# Separating Rank Logic from Polynomial Time

# Moritz Lichter RWTH Aachen University lichter@lics.rwth-aachen.de

August 15, 2023

#### Abstract

In the search for a logic capturing polynomial time, the most promising candidates are Choiceless Polynomial Time (CPT) and rank logic. Rank logic extends fixed-point logic with counting by a rank operator over prime fields. We show that the isomorphism problem for CFI graphs over  $\mathbb{Z}_{2^i}$  cannot be defined in rank logic, even if the base graph is totally ordered. However, CPT can define this isomorphism problem. We thereby separate rank logic from CPT and in particular from polynomial time.

### 1 Introduction

The quest for a logic capturing polynomial time (PTIME) is one of the central open questions in the field of descriptive complexity theory [18]. This question [8] asks whether there is a logic within which we can define exactly the polynomial-time computable properties of finite relational structures. The two most promising candidates for such a logic are Choiceless Polynomial Time and rank logic [16]. In this article we rule out rank logic as a candidate. We show that rank logic neither captures PTIME nor Choiceless Polynomial Time.

Rank logic was introduced in [11] and extends fixed-point logic with counting (IFP+C) by a rank operator. Using this rank operator, the rank of definable matrices can be accessed in the logic. Multiple variants of rank logic were proposed. In its first version [11], rank logic comes with a rank operator  $\mathsf{rk}_p$  for each prime p. If the universe of a finite structure is A, then an  $A^k \times A^k$  matrix is defined by a term  $s(\bar{x}, \bar{y})$  by setting the entry indexed by  $(\bar{u}, \bar{v})$  to the value  $s(\bar{u}, \bar{v})$ , to which s evaluates in the structure. The rank operator  $\mathsf{rk}_p$  evaluates to the rank of said matrix over  $\mathbb{F}_p$ . When considering  $A^k \times A^k$  matrices, we call the rank operator k-ary.

Crucially, rank logic defines the isomorphism problem of the so-called CFI graphs. These graphs were given by Cai, Fürer, and Immerman [6] to separate IFP+C from Ptime. From a base graph, one obtains a CFI graph by replacing every vertex with

The research leading to these results has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (EngageS: grant agreement No. 820148).

a particular gadget and by connecting the gadgets of adjacent vertices. A connection between two gadgets can either be straight or twisted. The essential point of the construction is that two CFI graphs over the same base graph are isomorphic if and only if they have the same parity of twisted connections. In particular, for each base graph there is a pair of non-isomorphic CFI graphs. The CFI query is the task of defining whether the parity of twisted connection of a CFI graph is zero. CFI graphs implicitly define a linear equation system over  $\mathbb{F}_2$ , which is solvable if and only if the parity of twists is zero. That is, the CFI query is decidable by checking these linear equations systems for solvability. Given a CFI graph, the matrix corresponding to the linear equation system is definable in rank logic and, by considering ranks, also whether it is solvable.

The CFI construction is not restricted to the field  $\mathbb{F}_2$  but can be generalized to other finite fields  $\mathbb{F}_p$  or even groups (see e.g. [3, 17, 28]). In the case of  $\mathbb{F}_p$ , there are p many non-isomorphic CFI graphs for a given base graph. Grädel and Pakusa [17] used CFI graphs over prime fields to show that extending IFP+C by the rank operators  $\mathsf{rk}_p$  is not sufficient to capture PTIME. If the CFI graphs are defined over  $\mathbb{F}_p$ , then the CFI query over  $\mathbb{F}_p$  is not definable only using rank operators  $\mathsf{rk}_q$  for other primes  $q \neq p$ . This implies that there is no single formula of this variant of rank logic defining the CFI query for all CFI graphs over an arbitrary prime field. An alternative variant of rank logic was proposed in [17, 23, 25, 30]. It replaces the rank operators  $\mathsf{rk}_p$  for fixed fields by a uniform rank operator  $\mathsf{rk}$ , which defines the prime p using a formula, i.e., p depends on the structure on which the formula is evaluated. This second variant of rank logic defines the CFI query over all prime fields.

