Responsibility and verification: Importance value in temporal logics

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Abstract—We aim at measuring the influence of the nondeterministic choices of a part of a system on its ability to satisfy a specification. For this purpose, we apply the concept of Shapley values to verification as a means to evaluate how important a part of a system is. The importance of a component is measured by giving its control to an adversary, alone or along with other components, and testing whether the system can still fulfill the specification. We study this idea in the framework of model-checking with various classical types of linear-time specification, and propose several ways to transpose it to branching ones. We also provide tight complexity bounds in almost every case.

I. INTRODUCTION

Classical model-checking algorithms try to detect undesired behaviors in a formal system with reference to a given specification, and the system is deemed correct if they cannot find one. However, simply knowing that the system satisfies the specification is in practice often unsatisfactory: we also want to know why it does, or does not. Especially in the case that the specification is violated, knowing where in the system to look for a potential model repair can significantly reduce troubleshooting times for both engineers and users.

To this end, Chockler, Halpern and Kupferman defined a notion of *causality* aimed at explaining which parts of a system are relevant for the satisfaction of a specification φ [1]. More specifically, a state s is considered a cause for φ with respect to an atomic proposition p if the value of p can be swapped in a subset of the states T such that further swapping the value of p in s turns φ from being satisfied to being violated (we say that (s,T) is *critical*). Counterfactual reasoning in this spirit (i.e., had the cause not occurred, then the event would not have happened) has a rich history in the philosophy and moral responsibility literature, and has been formalized in the framework of structural equation models [2], [3], on which the work [1] is based. Causes are further assigned a degree of responsibility by taking the inverse of the size of the smallest set $T \cup \{s\}$ such that (s, T) is critical. This numerical value, adapted from [4], is designed to measure the impact of the state on the specification: Causes with high degree of responsibility

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point to small changes of the system that have the power to crucially alter its behavior.

In this paper we define a novel measure for the influence of a state on a specification, called the importance. While it is related to the degree of responsibility of [1], a significant difference appears in how the counterfactuality principle is invoked. The degree of responsibility relies on hypothetical modifications of the structure and answers the question "Is the system still working if the truth value of this atomic proposition in that subset of states is switched?" In contrast, we never modify the system, but look at how its nondeterministic choices are resolved, thus tackling the question "Does the system yield a satisfying run if the subset of states is under control (i.e., behaving in a manner conducive to the functioning) while the others are not (i.e., behaving antagonistically)?" Hence, our definition of importance relies on a new viewpoint of what constitutes a critical pair, based on capturing the specific nondeterministic choices available in

The approach above determines the impact of a subset of states on the satisfaction of a specification. In order to turn this information into the *individual* importance of a state (or a component) we employ a solution concept from cooperative game theory, called the *Shapley value* [5]. In a context of collaborative multi-agent interaction, Shapley values aim at measuring how beneficial the participation of a specific agent is in reaching some objective. Translated to Kripke structures, the idea is to compute the probability that taking control over a particular state makes the system work as intended, where the control over states is taken in a (uniformly) random order. The importance distills those parts of the system whose choices are *crucial* for its functioning.

As an example, consider a system testing a server sv by sending regular requests. If the server does not respond correctly, the system retries to send a request; if it does respond correctly, then the system may wait before testing again. We represent this system by a Kripke structure, displayed on the left in Figure 1. Consider the specification stating that the system should make infinitely many tests and receive only finitely many incorrect answers (modeled by the LTL formula $\varphi = GF \operatorname{check} \wedge FG \neg \operatorname{fail}$). The system fails this condition if sv malfunctions and fails infinitely often or if ok waits indefinitely from some point on without rechecking the server. As the other states cannot enforce breaching φ without sv and

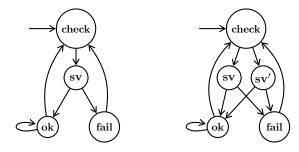


Fig. 1. Two simple systems, as used in the introductory example.

 \mathbf{ok} , the importance of these two states is 1/2 and that of the other states is 0.

Let us now add a backup server sv' with the same role as sv (as displayed on the right in Figure 1). Then the system succeeds if it loops infinitely often between ok, check and the set $\{sv, sv'\}$, which is only possible if ok and at least two of check, sv and sv' behave well. In this case we get an importance of 1/2 for ok and 1/6 for check, sv and sv' (these values are explained in detail in Example 1). This is a numerical interpretation of the fact that control over the behavior of ok is more critical to the functioning of the system: unfortunate choices made in ok (i.e., avoiding further tests forever) instantly make the system fail. The equal importance of check, sv, and sv' reflects the fact that – although their actual roles in the system differ – they play interchangeable parts when only the functioning is concerned: any two of them are needed to make the system work.

It is noteworthy that a variant of the degree of responsibility based on our notion of critical pair (and applied in reverse fashion, i.e., from violation of φ to satisfaction of φ) would not be able to distinguish **check**, **ok**, **sv**, and **sv**' as it evaluates to 1/3 for each of these states. Roughly speaking, the degree of responsibility only takes a *minimal* critical pair into account, whereas the importance computes a weighted average over the size of *all* critical pairs that a state belongs to. The rationale for this is that belonging to many critical pairs makes the state less dependent on behavior outside of its control, and hence more powerful.

The construction of the importance value as outlined above gives rise to the following three problems, whose complexity we study in this paper for a wide range of specifications. The *value problem* consists in determining if a subset of states of a given Kripke structure can guarantee that the specification is respected when the other states act in an adversarial way. The *importance problem* asks for the actual importance value. Finally, the *usefulness problem* asks whether a state of a system has positive importance, i.e., whether its behavior has any influence at all on the satisfaction of the specification. In fact we define the importance value in the presence of a prescribed partition of the state space, and study the complexity problems in this generalized setting. This allows us to capture more realistic scenarios such as the importance of a *system component* in a composite architecture.

Table I summarizes the complexity results obtained through-

out the paper. We write $\in \mathcal{C}$ when the problem is in class \mathcal{C} and we do not have a matching lower bound, and just \mathcal{C} when the problem is \mathcal{C} -complete. Since our examinations spread over a wide range of specifications, our results crucially rely on a diverse game-theoretic toolkit.

The paper is split into three parts: In the first part we define the notions in the general setup of turn-based two-player games on finite graphs. Then we apply these notions in order to define the importance on Kripke structures with respect to LTL specifications, and finally we look at the case of CTL specifications on modal transition systems. The proofs missing in the main document due to space constraints can be found in the appendix.

A. Related work

The complexity of computing the aforementioned degree of responsibility was examined for the general class of structural equation models in [4] and for Boolean circuits in [1]. They are closely related to the complexity results about deciding causality [12], [13].

Our work ties into a ubiquitous quest for powerful *explanations* of model-checking results. If a system satisfies a specification, then *coverage estimation* has been used to analyze which parts of the system are essential for the successful verification result [14]–[17]. As in the definition of the degree of responsibility, the idea is to apply small changes to the system (*mutants*) and check the resulting effect on the specification. *Vacuity detection*, on the other hand, applies the principle of small changes to the specification [18]–[20]. This strand of research aims at checking whether the specification is satisfied in an undesired, trivial fashion (typically due to insufficient modeling of the system). Coverage and vacuity have been shown to exhibit a formal duality [21], and recent work on the subject analyzes network formation games [22].

In the case of an unsuccessful verification process, one of the powerful features of many model checking approaches is the ability to generate a counterexample [23]. In order to extract further diagnostic information, there has been extensive work on *localizing* errors in faulty traces [24]–[29]. Typically, one compares an erroneous trace with a successful one that lies nearby with respect to a suitable metric. Early detection of error traces has been investigated in [30], where a gamelike description close to ours between a system module and its environment has been used. Explaining counterexamples using the notion of causality from [1] has been presented in [31].

The Shapley value is a classical solution concept in economics and has recently received considerable attention in the computer science literature. Shapley-like values have been used as explanations for machine learning models, where they estimate the impact of the input parameters on the outcome [32]–[34]. They have also been employed as a means by which centrality in networks can be measured [35] or responsibilities can be assigned in game-like structures [36]. Computational approaches for the Shapley value are given in [37]–[40]. For a variety of recent results and applications of Shapley values we refer to [41].

TABLE I
A SUMMARY OF THE RESULTS ON THE COMPLEXITY OF THE VALUE, USEFULNESS AND IMPORTANCE PROBLEMS FOR VARIOUS TYPES OF SPECIFICATIONS.

	Büchi	Rabin	Streett	Parity	Explicit Muller
Value	P [6]	NP [7]	coNP [7]	$\in NP \cap coNP \ [8]$	P [9] [6]
Usefulness	NP (Prop. IV.5)	$\Sigma_2^{\rm P}$ (Prop. IV.8)	$\Sigma_2^{\rm P}$ (Cor. IV.10)	NP (Prop. IV.5)	NP (Prop. IV.5)
Importance	#P (Thm. IV.6)	#P ^{NP} (Thm. IV.9)	#PNP (Cor. IV.10)	#P (Thm. IV.6)	#P (Thm. IV.6)
	Emerson-Lei	LTL	2-turn CTL	Concurrent CTL	
Value	Emerson-Lei PSPACE [10]	LTL 2ExpTime [11]	2-turn CTL $\Sigma_2^{\rm P}$ (Prop. V.2)	Concurrent CTL ∈ EXPTIME (Rmk. V.7)	
Value Usefulness			-		

II. PRELIMINARIES

A. Words and structures

a) Words and trees: Let A be an alphabet. We denote by A^* (resp. A^{ω}) the set of finite (resp. infinite) words over A. Given a word w, we write |w| for its length and, for all $0 \le i < |w|$, we write w_i for the (i+1)th letter of w.

An infinite tree t over A is a prefix-closed subset of A^* such that for all $p \in t$, there exists $a \in A$ such that $pa \in t$. The set of sons of a node p of the tree t is denoted by $Sons_t(p) = pA \cap t$.

- b) Kripke structures: A Kripke structure \mathcal{K} is a 5-tuple $(S, \operatorname{AP}, \Delta, \operatorname{init}, \lambda)$ where S is a finite set of states, AP is a finite set of atomic propositions, $\Delta \subseteq S \times S$ is a set of transitions, init is an initial state and $\lambda: S \to 2^{\operatorname{AP}}$ is a labeling function. For every $s \in S$, we define its image under Δ as $\Delta(s) = \{t \in S \mid (s,t) \in \Delta\}$, and we always assume $\Delta(s)$ to be nonempty for all s. A run of a Kripke structure $\mathcal{K} = (S, \operatorname{AP}, \Delta, \operatorname{init}, \lambda)$ is an infinite sequence $r \in S^\omega$ such that $r_0 = \operatorname{init}$ and for all $i \in \mathbb{N}$, we have $(r_i, r_{i+1}) \in \Delta$. To every run r we can associate a trace, which is the sequence of labelings $\lambda(r_0)\lambda(r_1)\cdots$. The set of runs of \mathcal{K} is denoted by $\mathcal{R}(\mathcal{K})$ while the set of traces it generates is called $\mathcal{L}(\mathcal{K})$.
- c) Modal transition systems: A modal transition system (MTS) [42] \mathcal{M} is a 6-tuple $(S, \operatorname{AP}, \Delta_{may}, \Delta_{must}, init, \lambda)$ where S is a finite set of states, AP is a set of atomic propositions, $\Delta_{must}, \Delta_{may} \subseteq S \times S$ are sets of transitions such that $\Delta_{must} \subseteq \Delta_{may}$, $init \in S$ is an initial state and $\lambda: S \to 2^{\operatorname{AP}}$ is a labeling function. We assume $\Delta_{must}(s)$ to be nonempty for every state s. We call a Kripke structure $\mathcal{K} = (S, AP, \Delta, init, \lambda)$ an implementation of \mathcal{M} if $\Delta_{must} \subseteq \Delta \subseteq \Delta_{may}$. This is in contrast to other works in the modal transition system literature which usually consider a more general notion of implementation based on refinement relations (see [43] for a recent overview).

B. Temporal logics

We now define the syntax of the two logics we will consider in this paper, LTL and CTL. For the semantics and basic properties of these logics we refer the reader to [44] or [45]. *a) Linear temporal logic:* The formulas of LTL are given by the grammar

$$\varphi ::= a \mid \varphi \vee \varphi \mid \neg \varphi \mid X\varphi \mid \varphi U\varphi$$

with a ranging over a finite set of atomic propositions AP.

LTL formulas are evaluated on infinite words over 2^{AP} . We extend the set of operators with \top, \bot, \land, F, G and R in the usual way.

b) Computation tree logic: The syntax of CTL is defined by the grammar

$$\varphi ::= a \mid \varphi \vee \varphi \mid \neg \varphi \mid EX\varphi \mid E\varphi U\varphi \mid A\varphi U\varphi$$

with a ranging over a finite set of atomic propositions AP.

CTL formulas are evaluated on infinite trees over 2^{AP} . We extend the set of operators with $\top, \bot, \land, EF, EG, ER, AX, AR, AF$ and AG in the usual way.

C. Games on graphs

A directed graph G is a pair (V, E) with V a set of vertices and $E \subseteq V^2$ a set of edges. An arena is a tuple $(G, S_{\mathtt{Sat}}, S_{\mathtt{Unsat}})$ with G a graph and $S_{\mathtt{Sat}}, S_{\mathtt{Unsat}}$ a partition of its vertices. We say that the vertices of $S_{\mathtt{Sat}}$ belong to player \mathtt{Sat} , or are controlled by player \mathtt{Sat} (and similarly for \mathtt{Unsat}). As the graphs we consider will be induced by Kripke structures, we will use from now on S to denote the set of vertices and Δ for the edges. We will also refer to vertices as states and edges as transitions.

A game \mathcal{G} is defined by an arena $((S,\Delta),S_{\mathtt{Sat}},S_{\mathtt{Unsat}})$, an initial vertex $init \in S$ and a winning condition (also called objective) $\Omega \subseteq S^{\omega}$. A play of \mathcal{G} is an infinite sequence $p \in S^{\omega}$ such that $p_0 = init$ and for all $i \in \mathbb{N}$, $(p_i,p_{i+1}) \in \Delta$. A strategy for Sat (resp. Unsat) is a function $\sigma_{\mathtt{Sat}}: S^*S_{\mathtt{Sat}} \to S$ (resp. $\sigma_{\mathtt{Unsat}}: S^*S_{\mathtt{Unsat}} \to S$). A play p is said to respect a strategy σ of Sat (resp. σ of Unsat) if for all $i \in \mathbb{N}$, if $p_i \in S_{\mathtt{Sat}}$ (resp. $p_i \in S_{\mathtt{Unsat}}$) then $p_{i+1} = \sigma(p_0 \dots p_i)$. A strategy σ of Sat (resp. Unsat) is winning if for every play p respecting σ we have $p \in \Omega$ (resp. $p \notin \Omega$). A game \mathcal{G} is determined if there exists a winning strategy for either Sat or Unsat.

For more information on infinite games played on finite graphs we refer to [46]. In particular, we will use several classical winning conditions on such games, whose definitions can be found in [46, Chapter 2].

D. Complexity classes

We consider mostly well-known and classical complexity classes, a description of which can be found, e.g., in [47]. We use logarithmic space reductions for the decision problems and Turing reductions for the counting complexity classes.

