle) \implies w\langle 1 \rangle = w\langle 2 \rangle.$$

Here, the adjacent relation adj(x(1), x(2)) means that two datasets x(1) and x(2) differs at most in one record. Explicitly, we define it by the following relation expression:

$$\operatorname{adj}(x\langle 1\rangle, x\langle 2\rangle) = \bigwedge_{1 \le i \le n} \left( (x[i]\langle 1\rangle \ne x[i]\langle 2\rangle) \implies \bigwedge_{1 \le j < i, i < j \le n} (x[j]\langle 1\rangle = x[j]\langle 2\rangle) \right).$$

The proof of this judgment follows by splitting AttMean into two commands LoopAM; NoiseG where NoiseG =  $w \leftarrow \text{Gauss}(z, 1/2n^2\rho)$ , and LoopAM is the rest of the program. Since the loop part LoopAM is deterministic, by standard reasoning, we obtain:

$$\Gamma \vdash \mathsf{LoopAM} \sim_0^{\alpha - \mathsf{RDP}} \mathsf{LoopAM}: \mathsf{adj}(x\langle 1 \rangle, x\langle 2 \rangle) \implies (|z\langle 1 \rangle - z\langle 2 \rangle| \le 1/n)$$

By applying [RDP-G], for the noise-adding step NoiseG we have:

$$\Gamma \vdash \text{NoiseG} \sim_{\alpha\rho}^{\alpha-\text{RDP}} \text{NoiseG:} (|z\langle 1\rangle - z\langle 2\rangle| \le 1/n) \implies (w\langle 1\rangle = w\langle 2\rangle).$$

Thus, by applying [seq] we complete the proof. A similar proof could have been carried out with both the rules for differential privacy, zCDP, and tCDP. Due to the simplicity of the example (that is, LoopAM is deterministic), the resulting guarantee would have been the same.

Algorithm 2 A mechanism estimates the attribute means with SinhNormal noise

1: **procedure** AMSinh(*n*: int,  $\rho$ : real (const.), *x*: bool<sup>*n*</sup> (dataset), *i*: int, *y*, *z*, *w*: real) 2:  $i \leftarrow 0; y \leftarrow 0;$ 3: **while** i < n **do** 4:  $y \leftarrow y + x[i]; i \leftarrow i + 1;$ 5:  $z \leftarrow y/n;$ 6:  $w \stackrel{\$}{\leftarrow} w + A \cdot \operatorname{arsinh}\left(\frac{1}{A}\operatorname{Gauss}(0, /2n^2\rho)\right);$ 

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

We change the noise in the algorithm AttMean from Gaussian noise to SinhNormal noise. Explicitly, we define a new algorithm AMSinh = LoopAM; NoiseSinh where the noise-adding part is changed to NoiseSinh =  $w \leftarrow w + A \cdot \operatorname{arsinh}\left(\frac{1}{A}\operatorname{Gauss}(0, /2n^2\rho)\right)$ , where A is a constant satisfying  $1 < 1/\sqrt{\rho} \leq A/n$ . In the similar way as the previous example AttMean, for the loop part LoopAM, we obtain:

 $\Gamma \vdash \text{LoopAM} \sim_{0}^{n \cdot A/8 - \text{tCDP}} \text{LoopAM}: \text{adj}(x\langle 1 \rangle, x\langle 2 \rangle) \implies (|z\langle 1 \rangle - z\langle 2 \rangle| \leq 1/n).$ 

By applying [tCDP-SinhG], the noise-adding part NoiseSinh satisfies

 $\Gamma \vdash \text{NoiseSinh} \sim_{16\rho}^{n \cdot A/8 - \text{tCDP}} \text{NoiseSinh} \colon (|z\langle 1\rangle - z\langle 2\rangle| \le 1/n) \implies (w\langle 1\rangle = w\langle 2\rangle).$ 

Thus, by applying [seq], we conclude that the algorithm AMSinh is  $(16\rho, n \cdot A/8)$ -tCDP.

# 7.2 Histograms

The following algorithm gives the histograms of dataset x over the finite set T with additive noise. We use a primitive data type T as a finite set of size T.

# Algorithm 3 A mechanism estimates the histogram

1: **procedure** Histogram(nint,  $\rho$ : real (const.), x:  $[T]^n$  (dataset), y, z: real<sup>T</sup>, i: int) 2:  $i \leftarrow 0; y \leftarrow (0, \dots, 0);$ 3: **while** i < n **do** 4:  $y[x[i]] \leftarrow y[x[i]] + 1; i \leftarrow i + 1;$ 5:  $i \leftarrow 0; z \leftarrow (0, \dots, 0);$ 6: **while** i < T **do** 7:  $z[i] \stackrel{\$}{\leftarrow} Gauss(y[i], 1/\rho); i \leftarrow i + 1;$ 

We show the zCDP of the algorithm Histogram. We set a typing context  $\Gamma$  by  $x: [T]^n$  (dataset),  $y, z: \text{real}^T$ , and i: int. We want to prove the validity of the following judgment:

 $\Gamma \vdash \text{Histogram} \sim_{0,\rho}^{\text{zCDP}} \text{Histogram} : \text{adj}(x\langle 1 \rangle, x\langle 2 \rangle) \implies z\langle 1 \rangle = z\langle 2 \rangle.$ 

Here,  $\operatorname{adj}(x\langle 1 \rangle, x\langle 2 \rangle)$  is defined in the similar way as the previous algorithm. We split the algorithm Histogram into Histogram = HGCalc; HGNoise where HGNoise is the second loop for adding noise, and HGCalc is the rest of the program that calculates a histogram without noise. We can now define two additional assertions for  $0 \le K \ne L < T$  and  $0 \le I < n$ :

$$\Phi_{I,K,L} = (x[I]\langle 1 \rangle \neq x[I]\langle 2 \rangle) \land (i \neq I \implies x[i]\langle 1 \rangle = x[i]\langle 2 \rangle) \land (x[I]\langle 1 \rangle = K) \land (x[I]\langle 2 \rangle = L)$$

$$\Psi_{K,L} = (y[K]\langle 1 \rangle = y[K]\langle 2 \rangle + 1) \land (y[L]\langle 1 \rangle + 1 = y[L]\langle 2 \rangle) \land (j \neq K,L \implies y[j]\langle 1 \rangle = y[j]\langle 2 \rangle).$$

It is easy to see that  $\operatorname{adj}(x\langle 1 \rangle, x\langle 2 \rangle) \iff \exists I, K, L. \Phi_{I,K,L}$ . Using this and some standard reasoning, we have

$$\Gamma \vdash \mathsf{HGCalc} \sim_{0,0}^{\mathsf{zCDP}} \mathsf{HGCalc} \colon \Phi(I,K,L)(x\langle 1\rangle,x\langle 2\rangle) \implies \Theta(K,L) \land (i\langle 1\rangle = 0)$$

where  $\Theta(K, L) = \Psi(K, L) \land (z\langle 1 \rangle = z\langle 2 \rangle) \land (i\langle 1 \rangle = i\langle 2 \rangle)$ . For proving the right judgment for HGNoise we also use the following additional axiom for zCDP that concludes (0, 0)-zCDP if both noises and inputs are the same (the soundness is rather straightforward):

$$\Gamma \vdash x_1 \stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_1, \sigma^2) \sim_{0,0}^{\mathsf{zCDP}} x_2 \stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_2, \sigma^2) \colon (e_1 \langle 1 \rangle = e_2 \langle 2 \rangle) \implies (x_1 \langle 1 \rangle = x_2 \langle 2 \rangle).$$

Now by using this axiom, [zCDP-G], and some basic reasoning for the loop we obtain:

 $\Gamma \vdash \mathsf{HGNoise} \sim^{\mathsf{ZCDP}}_{0,\rho} \mathsf{HGNoise} \colon \Theta(K,L) \land (i\langle 1 \rangle = 0) \implies \Theta(K,L) \land (i\langle 1 \rangle = T).$ 

Roughly speaking, we may regard HGNoise as a composition  $c[0]; c[1]; \dots; c[T-1]$  where c[j] is the *j*-th execution of the loop body of HGNoise. For  $j \neq K, L$  by using the new axiom,

$$\Gamma \vdash c[j] \sim_{0,0}^{\mathsf{zCDP}} c[j] \colon \Theta(K,L) \land (i\langle 1 \rangle = j) \implies \Theta(K,L) \land (i\langle 1 \rangle = j+1).$$

On the other hand, for j = K, L by applying [zCDP-G] (with  $\sigma^2 = \rho/2$ ), we obtain

$$\Gamma \vdash c[j] \sim_{0, \alpha/2}^{\mathsf{zCDP}} c[j] \colon \Theta(K, L) \land (\langle 1 \rangle = j) \implies \Theta(K, L) \land (i\langle 1 \rangle = j + 1).$$

Note that the second case occurs twice. The [seq] rule sums up the grading of each execution c[j], and we conclude  $\rho$ -zCDP of HGNoise. Finally, by using [seq] and some conditional computations, we complete the proof.

# 7.3 A k-fold Gaussian mechanism

Consider a type DATA of dataset and an predicate ADJ(-, =) of adjacency for the type DATA, and consider *K* queries q(i, -): DATA  $\rightarrow$  real ( $0 \le i < K$ ) with sensitivity 1, that is,

 $ADJ(D, D') \implies |q(i, D) - q(i, D')| \le 1.$ 

We want now to prove private the following *K*-fold Gaussian mechanism. Even though standard DP can already be handled by other verification techniques, our proof applies the conversion rules between DP and zCDP along with composition in zCDP, yielding a more precise analysis for standard DP.

Algorithm 4 Sum of K Gaussian mechanisms

1: **procedure** FoldG<sub>K</sub>(K: int,  $\sigma$ : real (const.), D: DATA, x, y, z: real, i: int ) 2:  $i \leftarrow 0; z \leftarrow 0;$ 3: **while** i < K **do** 4:  $x \leftarrow q(i, D); y \leftarrow Gauss(0, \sigma); z \leftarrow x + y + z; i \leftarrow i + 1;$ 

We set a typing context of  $FoldG_K$  by D: DATA, x, y, z: real, and i: int. Following sensitivity of queries q, for any  $0 \le i < K$  we may assume

$$\Gamma \vdash x \leftarrow q(i, D) \sim_{0,0}^{\mathsf{zCDP}} x \leftarrow q(i, D) \colon \mathsf{ADJ}(D\langle 1 \rangle, D\langle 2 \rangle) \implies |x\langle 1 \rangle - x\langle 2 \rangle| \le 1.$$

Thus, for the loop body *c* (line 4), by applying [zCDP-G], [seq] and [assn], we have

$$\Gamma \vdash c \sim_{0,1/2\sigma^2}^{z\mathsf{CDP}} c \colon \mathsf{ADJ}(D\langle 1 \rangle, D\langle 2 \rangle) \land (z\langle 1 \rangle = z\langle 2 \rangle) \implies z\langle 1 \rangle = z\langle 2 \rangle.$$

Then, by applying [assn], [seq], and [while] (the proof rule for while-loop) rules, we conclude

$$\Gamma \vdash \mathsf{FoldG}_K \sim_{0, K/2\sigma^2}^{\mathsf{zCDP}} \mathsf{FoldG}_K \colon \mathsf{ADJ}(D\langle 1 \rangle, D\langle 2 \rangle) \implies z\langle 1 \rangle = z\langle 2 \rangle.$$

Hence, the algorithm  $\operatorname{FoldG}_K$  is  $(0, K/2\sigma^2)$ -zCDP. Furthermore, by applying [z/D], we conclude that the algorithm  $\operatorname{FoldG}_K$  is  $\left(\frac{K}{2\sigma^2} + \frac{\sqrt{2K \log(1/\delta)}}{\sigma}, \delta\right)$ -DP for any  $0 < \delta < 1/2$ .

This analysis gives a more precise bound compared to reasoning in terms of standard differential privacy. First, by [DP-G], [seq] and [assn], for any  $0 < \delta_1 < 1/2$ , the loop body *c* satisfies

$$\Gamma \vdash c \sim_{\max((1+\sqrt{3})/2\sigma,\sqrt{2\log(0.66/\delta_1)/\sigma)},\delta_1}^{\mathsf{DP}} c : \operatorname{adj}(D\langle 1\rangle, D\langle 2\rangle) \land (z\langle 1\rangle = z\langle 2\rangle) \implies z\langle 1\rangle = z\langle 2\rangle.$$

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

Let  $\varepsilon = \max((1 + \sqrt{3})/2\sigma, \sqrt{2\log(0.66/\delta_1)}/\sigma)$ . The algorithm FoldG<sub>K</sub> can be seen as *K*-fold adaptive composition of the loop body  $c; \dots; c$ . By applying the advanced composition theorem [Dwork and Roth 2013, Theorem 3.20], the algorithm FoldG<sub>K</sub> is

$$\left(\varepsilon \cdot \sqrt{2K\log(1/\delta_2)} + K\varepsilon^2, K\delta_1 + \delta_2\right)$$
-DP for any  $0 < \delta_1, \delta_2 < 1/2$ .

We compare this bound and the bound given in the avove. When  $\delta_2 < 0.4$ , we have  $2 \log(0.66/\delta_2) > 1$ . We also have  $\varepsilon > 1.36/\sigma$  by the definition. Then, we can compute:

$$\frac{K}{2\sigma^2} + \frac{\sqrt{2K\log(1/\delta_2)}}{\sigma} < \frac{K}{2\sigma^2} + \frac{\sqrt{2K\log(1/\delta_2)}}{\sigma} \cdot \sqrt{2\log(0.66/\delta_1)} \le \varepsilon \cdot \sqrt{2K\log(1/\delta_2)} + K\varepsilon^2.$$

Hence,  $\varepsilon \cdot \sqrt{2K \log(1/\delta_2)} + K\varepsilon^2 > \frac{K}{2\sigma^2} + \frac{\sqrt{2K \log(1/\delta)}}{\sigma}$  whenever  $\delta = K\delta_1 + \delta_2$  and  $\delta_2 < 0.4$ . We can conclude that verification via zCDP is actually better than advanced composition for the

We can conclude that verification via zCDP is actually better than advanced composition for the algorithm FoldG. First, in the verification via zCDP, the approximation error  $\delta$  is given regardless of the number of queries *K*. Second, if the approximation error satisfies  $\delta < 0.4$  then the verification is significantly better than advanced composition. The restriction  $\delta < 0.4$  is quite weak since the approximation error  $\delta$  in the ( $\varepsilon$ ,  $\delta$ )-DP is thought as the probability of failure of  $\varepsilon$ -DP. Moreover in practical use of ( $\varepsilon$ ,  $\delta$ )-DP, the parameter  $\delta$  is usually taken to be quite small (e.g.,  $\delta \approx 10^{-5}$ ).

### 8 RELATED WORKS

## 8.1 Relational liftings for *f*-divergences

As we have mentioned, our work is inspired by work on verifying probabilistic relational properties involving f-divergences by Barthe and Olmedo [2013]; we generalize their results to a broader class of divergences and also to handle continuous distributions. Barthe and Olmedo also consider f-divergences that satisfy a more limited version of composability, called *weak composability*. Roughly, these composition results only apply when corresponding pairs of distributions have equal weight; the KL-divergence, Hellinger distance, and  $\chi^2$  divergences only satisfy this weaker version of composability. While we do not detail this extension, our framework can naturally handle weakly composable divergences in the continuous case.