Another example that demonstrates the expressiveness of rank logic are multipedes. These structures come also with an isomorphism problem, which cannot be defined in IFP+C but in rank logic [21, 23]. Moreover, rank logic captures PTIME on the class of structures with color class size two [33]. An open question in [9] is whether rank logic can express the solvability of linear equation systems over finite rings rather than only over finite fields.

In this article, we show that rank logic fails to define the CFI query over the rings  $\mathbb{Z}_{2^i}$  for every  $i \in \mathbb{N}$ . As in the case for fields, we consider the class of CFI graphs over all rings  $\mathbb{Z}_{2^i}$  and not a fixed one. This result eliminates rank logic as a candidate for a logic capturing PTIME. As for  $\mathbb{F}_2$ , the isomorphism problem for CFI graphs over  $\mathbb{Z}_{2^i}$  can be translated to a linear equation system over  $\mathbb{Z}_{2^i}$ . Hence, we also answer the question for solvability of linear equation systems over finite rings, where the ring is part of the input and so encoded in a relational structure, in the negative. Even more, we do not only separate rank logic from PTIME but also from the logic of Choiceless Polynomial Time.

Choiceless Polynomial Time (CPT) was introduced in [5]. It is a logic manipulating hereditarily finite sets and expresses all common operations on finite sets. The key point is, that by definition of CPT it is impossible to pick an arbitrary element out of a set. If one wants to process an element in a set, one has to process all of them. This makes CPT choiceless and thereby isomorphism-invariant. CPT defines the isomorphism problem of CFI graphs over  $\mathbb{F}_2$  if the base graph is totally ordered [14] (which is sufficient to separate IFP+C from PTIME). More generally, CPT captures PTIME on the class of structures with bounded color class size, where the automorphism group of each color class is abelian [33]. This result is established by showing that CPT defines the solvability problem of a certain class of linear equation systems. Grädel and Grohe suggested

that this class of equation systems might be a candidate for separating CPT from rank logic [16]. The result on bounded abelian color classes in [33] can be strengthened to not necessarily bounded color classes, as long as a total order of the automorphism group of each color class is given [30]. Using this result we show that CPT indeed defines the CFI query over  $\mathbb{Z}_{2^i}$  for every  $i \in \mathbb{N}$  for totally ordered base graphs. Hence, we separate rank logic not only from PTIME but also from CPT.

Our Techniques. We consider CFI graphs over  $\mathbb{Z}_{2^i}$ . The automorphism groups of these graphs are 2-groups. We show that in this case formulas containing the uniform rank operator  $\mathsf{rk}$  without a fixed prime can be translated to formulas only using the  $\mathsf{rk}_2$  operator. To do so, we use the result of [17] stating that on CFI graphs, whose automorphism groups are p-groups, rank logic formulas only using rank operators  $\mathsf{rk}_q$  for  $q \neq p$  can be simulated by IFP+C. Hence, if there is a formula defining the CFI query over  $\mathbb{Z}_{2^i}$ , we can assume that it only uses the rank operator  $\mathsf{rk}_2$ .

To prove that rank logic cannot define the CFI query over  $\mathbb{Z}_{2^i}$ , we use game-based methods. For IFP+C there is the well-known Ehrenfeucht-Fraïssé-like pebble game, called the bijective pebble game [22]. It can be used to show that a property is not IFP+C definable. Such a game also exists for the extension by rank operators [13]. During the game, ranks of matrices are computed, which are defined over the two graphs, on which the game is played. To show that a property is not definable in rank logic, it must always be possible to play in a way that the ranks of corresponding matrices for the two graphs are equal.

During the rank pebble game the two matrices (one for each graph) are partitioned into classes. Then one has to consider all labelings of the corresponding classes with values (0 and 1 in the case of  $\mathbb{F}_2$ ). This makes it in particular hard to prove that for every such labeling the two matrices have the same rank. To overcome this problem, we actually prove a stronger result and consider matrix similarity. Dawar and Holm [13] introduced the invertible-map game, which requires matrix similarity instead of matrix equivalence. In fact, simultaneous similarity of two sequences of matrices is required: each class in the partition gives rise to one matrix in the sequence, which only labels that that particular class with 1 and all others with 0. Every matrix obtained by labeling the classes can be expressed as a sum of these matrices labeling exactly one class with 1. Because matrix similarity implies matrix equivalence (and so equality of ranks), this game is potentially more expressive in the sense that it possibly distinguishes more structures.