III. A GENERAL DEFINITION OF IMPORTANCE IN TWO-PLAYER GAMES

Let \mathcal{G} be a two-player game between Sat and Unsat on an arena $((S,\Delta),S_{\mathtt{Sat}},S_{\mathtt{Unsat}})$. Let $\Omega\subseteq S^\omega$ be Sat's objective (i.e. the set of plays of \mathcal{G} she wins). In order for the game to be determined, we assume Ω to be a Borel set.

We start by defining a general notion of *importance* of a state (or a set of states, in a given partition), which is a measure of how much a state contributes towards Sat winning the game. In other words, if Sat is restricted to controlling only some of her states (for example, due to resource constraints) she should opt to control the ones with high importance in order to win the game.

Definition III.1. For all sets of states $S'_{\mathtt{Sat}} \subseteq S$, we define $\mathcal{G}_{S'_{\mathtt{Sat}}}$ as the game between Sat and Unsat played on the arena $((S, \Delta), S'_{\mathtt{Sat}}, S \setminus S'_{\mathtt{Sat}})$ with the same initial state and the same objective Ω for Sat.

Definition III.2 (Value of a state subset). For all sets of states $S'_{Sat} \subseteq S$, we define the value of the set S'_{Sat} as

$$val(S_{\mathtt{Sat}}') = \begin{cases} 1 \text{ if Sat has a winning strategy for } \mathcal{G}_{S_{\mathtt{Sat}}'} \\ 0 \text{ if Unsat has a winning strategy for } \mathcal{G}_{S_{\mathtt{Sat}}'} \end{cases}$$

Note that the value is defined with respect to a game, but that game does not appear in the notation as we will always make it clear from context. The value is well-defined for all $S'_{\mathtt{Sat}}$ as we assumed the objective to be a Borel set, thus the game is determined.

With the definition of value of a subset of states, we are in the position of defining the importance of a state. This definition corresponds to the classical formula for the Shapley value [5]. In our context it can be explained as follows: for a given state s, it counts the number of orderings of the states in $S_{\mathtt{Sat}}$ such that if \mathtt{Sat} gives up control of her states one by one in that order, then \mathtt{Sat} loses the game for the first time after giving up s. The number obtained is then divided by the total number of such orderings. We can also look at this definition from a probabilistic point of view: The importance of a state s is the probability that, if \mathtt{Sat} gives up control of the states sequentially in an order drawn uniformly at random, the first time \mathtt{Sat} is no longer able to win the game is when she gives up control of s. This is what we call *switching* in Definition III.3 below.

Definition III.3 (Importance). The *importance* for Sat of a state $s \in S_{\mathtt{Sat}}$ with respect to a game $\mathcal G$ on an arena $((S,\Delta),S_{\mathtt{Sat}},S_{\mathtt{Unsat}})$ is defined as

$$\mathcal{I}(s) = \frac{1}{n!} \sum_{\pi \in \Pi_{S_{\text{Sat}}}} val(S_{\geq s}^{\pi}) - val(S_{\geq s}^{\pi} \setminus \{s\})$$

where $n=|S_{\mathtt{Sat}}|$, $\Pi_{S_{\mathtt{Sat}}}$ is the set of bijections from $S_{\mathtt{Sat}}$ to $\{1,\ldots,n\}$, and $S^\pi_{>s}=\{s'\in S_{\mathtt{Sat}}\mid \pi(s')\geq \pi(s)\}.$

An equivalent definition, obtained by deleting the null terms from the sum, is obtained through the notion of *critical pair*. A pair $(s,T) \in S_{\mathtt{Sat}} \times 2^{S_{\mathtt{Sat}}}$ is *critical* if $val(T \cup \{s\}) = 1$ and val(T) = 0. Then we set

$$\mathcal{I}(s) = \frac{1}{n!} \sum_{(s,T) \text{ critical}} (|T|)!(n-|T|-1)!$$

as (|T|)!(n-|T|-1)! is the number of $\pi \in \Pi_{S_{\text{sat}}}$ such that $S^{\pi}_{>s} \setminus \{s\} = T$.

We say that s switches the value in π if $(s, S_{\geq s}^{\pi} \setminus \{s\})$ is a critical pair. The importance of a state s can then be seen as the proportion of orderings π of the states in which s switches the value.

The following lemma states that if a state s needs another state s' (meaning that a set of states containing s but not s' always has value 0), then the importance of s' is at least as large as the one of s.

Lemma III.4. Let s, s' be two states of \mathcal{G} . If for all $T \subseteq S_{\mathtt{Sat}}$ such that $s \in T$ and $s' \notin T$, we have val(T) = 0 then $\mathcal{I}(s) \leq \mathcal{I}(s')$.

Proof. Let $T \subseteq S_{\mathtt{Sat}}$ and suppose that (s,T) is critical. Then T necessarily contains s' (as $val(T \cup \{s\}) = 1$) and therefore $(s',T \cup \{s\} \setminus \{s'\})$ is critical. We can thus construct an injection associating to each T such that (s,T) is critical the set $T' = T \cup \{s\} \setminus \{s'\}$ of equal size such that (s',T') is critical.

We conclude using the second formula for the importance in Definition III.3.

We now assume that we are given a partition S_1, \ldots, S_n of S_{Sat} . We generalize the previous definitions in a straightforward manner. We simply replace states with parts of the partition in the definitions, considering the S_i as atomic elements. In all of our complexity proofs we will show the lower bounds for the previous case (in which states are partitioned in singletons) and the upper bounds for the general case. Thus all complexity results hold for both cases.

Definition III.5 (Importance for partitions). The *importance* for Sat of a set of states S_i with $1 \le i \le n$ is defined as

$$\mathcal{I}(S_i) = \frac{1}{n!} \sum_{\pi \in \Pi_n} val(S_{\geq i}^{\pi}) - val(S_{\geq i}^{\pi} \setminus S_i)$$

where Π_n stands for the set of permutations of $\{1, \ldots, n\}$, and

$$S_{\geq i}^{\pi} = \bigcup_{\substack{1 \leq j \leq n \\ \pi(j) \geq \pi(i)}} S_j$$

We define a pair $(i, J) \in \{1, \dots, n\} \times 2^{\{1, \dots, n\}}$ to be *critical* if $val(\bigcup_{j \in J \cup \{i\}} S_j) = 1$ and $val(\bigcup_{j \in J} S_j) = 0$. Then we have:

$$\mathcal{I}(S_i) = \frac{1}{n!} \sum_{(i,J) \text{ critical}} |J|!(n-|J|-1)!$$

Now let us show some basic results stating that parts with importance 0 can be ignored in the computation of the importance of the other parts.

Remark 1. Let $1 \leq i \leq n$. If $\mathcal{I}(S_i) = 0$ then there is no $J \subseteq \{1,\ldots,n\}$ such that (i,J) is critical. As a consequence, for all $J \subseteq \{1,\ldots,n\}$, $val(\bigcup_{j\in J}S_j) = val(\bigcup_{j\in J\cup\{i\}}S_j)$. This means that if $\mathcal{I}(S_i) = 0$, then Sat can always give up control of states S_i without any effect on whether she wins the game.

Lemma III.6 (Restriction to useful parts). Let $I \subseteq \{1, ..., n\}$ be such that for all $j \notin I$, $\mathcal{I}(S_j) = 0$. Then we have for all $i \in I$

$$\mathcal{I}(S_i) = \frac{1}{|I|!} \sum_{\pi \in \Pi_I} val(S_{\geq i}^{\pi}) - val(S_{\geq i}^{\pi} \setminus S_i),$$

with Π_I the set of bijections from I to $\{1, \ldots, |I|\}$.

Proof. For all $\pi \in \Pi_n$, let us denote by $\pi\big|_I : I \to \{1, \dots, |I|\}$ the bijection such that for all $i, j \in I$, $\pi(i) < \pi(j)$ if and only if $\pi\big|_I(i) < \pi\big|_I(j)$.

Note that for all $i \in I$ and $\pi \in \Pi$ we have

$$S_{\geq i}^{\pi} \setminus S_{\geq i}^{\pi|_I} \subseteq \bigcup_{j \in \{1, \dots, n\} \setminus I} S_j.$$

As a consequence, $val(S_{\geq i}^{\pi}) = val(S_{\geq i}^{\pi|_I})$, using Remark 1. Similarly we get $val(S_{\geq i}^{\pi} \setminus S_i) = val(S_{\geq i}^{\pi|_I} \setminus S_i)$. This allows us to rewrite the importance of S_i as

$$\mathcal{I}(S_i) = \frac{1}{n!} \sum_{\pi \in \Pi_n} val(S_{\geq i}^{\pi}) - val(S_{\geq i}^{\pi} \setminus S_i)$$

$$= \frac{1}{n!} \sum_{\pi \in \Pi_n} val(S_{\geq i}^{\pi|I}) - val(S_{\geq i}^{\pi|I} \setminus S_i)$$

$$= \frac{1}{n!} \sum_{\pi' \in \Pi_I} \frac{n!}{|I|!} \cdot (val(S_{\geq i}^{\pi'}) - val(S_{\geq i}^{\pi} \setminus S_i))$$

$$= \frac{1}{|I|!} \sum_{\pi' \in \Pi_I} val(S_{\geq i}^{\pi'}) - val(S_{\geq i}^{\pi'} \setminus S_i)$$

as for all $\pi' \in \Pi_I$ there are $\frac{n!}{|I|!}$ permutations $\pi \in \Pi_n$ such that $\pi|_I = \pi'$.

Corollary III.7. Just as in Definition III.5, by deleting the null terms from the sum we can rewrite the sum from Lemma III.6. Let $I \subseteq \{1...,n\}$ be such that for all $j \notin I$, $\mathcal{I}(S_j) = 0$. Then we have

$$\mathcal{I}(S_i) = \frac{1}{|I|!} \sum_{(i,J) \text{ critical}, J \subseteq I} (|J|)!(n-|J|-1)!$$

for all $i \in I$.

We will also need the following lemma, stating that the importance of a part of a system remains unchanged when the specification is replaced with its complement.

Lemma III.8 (Complement objective). Let $\overline{\mathcal{G}}$ be the game with the same arena and initial state as \mathcal{G} but the complement objective $\overline{\Omega} = S^{\omega} \setminus \Omega$. Then for all $1 \leq i \leq n$, the importance of S_i is the same for games \mathcal{G} and $\overline{\mathcal{G}}$.

We now define the four computational problems which we will study throughout this paper. The three first are decision problems, the fourth is a counting one:

Value problem

Input: A game
$$\mathcal{G}$$
, a subset $S'_{\mathtt{Sat}} \subseteq S_{\mathtt{Sat}}$
Output: Do we have $val(S'_{\mathtt{Sat}}) = 1$?

Usefulness problem

Input: A game
$$\mathcal{G}$$
, a partition S_1, \dots, S_n of the states, an index i
Output: Do we have $\mathcal{I}(S_i) > 0$?

Importance threshold problem

Input: A game
$$\mathcal{G}$$
, a partition S_1, \dots, S_n of the states, an index $i, \eta \in \mathbb{Q}$ Output: Do we have $\mathcal{I}(S_i) > \eta$?

Importance computation problem

$$\begin{cases} \text{Input:} & \text{A game } \mathcal{G}, \text{ a partition } S_1, \dots, S_n \\ & \text{of the states, an index } i \\ \text{Output:} & n! \cdot \mathcal{I}(S_i) \end{cases}$$

The way the game is encoded is left open at this point, as it will depend on the specific kind of game in question, especially when it comes to the encoding of the objective.

The two importance problems characterize the complexity of computing the importance of a state in a game. We will generally use the counting problem, except in cases where the complexity class obtained is more natural for the threshold version. For instance, if verifying some condition is already ExpTime-complete, then we want to say that the problem of computing how many elements of a set of exponential size respect that condition is also ExpTime-complete. However in order to do that we have to formulate the problem as a decision one. For the importance computation problem, the multiplication by n! ensures that the output is always an integer, which is necessary in order for this to be a counting problem.

The usefulness problem is a restricted version of the importance threshold problem, only focusing on whether some part of the system may become necessary to the satisfaction of the specification when some other parts malfunction. A similar problem for voting games, called the pivot problem, has been studied in [48].

IV. IMPORTANCE VALUES IN LTL

We now apply the theory developed in the preceding section to linear time specifications in Kripke structures. It turns out that the three decision problems defined above are 2EXPTIME-complete for LTL specifications. As this renders practical applications essentially impossible, we then go on to investigate the problems when specifications are restricted to fragments of LTL, for which we obtain more tractable complexity classes.

A. The full logic

Let $\mathcal{K}=(S,\operatorname{AP},\Delta,init,\lambda)$ be a Kripke structure and φ an LTL formula over AP.

Definition IV.1. Given a subset of states $S_{\mathtt{Sat}} \subseteq S$, let $\mathcal{G}_{S_{\mathtt{Sat}}}$ be the game between players \mathtt{Sat} and \mathtt{Unsat} over the arena $((S,\Delta),S_{\mathtt{Sat}},S_{\mathtt{Unsat}})$ with $S_{\mathtt{Unsat}}=S\backslash S_{\mathtt{Sat}}$ and init as initial state. The winning condition for player \mathtt{Sat} is the set of runs of $\mathcal K$ whose labeling satisfies φ , i.e. $\{r\in\mathcal R(\mathcal K)\mid \lambda(r)\models\varphi\}$. The $value\ val(S_{\mathtt{Sat}})$ of $S_{\mathtt{Sat}}\subseteq S$ is then defined as the value of $S_{\mathtt{Sat}}$ in the game $\mathcal G_{S_{\mathtt{Sat}}}$ (see Definition III.2).

Note that if one of the players owns all the states, then the game comes down to that player selecting a run in the structure. As a consequence, val(S)=1 if and only if $\mathcal K$ has a run satisfying φ , and $val(\emptyset)=1$ if and only if all runs in $\mathcal K$ satisfy φ .

Definition IV.2. Given a partition S_1, \ldots, S_n of S, we define the *importance* of a set of states S_i with respect to LTL formula φ as the importance of S_i in game \mathcal{G}_S under the same partition (see Definition III.5).

A straightforward telescope sum argument shows that $\sum_{i=1}^{n} \mathcal{I}(S_i) = val(S) - val(\emptyset)$. Therefore we have $\sum_{i=1}^{n} \mathcal{I}(S_i) = 1$ if and only if there exists a run in \mathcal{K} that satisfies φ , but not all runs satisfy φ . Otherwise the sum is 0.

The intuition behind these definitions is that the value of a subset of states is 1 if its elements can cooperate to guarantee the satisfaction of the specification no matter how the other states behave. The importance of a state is high if it is critical in small subsets, or numerous subsets. We now illustrate our importance notion with a number of examples.