A similar approach has also been used by Barthe et al. [2016a] in the context of an higher order functional language for reasoning about Bayesian inference. Their type system uses a graded monad to reason about f-divergences. The graded monad supports only discrete distributions and is interpreted via a set-theoretic semantics, again using the lifting by Barthe and Olmedo [2013].

### 8.2 Relational liftings for differential privacy

Approximate relational liftings were originally proposed for program logics targeting differential privacy. The first such system used a one-witness definition of lifting [Barthe et al. 2013], which was subsequently refined to several notions of two-witness lifting [Barthe et al. 2016b; Barthe and Olmedo 2013]. Sato [2016] developed approximate liftings and a program logic for continuous distribution using witness-free lifting based on a categorical monad lifting [Katsumata 2005; Katsumata and Sato 2015]. A *witness-free* relational lifting for differential privacy was introduced by Sato [2016]. This can be seen as an application of the general construction of *graded relational lifting* [Katsumata 2014, Section 5] to the Giry monad, using the technique of *codensity lifting* [Katsumata and Sato 2015, Section 3.3] instead of  $\top \top$ -lifting. The witness-free relational lifting by Sato [2016]

sends a binary relation *R* between measurable spaces *X*, *Y* to the following one between  $\mathcal{G}X, \mathcal{G}Y$ :

$$R^{\top\top(\varepsilon,\delta)} = \bigcap_{(k,l): R \to S^{(\varepsilon',\delta')}} (k^{\sharp} \times l^{\sharp})^{-1} S^{(\varepsilon+\varepsilon',\delta+\delta')}$$
  
where  $S^{(\varepsilon',\delta')} = \left\{ (x,y) \in \mathcal{G}1 \times \mathcal{G}1 \mid x \le e^{\varepsilon'}y + \delta' \right\}$ 

where  $\mathcal{G}$  is the sub-Giry monad,  $k^{\sharp}$  and  $l^{\sharp}$  denote the Kleisli extensions of k and l respectively,  $\rightarrow$  denotes a relation-preserving map, and  $\top \top$  is used to denote the codensity lifting and to distinguish it from our 2-witness lifting. Here, the intersection is taken over all measurable functions  $k : X \rightarrow \mathcal{G}1, l : Y \rightarrow \mathcal{G}1$  mapping pairs related by R to those related by  $S^{(\varepsilon', \delta')}$ . We note that the binary relation  $S^{(\varepsilon', \delta')}$  is a parameter of this witness-free lifting, and by changing it, we can derive other graded relational liftings of  $\mathcal{G}$ .

Checking the membership for  $R^{\top\top(\varepsilon,\delta)}$  is complex: we have to test the pair (x, y) against every pair (k, l) of measurable functions such that  $(k, l): R \rightarrow S^{(\varepsilon,\delta)}$ . Fortunately, since the divergence  $\Delta^{\mathsf{DP}(\varepsilon)}$  is defined by a linear inequality of measures, the witness-free lifting  $R^{\top\top(\varepsilon,\delta)}$  can be *simplified* to the following

$$R^{\top \top(\varepsilon,\delta)} = \{ (d_1, d_2) \in \mathcal{G}X \times \mathcal{G}Y \mid \forall A \subseteq \Sigma_X. \ d_1(A) \le e^{\varepsilon} d_2(R(A)) + \delta \}.$$

While we would like to generalize this lifting construction to handle more general divergences for RDP, zCDP, and tCDP, there are at least two obstacles. First, it is not clear how to find a parameter S to derive the suitable graded relational lifting for a given general divergence. Second, even if we can find a suitable parameter S, it is awkward to work with the lifting unless we can simplify the large intersection into a more convenient form. In contrast, 2-witness liftings seem more concrete and easier to work with: It suffices to give witness distributions to check the membership of lifted relations.

In the discrete case, witness-free liftings are equivalent to the witness-/span-based liftings by Barthe et al. [2017]. Recent work also considers liftings with more fine-grained parameters that can vary over different pairs of samples [Albarghouthi and Hsu 2018].

# 8.3 Other techniques for verifying privacy

Rényi and zero-concentrated differential privacy were recently proposed in the differential privacy literature; to the best of our knowledge, we are the first to verify these properties. In contrast, there are now numerous systems targeting differential privacy using a wide range of techniques beyond program logics, including dynamic analyses [McSherry 2009], linear [Azevedo de Amorim et al. 2014; Gaboardi et al. 2013; Reed and Pierce 2010] and dependent [Barthe et al. 2015] type systems, product programs [Barthe et al. 2014], partial evaluation [Winograd-Cort et al. 2017], and constraint-solving [Albarghouthi and Hsu 2018; Zhang and Kifer 2017]; see the recent survey [Barthe et al. 2016c] for more details.

# 9 CONCLUSION AND FUTURE WORK

We have developed a framework for reasoning about three relaxations of differential privacy: Rényi differential privacy, zero concentrated differential privacy, and truncated concentrated differential privacy. We extended the notion of divergences to a more general class, and to support subprobability measures. Additionally, we have introduced a novel notion of approximate span-lifting supporting these divergences and continuous distributions.

One promising direction for future work is to study the moment-accountant composition method [Abadi et al. 2016]. This composition method tracks the moments of the privacy loss

random variable, although it does not directly correspond to composition for RDP or zCDP. Another interesting direction would be to analyze recently-proposed RDP mechanisms for posterior sampling [Geumlek et al. 2017], and the GAP-Max tCDP algorithm by Bun et al. [2018].

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#### A CONTINUITY OF *f*-DIVERGENCES OF SUBPROBABILITY MEASURES

In this section we show the subprobability version of continuity of f-divergences [Liese and Vajda 2006, Theorem 16] in a different way from the paper [Liese and Vajda 2006].

THEOREM A.1 (THEOREM 4.7 / SUBPROBABILITY VERSION OF [LIESE AND VAJDA 2006, THEOREM 16]). For any weight function f, the f-divergence  $\Delta^f$  is continuous: for any subprobability measures  $\mu_1, \mu_2 \in \mathcal{G}X$  on X, we have

$$\Delta_X^f(\mu_1,\mu_2) = \sup\left\{\sum_{i=0}^n \mu_2(A_i) f\left(\frac{\mu_1(A_i)}{\mu_2(A_i)}\right) \mid \{A_i\}_{i=0}^n \text{ is a measurable finite partition of } X\right\}.$$

To prove this proposition, we introduce the singularity of measures. Two measures  $\mu_1$  and  $\mu_2$  on X are said to be mutually singular (written  $\nu_1 \perp \nu_2$ ) if there are partition  $A_1, A_2 \in \Sigma_X$  of X such that  $\mu_i(E) = \mu_i(A_i \cap E)$  for any  $E \in \Sigma_X$  (i = 1, 2).

LEMMA A.2 (LEBESGUE'S DECOMPOSITION THEOREM). Let  $\mu_1$  and  $\mu_2$  be  $\sigma$ -finite measures on X. There are unique finite measures  $\mu_1^{\bullet}$  and  $\mu_1^{\perp}$  on X such that  $\mu_1^{\bullet} \ll \mu_2$  and  $\mu_1^{\perp} \perp \mu_2$ .

We recall that the *f*-divergence for subprobability measures is defined by for any  $\mu_1, \mu_2, \mu \in \mathcal{G}X$  such that  $\mu_1, \mu_2 \ll \mu$ ,

$$\Delta_X^f(\mu_1,\mu_2) = \int_X \frac{d\mu_2}{d\mu} f\left(\frac{d\mu_1/d\mu}{d\mu_2/d\mu}\right) d\mu.$$

We remark that  $\mu$  satisfying  $\mu_1, \mu_2 \ll \mu$  always exists (e.g.  $(\mu_1 + \mu_2)/2$ ), and the  $\Delta_X^f(\mu_1, \mu_2)$  does not depend on the choice of  $\mu$ . We want to prove the continuity:

$$\Delta_X^f(\mu_1,\mu_2) = \sup\left\{\sum_{i=0}^n \mu_2(A_i) f\left(\frac{\mu_1(A_i)}{\mu_2(A_i)}\right) \mid \{A_i\}_{i=0}^n \text{ is a finite measurable partition of } X\right\}.$$

We define the following restricted sum of *f*-divergences. For any measurable subset  $D \in \Sigma_X$ ,

$$\begin{split} \Delta_X^f(\mu_1,\mu_2)|_D &= \int_D \frac{d\mu_2}{d\mu} f\left(\frac{d\mu_1/d\mu}{d\mu_2/d\mu}\right) d\mu.\\ \overline{\Delta}_X^f(\mu_1,\mu_2)|_D &= \sup\left\{\sum_{i=0}^n \mu_2(A_i) f\left(\frac{\mu_1(A_i)}{\mu_2(A_i)}\right) \mid \{A_i\}_{i=0}^n \text{ is a finite measurable partition of } D\right\}\\ &= \sup\left\{\sum_{i\in I} \mu_2(k^{-1}(i)) f\left(\frac{\mu_1(k^{-1}(i))}{\mu_2(k^{-1}(i))}\right) \mid I \in \mathbf{Fin}, k \colon D \to I\right\} \end{split}$$

Of course,  $\Delta_X^f(\mu_1, \mu_2) = \Delta_X^f(\mu_1, \mu_2)|_X$ . We write  $\overline{\Delta}_X^f(\mu_1, \mu_2) = \overline{\Delta}_X^f(\mu_1, \mu_2)|_X$ We temporary consider a positive weight function f.

LEMMA A.3. If  $\mu_1 \ll \mu_2$  then  $\Delta_X^f(\mu_1, \mu_2)|_D \leq \overline{\Delta}_X^f(\mu_1, \mu_2)|_D$  for any  $D \in \Sigma_X$ .

PROOF. Since  $\mu_1 \ll \mu_2$ , we may assume  $\mu = \mu_2$  (hence  $d\mu_2/d\mu = 1$ ). Then, we have  $\Delta_X^f(\mu_1, \mu_2)|_D = \int_D f(\frac{d\mu_1}{d\mu_2})d\mu_2$ . Since f is convex, there is  $\alpha \in \mathbb{R}_{\geq 0}$  which makes that f is monotone increasing on the interval  $[0, \alpha)$  and monotone decreasing on  $[\alpha, \infty)$ . Let  $\{A_i\}_{i=0}^n$  be an arbitrary finite partition of D which is finer than the partition  $\{\frac{d\mu_1^{-1}}{d\mu_2}([0, \alpha)) \cap D, \frac{d\mu_1^{-1}}{d\mu_2}([\alpha, \infty)) \cap D\}$ . The function  $f \circ \frac{d\mu_1}{d\mu_2}$  is either monotone increasing or monotone decreasing, on each partition  $A_i$ . Hence,  $\inf_{x \in A_i} f(\frac{d\mu_1}{d\mu_2})(x)$ 

is either  $f(\inf_{x \in A_i} \frac{d\mu_1}{d\mu_2}(x))$  or  $f(\sup_{x \in A_i} \frac{d\mu_1}{d\mu_2}(x))$ . From the mean-value theorem for measures, we obtain

$$\inf_{x\in A_i}\frac{d\mu_1}{d\mu_2}(x)\leq \frac{\mu_1(A_i)}{\mu_2(A_i)}\leq \sup_{x\in A_i}\frac{d\mu_1}{d\mu_2}(x).$$

Hence,

$$\sum_{i=0}^{n} \mu_2(A_i) \inf_{x \in A_i} f(\frac{d\mu_1}{d\mu_2})(x) \le \sum_{i=0}^{n} \mu_2(A_i) f(\frac{\mu_1(A_i)}{\mu_2(A_i)})$$

Since  $\{A_i\}_{i=0}^n$  is arbitrary, we conclude  $\Delta_X^f(\mu_1, \mu_2)|_D \le \overline{\Delta}_X^f(\mu_1, \mu_2)|_D$ .

LEMMA A.4. If  $\mu_1 \ll \mu_2$  and the Radon-Nikodym derivative  $d\mu_1/d\mu_2$  is bounded on D then  $\Delta_X^f(\mu_1,\mu_2)|_D = \overline{\Delta}_X^f(\mu_1,\mu_2)|_D$ .

PROOF. We fix a positive integer  $1 \le K \in \mathbb{N}$  such that  $0 \le \frac{d\mu_1}{d\mu_2} \le M$ . For given  $N \in \mathbb{N}$ , we define the partition  $\{A_i\}_{i=0}^{2^N K}$  of *D* by

$$A_{i} = \left( \left( \frac{d\mu_{1}}{d\mu_{2}} \right)^{-1} (B_{i}) \right) \cap D, \quad B_{i} = \begin{cases} \left[ \frac{i}{2^{N}}, \frac{i+1}{2^{N}} \right) & 0 \le i < 2^{N} \\ \{1\} & i = 2^{N} \\ \left( \frac{i-1}{2^{N}}, \frac{i}{2^{N}} \right] & 2^{N} < i \le 2^{N} K. \end{cases}$$

Since  $\mu_1 \ll \mu_2$  and 0f(0/0) = 0, if  $\mu_2(A_i) = 0$  then  $\mu_2(A_i)\frac{\mu_1(A_i)}{\mu_2(A_i)} = 0$ . If  $\mu_2(A_i) > 0$  then  $\left|\frac{d\mu_1}{d\mu_2}(x) - \frac{\mu_1(A_i)}{\mu_2(A_i)}\right| \le 2^{-(N-1)}$  for all  $x \in A_i$ , from the definition of  $\{A_i\}_{i=0}^{2^N K}$ ,

$$\frac{i-1}{2^N} \leq \inf_{x \in A_i} \frac{d\mu_1}{d\mu_2}(x) \leq \frac{\mu_1(A_i)}{\mu_2(A_i)} \leq \sup_{x \in A_i} \frac{d\mu_1}{d\mu_2}(x) \leq \frac{i+1}{2^N}.$$

Consider an arbitrary  $\varepsilon > 0$ . Since f is uniformly continuous on the closed interval [0, K], there are large enough  $N_2 \in \mathbb{N}$  and the corresponding partition  $\{A_i\}_{i=0}^{2^N K}$  such that

$$\mu_2(A_i) > 0 \implies \left| \inf_{x \in A_i} f(\frac{d\mu_1}{d\mu_2})(x) - f(\frac{\mu_1(A_i)}{\mu_2(A_i)}) \right| < \varepsilon$$

Hence, for any partition  $\{C_i\}_{i=0}^n$  of *D* finer than  $\{A_i\}_{i=0}^{2^N K}$ , we obtain

$$\sum_{i=0}^{n} \mu_2(C_i) f\left(\frac{\mu_1(C_i)}{\mu_2(C_i)}\right) \leq \sum_{i=0}^{n} \mu_2(C_i) \left(\inf_{x \in C_i} f\left(\frac{d\mu_1}{d\mu_2}\right)(x)\right) + \varepsilon.$$

This implies  $\overline{\Delta}_X^f(\mu_1, \mu_2)|_D \le \Delta_X^f(\mu_1, \mu_2)|_D + \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, we conclude  $\overline{\Delta}_X^f(\mu_1, \mu_2)|_D \le \Delta_X^f(\mu_1, \mu_2)|_D$ .

LEMMA A.5. We have  $\Delta_X^f(\mu_1, \mu_2)|_D = \overline{\Delta}_X^f(\mu_1, \mu_2)|_D$  when  $\mu_1 \ll \mu_2$ .