We use the invertible-map game to prove that there are non-isomorphic CFI graphs that cannot be distinguished by rank logic. We partition the matrices into orbits and show that the induced sequences of matrices are simultaneously similar. Indeed, we show that for every k there is an i such that Duplicator has a winning strategy in the k-ary invertible-map game played on CFI graphs over  $\mathbb{Z}_{2^i}$  whenever the base graph is sufficiently connected and its girth is sufficiently large. Requiring large connectivity is common for these arguments [6, 17], but the girth condition is specific to our construction.

The challenge for Duplicator in the invertible-map game is to come up with an invertible matrix proving simultaneous similarity of two sequences of matrices. To construct such matrices, we use two ingredients. The first consists of sets of local automorphisms we call *blurrers*. They satisfy some symmetry properties but contain a certain asymmetry. We use this asymmetry to "blur" the twist between two non-isomorphic CFI graphs,

that is, we "distribute" it among multiple edges in the graphs. Because of the symmetry of the blurrers, it is "hard" to detect the blurred twist in the invertible-map game. In particular, if we only consider 1-ary rank operators, blurrers suffice to construct a winning strategy for Duplicator. Considering arity k becomes inherently more difficult. While the argument for the 1-ary case is a local argument, in the k-ary case we have to consider k-tuples combining vertices scattered in the graph. However, we can use the blurrers such that only for k-tuples containing vertices of a single "problematic" gadget (and possibly other vertices far apart) the blurrer is not sufficient to prove simultaneous similarity. Here we use the second ingredient. We make an arbitrary vertex of the problematic gadget a parameter. This fixes the problematic gadgets and it suffices to consider the (k-1)-tuples where the vertex of the problematic gadgets is removed. We recurse on the arity and obtain for every edge, between which we blurred the twist, a similarity matrix for arity (k-1). Using the large girth of the graph, these edges are chosen sufficiently far apart each other. This is important to combine the (k-1)-ary similarity matrices with the blurrer to obtain a similarity matrix for the k-ary case.

**Related Work.** Hella [22] showed that for generalized Lindström quantifiers the expressiveness strictly increases with the arity. A similar result can also be given for rank logic [11, 23, 25]. In that light, the increased complexity of our approach for the k-ary case compared to the 1-ary case is not surprising.

We use the already mentioned result of Grädel and Pakusa [17] to argue that it suffices to consider the rank operator  $\mathsf{rk}_2$  over  $\mathbb{F}_2$  for CFI graphs over  $\mathbb{Z}_{2^i}$ . Consequently, we have to consider the invertible-map game [13] for  $\mathbb{F}_2$  only. Indeed, it was shown by Dawar, Grädel, and Pakusa [10] that a similar result also holds for the invertible-map game and the equally expressive linear-algebraic logic: When considering CFI graphs over  $\mathbb{F}_2$ , arbitrary linear-algebraic operators over  $\mathbb{F}_p$  for  $p \neq 2$  do not define the CFI query. Recently, this result was combined with the results of this article by Dawar, Grädel, and Lichter [12] to show that linear-algebraic logic does not capture PTIME, either.

Closely related to computing ranks is checking linear equation systems for solvability. Atserias, Bulatov, and Dawar proved that IFP+C does not define solvability of linear equation systems over finite rings [1]. Solvability of linear equation systems of prime-power fields is definable in rank logic [23]. So, because prime-power fields reduce to prime fields in rank logic, it is conceivable that a variant of rank operators using prime-power fields does not define the CFI query over  $\mathbb{Z}_{2^i}$ , either.