Example 1. Let us first consider the examples given in the introduction and depicted in Figure 1, with states partitioned into singletons. Again we consider the specification $\varphi = GF \operatorname{check} \wedge FG \neg \operatorname{fail}$, and we begin with the left-hand system involving only a single server sv . Then Sat wins the game $\mathcal{G}_{S_{\operatorname{Sat}}}$ if and only if $\{\operatorname{sv},\operatorname{ok}\}\subseteq S_{\operatorname{Sat}}\colon$ if Sat is not in control of sv , then Unsat can respond fail forever, and if Sat is not in control of ok , then Unsat can avoid further checks forever. Thus (sv,T) with $\operatorname{ok}\in T$ and (ok,T) with $\operatorname{sv}\in T$ are the only critical pairs, and it is straightforward to compute $\mathcal{I}(\operatorname{ok})=\mathcal{I}(\operatorname{sv})=1/2$.

Next consider the right-hand example of Figure 1 involving two servers \mathbf{sv} and $\mathbf{sv'}$. In this case Sat wins the game $\mathcal{G}_{S_{\mathrm{Sat}}}$ if and only if $\mathbf{ok} \in S_{\mathrm{Sat}}$ and $|\{\mathbf{sv}, \mathbf{sv'}, \mathbf{check}\} \cap S_{\mathrm{Sat}}| \geq 2$. Namely, in this case \mathbf{ok} can initiate infinitely many checks; if both servers can be controlled to respond correctly, then this automatically results in infinitely many successful checks, and if one server and **check** can be controlled, then **check** can choose the functioning server infinitely often. As fail only has one outgoing transition, $\mathcal{I}(\mathbf{fail}) = 0$, thus \mathbf{fail} can be ignored by Lemma III.6. Each $s \in \{\mathbf{sv}, \mathbf{sv'}, \mathbf{check}\}$ yields two critical pairs (s, T), where |T| = 2, and so $\mathcal{I}(s) = 1/6$. On the other hand, \mathbf{ok} is the left part of every other critical pair and one then calculates $\mathcal{I}(\mathbf{ok}) = 1/2$.

Example 2. In the three following examples we consider $\varphi = aUb$, and the states are partitioned into singletons.

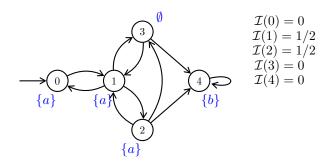


Fig. 2. Kripke structure of Example 2 (1), where atomic propositions are displayed in blue, and importance values for $\varphi=aUb$

(1) In the example of Figure 2 if 1 and 2 belong to Sat, then as every game starts with the transition from 0 to 1, she can then go from 1 to 2 and then to 4, satisfying the specification.

However if 1 belongs to Unsat, then Unsat can win by indefinitely going back to 0 from 1. Similarly, if 2 belongs to Unsat, then he can win by going from 2 to 3 if the game reaches 2, leaving no possibility for Sat to satisfy aUb.

As a result, a set of states will allow Sat to win if and only if it contains 1 and 2, thus 1 will be the one switching the value from 1 to 0 whenever it appears before 2 in a permutation. This happens in half of the permutations, thus state 1 has importance 1/2 (see Definition III.3 for what we mean by switching the value). Similarly, 2 also has importance 1/2.

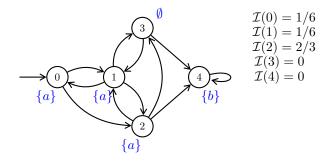


Fig. 3. Kripke structure of Ex. 2 (2) and importance values for $\varphi = aUb$

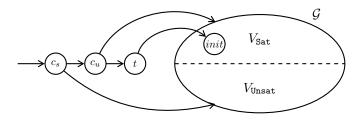


Fig. 4. Illustration for the proof of Theorem IV.3. Every state has a transition to a sink state which is not shown here.

(2) In the example of Figure 3 one can check that a set of states is allowing Sat to win if and only if it contains 2 and at least one of 0 and 1. Then 2 will be the one switching the value in permutations where it appears before either 1 or 0, i.e. in 2/3 of the permutations. In the other permutations the one switching the value is the second one to appear between 0 and 1.

We start our complexity results with the general case of an LTL specification. The complexities of the problems we consider is inferred from the 2EXPTIME-completeness of solving LTL games [11], which is inherited by the value problem.

Theorem IV.3. The usefulness and importance threshold problems for LTL with respect to Kripke structures are 2EXPTIME-complete. Further, one can compute the importance of a set of states in doubly exponential time.

Proof sketch. The upper bound comes from the 2EXPTIME upper bound on solving LTL games and the fact that enumerating exponentially many permutations still stays within that class. The idea for the lower bound is to reduce the problem of solving an LTL game to the usefulness problem (with states partitioned into singletons). We consider an LTL game with states split between $S_{\mathtt{Sat}}$ and $S_{\mathtt{Unsat}}$. We add states c_s, c_u and t which are visited at the beginning of the game, and we add transitions from c_s to states of $S_{\mathtt{Unsat}}$ and c_u to states of $S_{\mathtt{Sat}}$. Finally, we add a sink state and a transition to it from every state. We partition states into singletons. See Figure 4 for an illustration.

We encode in the specification that one of the player wins automatically as soon as c_s does not belong to Sat or c_u to Unsat. Let T be a set of states of the game and assume that one of the states of S_{Sat} is not in T. Then we also encode in the specification that Unsat can win by jumping from c_u to that state and then to sink, making Sat lose with both T and $T \cup \{t\}$. Similarly we ensure that in order for (t,T) to be critical, T has to be disjoint from S_{Unsat} . The only case in which (t,T) can be critical is then the case where states are correctly distributed between the players, and the usefulness of t is then equivalent to Sat winning the original game. \Box

B. Fragments of LTL

Considering the high complexity of the computation of the importance in the case of LTL, we now look at fragments of the logic in order to get more tractable problems. We therefore explore several classical winning conditions which can be expressed as LTL formulas. The value problem over Kripke structures with respect to some kind of specification is precisely the problem of deciding the winner of a game on a finite graph with such a specification as winning condition.

For the usefulness and importance problems, if the value problem has a complexity at least PSPACE, we can enumerate permutations of states while keeping the same complexity. However, if the value problem is for instance in P or NP, then the complexity of the usefulness and importance problems is more involved.

Below, we study various types of winning conditions. We start with the basic case of reachability conditions, which allows us to also prove tight complexity bounds for Büchi, Muller and parity conditions. We consider here explicit Muller conditions, i.e., the condition is encoded as a list of sets of states. Muller conditions are sometimes encoded in more concise forms, such as a coloring function. We will give the complexity of that version as a consequence of the Emerson-Lei case, studied later in the paper.

Proposition IV.4. The value problems for reachability, Büchi and explicit Muller conditions are P-complete.

Proof. Reachability, Büchi and explicit Muller conditions are all known to be in P [9]. Furthermore, solving reachability games is known to be P-hard [6].

As we can encode the reachability condition reaching f in all three winning conditions we consider here, we obtain Phardness for those conditions.

Remark 2. Solving games with parity conditions is in NP \cap *co*NP [8], but tight complexity bounds are not known, thus the same can be said about the value problem for parity conditions.

Proposition IV.5. The usefulness problems for reachability, Büchi, parity and explicit Muller conditions with respect to Kripke structures are NP-complete.

Proof. The problem is clearly in NP in the case of reachability, Büchi or Muller conditions as one can nondeterministically guess $J\subseteq\{1,\ldots,n\}$ and check in polynomial time whether $val(\bigcup_{i\in J}S_j)=0$ and $val(\bigcup_{j\in J\cup\{i\}}S_j)=1$ hold.

For parity conditions we also have to guess positional strategies for Sat and Unsat along with J and check in polynomial time that those strategies allow Sat to win when she owns $\bigcup_{j\in J} S_j$ and Unsat to win when Sat owns $\bigcup_{j\in J} S_j$.

We obtain NP-hardness through a reduction from 3SAT. Let $\psi = C_1 \wedge C_2 \wedge \cdots \wedge C_k$ be a 3SAT instance, with $C_j = (\ell_j^1 \vee \ell_j^2 \vee \ell_j^3)$ for all j, and let $\{x_1, \ldots, x_n\}$ be the set of variables appearing in ψ .

We consider the Kripke structure $\mathcal{K} = (S, AP, \Delta, c_1, \lambda)$ with states partitioned into singletons, and

- $S = \{f, s, sink\} \cup \{c_i \mid 1 \le i \le k\}$ $\cup \{\ell_i^p \mid 1 \le i \le k, 1 \le p \le 3\} \cup \{x_j', \neg x_j' \mid 1 \le j \le n\}$
- $\lambda(f) = \{f\}$ and $\lambda(q) = \emptyset$ for all $q \neq f$

$$\Delta = \{(c_i, \ell_i^p) \mid 1 \le p \le 3, 1 \le i \le k\}$$

$$\cup \{(\ell_i^p, c_{i+1}) \mid 1 \le p \le 3, 1 \le i \le k - 1\}$$

$$\cup \{(\ell_k^p, s) \mid 1 \le p \le 3\} \cup \{(s, x_1'), (s, \neg x_1')\}$$

$$\cup \{(x_j', x_{j+1}'), (\neg x_j', x_{j+1}') \mid 1 \le j \le n - 1\}$$

$$\cup \{(x_j', \neg x_{j+1}'), (\neg x_j', \neg x_{j+1}') \mid 1 \le j \le n - 1\}$$

$$\cup \{(x_n', f), (\neg x_n', f)\} \cup \{(q, sink) \mid q \in S\}$$

$$\cup \{(\ell_i^p, x_m') \mid \ell_i^p \equiv \neg x_m\}$$

Note that every literal in the clauses has a transition towards its negation in the variables. Player Sat wins if and only if f is reached, which can be expressed as a reachability, Büchi, parity or Muller condition. The construction can be done in logarithmic space. See Figure 5 for an illustration of the construction.

We are now going to show that state s is useful if and only if the 3SAT formula is satisfiable, thus proving NPhardness of the usefulness problem for all four types of winning conditions.

As every state has a transition to a sink state, if at some point a state belonging to Unsat is reached before reaching f, then Sat loses. As a consequence, Sat wins with a set of states if and only if there is a path in this set of states from c_1 to f (possibly not including f).

Suppose there exists a valuation ν satisfying ψ . We extend ν to literals in the natural way, i.e. $\nu(\neg x_i) = \bot$ if $\nu(x_i) = \top$ and $\nu(\neg x_i) = \top$ otherwise. Then we set

$$T = \{x'_m \mid \nu(x_m) = \top\} \cup \{\neg x'_m \mid \nu(x_m) = \bot\} \\ \cup \{\ell_j^p \mid \nu(\ell_j^p) = \top\} \cup \{c_i \mid 1 \le i \le k\}$$

Clearly there is a path from c_1 to f in $T \cup \{s\}$, as for all $1 \le i \le k$ there is at least one l_i^p satisfied by ν (and thus in T), and for all $1 \leq i \leq n$ one of $x'_i, \neg x'_i$ is in T. However, for all $(\ell^p_i, x'_j) \in \Delta$ (resp. $(\ell^p_i, \neg x'_j)$), if $\ell^p_i \in T$ then $\nu(\ell^p_i) = \top$ thus, as $\ell^p_i \equiv \neg x_j$ (resp. x_j), $\nu(x_j) = \bot$ (resp. \top) and $x'_j \notin T$ (resp. $\neg x'_i$). Therefore there is no path in T from c_1 to f.

Now suppose there exists T such that there is a path from c_1 to f in $T \cup \{s\}$ but not in T. Then there is a path in $T \cup \{s\}$ from c_1 to f going through s. In particular for all $1 \le i \le n$ at least one of x_i' , $\neg x_i'$ is in T. Let ν be a valuation such that for all i, if $\nu(x_i) = \top$ then $x_i' \in T$ and $\neg x_i' \in T$ otherwise. There is also a path from c_1 to s in T, hence for all i there is a p_i such that $\ell_i^{p_i} \in T$. Then for all $\ell_i^{p_i}$ of the form x_j for some j, the state $\neg x'_j$ cannot be in T as otherwise there would be a path from c_1 to $\ell_i^{p_i}$ then to $\neg x_j'$ and finally to f in T, not going through s. As a result we have $x_i' \in T$ and thus $\nu(l_i^{p_i}) = \nu(x_i) = \top$. By a similar argument, if $\ell_i^{p_i} = \neg x_i$ then $\nu(x_i) = \bot$. Hence for every i there is a literal in the ith clause satisfied by ν , thus the 3SAT instance is satisfiable.

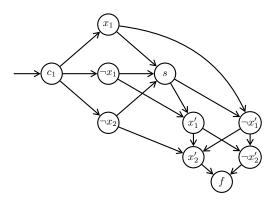


Fig. 5. Construction for $(x_1 \vee \neg x_1 \vee \neg x_2)$. All states have a transition to a sink state, not shown here.

Theorem IV.6. The importance computation problems for reachability, Büchi, parity and explicit Muller conditions with respect to Kripke structures are #P-complete.

Proof sketch. The idea is to reduce the problem of counting the valuations satisfying exactly one literal of every clause of a 3SAT formula φ , known to be #P-complete [49]. First we transform the formula φ into another one ψ that is satisfied by a valuation ν if and only if ν satisfies one literal per clause in φ . We then reuse the construction of the usefulness proof, and notice that the set of sets of states T making (s,T)critical in the structure can be split into parts of (up to some details) equal size, each one matching a valuation satisfying the formula. Further, all such sets are of (again, up to some details) the same size. This allows us to compute the number of valuations satisfying the formula from the importance of s.

Remark 3. One can show with nearly identical proofs that those problems keep the same complexity with co-Büchi, safety or co-safety conditions.

Now we consider not only Büchi conditions, but Boolean combinations of them, called Emerson-Lei conditions. As expected, we get an intermediate complexity between those for Büchi and LTL conditions.

Theorem IV.7. The value, usefulness and importance threshold problems for Emerson-Lei conditions are PSPACEcomplete. Further, one can compute the importance of a set of states in polynomial space.

Proof sketch. As Emerson-Lei games are known to be in PSPACE [10], the upper bound follows easily. We prove the lower bound by reduction of OSAT. We construct a structure encoding a sequence of choices of the values of the variables and partition its states into singletons. The structure contains a state s, which we will prove to be critical if and only if the QSAT formula is valid. We ensure that for all T, (s, T) can only be critical if T contains the states choosing the values of the existential variables and not the other ones, by making one of the players win without using s for sets T not satisfying this condition.

We also ensure that Unsat wins if he owns s, thus s is useful if and only if Sat wins with s. We make players choose valuations of the variables infinitely many times, and we encode in the specification that the player owning the first variable x_i such that x_i and $\neg x_i$ are chosen infinitely often loses. If both players play consistently, the game is decided by the satisfaction of the QSAT formula.

Remark 4. It was proven by Hunter and Dawar that Emerson-Lei conditions are more succinct than Muller conditions encoded with a coloring of the states and a list of sets of colors [10]. As a result, the PSPACE lower bounds we obtained for Emerson-Lei transfer to these succinct Muller conditions. Hunter and Dawar also show that solving games with those Muller conditions is PSPACE-complete, from which we can easily infer the PSPACE-completeness of the value, usefulness and importance threshold problem for this type of condition.

We continue our exploration with a more complicated case, the Rabin and Streett conditions. We treat both cases simultaneously as they are symmetric.

Remark 5. As solving Rabin (resp. Streett) games is NP-complete (resp. coNP-complete), so is the value problem for Rabin (resp. Streett) conditions [7].