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

PROOF. Let  $D_n = \left( \left( \frac{d\mu_1}{d\mu_2} \right)^{-1} [n, n+1) \right) \cap D$   $(n \in \mathbb{N})$ . From Jensen's inequality, we obtain for any partition  $\{A_i\}_{i=0}^m$  of  $D_i$ 

$$\begin{split} \sum_{i=0}^{m} \mu_2(A_i) f\left(\frac{\mu_1(A_i)}{\mu_2(A_i)}\right) &= \sum_{i=0}^{m} (\sum_{n \in \mathbb{N}} \mu_2(D_n \cap A_i)) f\left(\frac{\sum_{n \in \mathbb{N}} \mu_1(D_n \cap A_i)}{\sum_{n \in \mathbb{N}} \mu_2(D_n \cap A_i)}\right) \\ &\leq \sum_{i=0}^{m} \sum_{n \in \mathbb{N}} \mu_2(D_n \cap A_i) f\left(\frac{\mu_1(D_n \cap A_i)}{\mu_2(D_n \cap A_i)}\right) \\ &= \sum_{n \in \mathbb{N}} \sum_{i=0}^{m} \mu_2(D_n \cap A_i) f\left(\frac{\mu_1(D_n \cap A_i)}{\mu_2(D_n \cap A_i)}\right) \end{split}$$

This implies  $\overline{\Delta}_X^f(\mu_1, \mu_2)|_D \leq \sum_{n=0}^{\infty} \overline{\Delta}_X^f(\mu_1, \mu_2)|_{D_n}$  for each  $n \in \mathbb{N}$ . Since the Radon-Nikodym derivative  $\frac{d\mu_1}{d\mu_2}$  is bounded on each  $D_n$ , by Lemmas A.3 and A.4,  $\Delta_X^f(\mu_1,\mu_2)|_{D_n} = \overline{\Delta}_X^f(\mu_1,\mu_2)|_{D_n}$  for each  $n \in \mathbb{N}$ . Hence,

$$\overline{\Delta}_X^f(\mu_1,\mu_2) \le \sum_{n=0}^{\infty} \overline{\Delta}_X^f(\mu_1,\mu_2)|_{D_n} = \sum_{n=0}^{\infty} \Delta_X^f(\mu_1,\mu_2)|_{D_n} = \Delta_X^f(\mu_1,\mu_2) \le \overline{\Delta}_X^f(\mu_1,\mu_2).$$
uplies  $\Delta_X^f(\mu_1,\mu_2) = \overline{\Delta}_X^f(\mu_1,\mu_2).$ 

This implies  $\Delta_X^j(\mu_1, \mu_2) = \Delta_X^j(\mu_1, \mu_2)$ .

THEOREM 4.7, POSITIVE CASE. We show that for any positive weight function f, the continuity  $\overline{\Delta}_X^f(\mu_1,\mu_2) = \Delta_X^f(\mu_1,\mu_2)$  holds. Let  $(\mu_1^{\bullet},\mu_1^{\perp})$  be the Lebesgue decomposition of  $\mu_1$  with respect to  $\mu_2$ . Since  $(\mu_1^{\bullet}, \mu_1^{\perp})$  is the Lebesgue decomposition of  $\mu_1$  with respect to  $\mu_2$ , there is  $A \in \Sigma_X$ such that  $\mu_2(E) = \mu_2(E \setminus A)$  and  $\mu_1^{\perp}(E) = \mu_1^{\perp}(E \cap A)$  for any  $E \in \Sigma_X$ . The subset A also satisfies  $\mu_1(E \setminus A) = \mu_1^{\bullet}(E \setminus A)$  for any  $E \in \Sigma_X$ . We then obtain

$$\Delta_X^f(\mu_1,\mu_2) = \Delta_X^f(\mu_1,\mu_2)|_{X\setminus A} + \Delta_X^f(\mu_1,\mu_2)|_A = \Delta_X^f(\mu_1^{\bullet},\mu_2)|_{X\setminus A} + \Delta_X^f(\mu_1^{\bot},\mu_2)|_A = \overline{\Delta}_X^f(\mu_1^{\bullet},\mu_2)|_{X\setminus A} + \overline{\Delta}_X^f(\mu_1^{\bot},\mu_2)|_A = \overline{\Delta}_X^f(\mu_1,\mu_2)|_{X\setminus A} + \overline{\Delta}_X^f(\mu_1,\mu_2)|_A = \overline$$

From Lemma A.5,  $\Delta_X^f(\mu_1^{\bullet}, \mu_2)|_{X \setminus A} = \overline{\Delta}_X^f(\mu_1^{\bullet}, \mu_2)|_{X \setminus A}$  holds, and using the dual  $f^*$  we have

$$\Delta_X^f(\mu_1^{\perp},\mu_2)|_A = \int_A \frac{d\mu_2}{d\mu} f\left(\frac{d\mu_1^{\perp}/d\mu}{d\mu_2/d\mu}\right) d\mu = \int_A f^*(0) \frac{d\mu_1^{\perp}}{d\mu} d\mu = f^*(0)\mu_1(A) = \overline{\Delta}_X^f(\mu_1^{\perp},\mu_2)|_A.$$

THEOREM 4.7, GENERAL CASE. We show the continuity of  $\Delta^f$  for arbitrary weight function f. Let  $\alpha, \beta \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$  the functions be defined by  $\alpha(t) = a$  and  $\beta(t) = bt$  respectively where  $a, b \geq 0$ . Since *f* is convex, there are  $\alpha$  and  $\beta$  that makes  $f + \alpha + \beta$  positive. Hence,

$$\begin{split} \overline{\Delta}_{X}^{f}(\mu_{1},\mu_{2}) + a\mu_{2}(X) + b\mu_{1}(X) &= \overline{\Delta}_{X}^{f+\alpha+\beta}(\mu_{1},\mu_{2}) \\ &= \Delta_{X}^{f+\alpha+\beta}(\mu_{1},\mu_{2}) \\ &= \Delta_{X}^{f}(\mu_{1},\mu_{2}) + a\mu_{2}(X) + b\mu_{1}(X). \end{split}$$

This completes the proof.

# **B** OMITTED STRUCTURES OF THE PROGRAM LOGIC

# B.1 Typing Rules for Expressions and Programs

Before we give the semantics of programs, we first give a type system for expressions, distributions, and programs. A typing context is a finite set  $\Gamma = \{x_1 : \tau_1, x_2 : \tau_2, \ldots, x_n : \tau_n\}$  of pairs of a variable and a value type such that each variable occurs only once in the context. The type system is largely standard, with two kinds of judgments:  $\Gamma \vdash^t e : \tau$  states that expression *e* has type  $\tau$  in context  $\Gamma$ , while  $\Gamma \vdash^p v : \tau$  states that v is a distribution over  $\tau$  in context  $\Gamma$ . The third judgment  $\Gamma \vdash c$  states that program *c* is well-typed in context  $\Gamma$ , e.g., all guards are booleans, assignments are well-typed, etc. The expression typing rules are as follows:

$$\frac{x:\tau \in \Gamma}{\Gamma \vdash^{t} x:\tau} = \frac{\Gamma \vdash^{t} e_{1}:\tau}{\Gamma \vdash^{t} e_{1} \oplus e_{2}:\tau} = \frac{\Gamma \vdash^{t} e_{1}:\tau}{\Gamma \vdash^{t} e_{1} \bowtie e_{2}: t} = \frac{\Gamma \vdash^{t} e_{1}:\tau}{\Gamma \vdash^{t} e_{1} \bowtie e_{2}: t} = \frac{\Gamma \vdash^{t} e_{1}:\tau}{\Gamma \vdash^{t} e_{1} \bowtie e_{2}: t}$$

$$\frac{\Gamma \vdash^{t} e: real}{\Gamma \vdash^{p} Bern(e): bool} = \frac{\Gamma \vdash^{t} e_{1}: real}{\Gamma \vdash^{p} Lap(e_{1}, e_{2}): real} = \frac{\Gamma \vdash^{t} e_{1}: real}{\Gamma \vdash^{p} Gauss(e_{1}, e_{2}): real}$$

$$\frac{\Gamma \vdash^{t} e: \tau}{\Gamma \vdash^{p} Dirac(e): \tau} = \frac{\Gamma \vdash c_{1}}{\Gamma \vdash skip} = \frac{\Gamma, x: \tau \vdash^{p} v: \tau}{\Gamma, x: \tau \vdash x \xleftarrow v} = \frac{\Gamma \vdash c_{1}}{\Gamma \vdash c_{1}; c_{2}}$$

$$\frac{\Gamma \vdash^{t} b: bool}{\Gamma \vdash c_{1}} = \frac{\Gamma \vdash^{t} b: bool}{\Gamma \vdash b then c_{1} else c_{2}} = \frac{\Gamma \vdash^{t} b: bool}{\Gamma \vdash b then c_{1} else c_{2}}$$

*B.1.1* Forming Relation Expressions. The judgment  $\Gamma \vdash^R \Phi$  states that the relation expression  $\Phi$  is well-formed in context  $\Gamma$ .

$$\frac{\Gamma \vdash^{t} e_{1} \bowtie e_{2} : \text{bool}}{\Gamma \vdash^{R} e_{1}\langle 1 \rangle \bowtie e_{2}\langle 2 \rangle} \qquad \frac{\Gamma \vdash^{t} (e_{1} \oplus_{1} e_{2}) \bowtie (e_{3} \oplus_{2} e_{4}) : \text{bool}}{\Gamma \vdash^{R} (e_{1}\langle 1 \rangle \oplus_{1} e_{2}\langle 2 \rangle) \bowtie (e_{3}\langle 1 \rangle \oplus_{2} e_{4}\langle 2 \rangle)}$$
$$\frac{\Gamma \vdash^{R} \Phi \ \Gamma \vdash^{R} \Psi}{\Gamma \vdash^{R} \Phi \land \Psi} \qquad \frac{\Gamma \vdash^{R} \Phi \ \Gamma \vdash^{R} \Psi}{\Gamma \vdash^{R} \Phi \lor \Psi} \qquad \frac{\Gamma \vdash^{R} \Phi}{\Gamma \vdash^{R} \neg \Phi}$$

*B.1.2 Basic proof rules.* The basic proof rules are given in Figure 5.

#### **B.2** mechanism rules

Figure 6 is the list of mechanism rules in span-apRHL.

# **B.3 Denotational Semantics of pWHILE**

To prove the soundness of span-apRHL we interpret pWHILE in **Meas** using the sub-Giry monad  $\mathcal{G}$ . First, we interpret the value types bool, int, and real as the finite discrete space  $\mathbb{B} = 1 + 1 = \{\text{true}, \text{false}\}$ , the countable discrete space  $\mathbb{Z} = \{0, 1, \ldots\}$ , and the Lebesgue measurable space  $\mathbb{R}$  respectively. We interpret  $\tau^d$  as the product  $[\![\tau]\!]^d$  and we interpret a typing context  $\Gamma = \{x_1: \tau_1, x_2: \tau_2, \ldots, x_n: \tau_n\}$  as a product  $[\![\tau_1]\!] \times [\![\tau_2]\!] \times \cdots \times [\![\tau_n]\!]$ .

To give a semantics to expressions, distribution expressions, and commands, we interpret their associated typing/well-formedness judgments in a context  $\Gamma$ . We interpret an expression judgment  $\Gamma \vdash^t e: \tau$  as a measurable function  $[\![\Gamma \vdash^t e: \tau]\!]: [\![\Gamma]\!] \rightarrow [\![\tau]\!];$  for instance, the variable case  $\Gamma \vdash^t x: \tau$  is interpreted as the projection  $\pi_x: [\![\Gamma]\!] \rightarrow [\![\tau]\!].$ 

We interpret a reference  $\llbracket \Gamma \vdash^t e_1 \llbracket e_2 \rrbracket$ :  $\tau \rrbracket$  of an element by ref $\langle \tau, n \rangle (\llbracket \Gamma \vdash^t e_1 \rrbracket, \llbracket \Gamma \vdash^t e_2 \rrbracket)$  where ref $\langle \tau, n \rangle : \llbracket \tau \rrbracket^n \times \mathbb{Z} \to \llbracket \tau \rrbracket$  is defined by ref $\langle \tau, n \rangle ((x_0, \ldots, x_{n-1}), k) = x_{\min(\max(k,0),n)}$ .<sup>7</sup>

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

<sup>&</sup>lt;sup>7</sup>We can describe it categorically by using products and coproducts in Meas.

$$\begin{split} \Gamma \vdash x_{1} \leftarrow e_{1} \sim_{1_{A},0}^{\Lambda} x_{2} \leftarrow e_{2} : \Phi\{e_{1}\langle 1 \rangle, e_{2}\langle 2 \rangle / x_{1}\langle 1 \rangle, x_{2}\langle 2 \rangle\} \implies \Phi \quad [assn] \\ & \frac{\Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{1}' : \Phi \implies \Phi' \quad \Gamma \vdash c_{2} \sim_{\beta,\gamma}^{\Lambda} c_{2}' : \Phi' \implies \Psi}{\Gamma \vdash c_{1}; c_{2} \sim_{\alpha,\beta,\delta+\gamma}^{\Lambda} c_{1}'; c_{2}' : \Phi \implies \Psi} \quad [seq] \\ & \Gamma \vdash skip \sim_{1_{A},0}^{\Lambda} skip : \Phi \implies \Phi \quad [skip] \\ \hline \Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{1}' : \Phi \land b\langle 1 \rangle \implies \Psi \quad \Gamma \vdash c_{2} \sim_{\alpha,\delta}^{\Lambda} c_{2}' : \Phi \land \neg b\langle 1 \rangle \implies \Psi \quad [cond] \\ & \vdash if \ b \ then \ c_{1} \ else \ c_{2} \sim_{\alpha,\delta}^{\Lambda} if \ b' \ then \ c_{1}' \ else \ c_{2}' : \Phi \implies \Psi \quad [cond] \\ \hline \forall 0 \le k \le n - 1. \ \Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Theta \land (e\langle 1 \rangle \ge n) \implies \Theta \land (e\langle 1 \rangle > k) \\ \hline \Gamma \vdash while \ b_{1} do \ c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi_{1} \implies W \quad \Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi_{2} \implies \Psi \quad [case] \\ \hline \frac{\Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi_{1} \implies \Psi \quad \Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi_{2} \implies \Psi \quad [case] \\ \hline \frac{\Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi_{1} \implies \Psi \quad \Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi_{2} \implies \Psi \quad [case] \\ \hline \Gamma \vdash while \ b_{1} do \ c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi_{1} \implies \Psi' \quad \Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi_{2} \implies \Psi \quad [case] \\ \hline \Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi_{1} \implies \Psi' \quad \Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi_{2} \implies \Psi \quad [case] \\ \hline \Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi' \implies \Psi' \quad [case] \\ \hline \Gamma \vdash c_{1} \sim_{\beta,\gamma}^{\Lambda} c_{2} : \Phi' \implies \Psi' \quad [case] \\ \hline \Gamma \vdash c_{1} \sim_{\beta,\gamma}^{\Lambda} c_{2} : \Phi' \implies \Psi' \quad [case] \\ \hline \Gamma \vdash c_{1} \sim_{\beta,\gamma}^{\Lambda} c_{2} : \Phi' \implies \Psi' \quad [case] \\ \hline \Gamma \vdash c_{1} \sim_{\beta,\gamma}^{\Lambda} c_{2} : \Phi' \implies \Psi' \quad [case] \\ \hline \Gamma \vdash c_{1} \sim_{\beta,\gamma}^{\Lambda} c_{2} : \Phi' \implies \Psi' \quad [case] \\ \hline \Gamma \vdash c_{1} \sim_{\alpha,\delta}^{\Lambda} c_{2} : \Phi' \implies \Psi' \quad [case] \\ \hline \Gamma \vdash c_{1} \sim_{\beta,\gamma}^{\Lambda} c_{2} : \Phi' \implies \Psi' \quad [case] \\ \hline \Gamma \vdash c_{1} \sim_{\beta,\gamma}^{\Lambda} c_{2} : \Phi' \implies \Psi' \quad [case] \quad [case$$