While IFP+C fails to capture PTIME for CFI graphs, there are many other graph classes on which IFP+C captures PTIME. These include, e.g., graphs with excluded minors [19] and graphs with bounded rank width [20]. While showing that rank logic defines the CFI query for prime fields is rather simple [11], this is a non-trivial result for CPT. The already mentioned result by Dawar, Richerby, and Rossman [14] uses deeply nested sets and is restricted to totally ordered base graphs. This result was strengthened by Pakusa, Schalthöfer, and Selman [31] to base graphs with logarithmic color class size. Recently, the result for bounded abelian color classes by Abu Zaid, Grädel, Grohe, and Pakusa [33] was extended by Lichter and Schweitzer [27] to graphs with bounded color classes with dihedral colors and also for certain structures of arity 3.

Structure of this Article. After providing some preliminaries in Section 2, we introduce variants of rank logic in Section 3 and the invertible-map game in Section 4. Then we give a CFI construction suitable for our arguments in Section 5. Next, we discuss matrices defined over CFI structures in Section 6 and develop a criterion for invertibility of such matrices over  $\mathbb{F}_2$ . This will be used in Section 7, where we treat the case of 1-ary rank operators and introduce blurrers. Section 8 defines the notion of the active region of a matrix. This is used in the case of general k-ary rank operators in Section 9 to successfully combine the matrices obtained from recursion. There we also generalize blurrers to the k-ary case. Finally, we separate rank logic from CPT in Section 10 and conclude with a discussion in Section 11.

### 2 Preliminaries

We denote the set  $\{1,\ldots,k\}$  by [k]. Let N and I be finite sets. The set of I-indexed tuples over N is denoted by  $N^I$ . For a tuple  $\bar{t} \in N^I$  the entry for index  $i \in I$  is written as  $\bar{t}(i)$ . For  $k \in \mathbb{N}$ ,  $\bar{t} \in N^k = N^{[k]}$ , and  $i \leq k$  we also write  $t_i$  for the i-th entry. The concatenation of two tuples  $\bar{s} \in N^k$  and  $\bar{t} \in N^\ell$  is denoted by  $\bar{s}\bar{t} \in N^{k+\ell}$ . The restriction of  $\bar{t} \in N^I$  to  $K \subseteq I$  is denoted by  $\bar{t}|_K \in N^K$ .

For two finite index sets I and J, an  $I \times J$  matrix M over N is a map  $M: I \times J \to N$ . We write M(i,j) for the entry at position (i,j). The identity matrix is denoted by  $\mathbb{1}$  and the zero matrix by  $\mathbb{0}$ .

We write  $\mathbb{Z}_j$  for the ring of integers modulo j. Its elements are  $\{0,\ldots,j-1\}$ . For a tuple  $\bar{a} \in \mathbb{Z}_j^I$  we set  $\sum \bar{a} := \sum_{i \in I} \bar{a}(i)$  and likewise for a function  $f : I \to \mathbb{Z}_j$ .

A (relational) signature  $\tau = \{R_1, \dots, R_\ell\}$  is a set of relation symbols with associated arities  $r_i \in \mathbb{N}$  for each  $i \in [\ell]$ . A  $\tau$ -structure  $\mathfrak{A}$  is a tuple  $\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \dots, R_\ell^{\mathfrak{A}})$  where  $R_i^{\mathfrak{A}} \subseteq A^{r_i}$  for all  $i \in [\ell]$ . The universe of  $\mathfrak{A}$  is always denoted by A. In this article, we only consider finite structures. A **pebbled structure** is a pair  $(\mathfrak{A}, \bar{u})$  of a relational structure and a tuple  $\bar{u} \in A^k$ . Two pebbled structures  $(\mathfrak{A}, \bar{u})$  and  $(\mathfrak{B}, \bar{v})$  are isomorphic, if there is an isomorphism  $\varphi \colon A \to B$  such that  $\varphi((\mathfrak{A}, \bar{u})) = (\mathfrak{B}, \bar{v})$ . That is, every isomorphism has to map  $\bar{u}$  to  $\bar{v}$ . An automorphism of  $(\mathfrak{A}, \bar{u})$  is an isomorphism  $(\mathfrak{A}, \bar{u}) \to (\mathfrak{A}, \bar{u})$ . The automorphism group of  $(\mathfrak{A}, \bar{u})$  is denoted by  $\operatorname{Aut}((\mathfrak{A}, \bar{u}))$ .