Proposition IV.8. The usefulness problem for Rabin conditions is Σ_2^P -complete.

Proof sketch. The complete proof is in the appendix. We reduce the dual of the $\forall\exists$ 3SAT problem to the usefulness problem in the case when states are partitioned in singletons. The set of states T witnessing the usefulness of state s will encode the valuation of the first set of variables, with a trick similar to the one used in the proof of Proposition IV.5 to ensure that the encoded valuation is correct.

As Sat plays for a Rabin objective, she has a positional strategy, with which she has to choose for each clause a satisfied literal. We use the Rabin condition to make sure that Sat does not pick a literal and its negation. We also ensure that Sat wins automatically with $T \cup \{s\}$ as soon as T encodes a correct valuation, and then s is useful if and only if there exists a set of states T (i.e. a valuation of the first variables) such that for all positional strategy of Sat over T (i.e. valuation of the second variables), Sat loses the game (i.e. the formula is not satisfied).

The theorem below uses the complexity class $\#P^{NP}$, which is the class of counting problems P such that there exists a nondeterministic polynomial-time Turing machine with an NP oracle such that the answer of P on an input is the number of accepting runs of the machine on that input.

Theorem IV.9. The importance computation problem for Rabin conditions is P^{NP} -complete.

Proof sketch. The idea is simply to observe that in the construction for Proposition IV.8, the sets of states witnessing the usefulness of s are in bijection with the valuations of the universal variables witnessing the non-validity of the $\forall \exists 3Sat$

formula (up to some technical details). In the appendix we show that counting such valuations is $\#P^{NP}$ -complete, from which one can infer $\#P^{NP}$ -completeness of the importance computation problem.

Corollary IV.10. As Streett conditions are exactly the complements of Rabin ones, by Lemma III.8 and Proposition IV.8, the usefulness problem for Streett conditions is $\Sigma_2^{\rm P}$ -complete.

By the same argument, by Lemma III.8 and Theorem IV.9, the importance computation problem for Streett conditions is #P^{NP}-complete.

V. IMPORTANCE VALUES IN CTL

We now adapt the definitions to deal with CTL specifications. A notion of degree of responsibility of a state in a Kripke structure for the satisfaction of a CTL formula was already given by Chockler, Halpern, and Kupferman [1]. While in their approach the responsibility of a state was based on the set of atomic propositions it chooses to satisfy, in ours it is based on the set of outgoing transitions it chooses to allow.

In contrast to the previous sections, CTL has the additional challenge that the formulas are evaluated on trees and not on words. The first question that arises is the nature of the nondeterministic choices in this setting. Our definitions rely on the fact that the nondeterminism of a state may be resolved in different ways by the two players. However, due to the branching time nature of CTL, directly applying this methodology does not make sense, as CTL formulas already take the nondeterminism into account. This is why we consider modal transition systems (MTS, as introduced in Section II-A), in which there is another layer of choice: namely determining the subset of may transitions that are present in any state. Modal transition systems have been widely studied as a formalism to capture the refinement of processes from abstract specifications to concrete implementations [42], [43]. They have been extended in various ways, and the corresponding synthesis and verification problems have been considered [50]-[52].

The second, and related, issue is that letting the players construct the tree turn-by-turn runs into the problem that the order in which different branches are considered will often make a difference. In Section V-A we explain the difficulties of defining a game which allows both players to construct a tree generated by an MTS in a similar fashion as for LTL.

In Section V-B we define a notion of importance for CTL on MTS, which we call *two-turn CTL*, where both players choose once in the beginning which *may*-transitions they allow in the states under their control. This choice induces a Kripke structure on which the CTL formula can be evaluated.

However the order in which the choices are made affects the importance values. Therefore, in Section V-C we consider also the concurrent setting in which randomized strategies become important, which we call *Concurrent CTL*.

Throughout, let $\mathcal{M} = (S, AP, \Delta_{must}, \Delta_{may}, init, \lambda)$ be a modal transition system and let φ be a CTL formula.

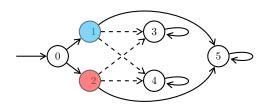


Fig. 6. A modal transition system with *must*-transitions depicted as solid lines and *may*-transitions depicted as dashed lines. This example illustrates the problem that has to be faced when defining turn-based CTL values (cf. Example 3). The state in blue belongs to Sat, the state in red to Unsat.

A. Importance in an MTS with respect to a CTL specification

It is appealing to define a notion of importance that relies on the ability of a set of states to *guarantee* the satisfaction of a specification. However, in the case of CTL the fact that formulas are evaluated on trees and not on runs forbids us to make the players construct a tree in a similar fashion as for LTL, as the winning player would heavily depend on the order in which branches are constructed. This is illustrated in the following example.

Example 3. Consider the formula $[EF3 \land EF4 \land AG(\neg(1 \land EX3 \land EX4))] \lor EF(2 \land AX5)$ and the MTS in Figure 6. Say we want to make the players construct a tree, and Sat wins if this tree satisfies the above formula. We would have to decide in which order the players make their choices. Say we make Sat choose the successors of 1 before Unsat chooses the ones of 2. Then Unsat wins by picking at least one of 3 and 4 as successor if 1 has neither of them as successor, and the same set of successors as 1 otherwise. If Unsat chooses first, then Sat can pick 3 and not 4 if Unsat chose 4, and vice-versa (and pick any one of the two if Unsat chose both or none). Thus the winner depends on the (arbitrary) order in which we let the players construct the tree.

We could define a success value where some sets of states are seen as neutral, meaning that when Sat has this group of states the game is undetermined, similarly to what was done in [53]. However we wish to define the importance as a numerical value, thus it is more practical that the success value can only be 0 or 1.

B. Two-turn CTL importance

The idea of two-turn importance values is that a set of states has value one if it can choose sets of outgoing transitions such that the specification is satisfied no matter which sets of outgoing transitions are chosen by the other states. This definition puts more burden on the satisfier, but matches a vision of the MTS as a way to represent a set of Kripke structures (possible implementations of a system) rather than a language of trees.

Definition V.1 (Two-turn importance values). Let $\mathcal{M} = (S, \operatorname{AP}, \Delta_{must}, \Delta_{may}, init, \lambda)$ be a modal transition system, let $S_{\mathtt{Sat}} \subseteq S$ and let φ be a CTL specification. A

pure strategy for Sat is a function $\sigma_{\text{Sat}}: S_{\text{Sat}} \to 2^{\Delta_{may}}$ such that for all $v \in S_{\text{Sat}}$, $\Delta_{must}(v) \subseteq \sigma_{\text{Sat}}(v) \subseteq \Delta_{may}(v)$. We define pure strategies σ_{Unsat} for Unsat symmetrically.

Two pure strategies $\sigma_{\mathtt{Sat}}$, $\sigma_{\mathtt{Unsat}}$ yield a Kripke structure, whose states are the ones of \mathcal{M} and transitions from a state are given by the strategy of the player owning that state. We call that Kripke structure $\mathcal{K}(\sigma_{\mathtt{Sat}}, \sigma_{\mathtt{Unsat}})$.

The value $val_{2turn}(S_{\mathtt{Sat}})$ of $S_{\mathtt{Sat}}$ is defined as 1 if there exists a pure strategy $\sigma_{\mathtt{Sat}}$ of \mathtt{Sat} such that for all pure strategies $\sigma_{\mathtt{Unsat}}$ of \mathtt{Unsat} , $\mathcal{K}(\sigma_{\mathtt{Sat}}, \sigma_{\mathtt{Unsat}})$ satisfies φ , and 0 otherwise.

The *importance* is defined analogously to Definition III.5: Given a partition S_1, \ldots, S_n of S, the importance of S_i is defined as:

$$\mathcal{I}_{2turn}(S_i) = \frac{1}{n!} \sum_{\pi \in \Pi_n} val_{2turn}(S_{\geq i}^{\pi}) - val_{2turn}(S_{\geq i}^{\pi} \setminus S_i)$$

Example 4. (1) Consider the formula $\varphi = A(EFa)Ub$ and the modal transition system displayed in Figure 7. Observe that the two ways in which φ may be violated are

- Unsat owns 0, 2 and 5 and allows the transition from 0 to 5 but not from 5 to 4 or 2 to 4, so that there is a path labeled {a}∅ to 5, but no path from 5 to a state labeled by an a.
- Unsat owns 1 and 2 and chooses transitions so that there is no transition from 1 to 3 or from 2 to 4.

The importance is therefore distributed as follows:

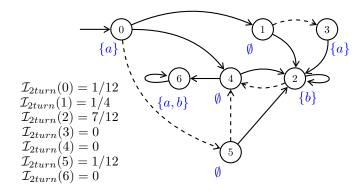


Fig. 7. MTS of Ex. 4 (1) and 2-turn importance values for $\varphi = A(EFa)Ub$

(2) In the example of Figure 8 we want to illustrate a limitation of this notion with respect to what was discussed in Section V-A. Such a mechanism can be illustrated by trying to prove $AG(a \Rightarrow EX(EFb))$ on the following structure:

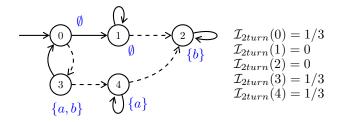


Fig. 8. MTS of Ex. 4 (2) and 2-turn importance values for $\varphi=AG(a\Rightarrow EX(EFb))$

In the 2-turn CTL framework, Unsat wins if and only if he owns 0, 3 and 4 in order to create a path to 4, but no transition from 4 to 2. Thus in any ordering of the states the last one between 0, 3 and 4 will be the one switching the value.

However, one might want to design a richer model in which we would also give the victory to Unsat when he owns either 0, 1 and 3 or 0, 1 and 4. The reason is that after allowing the transition from 0 to 3 at the start, we would like to let Unsat delete it. Then Unsat can not allow the transition from 1 to 2, and ensure that there is no path from 3 to 2. Therefore there is no path from a successor of 3 reaching a state labeled b.

This observation motivates the study of turn-based definitions of CTL importance in MTS for restricted sets of formulas, which we leave open for future work.

In the appendix we prove the following results. The hardness proofs consist in encoding choices of valuations of variables in SAT formulas as the players' choices of transitions.

Proposition V.2. The value problem for two-turn CTL is Σ_2^P -complete.

Proposition V.3. The usefulness problem for two-turn CTL is Σ_3^P -complete.

Theorem V.4. The importance computation problem for twoturn CTL is $\#P^{\sum_{2}^{p}}$ -complete.

Remark 6. We can define a dual game, in which Unsat plays first, and then Sat. While in the former game Sat was at a disadvantage, in this version Unsat is, as he is the one who has to choose his strategy without knowing the adversary's.

Let \mathcal{M} be an MTS with a set of states S, let $S_{\mathtt{Sat}} \subseteq S$ and let φ be a specification, the value of $S_{\mathtt{Sat}}$ with respect to φ in the game where Unsat starts is $1 - val_{2turn}(S \setminus S_{\mathtt{Sat}})$ with $val_{2turn}(S \setminus S_{\mathtt{Sat}})$ the value of $S \setminus S_{\mathtt{Sat}}$ with respect to $\neg \varphi$ in the game where Sat starts. From this one infers easily that the value problem for the game where Unsat starts is Π_2^P -complete and that, by an argument similar to the proof of III.8, the usefulness problem is Σ_3^P -complete and the importance computation problem $\#P^{\Sigma_2^P}$ -complete.

C. Concurrent CTL importance

The previous version of the game breaks the symmetry between the two players: We have to pick either Sat or Unsat to play first (we chose Sat in the definition above). One may prefer a version of this game in which we do not give any such advantage to a player.

We now introduce a concurrent game, in which both players choose a mixed strategy, in the form of a distribution over all the possible choices of sets of transitions from their respective states. The value of a set of states is the highest probability such a mixed strategy can guarantee for Sat with this set of states. The Nash Theorem guarantees the existence of a Nash equilibrium, which means that the highest probability a set of states can achieve for Sat is one minus the highest probability its complement can achieve for Unsat. For an introductory account on non-cooperative concurrent games, we refer to [54].

Definition V.5 (Concurrent game induced by CTL formula). Let $\mathcal{M}=(S,\operatorname{AP},\Delta_{must},\Delta_{may},init,\lambda)$ be a modal transition system, let $S_{\mathtt{Sat}}\subseteq S$ and let φ be a CTL specification. Let $C_S(\mathcal{M},S_{\mathtt{Sat}})$ be the set of pure strategies for Sat. A mixed strategy for Sat is a probability distribution $p_S:C_S(\mathcal{M},S_{\mathtt{Sat}})\to [0,1].$ We define $C_U(\mathcal{M},S_{\mathtt{Unsat}})$ and p_U in a similar way. Let M_S and M_U denote the set of mixed strategies of Sat and Unsat, respectively.

We consider the concurrent game with the payoff functions

$$\rho_{\mathtt{Sat}}(\sigma_{\mathtt{Sat}}, \sigma_{\mathtt{Unsat}}) = \begin{cases} 1 \text{ if } \mathcal{K}(\sigma_{\mathtt{Sat}}, \sigma_{\mathtt{Unsat}}) \text{ satisfies } \varphi \\ 0 \text{ otherwise} \end{cases}$$

and $\rho_{\mathtt{Unsat}}(\sigma_{\mathtt{Sat}}, \sigma_{\mathtt{Unsat}}) = 1 - \rho_{\mathtt{Sat}}(\sigma_{\mathtt{Sat}}, \sigma_{\mathtt{Unsat}})$, for all $\sigma_{\mathtt{Sat}} \in C_S(\mathcal{M}, S_{\mathtt{Sat}}), \sigma_{\mathtt{Unsat}} \in C_U(\mathcal{M}, S_{\mathtt{Unsat}})$. Given two mixed strategies p_S, p_U , the expected payoff of Sat is

$$E_{\mathtt{Sat}}(\mathcal{M}, \varphi, p_S, p_U) = \sum_{\substack{\sigma \in C_S(\mathcal{M}, S_{\mathtt{Sat}}) \\ \sigma' \in C_U(\mathcal{M}, S_{\mathtt{Unsat}})}} p_S(\sigma) p_U(\sigma') \rho_{\mathtt{Sat}}(\sigma, \sigma')$$

The expected payoff of Unsat is

$$E_{\mathtt{Unsat}}(\mathcal{M}, \varphi, p_S, p_U) = 1 - E_{\mathtt{Sat}}(\mathcal{M}, \varphi, p_S, p_U)$$

. Finally, we define the value of a set of states $S_{\mathtt{Sat}}$ as

$$val_{concur}(S_{\mathtt{Sat}}) = \sup_{p_S \in M_S} \inf_{p_U \in M_U} E_{\mathtt{Sat}}(\mathcal{M}, \varphi, p_S, p_U)$$

It is a direct consequence of Nash's Theorem [55] that $val_{concur}(S_{\mathtt{Sat}})$ is the payoff of Sat obtained in any Nash equilibrium of the concurrent game defined above. In particular we have

$$val_{concur}(S_{\mathtt{Sat}}) = \inf_{p_U \in M_U} \sup_{p_S \in M_S} E_{\mathtt{Sat}}(\mathcal{M}, \varphi, p_S, p_U)$$

Definition V.6 (Concurrent importance values). Given a partition of the states S_1, \ldots, S_n , we define the importance of a set of states S_i as usual:

$$\mathcal{I}_{concur}(S_i) = \frac{1}{n!} \sum_{\pi \in \Pi_n} val_{concur}(S_{\geq i}^{\pi}) - val_{concur}(S_{\geq i}^{\pi} \setminus S_i)$$

Lemma V.7. The importance value of a set of states of an MTS for a CTL specification with respect to Definition V.6 can be computed in exponential time.