All operators  $\oplus$  and comparisons  $\bowtie$  are interpreted as measurable functions  $\oplus : [\![\tau]\!] \times [\![\tau]\!] \rightarrow [\![\tau]\!]$ and  $\bowtie : [\![\tau]\!] \times [\![\tau]\!] \rightarrow [\![bool]\!]$  respectively. Likewise, we interpret a distribution expression judgment  $\Gamma \vdash^{p} v : \tau$  as a measurable function  $[\![\Gamma \vdash^{p} v : \tau]\!] : [\![\Gamma]\!] \rightarrow \mathcal{G}[\![\tau]\!]$  as follows:

$$\begin{split} & \left[\!\left[\Gamma \vdash^{p} \mathsf{Dirac}(e) \colon \tau\right]\!\right] = \eta_{\left[\!\left[\tau\right]\!\right]} \circ \left[\!\left[\Gamma \vdash^{t} e \colon \tau\right]\!\right], \\ & \left[\!\left[\Gamma \vdash^{p} \mathsf{Bern}(e) \colon \mathsf{bool}\right]\!\right] = \mathsf{Bern}(\left[\!\left[\Gamma \vdash^{t} e \colon \mathsf{real}\right]\!\right]), \\ & \left[\!\left[\Gamma \vdash^{p} \mathsf{Lap}(e_{1}, e_{2}) \colon \mathsf{real}\right]\!\right] = \mathsf{Lap}(\left[\!\left[\Gamma \vdash^{t} e_{1} \colon \mathsf{real}\right]\!\right], \left[\!\left[\Gamma \vdash^{t} e_{2} \colon \mathsf{real}\right]\!\right]), \\ & \left[\!\left[\Gamma \vdash^{p} \mathsf{Gauss}(e_{1}, e_{2}) \colon \mathsf{real}\right]\!\right] = \mathcal{N}(\left[\!\left[\Gamma \vdash^{t} e_{1} \colon \mathsf{real}\right]\!\right], \left[\!\left[\Gamma \vdash^{t} e_{2} \colon \mathsf{real}\right]\!\right]). \end{split}$$

Finally, we interpret a command judgment  $\Gamma \vdash c$  inductively as a measurable function  $[\Gamma \vdash c]: [\Gamma] \rightarrow \mathcal{G}[\Gamma]$  by

$$\begin{split} \llbracket \Gamma \vdash x & \stackrel{\$}{\leftarrow} \nu \rrbracket = \mathcal{G}(\operatorname{rw}\langle \Gamma \mid x \colon \tau \rangle) \circ \operatorname{st}_{\llbracket \Gamma \rrbracket, \llbracket \tau \rrbracket} \circ \langle \operatorname{id}_{\llbracket \Gamma \rrbracket, \llbracket \nu \rrbracket} \rangle, \\ \llbracket \Gamma \vdash c_1; c_2 \rrbracket &= \llbracket \Gamma \vdash c_2 \rrbracket^{\sharp} \circ \llbracket \Gamma \vdash c_1 \rrbracket, \\ \llbracket \Gamma \vdash \operatorname{skip} \rrbracket &= \eta_{\llbracket \Gamma \rrbracket} \\ \llbracket \Gamma \vdash \operatorname{if} b \text{ then } c_1 \text{ else } c_2 \rrbracket &= [\llbracket \Gamma \vdash c_1 \rrbracket, \llbracket \Gamma \vdash c_2 \rrbracket] \circ \operatorname{br}\langle \Gamma \rangle \circ \langle \llbracket \Gamma \vdash b \rrbracket, \operatorname{id}_{\llbracket \Gamma \rrbracket} \rangle \end{split}$$

Here,  $\operatorname{rw}(\Gamma \mid x: \tau) : \llbracket \Gamma \rrbracket \times \llbracket x: \tau \rrbracket \to \llbracket \Gamma \rrbracket (x: \tau \in \Gamma)$  is an overwriting operation of memories mapping  $((a_1, \ldots, a_k, \ldots, a_n), b_k) \mapsto (a_1, \ldots, b_k, \ldots, a_n)$ ; this is given by the Cartesian products in **Meas**. The function  $\operatorname{br}(\Gamma) : 2 \times \llbracket \Gamma \rrbracket \to \llbracket \Gamma \rrbracket + \llbracket \Gamma \rrbracket$  comes from the canonical isomorphism  $2 \times \llbracket \Gamma \rrbracket \cong \llbracket \Gamma \rrbracket + \llbracket \Gamma \rrbracket$  from the distributivity of **Meas**.

To interpret loops, we introduce the dummy "abort" command  $\Gamma \vdash \text{null}$  that is interpreted by the null/zero measure  $[\Gamma \vdash \text{null}] = 0$ , and the following commands corresponding to the finite

$$\begin{split} \Gamma + x_{1} \stackrel{\$}{\leftarrow} & \text{Bern}(e_{1}) \sim_{\log \max(p, 1-p)-\log \min(p, 1-p), 0}^{\text{DP}} x_{2} \stackrel{\$}{\leftarrow} & \text{Bern}(e_{2}): \\ & ((e_{1}(1) = p) \land (1 - e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = p) \land (1 - e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = p) \land (1 - e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) = e_{2}(2)) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le r) \implies (x_{1}(1) = x_{2}(2)) \\ & (e_{1}(1) - e_{2}(2)| \le$$

$$(|e_1\langle 1 \rangle - e_2\langle 2 \rangle| \le r) \implies (x_1\langle 1 \rangle = x_2\langle 2 \rangle)$$
 [tCDP-G]  
$$\exists c > \frac{1+\sqrt{3}}{2}. (2\log(0.66/\delta) \le c^2) \land (\frac{cr}{c} \le \sigma)$$

$$\frac{1}{\Gamma \vdash x_1} \stackrel{\$}{\leftarrow} \operatorname{Gauss}(e_1, \sigma^2) \sim^{\mathsf{DP}}_{e, \delta} x_2 \stackrel{\$}{\leftarrow} \operatorname{Gauss}(e_2, \sigma^2): \\ (|e_1\langle 1 \rangle - e_2\langle 2 \rangle| \le r) \implies (x_1\langle 1 \rangle = x_2\langle 2 \rangle) \\ 1 < 1/\sqrt{\rho} \le A/\delta$$
[DP-G]

$$\Gamma \vdash x_{1} \xleftarrow{\begin{subarray}{c}{l}} e_{1} + A \cdot \operatorname{arsinh}\left(\frac{1}{A}\operatorname{Gauss}(0, \delta^{2}/2\rho)\right) \qquad [tCDP-SinhG] \\ \sim^{tCDP}_{16\rho, A/8\delta} x_{2} \xleftarrow{\begin{subarray}{c}{l}} e_{2} + A \operatorname{arsinh}\left(\frac{1}{A}\operatorname{Gauss}(0, \delta^{2}/2\rho)\right): \\ (|e_{1}\langle 1 \rangle - e_{2}\langle 2 \rangle| \leq \delta) \implies (x_{1}\langle 1 \rangle = x_{2}\langle 2 \rangle) \end{cases}$$

Fig. 6. Rules for basic mechanisms for DP, RDP, zCDP, and tCDP in span-apRHL. Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

unrollings of the loop:

$$[\text{while } b \text{ do } c]_n = \begin{cases} \text{if } b \text{ then null else skip,} & \text{if } n = 0 \\ \text{if } b \text{ then } c; [\text{while } b \text{ do } c]_k, & \text{if } n = k + 1 \end{cases}$$

We then interpret loops as the supremum of interpretations of finite executions:<sup>8</sup>

$$\Gamma \vdash \mathsf{while} \ b \ \mathsf{do} \ c] = \sup_{n \in \mathbb{N}} [\Gamma \vdash [\mathsf{while} \ e \ \mathsf{do} \ c]_n].$$

# **B.4** Proof of Soundness of the Program Logic

LEMMA B.1. The [assn] rule is sound.

**PROOF.** We may assume  $x_1 \neq x_2$  without loss of generality. Let

$$((\phi^{1}, a_{1}^{1}, a_{2}^{1}), (\phi^{2}, a_{1}^{2}, a_{2}^{2})) \in (\Gamma \vdash^{R} \Phi\{e_{1}\langle 1 \rangle, e_{2}\langle 2 \rangle / x_{1}\langle 1 \rangle, x_{2}\langle 2 \rangle\})$$

where  $a_j^i$  is a value of variable  $x_j$  (i = 1, 2). Since  $x_1$  and  $x_2$  are not free variables in  $e_1$  and  $e_2$  respectively, we have

$$((\phi_1, \llbracket\Gamma \vdash^t e_1 \colon \tau \rrbracket)(\phi^1, a_1^1, a_2^1), a_2^1), (\phi^2, a_1^2, \llbracket\Gamma \vdash^t e_2 \colon \tau \rrbracket)(\phi^2, a_1^2, a_2^2)) \in (\Gamma \vdash^R \Phi).$$

Therefore,

$$(f_1(\phi^1, a_1^1, a_2^1), f_2(\phi^2, a_1^2, a_2^2)) \in (\![\Gamma \vdash^R \Phi]\!]$$

where  $f_i = \operatorname{rw}\langle \Gamma | x_i : \tau \rangle \circ \langle \operatorname{id}_{\llbracket \Gamma \rrbracket}, \llbracket \Gamma \vdash^t e_i : \tau \rrbracket \rangle$  (i = 1, 2). Therefore, we obtain the following morphism of spans (note that both  $\llbracket \Gamma \vdash^R \Phi\{e_1\langle 1 \rangle, e_2\langle 2 \rangle / x_1\langle 1 \rangle, x_2\langle 2 \rangle\}$  and  $\llbracket \Gamma \vdash^R \Phi \rrbracket$  are binary relation converted to spans),

$$\begin{split} (f_1, f_2, (f_1 \times f_2)|_{(\Gamma \vdash^R \Phi\{e_1\langle 1 \rangle, e_2\langle 2 \rangle / x_1 \langle 1 \rangle, x_2 \langle 2 \rangle\})}) : \\ & [\![\Gamma \vdash^R \Phi\{e_1\langle 1 \rangle, e_2\langle 2 \rangle / x_1 \langle 1 \rangle, x_2 \langle 2 \rangle\}]\!] \to [\![\Gamma \vdash^R \Phi]\!]. \end{split}$$

Letting  $g_i = \eta_{\llbracket \Gamma \rrbracket} \circ f_i = \llbracket \Gamma \vdash x_i \leftarrow e_i \rrbracket$ , we conclude

$$\begin{aligned} (g_1, g_2, \langle \eta_{\Phi}, \eta_{\Phi} \rangle \circ (g_1 \times g_2)|_{(\Gamma \vdash^R \Phi\{e_1 \langle 1 \rangle, e_2 \langle 2 \rangle / x_1 \langle 1 \rangle, x_2 \langle 2 \rangle\})}) : \\ & [\![\Gamma \vdash^R \Phi\{e_1 \langle 1 \rangle, e_2 \langle 2 \rangle / x_1 \langle 1 \rangle, x_2 \langle 2 \rangle\}]\!] \to [\![\Gamma \vdash^R \Phi]\!]^{\sharp(\Delta, 1_A, 0)}. \end{aligned}$$

LEMMA B.2. The [seq] rule is sound.

**PROOF.** Since the judgments  $\Gamma \vdash c_1 \sim_{\alpha,\delta}^{\Delta} c'_1 \colon \Phi \implies \Phi'$  and  $\Gamma \vdash c_2 \sim_{\beta,\gamma}^{\Delta} c'_2 \colon \Phi' \implies \Psi$  are valid, we obtain the following two morphisms in **Span(Meas**) for witness functions  $l_1$  and  $l_2$ :

$$(\llbracket \Gamma \vdash c_1 \rrbracket, \llbracket \Gamma \vdash c'_1 \rrbracket, l_1) \colon \llbracket \Gamma \vdash^R \Phi \rrbracket \to \llbracket \Gamma \vdash^R \Phi' \rrbracket^{\sharp(\Delta, \alpha, \delta)}$$
$$(\llbracket \Gamma \vdash c_2 \rrbracket, \llbracket \Gamma \vdash c'_2 \rrbracket, l_2) \colon \llbracket \Gamma \vdash^R \Phi' \rrbracket \to \llbracket \Gamma \vdash^R \Psi \rrbracket^{\sharp(\Delta, \beta, \gamma)}$$

By taking the graded Kleisli lifting of the second morphism  $(\llbracket \Gamma \vdash c_2 \rrbracket, \llbracket \Gamma \vdash c'_2 \rrbracket, l_2)$ , for some witness function  $l_3$ , we have a **Span(Meas**)-morphism

$$(\llbracket \Gamma \vdash c_2 \rrbracket^{\sharp}, \llbracket \Gamma \vdash c'_2 \rrbracket^{\sharp}, l_3) \colon \llbracket \Gamma \vdash^R \Phi' \rrbracket^{\sharp(\Delta, \alpha, \delta)} \to \llbracket \Gamma \vdash^R \Psi \rrbracket^{\sharp(\Delta, \alpha\beta, \delta+\gamma)}.$$

Composing, we have a span-morphism giving validity of  $\Gamma \vdash c_1; c_2 \sim^{\Delta}_{\alpha\beta,\delta+\gamma} c'_1; c'_2: \Phi \implies \Psi:$ 

$$(\llbracket \Gamma \vdash c_2 \rrbracket^{\sharp} \circ \llbracket \Gamma \vdash c_1 \rrbracket, \llbracket \Gamma \vdash c'_2 \rrbracket^{\sharp} \circ \llbracket \Gamma \vdash c'_1 \rrbracket, l_3 \circ l_1) \colon \llbracket \Gamma \vdash^R \Phi \rrbracket \to \llbracket \Gamma \vdash^R \Psi \rrbracket^{\sharp(\Delta, \alpha\beta, \delta+\gamma)}.$$

<sup>&</sup>lt;sup>8</sup>This is well-defined, since the family { $[\Gamma \vdash [while \ e \ do \ c]_n]$ } is an  $\omega$ -chain with respect to the  $\omega CPO_{\perp}$ -enrichment  $\sqsubseteq$  of  $Meas_{\mathcal{G}}$ .