Let G = (V, E) be a simple graph. For vertices  $x, y \in V$  we denote their distance in G by  $\operatorname{dist}_G(x, y)$ . For two sets  $X, Y \subseteq V$  we set  $\operatorname{dist}_G(X, Y) := \min_{x \in X, y \in Y} \operatorname{dist}_G(x, y)$  and likewise  $\operatorname{dist}_G(x, Y) := \operatorname{dist}_G(\{x\}, Y)$  for a vertex x and a set  $Y \subseteq V$ . The set of neighbors of a vertex  $x \in V$  is denoted by  $N_G(x)$ . The k-neighborhood of x in G is  $N_G^k(x) := \{y \in V \mid \operatorname{dist}_G(x, y) \leq k\}$ . The induced subgraph of G by  $W \subseteq V$  is G[W]. The graph G is k-connected, if  $|V| \geq k$  and for every  $V' \subseteq V$  of size at most k-1,  $G \setminus V'$  is connected. That is, after removing k-1 vertices, G is still connected. The girth of G is the length of the shortest cycle in G.

Let  $\Gamma$  be a finite permutation group with domain N and let p be a prime. If for every  $\sigma \in \Gamma$  there is an  $\ell$  such that  $\sigma$  is of order  $p^{\ell}$ , i.e,  $\sigma^{(p^{\ell})}$  is the identity, then  $\Gamma$  is called a p-group. The orbit of  $n \in N$  is the set  $\{\sigma(n) \mid \sigma \in \Gamma\}$ . In this way, N is partitioned into orbits. This notation generalizes to k-tuples. A k-orbit is a maximal set  $P \subseteq N^k$ , such that for every  $\bar{n}, \bar{m} \in P$ , there is a  $\sigma \in \Gamma$  such that  $\sigma(\bar{n}) = \bar{m}$ . We write orbs<sub>k</sub>( $\Gamma$ ) for the set of k-orbits of  $\Gamma$ . The group  $\Gamma$  is **transitive** if  $|\operatorname{orbs}_1(\Gamma)| = 1$ . If additionally

 $|\Gamma| = |N|$ , then  $\Gamma$  is called **regular**. The k-orbits of a pebbled relational structure  $(\mathfrak{A}, \bar{u})$  are  $\mathsf{orbs}_k((\mathfrak{A}, \bar{u})) := \mathsf{orbs}_k(\mathsf{Aut}((\mathfrak{A}, \bar{u})))$ .

### 3 Rank Logic

In this section we consider rank logic, an extension of inflationary fixed-point logic with counting by a rank operator. Let  $\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \dots, R_{\ell}^{\mathfrak{A}})$  be a relational  $\tau$ -structure. We set  $\tau^{\#} := \tau \uplus \{\cdot, +, 0, 1\}$  and  $\mathfrak{A}^{\#} := (A, R_1^{\mathfrak{A}}, \dots, R_{\ell}^{\mathfrak{A}}, \mathbb{N}, \cdot, +, 0, 1)$  to be the two-sorted  $\tau^{\#}$ -structure<sup>1</sup> that is the disjoint union of  $\mathfrak{A}$  and  $\mathbb{N}$ .

**Fixed-Point Logic with Counting.** We introduce IFP+C, the fixed-point logic with counting (proposed in [24], also see [29]). Let  $\tau$  be a signature. IFP+C is a two-sorted logic using the signature  $\tau^{\#}$  with *element* variables ranging over the universe of the input structure and *number* variables ranging over the natural numbers. We use the letters x and y for element variables, Greek letters  $\nu$  and  $\mu$  for numeric variables,  $\Phi$  and  $\Psi$  for formulas, and letters s and t for numeric terms. For a tuple of variables or terms we write  $\bar{x}$ ,  $\bar{\nu}$ , and  $\bar{s}$  respectively.