Proof. We can check whether a Kripke structure satisfies a CTL specification in polynomial time. Hence we can compute the winner for all (exponentially many) pairs of pure strategies.

Computing the value of a set of states then amounts to solving a linear optimization problem with exponential input [56]. As the latter problem can be solved in polynomial time, the former is in EXPTIME [57].

Lemma V.8. For each set of states $S_{\mathtt{Sat}}$, we have $val_{concur}(S_{\mathtt{Sat}}) = 1$ if and only if $val_{2turn}(S_{\mathtt{Sat}}) = 1$. In particular, as $val_{2turn}(S_{\mathtt{Sat}}) \in \{0,1\}$ for all $S_{\mathtt{Sat}}$, the value $val_{2turn}(S_{\mathtt{Sat}})$ is entirely determined by $val_{concur}(S_{\mathtt{Sat}})$ (it is its integer part).

Proof. Suppose $val_{2turn}(S_{\mathtt{Sat}}) = 1$, then Sat has a winning pure strategy, thus wins with probability 1 if she applies it in the concurrent game. Hence $val_{concur}(S_{\mathtt{Sat}}) = 1$.

Now suppose $val_{2turn}(S_{\mathtt{Sat}})=0$, then for every pure strategy σ of Sat, Unsat has a winning strategy against σ . As a result, by taking a uniform distribution over its strategies, Unsat can achieve a positive probability to win. As a result, $val_{concur}(S_{\mathtt{Sat}})<1$.

Remark 7. We can make a similar statement about the dual of the two-turn CTL game, described in Remark 6. For all sets of states and specifications the value given by the dual game is 0 if and only if the concurrent value is.

Proposition V.9 (2-turn versus concurrent importance values). Let S_1, \ldots, S_n be a partition of the states of an MTS. If a set of states S_i is useful with respect to the 2-turn Definition V.1, then it is useful with respect to the concurrent Definition V.6.

Proof. Suppose $\mathcal{I}_{concur}(S_i) = 0$. Then for all $J \subseteq \{1, \ldots, n\}$ we have $val_{concur}(\bigcup_{j \in J} S_j) = val_{concur}(\bigcup_{j \in J \cup \{i\}} S_j)$. Then by Lemma V.8, for all J we have

$$val_{2turn}(\bigcup_{j \in J} S_j) = val_{2turn}(\bigcup_{j \in J \cup \{i\}} S_j)$$

and thus $\mathcal{I}_{2turn}(S_i) = 0$.

The converse of Proposition V.9 does not hold as shown by the following example.

Example 5. We consider the MTS displayed in Figure 9 and the formula $\varphi_1 \vee \varphi_2 \vee \varphi_3$, with:

$$\varphi_1 = EX(b \land EXc) \land AX(\neg c \land \neg(a \land EXc))$$

$$\varphi_2 = AXa \land EXEXc$$

$$\varphi_3 = EXc \land EX(b \land EXc) \land EX(a \land EXc)$$

In this system, φ_1 expresses that the only path from 0 to 3 is through 2, φ_2 that the only path is through 1 and φ_3 that all three paths exist.

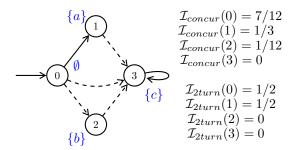


Fig. 9. MTS of Ex. 5

The computation of the concurrent game importance values is lengthy, but straightforward. We observe that Sat has a pure winning strategy whenever she has states 0 and 1, and Unsat has a pure winning strategy whenever he has either 0 or 1 and 2. The remaining case is when Sat has 0 and 2 and Unsat has 1, so Sat can choose to allow or not the paths 0,3 and 0,2,3, and Unsat can choose to allow or not path 0,1,3.

Then one can observe that the case where Unsat allows path 0,1,3 with probability 1/2 and Sat never allows 0,3 and allows 0,2,3 with probability 1/2 is a Nash equilibrium, thus the set $\{1,3\}$ has value 1/2. This example shows that in some cases some sets of states may be useless from the 2-turn CTL point of view but not from the Concurrent CTL one.

VI. CONCLUSION

We have introduced a new measure of the influence that a part of a system has on whether a given specification is satisfied. We studied it in the context of two model-checking frameworks, LTL formulas against Kripke structures and CTL formulas against modal transition systems. In most of the cases we provided tight complexity bounds for the corresponding computational problems. A general conclusion is that the notion of importance value is natural, but still costly in terms of complexity, especially in the case of LTL. This problem can be mitigated by considering sets of states rather than single states, and formulas from weaker logics.

We expect that the principle of designing a game and computing the importance of a part of the system by shifting its control from one player to the other can be easily adapted to many model-checking problems. We have studied here classical and basic logics, but one could try to find or design logics more well-suited to the computation of the importance, yielding lower complexities.

Another continuation of this work would be a fairer definition of the importance in the case of CTL model-checking. Some subsets of CTL formulas may allow us to design a game in which the players can simultaneously choose transitions on the structure and prove the formula without disadvantaging one of the two. This could be related to the notion of goodfor-games automata.

Finally we can extend the definition of value to probabilistic games, by defining the value as the maximal probability of success that Sat can achieve. This gives us a natural notion of importance in probabilistic games that calls for study.

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APPENDIX

A. General properties

Proof of Lemma III.8. Let $\overline{\mathcal{G}}$ be the game with the same arena and initial state as \mathcal{G} , but the complement objective $\overline{\Omega} = S^{\omega} \setminus \Omega$. Then for all $1 \leq i \leq n$, the importance of S_i is the same for games \mathcal{G} and $\overline{\mathcal{G}}$.

Proof. For all $S'\subseteq S$ let $\overline{val}(S')$ be the value of S' in $\overline{\mathcal{G}}$ and let $\overline{\mathcal{I}}(S_i)$ be the importance of S_i in $\overline{\mathcal{G}}$ for all i. For all permutations $\pi\in\Pi_n$ let $\tilde{\pi}$ be the mirror permutation, such that for all $1\leq i\leq n,\ \tilde{\pi}(i)=\pi(n+1-i).$ As the function associating its mirror to each permutation is a bijection from Π_n to itself, we can rewrite $\mathcal{I}(S_i)$ as

$$\mathcal{I}(S_i) = \frac{1}{n!} \sum_{\pi \in \Pi_n} val(S_{\geq i}^{\tilde{\pi}}) - val(S_{\geq i}^{\tilde{\pi}} \setminus S_i)$$

As $S_{>i}^{\tilde{\pi}} = S \setminus (S_{>i}^{\pi} \setminus S_i)$, we have

$$val(S_{>i}^{\tilde{\pi}}) = 1 - \overline{val}(S_{>i}^{\pi} \setminus S_i)$$

(Sat wins with states $S_{\geq i}^{\tilde{\pi}}$ for objective Ω if and only if she loses with states $S \setminus S_{>i}^{\tilde{\pi}}$ for objective $\overline{\Omega}$). Similarly, we have

$$val(S_{\geq i}^{\tilde{\pi}} \setminus S_i) = 1 - \overline{val}(S_{\geq i}^{\pi})$$

As a result, for all $\pi \in \Pi_n$,

$$\overline{val}(S_{>i}^{\pi}) - \overline{val}(S_{>i}^{\pi} \setminus S_i) = val(S_{>i}^{\tilde{\pi}}) - val(S_{>i}^{\tilde{\pi}} \setminus S_i)$$

Finally, we obtain

$$\mathcal{I}(S_i) = \frac{1}{n!} \sum_{\pi \in \Pi_n} \overline{val}(S_{\geq i}^{\pi}) - \overline{val}(S_{\geq i}^{\pi} \setminus S_i) = \overline{\mathcal{I}}(S_i) \quad \square$$

B. LTL Proofs

Proof of Theorem IV.3. The usefulness and importance threshold problems for LTL with respect to Kripke structures are 2EXPTIME-complete.

Proof. First, as one can solve LTL games in doubly exponential time, one can compute the value of any subset of the states of \mathcal{K} in doubly exponential time as well. There are exponentially many such subsets, thus the computation of all those values takes again doubly exponential time. The computation of the importance then comes down to enumerating orderings of the states and computing the sum along the way. As a result, one can compute the importance and compare it with τ in doubly exponential time, thus the importance threshold problem (and thus also the usefulness one) is in 2EXPTIME.

For the hardness, we prove that the usefulness problem is 2EXPTIME-hard in the case where the states are partitioned in singletons. The hardness of the usefulness and importance threshold problems follow directly. We reduce the problem of solving LTL games. Let $\mathcal{K}=(S,\operatorname{AP},\Delta,init,\lambda)$ be a Kripke structure, let φ be an LTL formula, and let $S_{\mathtt{Sat}}\sqcup S_{\mathtt{Unsat}}=S$ be a partition of S between states of Sat and Unsat. We consider the LTL game $\mathcal G$ induced by those parameters.

Consider the Kripke structure $\mathcal{K}' = (S', AP', \Delta', c_s, \lambda')$ with $S' = S \cup \{c_s, c_u, sink, t\}, AP' = AP \cup S'$, and

$$\Delta' = \Delta \cup \{(s, sink) \mid s \in S'\}$$

$$\cup \{(c_s, s) \mid s \in S_{\texttt{Unsat}}\} \cup \{(c_u, s) \mid s \in S_{\texttt{Sat}}\}$$

$$\cup \{(c_s, c_u), (c_u, t), (t, init)\}$$

and for all $s \in S'$, $\lambda'(s) = \lambda(s) \cup \{s\}$ if $s \in S$ and $\lambda'(s) = \{s\}$ otherwise. In other words, every state is labeled with its own name. See Figure 4 for an illustration of the construction.

Let
$$\varphi' = \neg \varphi_{checkUnsat} \lor (\varphi_{checkSat} \land X^3 \varphi)$$
 with

$$\varphi_{checkSat} = \neg X sink$$

$$\wedge X \neg c_u \Rightarrow X^2 sink$$

$$\wedge X^2 t \Rightarrow X^3 init$$

$$\wedge X^3 G(\bigvee_{s \in S_{\mathtt{Sat}}} s \Rightarrow \neg X sink)$$

$$\varphi_{checkUnsat} = [Xc_u \Rightarrow (\neg X^2 sink \land (\neg X^2 t \Rightarrow X^3 sink))]$$
$$\land X^3 G(\bigvee_{s \in S_{\texttt{Unsat}}} s \Rightarrow \neg X sink)$$

This construction can be done in logarithmic space. The intuition is that if some state in S_{Unsat} belongs to Sat, then she can win by going from c_s to that state and then to sink. Similarly if some state of S_{Sat} belongs to Unsat, then he can win by going to that state from c_u and then to sink. In both cases players win without using t. The remaining case is when Sat owns states of S_{Sat} and Unsat of S_{Unsat} . Then if Unsat owns t, he can win by going from there to sink, otherwise the players have to play the original game $\mathcal G$ from init. As a result, t is useful if and only if Sat wins $\mathcal G$. We will now prove that the state t is useful with respect to φ' if and only if Sat wins the original LTL game.

First suppose that Sat wins \mathcal{G} , then we consider $T = \{c_s\} \cup S_{\mathtt{Sat}}$. Player Sat loses with T:

- If she goes from c_s to sink she loses.
- If she goes from c_s to a state of S_{Unsat} , Unsat can then go to some state different from sink and not satisfy $\varphi_{checkSat}$ while satisfying $\varphi_{checkUnsat}$ (recall that in our definition of Kripke structure we assume every state to have at least one outgoing transition).
- If she goes from c_s to c_u , then Unsat can go to t then sink and not satisfy $\varphi_{checkSat}$ while satisfying $\varphi_{checkUnsat}$.

Moreover, player Sat wins with $T \cup \{t\}$, as she can start by going from c_s to c_u and:

- If Unsat goes to sink from c_u , he loses.
- If Unsat goes from c_u to a state of S_{Sat} , Sat can then go to some state different from sink and not satisfy $\varphi_{checkUnsat}$.
- If Unsat goes from c_u to s, then Sat can go to init and then win by playing a winning strategy for \mathcal{G} , thus satisfying $\varphi_{checkSat} \wedge X^3 \varphi$.

Thus t is useful.

Now suppose that t is useful, let $T \subseteq S'$ be a set of states such that (t,T) is critical. T has to contain c_s as otherwise Unsat can go from c_s to sink directly and make Sat lose with $T \cup \{t\}$. If Sat had a winning strategy with $T \cup \{t\}$ not going from c_s to c_u , then she would also win with just T by applying this strategy as t is then never reached.

As a result, Sat with $T \cup \{t\}$ has to go from c_s to c_u . As a consequence, T has to be disjoint from $S_{\mathtt{Unsat}}$, as otherwise Sat with T could go from c_s to a state in $T \cap S_{\mathtt{Unsat}}$ and from there to sink, unsatisfying $\varphi_{checkUnsat}$. Further, c_u cannot be in T as otherwise Sat could win by going from c_u to sink.

Finally, Unsat cannot win when Sat has T by going from c_u to a state different from t as otherwise he could win when Sat has $T \cup \{t\}$ with the same strategy. As a consequence, T has to contain S_{Sat} , as if not Unsat could go from c_u to a state in $S_{\text{Sat}} \setminus T$ and then sink, winning the game.

Whether sink is in T is irrelevant to the game as there is only one outgoing transition from sink. Thus we can assume that $T = \{c_s\} \cup S_{\mathtt{Sat}}$. Suppose Unsat wins \mathcal{G} , and consider the game where Sat has $T \cup \{t\}$. As Sat has to go from c_s to c_u to win, Unsat can then go from c_u to s, and Sat has to go to init. Then Unsat can apply his winning strategy for \mathcal{G} , as Sat loses if she goes to sink and thus cannot go out of S. This makes Unsat win, contradicting the hypothesis that Sat wins with $T \cup \{t\}$. In conclusion, Sat wins \mathcal{G} .

As a result, the usefulness and importance threshold problems are 2EXPTIME-complete for LTL.

Proof of Theorem IV.6. The importance computation problem for reachability, Büchi, parity and explicit Muller conditions with respect to Kripke structures are #P-complete.

Proof. First, the upper bound for reachability, Büchi and explicit Muller conditions is obtained by constructing a Turing machine guessing an ordering of $\{1,\ldots,n\}$, and accepting if the set J of indices coming after i in the ordering is such that (i,J) is critical, which can be checked in polynomial time. The number of accepting runs is the number of permutations satisfying this condition, i.e., $n!\mathcal{I}(S_i)$. The problem is therefore in #P.

For parity conditions, we rely on the result by Jurdziński that solving parity games can be done by a polynomial-time unambiguous Turing machine, i.e., a nondeterministic machine that has at most one accepting run on every input [58].