LEMMA B.3. The [weak] rule is sound

PROOF. Since the judgment  $\Gamma \vdash c_1 \sim^{\Delta}_{\alpha,\delta} c_2 \colon \Phi \implies \Psi$  is valid, we have a witness function  $l \colon (\![\Gamma \vdash^R \Phi]\!] \to W([\![\Gamma \vdash^R \Psi]\!], \Delta, \alpha, \delta)$  such that

$$(\llbracket\Gamma \vdash c_1\rrbracket, \llbracket\Gamma \vdash c_2\rrbracket, l) \colon \llbracket\Gamma \vdash^R \Phi\rrbracket \longrightarrow \llbracket\Gamma \vdash^R \Psi\rrbracket^{\sharp(\Delta, \alpha, \delta)}$$

From the inclusions  $\Gamma \vdash^{I} \Phi' \implies \Phi$  and  $\Gamma \vdash^{I} \Psi \implies \Psi'$  of relations, we have

$$\begin{aligned} (\mathrm{id}_{\llbracket\Gamma\rrbracket},\mathrm{id}_{\llbracket\Gamma\rrbracket},(\mathrm{id}_{\llbracket\Gamma\rrbracket}\times\mathrm{id}_{\llbracket\Gamma\rrbracket})|_{(\Gamma\vdash^{R}\Phi')}) \colon \llbracket\Gamma\vdash^{R}\Phi'\rrbracket \to \llbracket\Gamma\vdash^{R}\Phi\rrbracket \\ (\mathrm{id}_{\llbracket\Gamma\rrbracket},\mathrm{id}_{\llbracket\Gamma\rrbracket},(\mathrm{id}_{\llbracket\Gamma\rrbracket}\times\mathrm{id}_{\llbracket\Gamma\rrbracket})|_{(\Gamma\vdash^{R}\Psi)}) \colon \llbracket\Gamma\vdash^{R}\Psi\rrbracket \to \llbracket\Gamma\vdash^{R}\Psi'\rrbracket. \end{aligned}$$

Thanks to the inclusion structure of the span-lifting  $(-)^{\sharp(\Delta)}$ , we obtain

$$(\mathcal{G}\mathrm{id}_{\llbracket\Gamma\rrbracket},\mathcal{G}\mathrm{id}_{\llbracket\Gamma\rrbracket},(\mathcal{G}\mathrm{id}_{\llbracket\Gamma\rrbracket}\times\mathcal{G}\mathrm{id}_{\llbracket\Gamma\rrbracket})|_{W(\llbracket\Gamma\vdash^{R}\Phi\rrbracket,\Delta,\alpha,\delta)})\colon \llbracket\Gamma\vdash^{R}\Psi'\rrbracket^{\sharp(\Delta,\alpha,\delta)} \to \llbracket\Gamma\vdash^{R}\Psi'\rrbracket^{\sharp(\Delta,\beta,\gamma)}$$

Therefore, we conclude

$$(\llbracket \Gamma \vdash c_1 \rrbracket, \llbracket \Gamma \vdash c_2 \rrbracket, l|_{(\Gamma \vdash {}^{R}\Phi')}) \colon \llbracket \Gamma \vdash^{R} \Phi' \rrbracket \to \llbracket \Gamma \vdash^{R} \Psi' \rrbracket^{\sharp(\Delta, \beta, \gamma)}$$

LEMMA B.4. The [cond] rule is sound.

PROOF. Since the judgments  $\Gamma \vdash c_1 \sim_{\alpha,\delta}^{\Delta} c_2 : \Phi \land b \langle 1 \rangle \implies \Psi$  and  $\Gamma \vdash c'_1 \sim_{\alpha,\delta}^{\Delta} c'_2 : \Phi \land \neg b \langle 1 \rangle \implies \Psi$  are valid, we have two witness functions  $l_T : (\Gamma \vdash^R \Phi \land b \langle 1 \rangle) \rightarrow W([\Gamma \vdash^R \Psi]], \Delta, \alpha, \delta)$  and  $l_F : (\Gamma \vdash^R \Phi \land \neg b \langle 1 \rangle) \rightarrow W([\Gamma \vdash^R \Psi]], \Delta, \alpha, \delta)$  that make the following morphisms in **Span(Meas**):

$$(\llbracket \Gamma \vdash c_1 \rrbracket, \llbracket \Gamma \vdash c_2 \rrbracket, l_T) \colon \llbracket \Gamma \vdash^R \Phi \land b\langle 1 \rangle \rrbracket \to \llbracket \Gamma \vdash^R \Psi \rrbracket^{\sharp(\Delta, \alpha, \delta)}$$
$$(\llbracket \Gamma \vdash c_1' \rrbracket, \llbracket \Gamma \vdash c_2' \rrbracket, l_F) \colon \llbracket \Gamma \vdash^R \Phi \land \neg b\langle 1 \rangle \rrbracket \to \llbracket \Gamma \vdash^R \Psi \rrbracket^{\sharp(\Delta, \alpha, \delta)}.$$

By the coproduct structure of Span(Meas), we have the following span-morphism:

$$\begin{split} (\llbracket \llbracket \Gamma \vdash c_1 \rrbracket, \llbracket \Gamma \vdash c_1' \rrbracket], \llbracket \llbracket \Gamma \vdash c_2 \rrbracket, \llbracket \Gamma \vdash c_2' \rrbracket], \llbracket l_T, l_F \rrbracket) : \\ \llbracket \Gamma \vdash^R \Phi \land b\langle 1 \rangle \rrbracket \dotplus \llbracket \Gamma \vdash^R \Phi \land \neg b\langle 1 \rangle \rrbracket \to \llbracket \Gamma \vdash^R \Psi \rrbracket^{\sharp(\Delta, \alpha, \delta)}. \end{split}$$

We write  $g_1 = br\langle \Gamma \rangle \circ \langle [\![\Gamma \vdash^t b]\!], id_{[\![\Gamma]\!]} \rangle$  and  $g_2 = br\langle \Gamma \rangle \circ \langle [\![\Gamma \vdash^t \neg b]\!], id_{[\![\Gamma]\!]} \rangle$ . We construct the following morphism by using  $\Gamma \vdash^I \Phi \implies b\langle 1 \rangle = b'\langle 2 \rangle$ 

$$(g_1, g_2, H \circ (g_1 \times g_2)|_{(\Gamma \vdash^R \Phi)}) \colon \llbracket \Gamma \vdash^R \Phi \rrbracket \to \llbracket \Gamma \vdash^R \Phi \land b\langle 1 \rangle \rrbracket \dotplus \llbracket \Gamma \vdash^R \Phi \land \neg b\langle 1 \rangle \rrbracket, \tag{8}$$

where *H* is the composition  $H_3 \circ H_2 \circ H_1$  of

- *H*<sub>1</sub>: ([[Γ]] + [[Γ]]) × ([[Γ]] + [[Γ]]) ≅ 4 × ([[Γ]] × [[Γ]]) defined by (*ι<sub>i</sub>*(φ<sub>1</sub>), *ι<sub>j</sub>*(φ<sub>2</sub>)) ↦ ((*i*, *j*), (φ<sub>1</sub>, φ<sub>2</sub>)) where *i*, *j* ∈ 2,
   *H*<sub>2</sub>: 4 × ([[Γ]] × [[Γ]]) → 2 × ([[Γ]] × [[Γ]])
- defined by  $((b_1, b_2), \phi^1, \phi^2) \mapsto (b_1, \phi^1, \phi^2),$ •  $H_3: 2 \times (\llbracket \Gamma \rrbracket \times \llbracket \Gamma \rrbracket) \cong (\llbracket \Gamma \rrbracket \times \llbracket \Gamma \rrbracket) + (\llbracket \Gamma \rrbracket \times \llbracket \Gamma \rrbracket)$ 
  - defined by  $(b, (\phi^1, \phi^2)) \mapsto \iota_b(\phi^1, \phi^2)$ .

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

Here,  $\iota_i$  are coprojections  $\iota_1: A \to A + B$  and  $\iota_2: B \to A + B$ . The bijections  $H_1$  and  $H_3$  are given from the distributivity of products and coproducts in **Meas**, and  $H_2$  is given by a projection.

Now, let  $(\phi^1, \phi^2) \in (\Gamma \vdash^R \Phi)$ . Since we suppose  $\Gamma \vdash^I \Phi \implies b\langle 1 \rangle = b'\langle 2 \rangle$ , we have

$$(g_1(\phi^1), g_2(\phi^2)) = \begin{cases} ((1, \phi^1), (1, \phi^2)) & (\phi^1, \phi^2) \in (\Gamma \vdash^R \Phi \land b\langle 1 \rangle) = (\Gamma \vdash^R \Phi \land b'\langle 2 \rangle) \\ ((2, \phi^1), (2, \phi^2)) & (\phi^1, \phi^2) \in (\Gamma \vdash^R \Phi \land \neg b\langle 1 \rangle) = (\Gamma \vdash^R \Phi \land \neg b'\langle 2 \rangle) \end{cases}$$

We observe the role of *H* in the first case  $((\phi^1, \phi^2) \in (\Gamma \vdash^R \Phi \land b\langle 1 \rangle))$ ,

$$H(g_1(\phi^1), g_2(\phi^2)) = H_3 \circ H_2 \circ H_1((1, \phi^1), (1, \phi^2))$$
  
=  $H_3 \circ H_2((1, 1), (\phi^1, \phi^2)) = H_3(1, (\phi^1, \phi^2)) = \iota_1(\phi^1, \phi^2).$ 

In the same way, we have  $H(g_1(\phi^1), g_2(\phi^2)) = \iota_2(\phi^1, \phi^2)$  in the second case. Therefore, the measurable function  $H \circ (g_1 \times g_2)|_{(\Gamma \vdash^R \Phi)}$  forms a function from  $(\Gamma \vdash^R \Phi)$  to  $(\Gamma \vdash^R \Phi \land b\langle 1 \rangle) + (\Gamma \vdash^R \Phi \land \neg b\langle 1 \rangle)$  satisfying (8).

Since  $\llbracket \Gamma \vdash if b$  then c else  $c' \rrbracket = [\llbracket \Gamma \vdash c \rrbracket, \llbracket \Gamma \vdash c' \rrbracket] \circ br \langle \Gamma \rangle \circ \langle \llbracket \Gamma \vdash^t b \rrbracket, id_{\llbracket \Gamma \rrbracket} \rangle$ , we conclude the soundness.

REMARK B.1. Similarly, we have soundness of [case].

REMARK B.2. The soundness of the [while] rule is a consequence of the soundness of [seq], [weak], and the [case] rule since the [while] in our logic deal only with finite-loops.

LEMMA B.5. The rule [RDP-G] is sound.

PROOF. We assume  $x_1 \neq x_2$ . First, it can be directly checked that the function  $f = \mathcal{N}(-, \sigma^2)$ :  $\mathbb{R} \to \mathcal{G}\mathbb{R}$  is measurable. From Mironov [2017, Proposition 3], the function f satisfies  $D^{\alpha}(f(x)||f(y)) \leq \alpha r^2/2\sigma^2$  whenever  $|x - y| \leq r$ . Hence,  $(f, f, (f \times f)|_{\Phi})$  is a span-morphism  $\Phi \to \mathrm{Eq}_{\mathbb{R}}^{\sharp(D^{\alpha}, *, \alpha r^2/2\sigma^2)}$  where  $\Phi = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x - y| \leq r\}$  is regarded as a span.

We next construct a span-morphism  $(h_1, h_2, (h_1 \times h_2)|_{\Theta})$  mapping  $\Theta \to \operatorname{Eq}_{\mathbb{R}}^{\sharp(D^{\alpha}, *, \alpha r^2/2\sigma^2)}$  where  $\Theta = \llbracket \Gamma \vdash^R |e_1\langle 1 \rangle - e_2\langle 2 \rangle| \leq r \rrbracket$  and  $h_i = \llbracket \Gamma \vdash^P \operatorname{Gauss}(e_i, \sigma^2)$ : real  $\rrbracket$  (i = 1, 2). We write  $g_i = \llbracket \Gamma \vdash^t e_i$ : real  $\rrbracket$  (i = 1, 2). It is clear that  $(g_1, g_2, (g_1 \times g_2)|_{\Theta})$  is a span-morphism  $\Theta \to \Phi$ . Since  $h_i = f_i \circ g_i$  (i = 1, 2), the triple  $(h_1, h_2, (h_1 \times h_2)|_{\Theta})$  is a span-morphism  $\Theta \to \operatorname{Eq}_{\mathbb{R}}^{\sharp(D^{\alpha}, *, \alpha r^2/2\sigma^2)}$ .

Now, the triple  $(\operatorname{id}_{\llbracket\Gamma\rrbracket} \times h_1, \operatorname{id}_{\llbracket\Gamma\rrbracket} \times h_2, (\operatorname{id}_{\llbracket\Gamma\rrbracket} \times \operatorname{id}_{\llbracket\Gamma\rrbracket}) \times (h_1 \times h_2)|_{\Theta})$  is a morphism of spans  $\top_{\llbracket\Gamma\rrbracket} \dot{\times} \Theta \to$  $\top_{\llbracket\Gamma\rrbracket} \dot{\times} (\operatorname{Eq}_{\mathbb{R}})^{\sharp(D^{\alpha}, *, \alpha r^2/2\sigma^2)}$  where  $\top_{\llbracket\Gamma\rrbracket} = (\llbracket\Gamma\rrbracket, \llbracket\Gamma\rrbracket, \llbracket\Gamma\rrbracket, \llbracket\Gamma\rrbracket, \pi_1, \pi_2)$ . Thanks to the unit and the double strength of the span-lifting  $\{(-)^{\sharp(D^{\alpha}, *, \rho)}\}_{\rho}$ , the triple  $(\operatorname{st}_{\llbracket\Gamma\rrbracket, \mathbb{R}}, \operatorname{st}_{\llbracket\Gamma\rrbracket, \mathbb{R}}, (\operatorname{st}_{\llbracket\Gamma\rrbracket, \mathbb{R}}) \times (\pi_1 \times \pi_1), \operatorname{st}_{\llbracket\Gamma\rrbracket, \mathbb{R}}, (\pi_2 \times \pi_2))|_{(\llbracket\Gamma\rrbracket \times \llbracket\Gamma\rrbracket) \times W(\operatorname{Eq}_{\mathbb{R}}, D^{\alpha}, *, \alpha r^2/2\sigma^2)})$  is a morphism of spans  $\top_{\llbracket\Gamma\rrbracket} \dot{\times} (\operatorname{Eq}_{\mathbb{R}})^{\sharp(D^{\alpha}, *, \alpha r^2/2\sigma^2)} \to (\top_{\llbracket\Gamma\rrbracket} \dot{\times} \operatorname{Eq}_{\mathbb{R}})^{\sharp(D^{\alpha}, *, \alpha r^2/2\sigma^2)}$ .

We write  $k_i = \operatorname{rw} \langle \Gamma \mid x_i : \operatorname{real} \rangle$  (i = 1, 2). For any  $(((\phi^1, a_1^1, a_2^1), r), ((\phi^2, a_1^2, a_2^2), r)) \in \top_{\llbracket \Gamma \rrbracket} \times \operatorname{Eq}_{\mathbb{R}}$ where  $a_i^i$  is a value of variable  $x_j$  (i = 1, 2), we have

$$\begin{aligned} (\operatorname{rw}\langle \Gamma \mid x_1: \operatorname{real}\rangle((\phi^1, a_1^1, a_2^1), r^1), \operatorname{rw}\langle \Gamma \mid x_2: \operatorname{real}\rangle((\phi^2, a_1^2, a_2^2), r^2)) \\ &= ((\phi^1, r, a_2^1), (\phi^2, a_1^2, r)) \in \langle\!\!|\Gamma \vdash^R x_1\langle 1\rangle = x_2\langle 2\rangle\!\!| \end{aligned}$$

Hence, the triple  $(k_1, k_2, (k_1 \times k_2)|_{(\llbracket\Gamma \rrbracket \times \llbracket\Gamma \rrbracket) \times Eq_{\mathbb{R}}})$  forms a morphism of spans  $(\top_{\llbracket\Gamma \rrbracket} \dot{\times} Eq_{\mathbb{R}}) \rightarrow \llbracket\Gamma \vdash^R x_1 \langle 1 \rangle = x_2 \langle 2 \rangle \rrbracket$ . (Note that  $(\top_{\llbracket\Gamma \rrbracket} \dot{\times} Eq_{\mathbb{R}})$  and  $\llbracket\Gamma \vdash^R x_1 \langle 1 \rangle = x_2 \langle 2 \rangle \rrbracket$  are binary relations converted to spans.)