IFP+C-formulas are built from first-order formulas, a fixed-point operator, and counting terms. To ensure polynomial-time evaluation, quantification over numeric variables needs to be bounded: Whenever  $\Phi$  is an IFP+C-formula,  $\nu$  is a numeric, possibly free variable in  $\Phi$  and s is a closed numeric term, then

$$Q\nu < s. \Phi$$

is an IFP+C-formula, where  $Q \in \{\forall, \exists\}$ . We now consider (inflationary) fixed-points. Let R be a relation symbol to define using a fixed-point. The relation R can contain both elements of the universe and numbers as follows. Let  $\Phi$  be an IFP+C-formula,  $\bar{x}$  and  $\bar{\nu}$  be variables, and  $\bar{s}$  be a tuple of  $|\bar{\nu}|$  many closed numeric terms that bound the values of  $\bar{\nu}$ . Then

$$[\mathsf{ifp}R\bar{x}\bar{\nu} \leq \bar{s}.\ \Phi]\ (\bar{x}\bar{\nu})$$

is an IFP+C-formula. To relate element and numeric variables, IFP+C possesses counting terms that count the number of different values for some variables satisfying a formula. As before, let  $\Phi$  be an IFP+C-formula,  $\bar{x}$  and  $\bar{\nu}$  be variables, and  $\bar{s}$  be a tuple of  $|\bar{\nu}|$  many closed numeric terms which bound the values of  $\bar{\nu}$ . Then

$$\#\bar{x}\bar{\nu} < \bar{s}. \Phi$$

is a numeric IFP+C-term.

Let  $\mathfrak{A}$  be a  $\tau$ -structure. An IFP+C-formula (or term) is evaluated over  $\mathfrak{A}^{\#}$ . For a numeric term  $s(\bar{x}\bar{\nu})$  we denote by  $s^{\mathfrak{A}} \colon A^{|\bar{x}|} \times \mathbb{N}^{|\bar{\nu}|} \to \mathbb{N}$  the function that maps the possible values of the free variables of s to the value that s takes in  $\mathfrak{A}^{\#}$ . Similarly, for a formula  $\Phi(\bar{x}\bar{\nu})$  we write  $\Phi^{\mathfrak{A}} \subseteq A^{|\bar{x}|} \times \mathbb{N}^{|\bar{\nu}|}$  for the set of values for the free variables satisfying  $\Phi$ . Then, e.g., the evaluation of a counting term for a formula  $\Phi(\bar{y}\bar{x}\bar{\mu}\bar{\nu})$  is defined as

$$\underline{(\#\bar{x}\bar{\nu}\leq\bar{s}.\ \Phi)^{\mathfrak{A}}(\bar{u}\bar{m}):=\left|\left\{\bar{w}\bar{n}\in A^{|\bar{x}|}\times\mathbb{N}^{|\bar{\nu}|}\ \middle|\ n_{i}\leq s_{i}^{\mathfrak{A}}\ \text{for all}\ i\in[|\bar{\nu}|]\ \text{and}\ \bar{u}\bar{w}\bar{m}\bar{n}\in\Phi^{\mathfrak{A}}\right.\right\}\right|.}$$

<sup>&</sup>lt;sup>1</sup>This is the only non-finite structure in this article.

**Rank Logic.** We now consider the extension of IFP+C by the *uniform* rank operator. We follow the definition in [23]. Let  $s(\bar{x}, \bar{y})$  be a numeric term such that  $k := |\bar{x}| = |\bar{y}|$  and t be a closed numeric term. Then

$$\mathsf{rk}(\bar{x},\bar{y}).\;(s,t)$$

is a numeric term. We say for convenience that k is the **arity** of the operator, although it is actually 2k. The logic IFP+R is the extension of IFP+C by the uniform rank operator rk. We restricted the definition to square matrices, but this does not limit the expressive power. The rank operator is evaluated as follows. Let  $\mathfrak{A}$  be a  $\tau$ -structure. The term s defines an  $A^k \times A^k$  matrix  $M_s^{\mathfrak{A}}$  over  $\mathbb{N}$ :

$$M_s^{\mathfrak{A}}(\bar{u},\bar{v}) := s^{\mathfrak{A}}(\bar{u},\bar{v}).$$

Finally, we define  $(\mathsf{rk}(\bar{x},\bar{y}).(s,t))^{\mathfrak{A}}$ : The rank operator evaluates the rank of  $(M_s^{\mathfrak{A}} \bmod p)$  over  $\mathbb{F}_p$  if  $p := t^{\mathfrak{A}}$  is prime and to 0 otherwise. We omitted parameters for readability.