This allows us to build a machine that takes as input a Kripke structure \mathcal{K} , a partition S_1,\ldots,S_n of the states, an index i and a coloring c and guesses an ordering of $\{1,\ldots,n\}$. Let J be the set of indices coming after i in the permutation, our machine can simulate the unambiguous Turing machine in order to check that Sat wins with $\bigcup_{j\in J} \{1\} S_j$ and Unsat wins with $\bigcup_{j\in J} S_j$. The number of accepting runs of this machine is precisely $n!\mathcal{I}(S_i)$.

Our reduction to show #P-hardness is from the problem of counting solutions to a 1-in-3SAT instance, i.e., given a 3SAT formula, counting the number of valuations such that every clause has exactly one satisfied literal. This problem was shown to be #P-complete by Creignou and Hermann [59].

Let $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_k$ be a 3SAT formula, with $C_j = (\ell_j^1 \vee \ell_j^2 \vee \ell_j^3)$ for all j, and let $\{x_1, \dots, x_n\}$ be the set of variables appearing in φ . We first construct the formula $\psi = \bigwedge_{j=1}^k C_j \wedge \bigwedge_{j=1}^n (x_j \vee \neg x_j) \wedge \bigwedge_{j=1}^k C_j^{1,2} \wedge C_j^{2,3} \wedge C_j^{3,1}$ with $C_j^{i_1,i_2} = (\neg \ell_j^{i_1} \vee \neg \ell_j^{i_2})$.

One can check that a valuation $\nu: \{x_1, \ldots, x_n\} \to \{\bot, \top\}$ satisfies ψ if and only if it satisfies exactly one literal per clause in φ .

Furthermore if a valuation satisfies ψ , then it satisfies exactly one literal in every clause except for exactly one of $C_j^{1,2}, C_j^{2,3}, C_j^{3,1}$ for each j, in which it satisfies both literals.

We reuse the construction from the proof of Proposition IV.5, with ψ as our 3SAT instance. Recall that this construction used a reachability condition, easily expressible as a Büchi, parity or Muller condition, making the reduction work for all those winning conditions. As sink and f only have one outgoing transition, they have no influence on the satisfaction of a specification by a set of states, thus their importance is 0. As a consequence, by Corollary III.7 they can be ignored in the computation of the importance, thus we will only consider set of states containing neither. Then a team of states T makes (s,T) critical if and only if it contains all the c_i but not s and there exists a valuation s satisfying s such that s contains exactly the states associated literals satisfied by s, except in clauses s in which s satisfies both literals, in which s contains either one of the two states or both.

As a result, for every valuation ν satisfying ψ , we have exactly 3^k sets of states T making s critical and matching that valuation. Indeed, T is completely determined by ν except for one $C_j^{i_1,i_2}$ for each $1 \leq j \leq k$, in which it has three possibilities: contain the first literal, the second, or both.

A brief analysis shows that for each such valuation ν , there are, for each $0 \le i \le k$, $\binom{i}{k} 2^{k-i}$ corresponding teams of size 2i+k-i+2k+2n (those teams being the ones containing both literals in i out of the k clauses $C_j^{i_1,i_2}$ in which ν satisfies both literals), adding up to 3^k teams.

Let N be the total number of states in the Kripke structure. By Corollary III.7, the number of valuations satisfying φ with exactly one satisfied literal per clause is therefore $\frac{(N-2)!}{N!M}N!I(s)$, with

$$M = \sum_{i=1}^{k} {i \choose k} 2^{k-i} (i+3k+2n)! ([N-2]-i-3k-2n)!$$

As M can be computed in polynomial time, the problem is therefore #P-complete. \Box

Proof of Theorem IV.7. The value, usefulness and importance threshold problems for Emerson-Lei conditions are PSPACE-complete.

Proof. The upper bounds arise from the complexity of solving Emerson-Lei games, which are PSPACE-complete [10]. As enumerating permutations of the states can be done in linear

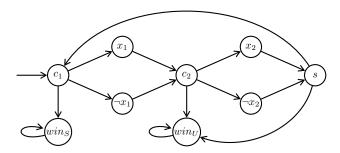


Fig. 10. Kripke structure corresponding to formula $\forall x_1, \exists x_2, x_1 \land \neg x_2$

space, one can compute the importance of a set of states in PSPACE.

For the lower bounds, we adapt a classic proof that Emerson-Lei games are PSPACE-hard to our framework. We only need to prove that the usefulness problem is PSPACE-hard as the importance threshold problem reduces to it. Further, we only use the particular case when the set of states is partitioned in singletons.

We reduce the QSAT problem. Let $Q_1x_1\cdots Q_kx_k\psi$ be a QSAT instance, we consider the following Kripke structure:

- $\{c_i, x_i, \neg x_i \mid 1 \le i \le k\} \cup \{s, win_S, win_U\}$ is the set of states, c_1 is the only initial state.
- For all $1 \leq i \leq k$ there are transitions $(c_i, x_i), (c_i, \neg x_i), (x_i, c_{i+1}), (\neg x_i, c_{i+1}),$ with $c_{k+1} = s$. There is also a transition (c_i, win_U) if $Q_i = \exists$ and (c_i, win_S) if $Q_i = \forall$. The remaining transitions are $(s, c_1), (s, win_U), (win_S, win_S), (win_U, win_U)$.

The labeling is irrelevant here. Figure 10 illustrates the construction.

For all $1 \le i \le k$ let

$$\varphi_i = Inf(x_i) \wedge Inf(\neg x_i) \wedge \bigwedge_{j=1}^{i} \neg (Inf(x_j) \wedge Inf(\neg x_j))$$

expressing that i is the minimal i such that both x_i and $\neg x_i$ are visited infinitely many times.

We take as winning condition for Sat the formula

$$(\psi' \vee Inf(win_S) \vee \bigvee_{Q_i = \forall} \varphi_i) \wedge \neg Inf(win_U) \wedge \bigwedge_{Q_i = \exists} \neg \varphi_i$$

where ψ' is ψ in which every x_i has been replaced with $Inf(x_i)$.

This construction can be done in logarithmic space. We will now prove that the QSAT formula is valid if and only if state s is useful.

Suppose the QSAT formula is valid, let $T = \{c_i \mid Q_i = \exists\}$. Clearly Sat loses with T as by taking the transition to win_U from s, Unsat can guarantee that every play reaches win_U and thus wins.

As the QSAT formula is valid, there exist functions $(f_i)_{Q_i=\exists}$ such that for all i $f_i: \{\top,\bot\}^{i-1} \to \{\top,\bot\}$ and for all $\nu: \{x_1,\ldots,x_k\} \to \{\top,\bot\}$ such that for all f_i we have $\nu(x_i)=f_i(\nu(x_1),\ldots,\nu(x_{i-1})), \, \nu$ satisfies ψ .

Further, as Sat makes all the existential choices, if Sat chooses according to f_i from every c_i she owns, and takes the transition to c_1 from s. Suppose Sat takes the transitions to x_i and $\neg x_i$ infinitely many times, then as Sat plays according to functions f_i , it means there exists a j < i such that x_j and $\neg x_j$ were visited infinitely many times.

As a consequence, the minimal i, if it exists, such that x_i and $\neg x_i$ are visited infinitely many times is such that $Q_i = \forall$. If it exists, then φ_i is satisfied, while φ_j is not satisfied for any other j, and as win_U is never visited, Sat wins.

If it does not exist, then for all j exactly one of $x_j, \neg x_j$ is visited infinitely many times, and as Sat plays according to the f_i , we have that ψ' is satisfied. As no φ_j is satisfied and win_U is never visited, Sat wins.

Now suppose the QSAT formula is not satisfiable, and suppose there exists T such that Sat wins with $T \cup \{s\}$ but not with T. If there exists $c_i \in T$ such that $Q_i = \forall$ or $c_i \notin T$ such that $Q_i = \exists$, then either Sat can reach win_S and win with T, or Unsat can reach win_U and win while Sat has $T \cup \{s\}$. Whether win_S, win_U or the $x_i, \neg x_i$ belong to T is irrelevant as they have only one outgoing transition.

Thus we can assume that $T = \{c_i \mid Q_i = \exists\}$. By similar arguments as above, there exist functions $(f_i)_{Q_i = \forall}$ such that f_i associates to the i-1 values of the previous literals a valuation of x_i , and any valuation respecting those functions does not satisfy ψ . And again by similar arguments as above, playing according to those functions allows Unsat to win the game while Sat has $T \cup \{s\}$, contradicting the hypothesis that Sat wins with $T \cup \{s\}$. As a result, s is not useful. \Box

Proof of Proposition IV.8. The usefulness problem for Rabin conditions is Σ_2^P -complete.

Proof. For the upper bound we simply consider a nondeterministic Turing machine guessing a set of indices J and calling an NP oracle twice to check that player Sat wins with $\bigcup_{j\in J\cup\{i\}} S_j$ as set of states and loses with just $\bigcup_{j\in J} S_j$.

For the lower bound, we reduce the dual of the $\forall \exists 3SAT$ problem, known to be Π_2^P -complete [60] [61]. Given a formula $\varphi = \forall (x_i)_{1 \leq i \leq n}, \exists (y_i)_{1 \leq i \leq p} \psi$ with $\psi = \bigvee_{i=1}^k Cl_i$ a $\forall \exists 3SAT$ instance, we are going to construct a Kripke structure \mathcal{K} (with states partitioned in singletons), a state s and a Rabin condition R such that s is useful to \mathcal{K} with respect to R if and only if this formula is *not* valid.

First of all note that we can assume that every clause contains an existential variable y_i . Indeed, any clause $(\ell_1 \lor \ell_2 \lor \ell_3)$ can be replaced by $(\ell_1 \lor \ell_2 \lor y) \land (\neg y \lor \ell_3) \land (\neg \ell_3 \lor y)$, with y a fresh variable which we add to the set of existential ones. One can check that we obtain a formula equisatisfiable to the previous one.

Consider the structure K whose states are elements of

$$\{c_{i}, x_{i}, \neg x_{i}, c'_{i}, x'_{i}, \neg x'_{i} \mid 1 \leq i \leq n\}$$

$$\cup \{sk_{x_{i}}, sk_{\neg x_{i}}, ret_{x_{i}}, ret_{\neg x_{i}} \mid 1 \leq i \leq n\}$$

$$\cup \{y_{j}, \neg y_{j} \mid 1 \leq j \leq p\} \cup \{Cl_{i} \mid 1 \leq i \leq k\} \cup \{s, sink\}$$

whose initial state is c_1 and whose transitions are as follows:

- There are transitions from init to itself, to c_1 and to every Cl_i .
- For all i there are transitions from c_i to x_i and $\neg x_i$ and from x_i and $\neg x_i$ to c_{i+1} , with $c_{n+1} = c'_1$.
- For all i, for all $\ell \in \{x_i, \neg x_i\}$, there are transitions $(\ell, sk_\ell), (sk_\ell, \neg \ell')$
- We have transitions $(c_i',x_i'),(c_i',\neg x_i'),(x_i',c_{i+1}'),(\neg x_i',c_{i+1}')$ for all $1\leq i\leq n$, with the convention $c_{n+1}'=s$.
- For all ℓ of the form x_i or $\neg x_i$, for all clause Cl_j containing ℓ there are transitions (Cl_j, ret_ℓ) and (ret_ℓ, ℓ) .
- For all ℓ of the form y_i or $\neg y_i$, for all Cl_j containing ℓ , there is a transition (Cl_j, ℓ) and a transition (ℓ, c_1) .
- For all clause Cl_j there is a transition (s, Cl_j) .
- There are transitions from all c_i, c'_i, x_i, ¬x_i, x'_i, ¬x'_i, Cl_j to sink.

Figure 11 illustrates the construction. There are transitions from the blue and white states to skip, which is omitted on the picture. The blue states are the ones hardcoded to belong to Sat, the grey ones are the ones that have only one outgoing transition, and the white ones are the ones encoding the valuation of the x_i .

As states $sink, ret_\ell, sk_\ell, y_i, \neg y_i$ have only one outgoing transition, whether they belong to Sat or Unsat has no consequence on the game. In the proof that follows we will ignore which player they belong to.

We take as Rabin condition

$$R = \{(\{y_i\}, \{\neg y_i\}), (\{\neg y_i\}, \{y_i\}) \mid 1 \le i \le p\}$$

$$\cup \{(\{sk_\ell\}, \emptyset), (\{ret_\ell\}, \emptyset) \mid 1 \le i \le n, \ell \in \{x_i, \neg x_i\}\}$$

$$\cup \{(\emptyset, \{c'_1, sink\}), (\{init\}, \emptyset)\}.$$

The construction can be done in logarithmic space. We will now show that the formula φ is not valid if and only if s is useful in the Kripke structure with respect to this Rabin condition.

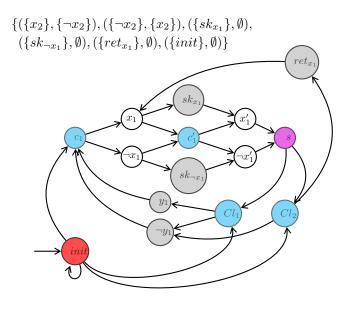


Fig. 11. Construction for the formula $\forall x_1 \exists x_2 (x_2 \lor \neg x_2) \land (x_1 \lor \neg x_2)$. The sink state is omitted. The Rabin condition is displayed at the top of the figure.

Suppose that φ is not valid, let ν be a valuation of the x_i not satisfying $\exists (y_i)\psi$. We take as set of states T all the c_i, c_i' , all the Cl_i , and the ℓ and ℓ' such that $\nu(\ell) = \top$.

If Sat has states $T \cup \{s\}$, then she can pick any clause Cl_j , any literal ℓ in Cl_j of the form y_i or $\neg y_i$.

If Unsat loops on init forever, then Sat wins. If he chooses to go to c_1 , then as there is a path P in T from c_1 to s, Sat can repeat indefinitely the cycle taking P from c_1 to s, then going through Cl_j , then ℓ , then back to c_1 . This allows Sat to win as she goes through ℓ infinitely many times without going through $\neg \ell$.

If Unsat goes to some Cl_j from init then Sat can simply go to some y_i or $\neg y_i$ (recall that we assumed every clause to contain an y_i or $\neg y_i$), then to c_1 and from there play as in the previous case.

If Sat has states T, then we proceed by contradiction. Suppose Sat has a winning strategy, then as she is the player with a Rabin winning condition, she has a positional one [62]. In particular from every Cl_j Sat picks either a successor ret_ℓ with ℓ in Cl_j or a successor y_i or $\neg y_i$ in Cl_j . In the first case, ℓ has to be satisfied by ν , otherwise after ret_ℓ the game reaches ℓ , from which Unsat goes to sink and wins.

Further, from every c_i Sat has to pick the successor x_i or $\neg x_i$ satisfied by ν , otherwise Unsat can then reach sink and win.

From an x_i belonging to Sat, she cannot go to sk_{x_i} as then she ends up in $\neg x_i'$, from where Unsat can reach sink. The same argument stops her from going to $sk_{\neg x_i}$ from $\neg x_i$. As a result, from c_1 Sat has to follow a path to s.