By the functoriality of the span-lifting  $\{(-)^{\sharp(D^{\alpha},*,\rho)}\}_{\rho}$ , we obtain in Span(Meas),

$$\begin{split} (k_1, k_2, (k_1 \times k_2)|_{(\llbracket \Gamma \rrbracket \times \llbracket \Gamma \rrbracket) \times \mathrm{Eq}_{\mathbb{R}}})^{\sharp(D^{\alpha}, *, \alpha r^2/2\sigma^2)} : \\ (\top_{\llbracket \Gamma \rrbracket} \dot{\times} \mathrm{Eq}_{\mathbb{R}})^{\sharp(D^{\alpha}, *, \alpha r^2/2\sigma^2)} \to \llbracket \Gamma \vdash^R x_1 \langle 1 \rangle = x_2 \langle 2 \rangle \rrbracket^{\sharp(D^{\alpha}, *, \alpha r^2/2\sigma^2)}. \end{split}$$

Since  $\llbracket \Gamma \vdash x_i \stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_i, \sigma^2) \rrbracket = \mathcal{G}k_i \circ \operatorname{st}_{\llbracket \Gamma \rrbracket, \mathbb{R}} \circ \langle \operatorname{id}_{\llbracket \Gamma \rrbracket}, h_i \rangle$  (*i* = 1, 2), we conclude the soundness of [RDP-G]:

$$\begin{split} (\llbracket \Gamma \vdash x_1 &\stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_1, \sigma^2) \rrbracket, \llbracket \Gamma \vdash x_2 &\stackrel{\$}{\leftarrow} \mathsf{Gauss}(e_2, \sigma^2) \rrbracket, l) \\ &= (k_1, k_2, (k_1 \times k_2) |_{(\llbracket \Gamma \rrbracket \times \llbracket \Gamma \rrbracket) \times \mathbb{E} q_{\mathbb{R}}})^{\sharp (D^{\alpha}, *, \alpha r^2/2\sigma^2)} \\ &\quad \circ (\mathsf{st}_{\llbracket \Gamma \rrbracket, \mathbb{R}}, \mathsf{st}_{\llbracket \Gamma \rrbracket, \mathbb{R}}, \langle \mathsf{st}_{\llbracket \Gamma \rrbracket, \mathbb{R}} \circ (\pi_1 \times \pi_1), \mathsf{st}_{\llbracket \Gamma \rrbracket, \mathbb{R}} \circ (\pi_2 \times \pi_2) \rangle |_{(\llbracket \Gamma \rrbracket) \times \llbracket \Gamma \rrbracket) \times W(\mathbb{E} q_{\mathbb{R}}, D^{\alpha}, *, \alpha r^2/2\sigma^2)} \\ &\quad \circ (\mathsf{id}_{\llbracket \Gamma \rrbracket } \times h_1, \mathsf{id}_{\llbracket \Gamma \rrbracket} \times h_2, (\mathsf{id}_{\llbracket \Gamma \rrbracket } \times \mathsf{id}_{\llbracket \Gamma \rrbracket}) \times (h_1 \times h_2) |_{\Theta}) \\ &\quad \circ (\langle \mathsf{id}_{\llbracket \Gamma \rrbracket}, \mathsf{id}_{\llbracket \Gamma \rrbracket}), \langle \mathsf{id}_{\llbracket \Gamma \rrbracket} \rangle, \langle \mathsf{id}_{\llbracket \Gamma \rrbracket} \rangle, \mathsf{did}_{\llbracket \Gamma \rrbracket} \rangle |_{\Theta}, \mathsf{id}_{\Theta} \rangle) \colon \\ \Theta \to \llbracket \Gamma \vdash^R x_1 \langle 1 \rangle = x_2 \langle 2 \rangle \rrbracket^{\sharp (D^{\alpha}, *, \alpha r^2/2\sigma^2)}. \end{split}$$

Soundness of other mechanism rules follows similarly using Mironov [2017, Propositions 5, 6, 7], Dwork et al. [2006, Proposition 1], Sato [2016, Lemma 4.2] (a refinement of Dwork and Roth [2013, Theorem 3.22]), the soundness of the transitivity rules are proved by Olmedo [2014, Lemma 4.2(iii)], Bun and Steinke [2016, Proposition 27] and Lemma 4.2, and the soundness of the conversion rules follows by Bun and Steinke [2016, Proposition 4], Mironov [2017, Proposition 3], and Bun and Steinke [2016, Lemmas 3.2, 3.5].

#### C OMITTED PROOFS

THEOREM C.1 (THEOREM 4.4). An A-graded family  $\Delta$  is additive if it is continuous and composable.

**PROOF.** From the continuity of  $\Delta$ ,

$$\Delta_{X\times Y}^{\alpha\beta}(\mu_1\otimes\mu_3,\mu_2\otimes\mu_4) = \sup\left\{\Delta_I^{\alpha\beta}(\mathcal{G}k(\mu_1\otimes\mu_3),\mathcal{G}k(\mu_2\otimes\mu_4)) \mid k:X\times Y\to I\right\}$$

We fix  $k: X \times Y \to I$ . For any  $\mu \in \mathcal{G}Y$ , we define  $K_{\mu}: X \to \mathcal{G}I$  by  $K_{\mu} = \mathcal{G}k \circ \operatorname{st}_{X,Y} \circ (\operatorname{id}_X \times \overline{\mu}) \circ \rho^{-1}_X$ where  $\overline{\mu}$  is the generalized element  $1 \to \mathcal{G}Y$  assigning  $\mu$ , and  $\rho_X$  is a canonical isomorphism  $X \cong X \times 1$ . We then obtain for any  $\mu \in \mathcal{G}X$ ,

$$\begin{split} K^{\sharp}_{\mu}(\mu') &= \mathcal{G}k \circ \mu_{X \times Y} \circ \mathcal{G}\operatorname{st}_{X,Y} \circ \mathcal{G}(\operatorname{id}_{X} \times \overline{\mu}) \circ \mathcal{G}\rho^{-1}_{X}(\mu') \\ &= \mathcal{G}k \circ \mu_{X \times Y} \circ \mathcal{G}\operatorname{st}_{X,Y} \circ \mathcal{G}(\operatorname{id}_{X} \times \overline{\mu}) \circ \operatorname{st}'_{X,1} \circ \rho^{-1}_{\mathcal{G}X}(\mu') \\ &= \mathcal{G}k \circ \mu_{X \times Y} \circ \mathcal{G}\operatorname{st}_{X,Y} \circ \operatorname{st}'_{X,\mathcal{G}Y} \circ (\operatorname{id}_{\mathcal{G}X} \times \overline{\mu}) \circ \rho^{-1}_{\mathcal{G}X}(\mu') \\ &= \mathcal{G}k \circ \operatorname{dst}_{X,Y}(\mu',\mu) = \mathcal{G}k(\mu' \otimes \mu). \end{split}$$

We also obtain  $K_{\mu}(x) = \mathcal{G}k(\mathbf{d}_x \otimes \mu)$  for any  $x \in X$ . This implies  $K_{\mu}(x) = \mathcal{G}k(x, -)(\mu)$  where  $k(x, -)Y \rightarrow I$  is measurable because  $(\mathbf{d}_x \otimes \mu)(k^{-1}(A)) = \mu((k^{-1}(A))|_x) = \mu(k(x, -)^{-1}(A))$  for any  $A \subseteq I$ . From the composability and continuity of  $\Delta$ , we have

$$\begin{split} \Delta_{I}^{\alpha\beta}(\mathcal{G}k(\mu_{1}\otimes\mu_{3}),\mathcal{G}k(\mu_{2}\otimes\mu_{4})) &= \Delta_{I}^{\alpha\beta}(K_{\mu_{3}}^{\sharp}(\mu_{1}),K_{\mu_{4}}^{\sharp}(\mu_{2})) \\ &\leq \Delta_{X}^{\alpha}(\mu_{1},\mu_{2}) + \sup_{x\in X} \Delta_{I}^{\beta}(K_{\mu_{3}}(x),K_{\mu_{4}}(x)) \\ &= \Delta_{X}^{\alpha}(\mu_{1},\mu_{2}) + \sup_{x\in X} \Delta_{I}^{\beta}(\mathcal{G}k(x,-)(\mu_{3}),\mathcal{G}k(x,-)(\mu_{4})) \\ &\leq \Delta_{X}^{\alpha}(\mu_{1},\mu_{2}) + \Delta_{Y}^{\beta}(\mu_{3},\mu_{4}). \end{split}$$

Since  $k: X \times Y \to I$  is arbitrary, we conclude the additivity of  $\Delta$ .

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

THEOREM C.2 (THEOREM 4.6). A continuous approximable A-graded family  $\Delta$  is composable if finite-composable.

PROOF. Let  $\mu_1, \mu_2 \in \mathcal{G}X$  and  $f, g: X \to \mathcal{G}Y$ . Since  $\Delta$  is continuous, approximable, and finite composable, we obtain,

$$\begin{split} &\Delta_{Y}^{\alpha\rho}(f^{\sharp}(\mu_{1}),g^{\sharp}(\mu_{2})) \\ &\leq \sup\left\{\Delta_{I}^{\alpha\beta}(\mathcal{G}k(f^{\sharp}(\mu_{1})),\mathcal{G}k(f^{\sharp}(\mu_{2}))) \mid I \in \operatorname{Fin}, k \colon X \to I\right\} \\ &\leq \sup\left\{\lim_{n \to \infty}\Delta_{I}^{\alpha\beta}((\mathcal{G}k \circ f \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{1}),(\mathcal{G}k \circ g \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{2})) \mid I \in \operatorname{Fin}, k \colon X \to I.\right\} \\ &\leq \sup\left\{\lim_{n \to \infty}\Delta_{J_{n}}^{\alpha}(\mathcal{G}m_{n}^{*}(\mu_{1}),\mathcal{G}m_{n}^{*}(\mu_{2})) \mid I \in \operatorname{Fin}, k \colon X \to I\right\} \\ &+ \sup\left\{\lim_{n \to \infty}\sup_{j \in J_{n}}\Delta_{I}^{\beta}(\mathcal{G}k \circ f \circ m_{n}(j),\mathcal{G}k \circ g \circ m_{n}(j)) \mid I \in \operatorname{Fin}, k \colon X \to I\right\} \end{split}$$

Regarding the first term of the last inequality, since  $m_n^* \colon X \to J_n$  where  $J_n \in Fin$ , and  $\Delta^{\alpha}$  is continuous, we have

$$\Delta_{J_n}^{\alpha}(\mathcal{G}m_n^*(\mu_1),\mathcal{G}m_n^*(\mu_2)) \leq \Delta_X^{\alpha}(\mu_1,\mu_2).$$

Concerning the second term, since  $m_n(j) \in X$  for any n and  $j \in J_n$ , and  $k: I \to X$  and  $\Delta^{\beta}$  is continuous, we obtain

$$\sup_{j\in J_n} \Delta_I^\beta(\mathcal{G}k \circ f \circ m_n(j), \mathcal{G}k \circ g \circ m_n(j)) \leq \sup_{x \in X} \Delta_I^\beta(\mathcal{G}k \circ f(x), \mathcal{G}k \circ g(x)) \leq \sup_{x \in X} \Delta_Y^\beta(f(x), g(x)).$$

This completes the proof.

THEOREM C.3 (THEOREM 4.8). The f-divergence  $\Delta^f$  is approximable for any weight function f.

PROOF. Consider  $h, k: X \to \mathcal{G}I$ . Let |I| = N. We may regard  $\mathcal{G}I \subseteq [0, 1]^N$ . We define a partition  $\{C_{j_1...,j_{2N}}^n\}_{j_1,...,j_{2N} \in \{0,1,...,2^{n}-1\}}$  of X by

$$C_{j_1...j_{2N}}^n = h^{-1}(B_{j_1...j_N}^n) \cap h^{-1}(B_{j_{N+1}...j_{2N}}^n)$$
  
where  $B_{j_1...j_N}^n = A_{j_1} \times \cdots \times A_{j_N}$ ,  $A_0^n = \{0\}$  and  $A_{l+1}^n = (l/2^n, (l+1)/2^n]$ 

We define  $J_n = \left\{ (j_1, \dots, j_{2N}) \mid j_1, \dots, j_{2N} \in \{0, 1, \dots, 2^n - 1\}, C_{j_1 \dots, j_{2N}}^n \neq \emptyset \right\}$ . We next define  $m_n^* \colon X \to J_n$  and  $m_n \colon J_n \to X$  as follows:  $m_n^*(x)$  is the unique element  $(j_1, \dots, j_{2N}) \in J_n$  satisfying  $x \in C_{j_1,\dots, j_{2N}}^n$ , and  $m_n(j_1, \dots, j_{2N})$  is an element of  $C_{j_1,\dots, j_{2N}}^n$ .

From the construction of  $\{C_{j_1...,j_{2N}}^n\}_{j_1,...,j_{2N}\in\{0,1,...,2^n-1\}}$ , for any  $n \in \mathbb{N}$ ,  $x \in X$ , and  $i \in I$ ,

$$|h(x)(i) - (h \circ m_n \circ m_n^*)(x)(i)| \le 2/2^n, \quad |k(x)(i) - (k \circ m_n \circ m_n^*)(x)(i)| \le 2/2^n$$

holds. In particular, for any  $i \in I$ , the sequences of functions  $\{(h \circ m_n \circ m_n^*)(-)(i)\}_{n \in \mathbb{N}}$  and  $\{(k \circ m_n \circ m_n^*)(-)(i)\}_{n \in \mathbb{N}}$  converge *uniformly* to h(-)(i) and k(-)(i) respectively. Hence, for any  $\mu_1, \mu_2 \in \mathcal{G}X$ , we have

$$\begin{aligned} h^{\sharp}(\mu_{1})(i) &= \int_{X} h(-)(i) \ d\mu_{1} = \lim_{n \to \infty} \int_{X} (h \circ m_{n} \circ m_{n}^{*})(-)(i) \ d\mu_{1} = \lim_{n \to \infty} (h \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{1})(i), \\ g^{\sharp}(\mu_{2})(i) &= \int_{X} k(-)(i) \ d\mu_{2} = \lim_{n \to \infty} \int_{X} (k \circ m_{n} \circ m_{n}^{*})(-)(i) \ d\mu_{2} = \lim_{n \to \infty} (k \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{2})(i). \end{aligned}$$

Therefore,

$$\begin{split} \Delta_{I}^{f}(h^{\sharp}(\mu_{1}),k^{\sharp}(\mu_{2})) &= \sum_{i \in I} g^{\sharp}(\mu_{2})(i) f\left(\frac{h^{\sharp}(\mu_{1})(i)}{g^{\sharp}(\mu_{2})(i)}\right) \\ &= \sum_{i \in I} (\lim_{n \to \infty} (k \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{2})(i)) f\left(\frac{\lim_{n \to \infty} (h \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{1})(i)}{\lim_{n \to \infty} (k \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{2})(i)}\right) \\ &= \lim_{n \to \infty} \sum_{i \in I} (k \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{2})(i) f\left(\frac{(h \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{1})(i)}{(k \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{2})(i)}\right) \\ &= \lim_{n \to \infty} \Delta_{I}^{f}((h \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{1}), (k \circ m_{n} \circ m_{n}^{*})^{\sharp}(\mu_{2})) \end{split}$$

Remark that the third equality in the above calculation is obtained from the continuity of the weight function f. We then conclude that  $\Delta^f$  is approximable.

THEOREM C.4 (THEOREM 4.11). For any  $\alpha > 1$ , the Rényi divergence  $D^{\alpha}$  of order  $\alpha$  is reflexive, continuous, approximable, composable, and additive (as a singleton-graded family).

PROOF. By Theorems 4.7 and 4.8, the *f*-divergence  $\Delta^{\mathbb{R}(\alpha)}$  of the weight function  $f(t) = t^{\alpha}$  is continuous and approximable. Since the function  $g: \mathbb{R}_{\leq 0} \to \overline{\mathbb{R}}$  defined by  $g(t) = \frac{1}{\alpha-1} \log(t)$  is monotone and continuous,  $D^{\alpha} = \frac{1}{\alpha-1} \log \Delta^{\mathbb{R}(\alpha)}$  is also continuous and approximable. Thus, it suffices to show the reflexivity and *finite-composability* of  $D^{\alpha}$ . The reflexivity is obvious:  $D_X^{\alpha}(\mu||\mu)_X = \frac{1}{\alpha-1} \log \mu(X) \leq 0$ . We show the finite-composability. Let  $I, J \in \text{Fin}, d_1, d_2 \in \mathcal{G}J$ , and  $h, k: J \to \mathcal{G}I$ . We calculate by Jensen's inequality:

$$\begin{split} \Delta_I^{\mathbf{R}(\alpha)}(h^{\sharp}d_1, k^{\sharp}d_2) &= \sum_{i \in I} \left( \sum_{j \in J} d_2(j) \cdot k(j)(i) \right) \left( \frac{\sum_{j \in J} d_1(j) \cdot h(j)(i)}{\sum_{j \in J} d_2(j) \cdot k(j)(i)} \right)^{\alpha} \\ &\leq \sum_{j \in J} d_2(j) \left( \frac{d_1(j)}{d_2(j)} \right)^{\alpha} \sum_{i \in I} k(j)(i) \left( \frac{h(j)(i)}{k(j)(i)} \right)^{\alpha} \\ &\leq \sum_{j \in J} d_2(j) \left( \frac{d_1(j)}{d_2(j)} \right)^{\alpha} \cdot \Delta_{\alpha}^{\mathbf{R}(\alpha)}(h(j), k(j)) \\ &\leq \Delta_J^{\mathbf{R}(\alpha)}(d_1, d_2) \cdot \sup_{i \in J} \Delta_I^{\mathbf{R}(\alpha)}(h(j), k(j)). \end{split}$$

This implies  $D_I^{\alpha}(h^{\sharp}d_1||k^{\sharp}d_2) \leq D_I^{\alpha}(d_1||d_2) + \sup_{i \in I} D_I^{\alpha}(h(i)||k(i)).$ 

PROPOSITION C.1 (PROPOSITION 4.1). If  $1 < \alpha \leq \beta$  then

$$D_X^{\alpha}(\mu_1||\mu_2) \le D_X^{\beta}(\mu_1||\mu_2).$$

PROOF. The proof is almost the same as Van Erven and Harremoës [2014, Theorem 3]. Since  $D^{\alpha}$  and  $D^{\beta}$  are continuous, it suffices to prove in finite discrete case. We denote by |p| the sum  $\sum_{i \in I} p_i$ . We may assume |p| > 0 since if |p| = 0 then  $D_I^{\alpha}(p||q) = D_I^{\beta}(p||q) = -\infty$ . We remark that

1:38

the function  $t \mapsto t^{\frac{\alpha-1}{\beta-1}}$  is concave. We have  $\left(\frac{1}{|p|}\right)^{\frac{\alpha-1}{\beta-1}} \leq \frac{1}{|p|}$  since  $1 \leq \frac{1}{|p|}$ . Therefore,

$$\begin{split} \frac{1}{\alpha - 1} \log \sum_{i \in I} p_i^{\alpha} q_i^{1 - \alpha} &= \frac{1}{\alpha - 1} \log \left( |p| \sum_{i \in I} \frac{p_i}{|p|} \left( \left( \frac{p_i}{q_i} \right)^{1 - \beta} \right)^{\frac{\alpha - 1}{\beta - 1}} \right) \\ &\leq \frac{1}{\alpha - 1} \log \left( |p| \left( \sum_{i \in I} \frac{p_i}{|p|} \left( \frac{p_i}{q_i} \right)^{1 - \beta} \right)^{\frac{\alpha - 1}{\beta - 1}} \right) \\ &\leq \frac{1}{\alpha - 1} \log \left( |p| \cdot \sum_{i \in I} \frac{p_i}{|p|} \left( \frac{p_i}{q_i} \right)^{1 - \beta} \right)^{\frac{\alpha - 1}{\beta - 1}} \\ &= \frac{1}{\beta - 1} \log \sum_{i \in I} p_i^{\beta} q_i^{1 - \beta} \end{split}$$

This completes the proof.

PROPOSITION C.2 (PROPOSITION 4.2). For any  $\alpha > 1$ ,  $\mu_1, \mu_2, \mu_3 \in GX$ , and p, q > 1 satisfying  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$D_X^{\alpha}(\mu_1||\mu_3) \le \frac{p\alpha - 1}{p(\alpha - 1)} D_X^{p\alpha}(\mu_1||\mu_2) + D_X^{\frac{q}{p}(p\alpha - 1)}(\mu_1||\mu_2).$$

PROOF. Recall that if  $\mu_1 \ll \mu_2$  then  $D_X^{\alpha}(\mu_1 || \mu_2) = \infty$ . Hence, we may assume  $\mu_1 \ll \mu_2 \ll \mu_3$  without loss of generality (if not so, the right-hand side should be infinity). By chain rule of Radon-Nikodym derivative and Hölder's inequality,

$$\begin{split} \Delta_X^{\mathrm{R}(\alpha)}(\mu_1,\mu_3) &= \int_X \left(\frac{d\mu_1}{d\mu_3}\right)^{\alpha} d\mu_3 \\ &= \int_X \left(\frac{d\mu_1}{d\mu_2} \cdot \frac{d\mu_2}{d\mu_3}\right)^{\alpha} d\mu_3 \\ &= \int_X \left(\frac{d\mu_1/d\mu_3}{d\mu_2/d\mu_3}\right)^{\alpha} \cdot \left(\frac{d\mu_2}{d\mu_3}\right)^{\frac{1}{p}} \cdot \left(\frac{d\mu_2}{d\mu_3}\right)^{\alpha-\frac{1}{p}} d\mu_3 \\ &\leq \left(\int_X \left(\frac{d\mu_1/d\mu_3}{d\mu_2/d\mu_3}\right)^{p\alpha} \cdot \left(\frac{d\mu_2}{d\mu_3}\right) d\mu_3\right)^{\frac{1}{p}} \cdot \left(\int_X \left(\frac{d\mu_2}{d\mu_3}\right)^{q(\alpha-\frac{1}{p})} d\mu_3\right)^{\frac{1}{q}} \\ &= \Delta_X^{\mathrm{R}(p\alpha)}(\mu_1||\mu_2)^{\frac{1}{p}} \cdot \Delta_X^{\mathrm{R}(q\alpha-\frac{q}{p})}(\mu_2||\mu_3)^{\frac{1}{q}} \end{split}$$

We then conclude  $D_X^{\alpha}(\mu_1||\mu_3) \le \frac{p\alpha-1}{p(\alpha-1)} D_X^{p\alpha}(\mu_1||\mu_2) + D_X^{\frac{q}{p}(p\alpha-1)}(\mu_1||\mu_2).$ 

THEOREM C.5 (THEOREM 4.12). The  $\mathbb{R}_{\geq 0}$ -graded family  $\Delta^{\text{zCDP}} = {\Delta^{\text{zCDP}(\xi)}}_{0 \leq \xi}$  is reflexive, continuous, composable, and additive.

PROOF. Consider any  $\alpha > 1$ . We consider a  $(\mathbb{R}_{\geq 0}, +, 0, \leq)$ -graded family  $\Delta^{\mathsf{zCDP}+(\alpha)} = {\Delta^{\mathsf{zCDP}+(\xi, \alpha)}}_{\xi \in \mathbb{R}_{\geq 0}}$  of the following divergences:

$$\Delta_X^{\text{zCDP}+(\xi,\alpha)}(\mu_1,\mu_2) = \frac{1}{\alpha} \left( D^{\alpha}(\mu_1||\mu_2) \right) - \xi \right).$$

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

By the previous theorem 4.11, this family is reflexive and continuous for any  $\alpha > 1$ . The composability of the family  $\Delta^{zCDP+(\alpha)} = {\Delta^{zCDP+(\xi,\alpha)}}_{\xi \in \mathbb{R}_{\geq 0}}$  is the direct consequence of the composability of  $\alpha$ -Rényi divergence: for any  $\mu_1, \mu_2 \in \mathcal{G}X$ , and  $f, g: X \to \mathcal{G}Y$ ,

$$\frac{1}{\alpha}(D_X^{\alpha}(f^{\sharp}(\mu_1)||g^{\sharp}(\mu_2)) - (\xi_1 + \xi_2)) \le \frac{1}{\alpha}(D_X^{\alpha}(\mu_1||\mu_2) - \xi_1) + \sup_{x \in X} \frac{1}{\alpha}(D_Y^{\alpha}(f(x)||g(y)) - \xi_2) \le \frac{1}{\alpha}(D_X^{\alpha}(f^{\sharp}(\mu_1)||g^{\sharp}(\mu_2)) - \xi_1) + \frac{1}{\alpha}(D_X^{\alpha}(f^{\sharp}(\mu_1)||g^{\sharp}(\mu_2)) - \xi_1) + \frac{1}{\alpha}(D_X^{\alpha}(f^{\sharp}(\mu_1)||g^{\sharp}(\mu_2)) - \xi_1) + \frac{1}{\alpha}(D_X^{\alpha}(f^{\sharp}(\mu_1)||g^{\sharp}(\mu_2)) - \xi_1) + \frac{1}{\alpha}(D_X^{\alpha}(\mu_1)||g^{\sharp}(\mu_2)) - \frac{1}{\alpha}(D_X^{\alpha}(\mu_1)||g^{\sharp}(\mu_2)) - \xi_1) + \frac{1}{\alpha}(D_X^{\alpha}(\mu_1)||g^{\sharp}(\mu_2)) - \frac{1}{\alpha}(D_X^{\alpha}(\mu_2)||g^{\sharp}(\mu_2)) - \frac{1}{\alpha}(D_X^{\alpha}(\mu_2)||g^{\sharp}(\mu_2)||g^{\sharp}(\mu_2)) - \frac{1}{\alpha}(D_X^{\alpha}(\mu_2)||g^{\sharp}(\mu_2)||g^{\sharp}(\mu_2)) - \frac{1}{\alpha}(D_X^{\alpha}(\mu_2)||g^{\sharp}(\mu_2)) - \frac{1}{\alpha}(D_$$

Since  $\Delta^{zCDP(\xi)} = \sup_{\alpha>1} \Delta^{zCDP+(\xi,\alpha)}$ , the graded family  $\Delta^{zCDP} = {\Delta^{zCDP(\xi)}}_{0 \le \xi}$  is reflexive, continuous, and composable.<sup>9</sup> The additivity is obtained from Theorem 4.4.

# C.1 Detailed Construction and Proof of Well-definedness of Approximate Span-lifting

Definition C.6 (Functors). If the family  $\Delta$  is functorial then the structure of endofunctor on **Span(Meas)** of the approximate span-lifting  $(-)^{\sharp(\Delta,\alpha,\delta)}$  is given as follows: for all  $\alpha \in A, \delta \in \mathbb{R}$ , and  $(h,k,l): (X, Y, \Phi, \rho_1, \rho_2) \to (X', Y', \Psi, \rho'_1, \rho'_2)$  in **Span(Meas**),

$$(\mathcal{G}h, \mathcal{G}k, (\mathcal{G}l \times \mathcal{G}l)|_{W(\Phi, \Delta, \alpha, \delta)}) \colon (X, Y, \Phi, \rho_1, \rho_2)^{\sharp(\Delta, \alpha, \delta)} \to (X', Y', \Psi, \rho_1', \rho_2')^{\sharp(\Delta, \alpha, \delta)}.$$
(9)

THEOREM C.7 (Well-DEFINEDNESS). If  $\Delta$  is functorial then the above structure  $(-)^{\sharp(\Delta, \alpha, \delta)}$  forms indeed an endofunctor on Span(Meas).

PROOF. We first show the well-definedness of (9). We fix  $(h, k, l): (X, Y, \Phi, \rho_1, \rho_2) \rightarrow (X', Y', \Psi, \rho'_1, \rho'_2)$ in **Span(Meas**) and parameters  $\alpha \in A$  and  $\delta \in \mathbb{R}$ . Let  $(\nu_1, \nu_2) \in W(\Phi, \Delta, \alpha, \delta)$ . The pair satisfies  $\Delta^{\alpha}_{\Phi}(\nu_1, \nu_2) \leq \delta$ . Since the divergence  $\Delta^{\alpha}$  is functorial, we have  $\Delta^{\alpha}_{\Psi}(\mathcal{G}(l)(\nu_1), \mathcal{G}(l)(\nu_2)) \leq \delta$ . Thus,  $(\mathcal{G}l \times \mathcal{G}l)|_{W(\Phi, \Delta, \alpha, \delta)}$  is a measurable function from  $W(\Phi, \Delta, \alpha, \delta)$  to  $W(\Psi, \Delta, \alpha, \delta)$ .<sup>10</sup> Since  $\mathcal{G}$  is a functor on **Meas**, we obtain,

$$\begin{split} \mathcal{G}\rho_{1}^{\prime} \circ \pi_{1} \circ (\mathcal{G}l \times \mathcal{G}l)|_{W(\Phi, \Delta, \alpha, \delta)} &= \mathcal{G}\rho_{1}^{\prime} \circ \mathcal{G}l \circ \pi_{1}|_{W(\Phi, \Delta, \alpha, \delta)} = \mathcal{G}h \circ \mathcal{G}\rho_{1} \circ \pi_{1}|_{W(\Phi, \Delta, \alpha, \delta)},\\ \mathcal{G}\rho_{2}^{\prime} \circ \pi_{2} \circ (\mathcal{G}l \times \mathcal{G}l)|_{W(\Phi, \Delta, \alpha, \delta)} &= \mathcal{G}\rho_{2}^{\prime} \circ \mathcal{G}l \circ \pi_{2}|_{W(\Phi, \Delta, \alpha, \delta)} = \mathcal{G}k \circ \mathcal{G}\rho_{2} \circ \pi_{2}|_{W(\Phi, \Delta, \alpha, \delta)}. \end{split}$$

Thus, the construction (9) is a mapping on Span(Meas)-morphisms.

The functoriality is obvious by definition.

Definition C.8 (Graded monad structures). If the family  $\Delta$  is reflexive and composable then the structure of  $A \times (\overline{\mathbb{R}}, +, 0, \leq)$ -graded monad on Span(Meas) is given as follows.