As every ret_{ℓ} to which Sat goes from a Cl_j is such that ν satisfies ℓ , if there were a valuation of the y_i satisfying every

literal y_i or $\neg y_i$ to which Sat goes to from a clause, then we could infer from the strategy of Sat a valuation μ such that ν and μ combined satisfy ψ . This would contradict the fact that ν does not satisfy $\exists (y_i)\psi$, thus there is no such valuation μ .

As a result, there exist i, j_1, j_2 such that Sat picks y_i from Cl_{j_1} and $\neg y_i$ from Cl_{j_2} . As Sat has to go to s from c_1 , Unsat can then alternately choose Cl_{j_1} and Cl_{j_2} as successors, thus making Sat go infinitely many times through y_i and $\neg y_i$ (and never through other y_i or $\neg y_i$). Then Unsat wins, contradicting the fact that Sat is playing a winning strategy.

In conclusion, if φ is not valid, then s is useful for \mathcal{K} with respect to R.

Now we have to prove that if s is useful for K with respect to R, then φ is not valid. Suppose the former, let T be a set of states such that Sat wins with $T \cup \{s\}$ but not with T.

As Sat loses with T, init cannot be in T, otherwise Sat could win by looping forever on init.

Suppose there exists a $Cl_j \notin T$, then Unsat can win by going to Cl_j from init and then to sink, contradicting the fact that Sat wins with $T \cup \{s\}$. Thus T contains all Cl_j . If Unsat loops on init forever then Sat wins. If Unsat goes from init to some Cl_j then Sat can go to some y_i or $\neg y_i$ and from there to c_1 . We can thus assume that the players always end up reaching c_1 .

As Sat loses with T, there cannot be any path in T from c_1 to s going through a sk_ℓ , otherwise Sat could go infinitely many times through that sk_ℓ . However as Sat wins with $T \cup \{s\}$, there has to be a path from c_1 to s in T (otherwise Sat would have to reach a state of Unsat with a transition to sink and lose). Thus for all i, c_i, c_i' belong to T, as well as one of $x_i, \neg x_i$ and one of $x_i', \neg x_i'$. Further, for all i, we cannot have both x_i and $\neg x_i'$, or both $\neg x_i$ and x_i' in T. As a result for all i either $x_i, x_i' \in T$ and $\neg x_i, \neg x_i' \notin T$ or $\neg x_i, \neg x_i' \in T$ and $x_i, x_i' \notin T$. Let ν be the valuation of the x_i such that $\nu(x_i) = \top$ if and only if $x_i \in T$.

Let μ be a valuation of the y_i , suppose for the sake of contradiction that the combination of ν, μ satisfies ψ . Then for all j Cl_j has a transition either to a ret_ℓ with $\nu(\ell) = \top$ or to an y_i with $\mu(y_i) = \top$ or to a $\neg y_i$ with $\mu(y_i) = \bot$.

Then by taking from each Cl_j the successor as stated above, Sat wins as she will necessarily either go through a ret_ℓ infinitely many times, or through an y_i or $\neg y_i$ infinitely many times while never visiting the opposite literal.

We obtain a contradiction as Unsat is supposed to win the game when Sat only owns T.

As a result, ν , μ cannot satisfy ψ , thus φ is not valid.

Proof of Theorem IV.9. The importance computation problem for Rabin conditions is $\#P^{NP}$ -complete.

Proof. We reduce the problem of counting, given a formula ψ in 3CNF over variables $x_1, \ldots, x_n, y_1, \ldots, y_p$, the number of valuations of the x_i such that for all valuations of the y_i , the combination of those does not satisfy ψ .

First, let us justify that this problem is $\#P^{NP}$ -hard. Let M be a non-deterministic Turing machine with an oracle solving an NP-complete problem (say SAT), let w be an input.

We can assume without loss of generality that M only makes one query to the oracle, and accepts if and only if the answer is negative.

Indeed, say M has to make queries ψ_1, \ldots, ψ_k to the oracle, all over existential variables y_1, \ldots, y_p . It can nondeterministically guess the answers of the oracle and delay the verification to the end of the run.

Now let us define the order \leq on valuations of the existential variables as the lexicographic order, a valuation ν being seen as the tuple $(\nu(y_1),\ldots,\nu(y_p))$ and with the convention $\bot \leq \top$. Then for the positive answers M can guess a minimal witness valuation for the y_i with respect to \leq , negate the formula. The problem, given a SAT instance and a valuation of the existential variables, of checking whether this is the minimal valuation witnessing the satisfiability of the formula, is clearly in coNP. As a result M can guess the minimal valuation of the y_i witnessing the satisfiability of the formula, and then turn it into a SAT formula unsatisfiable if and only if the guess is correct. As there is for every SAT formula a unique minimal valuation satisfying it, M can only make one correct guess, thus its number of runs is unchanged.

Finally, in the end M has to make the oracle check a disjunction of \exists formulas, it can rename variables in order to merge them all into one equivalent SAT instance, and accept if and only if the oracle rejects that formula.

We now use the classical encoding of Turing machines in 3CNF formulas to construct a 3CNF formula ψ over variables $x_1, \ldots, x_n, y_1, \ldots, y_p, r_1, \ldots, r_k$ such that for all valuations of the x_i, y_i (encoding respectively the nondeterministic choices of M and the ones of the oracle), there is a non-accepting run of M on w if and only if there exists a valuation of the r_i (encoding the runs of M and the oracle) satisfying the formula along with these valuations of x_i, y_i .

As a result, there is an accepting run of M on w if and only if the formula $\forall (x_i) \exists (y_i), (r_i) \varphi_1$ is not valid, and the valuations of the x_i witnessing non-validity are in bijection with the runs of M.

Hence the problem is #PNP-hard.

Now in order to prove the hardness for the importance computation problem for Rabin conditions, we use the same construction as in the proof of Proposition IV.8. Let φ be a 3CNF formula with k clauses over variables $x_1, \ldots, x_n, y_1, \ldots, y_p$, we consider the Kripke structure from that proof.

Furthermore, a set of states T makes (s,T) critical if and only if it contains the c_i, c_i', Cl_i , and the $x_i, \neg x_i, x_i' \neg x_i'$ encoding a valuation of the x_i such that for all valuation of the y_i , the combination of the two valuations does not satisfy φ .

Note that states $sk_{\ell}, ret_{\ell}, y_i, \neg y_i$ all have one outgoing transition and thus have importance 0.

As those sets T all have the same size k+4n, the formula from Corollary III.7 gives us that the number of valuations of the x_i such that for all valuation of the y_i , the combination of the two valuations does not satisfy φ is $\frac{P!}{N!(k+4n)!(P-k+4n)!}N!I(s)$ where N is the number of states

in the Kripke structure and P the number of states minus the $sk_{\ell}, ret_{\ell}, y_i, \neg y_i$

As $\frac{P!}{N!(k+4n)!(P-k+4n)!}$ can be computed in polynomial time, the importance computation problem for Rabin conditional time. tions is #P^{NP}-complete.

C. CTL proofs

Proof of Proposition V.2. The value problem for two-turn CTL is Σ_2^P -complete.

Proof. One can reformulate the problem as the existence of a subset of outgoing transitions from S_{Sat} such that for all subsets of outgoing transitions from S_{Unsat} , the structure yielded by those subsets of transitions satisfies φ .

As those subsets of transitions are of polynomial size, and as the satisfaction of a Kripke structure by a CTL formula can be checked in polynomial time, the problem is in $\Sigma_2^{\rm P}$.

We now prove the lower bound, by reducing $\exists \forall SAT$. Let $\exists (x_i)_{1 < i < n}, \forall (y_i)_{1 < i < k} \psi$ with ψ quantifier-free be a $\exists \forall SAT$ instance. Without loss of generality, we assume that all the negations in ψ have been pushed to the atomic propositions.

We consider the following modal transition system $\mathcal{M} = (S, AP, \Delta_{must}, \Delta_{may}, init, \lambda)$ with:

- $S = \{sink\} \cup \{c_i, x_i, \neg x_i \mid 1 \le i \le n + k\}$. The initial state is c_1 .
- AP = $\{x_i \mid 1 \le i \le n+k\}$ and $\lambda(x_i) = \{x_i\}$ for all $1 \le i \le n + k$ and $\lambda(s) = \emptyset$ for all other $s \in S$.
- $\Delta_{must} = \{(x_{n+k}, sink), (\neg x_{n+k}, sink), (sink, sink)\} \cup$ $\{(x_i, c_{i+1}), (\neg x_i, c_{i+1}) \mid 1 \le i \le n+k-1\}.$
- $\Delta_{may} = \{(c_i, x_i), (c_i, \neg x_i) \mid 1 \le i \le n + k\}.$

We split S into

$$S_{\mathtt{Sat}} = \{sink\} \cup \{x_i, \neg x_i \mid 1 \le i \le n+k\} \cup \{c_i \mid 1 \le i \le n\}$$

and $S_{\mathtt{Unsat}} = \{c_i \mid n+1 \le i \le n+k\}.$

Informally, we are going to make players choose valuations of the variables through their choices of transitions. The CTL formula will then ensure that the choices of transitions yield well-defined valuations, and that these valuations satisfy the SAT formula.

With that goal in mind, we define the specification as follows:

$$\varphi = (\varphi_{SAT} \land \varphi_{checkSat}) \lor \varphi_{checkUnsat}$$

$$\varphi_{checkSat} = \bigwedge_{i=1}^{n} EX^{2i-2} (AX(x_i) \vee AX(\neg x_i)) \wedge EX \top$$

$$\varphi_{checkUnsat} = EX^{2n} \bigvee_{i=1}^{n} EX^{2i-2}(EX(x_i) \wedge EX(\neg x_i)) \wedge AX \perp \text{ Proof. Let } \mathcal{M} = (S, AP, \Delta_{must}, \Delta_{may}, init, \lambda) \text{ be an MTS,}$$

$$\text{let } S_1 = S_2 \text{ be a partition of } S \text{ let } 1 \leq i \leq n \text{ let } \varphi \text{ be a}$$

and φ_{SAT} is ψ where every x_p has been replaced by $EX^{2p-1}x_p$ and every $\neg x_p$ replaced by $EX^{2p-1}\neg x_p$. Recall that we assumed that ψ only has negations in front of atomic propositions. This construction can be done in logarithmic space.

The idea is that φ_{SAT} mimics ψ in order to check that there exists a path in the structure obtained through the game matching a valuation satisfying ψ . Meanwhile, formulas $\varphi_{checkSat}$ and $\varphi_{checkUnsat}$ ensure that players never pick both x_i or neither.

Now for the formal proof, suppose there exists a valuation $\nu_1: \{x_1,\ldots,x_n\} \to \{\top,\bot\}$ such that for every valuation $\nu_2: \{x_{n+1}, \dots, x_{n+k}\} \to \{\top, \bot\}$, the combination of ν_1 and ν_2 satisfies ψ .

Then let
$$\sigma_1(c_i) = \begin{cases} x_i \text{ if } \nu_1(x_i) = \top \\ \neg x_i \text{ otherwise} \end{cases}$$
 for $1 \leq i \leq n$ and let σ_2 be a pure strategy for Unsat. Clearly as

 $|\sigma_1(c_i)| = 1$ for all i, the resulting structure satisfies $\varphi_{checkSat}$. If $|\sigma_2(c_i)| = 0$ for some i, then $\varphi_{checkUnsat}$ is satisfied, thus so is φ . If Unsat gives every c_i a successor, then there is a path from c_1 to sink, representing a valuation whose projection to $\{x_1, \cdots, x_n\}$ matches ν_1 . As a result, ψ is satisfied by this valuation, thus φ_{SAT} is satisfied by the structure yielded by σ_1 and σ_2 , hence so is φ .

Now suppose there exists a pure strategy σ_1 for Sat such that for every pure strategy σ_2 for Unsat, σ_1, σ_2 yield a structure satisfying φ . For all $1 \le i \le n$, if we had $|\sigma(c_i)| = 0$, then neither $\varphi_{checkSat}$ nor $\varphi_{checkUnsat}$ would be satisfied, and if we had $|\sigma(c_i)| > 1$, then $\varphi_{checkSat}$ would not be satisfied, and Unsat could win by choosing one outgoing transition for each c_i he owns, thereby unsatisfying $\varphi_{checkUnsat}$. As a result, σ_1 selects exactly one of $\{x_i, \neg x_i\}$ for each i, thus we can define ν_1 the valuation such that

$$\nu_1(x_i) = \begin{cases} \top \text{ if } \sigma_1(c_i) = x_i \\ \bot \text{ otherwise} \end{cases} \text{ for } 1 \le i \le n$$
 Let ν_2 : $\{x_{n+1}, \dots, x_{n+k}\} \rightarrow \{\top, \bot\}$, we define a

corresponding strategy for Unsat as
$$\sigma_2(c_i) = \begin{cases} x_i \text{ if } \nu_2(c_i) = \top \\ \neg x_i \text{ otherwise} \end{cases} \text{ for } n+1 \leq i \leq n+k.$$

As σ_1, σ_2 yield a structure satisfying φ , either φ_{SAT} is satisfied or $\varphi_{checkUnsat}$ is. Further, as in that structure every state has exactly one successor, $\varphi_{checkUnsat}$ is not satisfied, thus φ_{SAT} is. As a consequence, the combination of ν_1 and ν_2 satisfies ψ .

We have constructed in logarithmic space a CTL formula, a modal transition system and a subset S_{Sat} of states such that Sat has a pure winning strategy on S_{Sat} if and only if ψ with set of existential variables $\{x_1, \dots, x_n\}$ is in $\exists \forall SAT$.

As a result the value problem corresponding to definition V.1 is Σ_2^P -complete.

Proof of Proposition V.3. The usefulness problem for twoturn CTL is Σ_3^P -complete.

let S_1, \ldots, S_n be a partition of S, let $1 \le i \le n$, let φ be a CTL formula.

In order to check the usefulness of s, we can guess a set of indices J and a pure strategy $\sigma_1: \bigcup_{j \in J \cup \{i\}} S_j \to \Delta_{may}$, make an adversary choose pure strategies $\sigma_1^{(i)}: \bigcup_{j \in J} S_j \rightarrow$ Δ_{may} and $\sigma_2: S \setminus (\bigcup_{i \in J \cup \{i\}} S_i) \to \Delta_{may}$, and then guess a pure strategy $\sigma_2': S \setminus \bigcup_{j \in J} S_j \to \Delta_{may}$ such that the structure yielded by σ_1, σ_2 satisfies φ but the one yielded by σ_1', σ_2' does not

This shows that the problem is in Σ_3^P .

Now let us show hardness. We reduce the problem $\exists \forall \exists SAT$. Let $\exists x_1,\ldots,x_n, \forall y_1,\ldots,y_k, \exists z_1,\ldots,z_p\psi$ with ψ quantifier-free be a $\exists \forall \exists SAT$ instance. We assume without loss of generality that all negations have been pushed to the atomic propositions.