**Unit:** for any span ( $X, Y, \Phi, \rho_1, \rho_2$ ), we define

$$(\eta_X, \eta_Y, \langle \eta_\Phi, \eta_\Phi \rangle) \colon (X, Y, \Phi, \rho_1, \rho_2) \to (X, Y, \Phi, \rho_1, \rho_2)^{\sharp(\Delta, 1_A, 0)}.$$
(10)

**Kleisli extensions:** for any morphism (h, k, l):  $(X, Y, \Phi, \rho_1, \rho_2) \rightarrow (X', Y', \Psi, \rho'_1, \rho'_2)^{\sharp(\Delta, \alpha, \delta)}$  in **Span**(Meas), we define

$$(h^{\sharp}, k^{\sharp}, ((\pi_{1}|_{W(\Psi, \Delta, \alpha, \delta)} \circ l)^{\sharp} \times (\pi_{2}|_{W(\Psi, \Delta, \alpha, \delta)} \circ l)^{\sharp})|_{W(\Phi, \Delta, \beta, \gamma)}):$$

$$(X, Y, \Phi, \rho_{1}, \rho_{2})^{\sharp(\Delta, \beta, \gamma)} \to (X', Y', \Psi, \rho'_{1}, \rho'_{2})^{\sharp(\Delta, \alpha\beta, \delta+\gamma)}$$
(11)

**Inclusions:** for any  $\alpha \leq \beta$ ,  $\delta \leq \gamma$ , and  $(X, Y, \Phi, \rho_1, \rho_2)$  in **Span(Meas**), we define

$$(\mathrm{id}_{\mathcal{G}X}, \mathrm{id}_{\mathcal{G}Y}, (\mathrm{id}_{\mathcal{G}\Phi} \times \mathrm{id}_{\mathcal{G}\Phi})|_{W(\Phi, \Delta, \alpha, \delta)}) \colon (X, Y, \Phi, \rho_1, \rho_2)^{\sharp(\Delta, \alpha, \underline{\delta})} (X, Y, \Phi, \rho_1, \rho_2)^{\sharp(\Delta, \beta, \gamma)}.$$
(12)

We remark here that each  $(-)^{\sharp(\Delta, \alpha, \delta)}$  is also an endofunctor because  $\Delta$  is the functorial since it is both reflexive and composable.

<sup>9</sup>We obtain these properties from commutativity  $\sup_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \sup_{y \in Y} f(x, y)$  of supremums. We drop the approximability, which is not given by a supremum but rather by a limit.

Proc. ACM Program. Lang., Vol. 1, No. CONF, Article 1. Publication date: January 2018.

<sup>&</sup>lt;sup>10</sup>Strictly speaking, we consider the function  $W(\Phi, \Delta, \alpha, \delta) \xrightarrow{(\mathcal{G}l \times \mathcal{G}l)|_{W(\Phi, \Delta, \alpha, \delta)}} (Image) \xrightarrow{\text{inclusion}} W(\Phi, \Delta, \alpha, \delta)$  through the image (Image). Functoriality shows the existence of the inclusion.

THEOREM C.9 (WELL-DEFINEDNESS). If  $\Delta$  is reflexive and composable then the above structures  $(-)^{\sharp(\Delta,\alpha,\delta)}$  form indeed an  $A \times \overline{\mathbb{R}}$ -graded monad on Span(Meas).

PROOF. We first prove that the components are well-defined.

**Unit:** We show the well-definedness of (10). We fix  $(X, Y, \Phi, \rho_1, \rho_2)$  in **Span(Meas**). For any  $\phi \in \Phi$ , we have  $\langle \eta_{\Phi}, \eta_{\Phi} \rangle(\phi) = (\mathbf{d}_{\phi}, \mathbf{d}_{\phi})$ . Since  $\Delta$  is reflexive, we have  $\Delta^{1_A}(\mathbf{d}_{\phi}, \mathbf{d}_{\phi}) \leq 0$ . Thus,  $\langle \eta_{\Phi}, \eta_{\Phi} \rangle$  is indeed a measurable function from  $(X, Y, \Phi, \rho_1, \rho_2)$  to  $W(\Phi, \Delta, 1_A, 0)$ . Since  $\eta$  is a unit of the sub-Giry monad  $\mathcal{G}$ , we obtain

$$\begin{aligned} \mathcal{G}\rho_1 \circ \pi_1 |_{W(\Phi, \Delta, 1_A, 0)} \circ \langle \eta_{\Phi}, \eta_{\Phi} \rangle &= \mathcal{G}\rho_1 \circ \eta_{\Phi} = \eta_X \circ \rho_1, \\ \mathcal{G}\rho_2 \circ \pi_2 |_{W(\Phi, \Delta, 1_A, 0)} \circ \langle \eta_{\Phi}, \eta_{\Phi} \rangle &= \mathcal{G}\rho_2 \circ \eta_{\Phi} = \eta_Y \circ \rho_2. \end{aligned}$$

Thus (10) is well-defined.

Kleisli extensions: We show the well-definedness of (11). We fix a Span(Meas)-morphism

$$(h, k, l): (X, Y, \Phi, \rho_1, \rho_2) \rightarrow (X', Y', \Psi, \rho_1', \rho_2')^{\sharp(\Delta, \alpha, \delta)}$$

and parameters  $\beta \in A$  and  $\gamma \in \mathbb{R}$ . For any  $\phi \in \Phi$ , we have  $\Delta_{\Psi}^{\alpha}(\pi_1|_{W(\Psi, \Delta, \alpha, \delta)} \circ l(\phi), \pi_2|_{W(\Psi, \Delta, \alpha, \delta)} \circ l(\phi)) \leq \delta$ . Since  $\Delta$  is composable, we have for any  $(\nu_1, \nu_2) \in W(\Phi, \Delta, \delta, \gamma)$ ,

$$\Delta_{\Psi}^{\alpha\beta}((\pi_1|_{W(\Psi,\Delta,\alpha,\delta)} \circ l)^{\sharp}(\nu_1), (\pi_2|_{W(\Psi,\Delta,\alpha,\delta)} \circ l)^{\sharp}(\nu_2)) \le \delta + \gamma$$

This implies that  $((\pi_1|_{W(\Psi, \Delta, \alpha, \delta)} \circ l)^{\sharp} \times (\pi_2|_{W(\Psi, \Delta, \alpha, \delta)} \circ l)^{\sharp})|_{W(\Phi, \Delta, \beta, \gamma)}$  is indeed a measurable function from  $W(\Phi, \Delta, \beta, \gamma)$  to  $W(\Psi, \Delta, \alpha\beta, \delta + \gamma)$ . Since  $(-)^{\sharp}$  is the Kleisli lifting of the sub-Giry monad, we obtain

$$\begin{split} \mathcal{G}\rho_{1}^{\prime} \circ \pi_{1}|_{W(\Psi,\Delta,\alpha\beta,\delta+\gamma)} \circ \left(\left(\pi_{1}|_{W(\Psi,\Delta,\alpha,\delta)} \circ l\right)^{\sharp} \times \left(\pi_{2}|_{W(\Psi,\Delta,\alpha,\delta)} \circ l\right)^{\sharp}\right)|_{W(\Phi,\Delta,\beta,\gamma)} \\ &= \mathcal{G}\rho_{1}^{\prime} \circ \left(\pi_{1}|_{W(\Psi,\Delta,\alpha,\delta)} \circ l\right)^{\sharp} \circ \pi_{1}|_{W(\Phi,\Delta,\beta,\gamma)} = \left(\mathcal{G}\rho_{1}^{\prime} \circ \pi_{1}|_{W(\Psi,\Delta,\alpha,\delta)} \circ l\right)^{\sharp} \circ \pi_{1}|_{W(\Phi,\Delta,\beta,\gamma)} \\ &= h^{\sharp} \circ \mathcal{G}\rho_{1} \circ \pi_{1}|_{W(\Phi,\Delta,\beta,\gamma)} \\ \mathcal{G}\rho_{2}^{\prime} \circ \pi_{2}|_{W(\Psi,\Delta,\alpha\beta,\delta+\gamma)} \circ \left(\left(\pi_{1}|_{W(\Psi,\Delta,\alpha,\delta)} \circ l\right)^{\sharp} \times \left(\pi_{2}|_{W(\Psi,\Delta,\alpha,\delta)} \circ l\right)^{\sharp}\right)|_{W(\Phi,\Delta,\beta,\gamma)} \\ &= k^{\sharp} \circ \mathcal{G}\rho_{2} \circ \pi_{2}|_{W(\Phi,\Delta,\beta,\gamma)} \end{split}$$

Thus (11) is well-defined.

**Inclusions:** We show the well-definedness of (12). We fix  $(X, Y, \Phi, \rho_1, \rho_2)$  in **Span(Meas)** and parameters  $\alpha \leq \beta$  and  $\delta \leq \gamma$ . Since  $\Delta$  is an *A*-graded family of divergences, we have  $\Delta^{\beta} \leq \Delta^{\alpha}$ . This implies that there is the inclusion function  $W(\Phi, \Delta, \alpha, \delta) \hookrightarrow W(\Phi, \Delta, \beta, \gamma)$  in **Meas**. Hence, by treating the restrictions of functions, we obtain

$$\begin{split} &\mathrm{id}_{\mathcal{G}X}\circ\mathcal{G}\rho_{1}\circ\pi_{1}|_{W(\Phi,\Delta,\alpha,\delta)}=\mathcal{G}\rho_{1}\circ\pi_{1}\circ(\mathrm{id}_{\mathcal{G}\Phi}\times\mathrm{id}_{\mathcal{G}\Phi})|_{W(\Phi,\Delta,\alpha,\delta)}\\ &\mathrm{id}_{\mathcal{G}Y}\circ\mathcal{G}\rho_{2}\circ\pi_{2}|_{W(\Phi,\Delta,\alpha,\delta)}=\mathcal{G}\rho_{2}\circ\pi_{2}\circ(\mathrm{id}_{\mathcal{G}\Phi}\times\mathrm{id}_{\mathcal{G}\Phi})|_{W(\Phi,\Delta,\alpha,\delta)}. \end{split}$$

Therefore (12) is well defined.

Therefore, the components of graded monad structures are well-defined. It is easy to check the axioms of graded monad in Katsumata [2014, Definition 2.3] by using monad structure of the sub-Giry monad  $\mathcal{G}$  since the graded monad structure of the approximate span-lifting is given by using the monad structure of  $\mathcal{G}$  and restrictions.

Definition C.10 (Double strength). If the family  $\Delta$  is reflexive, composable, and *additive* then a *double strength* of the graded monad  $(-)^{\sharp(\Delta, \alpha, \delta)}$  is given as follows: for any pair  $(X, Y, \Phi, \rho_1, \rho_2)$  and

 $(X', Y', \Psi, \rho'_1, \rho'_2)$  of spans,

$$(\operatorname{dst}_{X,X'},\operatorname{dst}_{Y,Y'},\langle\operatorname{dst}_{\Phi,\Psi}\circ(\pi_1\times\pi_1),\operatorname{dst}_{\Phi,\Psi}\circ(\pi_2\times\pi_2)\rangle|_{W(\Phi,\Delta,\alpha,\delta)\times W(\Psi,\Delta,\beta,\gamma)}):$$
$$(X,Y,\Phi,\rho_1,\rho_2)^{\sharp(\Delta,\alpha,\delta)} \dot{\times} (X',Y',\Psi,\rho_1',\rho_2')^{\sharp(\Delta,\beta,\gamma)} \to (\Phi \dot{\times} \Psi)^{\sharp(\Delta,\alpha\beta,\delta+\gamma)}.$$
(13)

THEOREM C.11 (WELL-DEFINEDNESS (THEOREM 5.3)). If  $\Delta$  is reflexive, composable, and additive then the above structure forms indeed a double strength of the graded monad  $(-)^{\sharp(\Delta, \alpha, \delta)}$  on Span(Meas).

PROOF. Since  $\Delta$  is reflexive and composable,  $(-)^{\sharp(\Delta, \alpha, \delta)}$  forms an  $A \times \mathbb{R}$ -graded monad on **Span(Meas**). We show the well-definedness of (13). We fix spans  $(X, Y, \Phi, \rho_1, \rho_2)$  and  $(X', Y', \Psi, \rho'_1, \rho'_2)$  and parameters  $\alpha, \beta \in A$  and  $\gamma, \delta \in \mathbb{R}$ . Since  $\Delta$  is additive,  $\langle dst_{\Phi,\Psi} \circ (\pi_1 \times \pi_1), dst_{\Phi,\Phi'} \circ (\pi_2 \times \pi_2) \rangle|_{W(\Phi, \Delta, \alpha, \delta) \times W(\Psi, \Delta, \beta, \gamma)}$  is indeed a measurable function from  $W(\Phi, \Delta, \alpha, \delta) \times W(\Psi, \Delta, \beta, \gamma)$  to  $W(\Phi \times \Psi, \Delta, \alpha\beta, \delta + \gamma)$ . From the binaturality of the double strength dst of the sub-Giry monad  $\mathcal{G}$ , we have

$$\begin{split} &\mathcal{G}(\rho_{1}\times\rho_{1}')\circ\pi_{1}|_{W(\Phi\dot{\times}\Psi,\Delta,\alpha\beta,\delta+\gamma)}\circ\langle\mathrm{dst}_{\Phi,\Psi}\circ(\pi_{1}\times\pi_{1}),\mathrm{dst}_{\Phi,\Psi}\circ(\pi_{2}\times\pi_{2})\rangle|_{W(\Phi,\Delta,\alpha,\delta)\times W(\Psi,\Delta,\beta,\gamma)} \\ &=\mathcal{G}(\rho_{1}\times\rho_{1}')\circ\mathrm{dst}_{\Phi,\Psi}\circ(\pi_{1}\times\pi_{1})|_{W(\Phi,\Delta,\alpha,\delta)\times W(\Psi,\Delta,\beta,\gamma)} \\ &=\mathcal{G}(\rho_{1}\times\rho_{1}')\circ\mathrm{dst}_{\Phi,\Psi}\circ(\pi_{1}|_{W(\Phi,\Delta,\alpha,\delta)}\times\pi_{1}|_{W(\Psi,\Delta,\beta,\gamma)}) \\ &=\mathrm{dst}_{X,X'}\circ((\mathcal{G}\rho_{1}\circ\pi_{1}|_{W(\Phi,\Delta,\alpha,\delta)})\times(\mathcal{G}\rho_{1}'\circ\pi_{1}|_{W(\Psi,\Delta,\beta,\gamma)})), \\ &\mathcal{G}(\rho_{2}\times\rho_{2}')\circ\pi_{1}|_{W(\Phi\dot{\times}\Psi,\Delta,\alpha\beta,\delta+\gamma)}\circ\langle\mathrm{dst}_{\Phi,\Psi}\circ(\pi_{1}\times\pi_{1}),\mathrm{dst}_{\Phi,\Psi}\circ(\pi_{2}\times\pi_{2})\rangle|_{W(\Phi,\Delta,\alpha,\delta)\times W(\Psi,\Delta,\beta,\gamma)} \\ &=\mathrm{dst}_{Y,Y'}\circ((\mathcal{G}\rho_{2}\circ\pi_{1}|_{W(\Phi,\Delta,\alpha,\delta)})\times(\mathcal{G}\rho_{2}'\circ\pi_{1}|_{W(\Psi,\Delta,\beta,\gamma)})). \end{split}$$

Hence, (13) is well-defined. It is easy to check the axioms of double strength (modulo grading) by using double strength of the sub-Giry monad  $\mathcal{G}$ .