We define the MTS $\mathcal{M} = (S, AP, \Delta_{must}, \Delta_{may}, init, \lambda)$ as follows:

$$S = \{x_i, \neg x_i, c_i^x \mid 1 \le i \le n\}$$

$$\cup \{y_i, \neg y_i, c_i^y \mid 1 \le i \le k\}$$

$$\cup \{z_i, \neg z_i, c_i^z \mid 1 \le i \le p\}$$

$$\cup \{x_i', \neg x_i' \mid 1 \le i \le n\} \cup \{win_S, win_U, s\}$$

$$AP = S$$

$$\Delta_{must} = \{(x_{i}, c_{i+1}^{x}), (\neg x_{i}, c_{i+1}^{x}) \mid 1 \leq i \leq n-1\}$$

$$\cup \{(y_{i}, c_{i+1}^{y}), (\neg y_{i}, c_{i+1}^{y}) \mid 1 \leq i \leq k-1\}$$

$$\cup \{(z_{i}, c_{i+1}^{z}), (\neg z_{i}, c_{i+1}^{z}) \mid 1 \leq i \leq p-1\}$$

$$\cup \{(x_{n}, c_{1}^{y}), (\neg x_{n}, c_{1}^{y}), (y_{k}, c_{1}^{z}), (\neg y_{k}, c_{1}^{z})\}$$

$$\cup \{(x'_{i}, x'_{i+1}), (x'_{i}, \neg x'_{i+1}) \mid 1 \leq i \leq n-1\}$$

$$\cup \{(\neg x'_{i}, x'_{i+1}), (\neg x'_{i}, \neg x'_{i+1}) \mid 1 \leq i \leq n-1\}$$

$$\cup \{(x_{i}, \neg x'_{i}), (\neg x_{i}, x'_{i}) \mid 1 \leq i \leq n\}$$

$$\cup \{(x'_{n}, win_{S}), (win_{U}, win_{S})\}$$

$$\cup \{(win_{S}, win_{S}), (win_{U}, win_{U}), (z_{p}, s), (\neg z_{p}, s)\}$$

$$\cup \{(c_{i}^{y}, win_{U}) \mid 1 \leq i \leq k\}$$

$$\cup \{(c_{i}^{z}, win_{S}) \mid 1 \leq i \leq p\}$$

$$\Delta_{may} = \{ (c_i^x, x_i), (c_i^x, \neg x_i) \mid 1 \le i \le n \}$$

$$\cup \{ (c_i^y, y_i), (c_i^y, \neg y_i) \mid 1 \le i \le k \}$$

$$\cup \{ (c_i^z, z_i), (c_i^z, \neg z_i) \mid 1 \le i \le p \}$$

$$\cup \{ (x_i, win_U), (\neg x_i, win_U) \mid 1 \le i \le n \}$$

$$\cup \{ (x_i', win_U), (\neg x_i', win_U) \mid 1 \le i \le n \}$$

$$\cup \{ (s, x_1'), (s, \neg x_1') \}$$

 $\lambda(t)=\{t\}$ for every state t, and the initial state is $init=c_1^x$. We consider the formula

$$\varphi = (\neg \varphi_{SAT} \land \varphi_{checkSat} \land AG \neg win_U)$$
$$\lor (EFwin_S \land AG \neg win_U)$$
$$\lor \varphi_{checkUnsat}$$

with

$$\varphi_{checkSat} = AG(\bigwedge_{i=1}^{k} AX(y_i) \vee AX(\neg y_i)) \wedge (\bigwedge_{i=1}^{k} (EX^{2n+2i-2})EX\top)$$

 $\varphi_{checkUnsat} =$

$$EF(\bigvee_{i=1}^{p}EX(z_{i})\wedge EX(\neg z_{i}))\vee\bigvee_{i=1}^{p}(EX^{2n+2k+2i-2})AX\bot$$

and φ_{SAT} is ψ where every x_i, y_i, z_i has been replaced by respectively EFx_i, EFy_i and EFz_i , and every $\neg x_i, \neg y_i, \neg z_i$ by respectively $AG \neg x_i, AG \neg y_i, AG \neg z_i$. This construction can be done in logarithmic space. The formulas $\varphi_{checkUnsat}$ and $\varphi_{checkUnsat}$ ensure that the players never allow transitions to both or neither variables from a c_i^x, c_i^y or c_i^z state.

Suppose there exists T such that Sat wins with $T \cup \{s\}$ but loses with T. As all $x_i, \neg x_i, x_i', \neg x_i'$ have a may transition to win_U , there has to be either a path from c_1^x to c_1^y in T, or a path in T from c_1^x to some x_i or $\neg x_i$, from there a transition to some x_i' or $\neg x_i'$, and a path in T from there to win_S , otherwise Unsat wins both games. In the second case, Sat wins without s, thus we have to be in the first case. In particular for every $x_i \in T$, $x_i' \notin T$ and for every $\neg x_i \in T$, $\neg x_i' \notin T$.

As a result, there has to be a path in \mathcal{M} (using may and must transitions) to all c_i^x, c_i^y, c_i^z from c_1^x . In order for the games with T and $T \cup \{s\}$ to have different winners, every c_i^y has to be in T (as they have a may transition to win_U) and similarly every c_i^z has to not be in T. The formulas $\varphi_{checkSat}$ and $\varphi_{checkUnsat}$ force both players to pick exactly one outgoing transition from each c_i^x, c_i^y, c_i^z .

Now observe that the choice of transitions from s has no impact on the satisfaction of $\neg \varphi_{SAT}, \varphi_{checkSat}, AG \neg win_U$ or $\varphi_{checkUnsat}$. As Unsat has a winning strategy when Sat only has T, this same strategy will ensure that Sat can only win by satisfying $EFwin_S \wedge AG \neg win_U$ in the game with $T \cup \{s\}$. In order to satisfy $EFwin_S \wedge AG \neg win_U$, there has to be a path in T from s to win_S . As a result, at least one of $x_i', \neg x_i'$ has to be in T. As we have seen before, for every $x_i \in T$, $x_i' \notin T$ and for every $\neg x_i \in T, \neg x_i' \notin T$, thus at most one of $x_i, \neg x_i$ can be in T for all $1 \leq i \leq n$. Further, we have seen that at least one of $x_i, \neg x_i$ has to be in T.

As a result, the set of x_i in T with $1 \leq i \leq n$ matches a valuation ν_1 of x_1, \cdots, x_n . Let ν_2 be a valuation of y_1, \cdots, y_k , suppose Sat picks transitions matching ν_2 from the c_i^y . As Unsat wins the game in which Sat owns only T, and as the satisfaction of both $\varphi_{checkSat}$ and $AG \neg win_U$ is guaranteed by the strategy of Sat, the only possibility is that $\neg \varphi_{SAT}$ is dissatisfied, which Unsat can only achieve by picking transitions matching a valuation ν_3 of z_1, \cdots, z_p such that the combination of ν_1, ν_2 and ν_3 satisfies φ . As a result, the $\exists \forall \exists SAT$ instance is true.

Now for the converse, suppose there exists a valuation ν_1 such that for all ν_2 , there exists ν_3 such that their combination satisfies ψ . Let T be such that $T \cap \{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\}$

and $T \cap \{x'_1, \dots, x'_n, \neg x_1', \dots, \neg x_n\}$ both match ν_1 , T contains every c_i^x and c_i^y but does not contain c_i^z for any $1 \le i \le p$.

Let us first look at the game in which Sat has states $T \cup \{s\}$. As one of $\{x_i, \neg x_i\}$ belongs to T for all $1 \le i \le n$, Sat can choose transitions so that there is a path from c_1 to c_{n+k+1} , and a transition from s to win_S .

If Unsat gives no outgoing transition to one of the c_i with $n+k+1 \leq i \leq n+k+p$, then $\varphi_{checkUnsat}$ is satisfied, thus so is φ . As a result, there is a path from c_{n+k+1} to either win_S or s, and thus also win_S . Hence φ is satisfied in every case, Sat wins that game.

Now let us study the game in which Sat only owns T. No matter which strategy Sat chooses, Unsat can guarantee that $EFwin_S \wedge AG\neg win_U$ is not satisfied by allowing the transition to win_U from every $x_i, \neg x_i x_i', \neg x_i'$ it owns, and not allowing any transition from s. This way, all paths to win_S go through states with a transition towards win_U . Unsat can also ensure that $\varphi_{checkUnsat}$ is not satisfied by picking transitions matching some valuation from the c_i he owns.

Assume Sat has a winning strategy to ensure that $\varphi_{checkSat} \wedge \varphi_{checkSat} \wedge AG \neg win_U$ is satisfied. As $\varphi_{checkSat}$ and $AG \neg win_U$ are satisfied, the choices of transitions of Sat from the c_i have to match ν_1 and a valuation ν_2 of $\{x_{n+1}, \cdots, x_{n+k}\}$. There exists a valuation ν_3 of $\{x_{n+k+1}, \ldots, x_{n+k+p}\}$ such that the combination of ν_1, ν_2 and ν_3 satisfies ψ . Then if Unsat chooses transitions from the c_i he owns matching ν_3 , $\neg \varphi_{SAT}$ is not satisfied by the resulting structure, contradicting the existence of a winning strategy for Sat.

We have proven the proposition.

Proof of Theorem V.4. The importance computation problem associated to definition V.1 is $\#P^{\sum_{2}^{P}}$ -complete.

Proof. The upper bound is easily obtained by considering the machine which guesses an ordering π of the elements of the partition S_1,\ldots,S_n , computes the set J of indices appearing after s in π , and then calls a Σ_2^P -oracle twice to determine the winner when Sat owns $\bigcup_{j\in J} S_j$ and when Sat owns $\bigcup_{j\in J\cup\{i\}} S_j$. The number of accepting runs of the machine is then precisely the number of permutations π matching the above condition.

Now for the lower bound, we proceed in two steps. First we show that the following problem is $\#P^{\sum_{2}^{p}}$ -complete.

 $Count\exists \forall \exists SAT:$

Input: A SAT formula ψ over variables

 $\{x_1,\ldots,x_n,y_1,\ldots,y_m,z_1,\ldots,z_r\}.$

Output: The number of valuations of the x_i such that

for all valuations of the y_i

there exists a valuation of the z_i such that

the combination of those valuations satisfies ψ .

Then we show that the importance computation problem reduces to $Count\exists \forall \exists SAT$.

For the first part, let M be a nondeterministic Turing machine with an oracle solving a Σ_2^P -complete problem (say

 $\exists \forall SAT$). We can assume without loss of generality that M only makes one query to the oracle, and accepts if and only if the answer is negative.

Indeed, say M has to make queries ψ_1, \ldots, ψ_k to the oracle, all over existential variables y_1, \ldots, y_m and universal variables z_1, \ldots, z_r . It can nondeterministically guess the answers of the oracle and delay the verification to the end of the run.

Now let us define the order \leq on valuations of the existential variables as the lexicographic order, a valuation ν being seen as the tuple $(\nu(y_1),\ldots,\nu(y_m))$ and with the convention $\bot \leq \top$. Then for the positive answers M can guess a *minimal* witness valuation for the y_i with respect to \leq , negate the formula. The problem, given a $\exists \forall \mathsf{SAT}$ instance and a valuation of the existential variables, of checking whether this is the minimal valuation witnessing the satisfiability of the formula, is clearly in coNP, thus also in Π_2^P . As a result M can guess the minimal valuation of the y_i witnessing the satisfiability of the formula, and then turn it into a $\exists \forall \mathsf{SAT}$ formula unsatisfiable if and only if the guess is correct.

Finally, in the end M has to make the oracle check a disjunction of $\exists \forall$ formulas, it can rename variables in order to merge them all into one equivalent $\exists \forall$ formula, and accept if and only if the oracle rejects that formula.

In all the above transformations, the number of accepting runs of the machine stays the same as the non-deterministic transitions we added (in order to guess minimal valuations witnessing satisfiability of $\exists \forall SAT$ formulas) yield at most one accepting run (as the existence of such a valuation is equivalent to the existence of a single minimal one).

An adaptation of the classical construction proving that $\exists \forall \text{SAT}$ is Σ_2^{P} -complete allows us to construct in polynomial time, given an input w for M, a formula $\varphi_1((x_i), (q_i), (s_i))$ such that the following conditions are equivalent for all valuations ν of the x_i , q_i and s_i :

- ν satisfies $\varphi_1((x_i),(q_i),(s_i))$
- the $\nu(x_i)$ encode a sequence of non-deterministic choices of M, the $\nu(s_i)$ encode a correct run of M following those choices, and the $\nu(q_i)$ encode the query made to the oracle at the end of this run

We can also construct in polynomial time a formula $\varphi_2((y_i)_{1 \leq i \leq m}, (z_i)_{1 \leq i \leq r}, (q_i)_{1 \leq i \leq p}, (u_i)_{1 \leq i \leq k})$ simulating the oracle such that a valuation of the q_i satisfies $\exists (y_i), \forall (z_i), \varphi_2((y_i), (z_i), (q_i), (u_i))$ if and only if the q_i encode a valid instance of $\exists \forall SAT$. As a result the formula

$$\forall (y_i), \exists (z_i), (x_i), (q_i), \varphi_1((x_i), (q_i), (s_i) \land \varphi_2((y_i), (z_i), (q_i), (u_i))$$

is satisfied by a valuation of the x_i if and only if M has a run accepting w following the choices encoded by this valuation. Thus the number of accepting runs of M is precisely the number of valuations of the s_i witnessing the validity of

$$\exists (x_i), \forall (y_i), \exists (z_i), (s_i), (q_i), \\ \varphi_1((x_i), (q_i), (s_i) \land \varphi_2((y_i), (z_i), (q_i), (u_i))$$

The problem $Count\exists \forall \exists SAT$ is therefore $\#P^{\Sigma_2^P}$ -hard.

Finally, $Count\exists \forall \exists \mathsf{SAT}$ can be reduced to the importance computation problem for 2-turn CTL using the same construction as in the proof of Proposition V.3. Note that the $x_i, \neg x_i$ for all $n+1 \leq i \leq n+k+p$, as well as win_S, win_U , all have no outgoing may transitions, thus have importance 0 and thus, by a similar argument as in Lemma III.6, can be ignored in the computation of the importance. We will now only consider sets of states containing none of those. Then one can observe that the teams T allowing player Sat to win with $T \cup \{s\}$ but not with T are exactly the teams T such that

- T contains all the c_i for $\leq n + k$ and no other c_i .
- $T \cap \{x_i, \neg x_i \mid 1 \le i \le n\}$ and $T \cap \{x_i', \neg x_i' \mid 1 \le i \le n\}$ match a same valuation ν witnessing the validity of the $\exists \forall \exists \mathsf{SAT}$ formula.

Then we have that all the teams T such that (s,T) is critical (and containing none of the aforementioned states with importance 0) have the same size M.

We obtain that the number of valuations witnessing the validity of the $\exists \forall \exists \mathsf{SAT}$ formula is $\frac{P!}{N!M!(P-M-1)!}N!I(s)$, with N the number of states in the constructed MTS and P the number of states minus the $x_i, \neg x_i$ for $n+1 \leq i \leq n+k+p$, $s, \ win_U$ and win_S . Hence we have a reduction from $Count\exists \forall \exists \mathsf{SAT}$ to the importance problem for 2-turn CTL. As $\frac{P!}{N!M!(P-M-1)!}$ can be computed in polynomial time, the latter problem is $\#P^{\Sigma_2^p}$ -complete.