For a set of prime numbers  $\Omega$ , we set IFP+R $_{\Omega}$  to be the variant of IFP+R, in which we have instead of the uniform rank operator  $\mathsf{rk}$  a different rank operator  $\mathsf{rk}_p$  for every  $\mathbb{F}_p$  such that  $p \in \Omega$ . That is, we have to fix the field in the formula independently of the structure. This is not the case for the operator  $\mathsf{rk}$ , where we can determine the value for p by another term that evaluates differently for different structures.

Choiceless Polynomial Time. Choiceless Polynomial Time (CPT) is a logic different from IFP+C. CPT-formulas manipulate hereditarily finite sets. They are choiceless in the sense that they either process all elements of such sets or none. It is not possible to pick an arbitrary element from a set. By these conditions, all sets constructed by a CPT-term are closed under automorphisms of the input structure. Evaluation in polynomial time is guaranteed by explicit polynomial bounds on the number of steps and sizes of the constructed sets. We omit a formal definition of CPT here because it is not needed in this article. For a formal definition we refer to [16, 30].

We review two results for CPT: A relational  $\tau$ -structure  $\mathfrak A$  has q-bounded colors, if one relation  $\leq \tau$  is a total preorder partitioning the universe into  $\leq$ -equivalence classes, called **color classes**, of size at most q. The structure  $\mathfrak A$  has **abelian colors**, if the induced substructure of every color class has an abelian automorphism group.

**Theorem 1** ([33]). CPT captures PTIME on q-bounded relational structures with abelian colors.

This result can be strengthened from bounded color class size to ordered colors. A  $\tau$ -structure with **ordered colors** is a tuple  $(\mathfrak{A}, \mathbb{F})$ , where  $\mathfrak{A}$  is a relational  $\tau$ -structure with color classes  $C_1, \ldots, C_n$  and  $\mathbb{F} = \{(\Gamma_i, \leq_i) \mid i \in [n]\}$  is a family of ordered permutations groups such that  $\Gamma_i$  is a transitive group with domain  $C_i$  for every  $i \in [n]$ . Note that structures with ordered colors are, without further encoding, not relational structures because  $\mathbb{F}$  is a higher-order object. However, we only define given a relational  $\tau$ -structure  $\mathfrak{A}$  the family  $\mathbb{F}$  of ordered permutation groups in CPT and thus can represent  $\mathbb{F}$  as a hereditarily finite set.

**Theorem 2** ([30]). CPT captures PTIME on structures with ordered abelian colors.

### 4 The Invertible-Map Game

Undefinability results for IFP+C are often achieved by the embedding of IFP+C into infinitary bounded variable counting logic (see [29]) and exploiting an equally expressive Ehrenfeucht-Fraïssé-like pebble game. This game, called the bijective k-pebble game [22], is a game between two players called Spoiler and Duplicator played on two pebbled structures. The aim of Spoiler is to prove that the two structures can be distinguished in infinitary k-variable counting logic, where Duplicator tries to show the converse. Such a game also exists for rank logic (called matrix-equivalence game in [13]). It extends the bijective pebble game with ranks. Instead of looking at this pebble game, we consider the invertible-map game [13]. Its distinguishing power is at least as strong as the one of the rank-pebble game in the sense that if Duplicator has a winning strategy in the invertible-map game, then Duplicator has a winning strategy in the rank-pebble game, too. Hence, to show that rank logic cannot distinguish two structures, it suffices to show that Duplicator has a winning strategy in the invertible-map game. The game is defined as follows:

Let k and m be two positive integers such that  $2k \leq m$  and let  $\Omega$  be a finite and nonempty set of primes. The **invertible-map game**  $\mathcal{M}^{m,k,\Omega}$  is played on two pebbled structures  $(\mathfrak{A}, \bar{u})$  and  $(\mathfrak{B}, \bar{v})$  with  $|\bar{u}| = |\bar{v}| \leq m$  of the same signature. For each structure there are m pebbles labeled with  $1, \ldots, m$ , where on  $u_i$  and  $v_i$  there are pebbles with the same label for all  $i \in [|\bar{u}|]$ . That is, if  $|\bar{u}| < m$ , some of the pebbles are not used. There are two players called Spoiler and Duplicator. If  $|A| \neq |B|$ , then Spoiler wins the game. Otherwise a round of the game proceeds as follows:

- 1. Spoiler chooses a prime  $p \in \Omega$  and picks up 2k many pebbles from  $\mathfrak{A}$  and the corresponding pebbles (with the same labels) from  $\mathfrak{B}$ .
- 2. Duplicator picks a partition  $\mathbf{P}$  of  $A^k \times A^k$  and another one  $\mathbf{Q}$  of  $B^k \times B^k$  such that  $|\mathbf{P}| = |\mathbf{Q}|$ . Furthermore, Duplicator picks an invertible  $A^k \times B^k$  matrix S over  $\mathbb{F}_p$ , such that the matrix induces a total and bijective map  $\lambda \colon \mathbf{P} \to \mathbf{Q}$  defined by  $P \mapsto Q$  if and only if  $\chi^P = S \cdot \chi^Q \cdot S^{-1}$ . Here  $\chi^P$  (respectively  $\chi^Q$ ) is the characteristic  $A^k \times A^k$  matrix over  $\mathbb{F}_p$  of P (respectively the  $B^k \times B^k$  matrix over  $\mathbb{F}_p$  of Q) which satisfies that  $\chi^P(\bar{u}', \bar{v}') = 1$  if  $\bar{u}'\bar{v}' \in P$  and  $\chi^P(\bar{u}', \bar{v}') = 0$  otherwise. To say it differently, Duplicator has to pick a bijection  $\lambda \colon \mathbf{P} \to \mathbf{Q}$  and an invertible  $A^k \times B^k$  matrix S satisfying  $\chi^P = S \cdot \chi^{\lambda(P)} \cdot S^{-1}$  for all  $P \in \mathbf{P}$ , i.e., the characteristic matrices of  $\mathbf{P}$  and  $\mathbf{Q}$  are simultaneously similar.
- 3. Spoiler chooses a block  $P \in \mathbf{P}$ , a tuple  $\bar{w} \in P$ , and a tuple  $\bar{w}' \in \lambda(P)$ . Then for each  $i \in [2k]$  Spoiler places a pebble on  $w_i$  and the corresponding pebble on  $w_i'$ .

After a round, Spoiler wins the game if the pebbles do not define a local isomorphism or if Duplicator was not able to respond with a matrix satisfying the conditions above. Duplicator wins the game if Spoiler forever fails to win. Spoiler has a winning strategy if Spoiler can win the game starting at  $(\mathfrak{A}, \bar{u})$  and  $(\mathfrak{B}, \bar{v})$  in any case independently of the actions of Duplicator. Likewise, Duplicator has a winning strategy, if Duplicator can always win the game. In that case, we write  $(\mathfrak{A}, \bar{u}) \equiv_{\mathcal{M}}^{m,k,\Omega} (\mathfrak{B}, \bar{v})$ . Finally, we consider the game with a bounded number of rounds: The  $\ell$ -round invertible-map game  $\mathcal{M}_{\ell}^{m,k,\Omega}$  proceeds exactly as  $\mathcal{M}^{m,k,\Omega}$  but stops after  $\ell$  rounds. Duplicator wins, if Spoiler did not

win in these rounds. In the following, we use the invertible-map game instead of the rank-pebble game because it allows us to prove a stronger result and simplifies proofs.

**Lemma 3** ([13]). Let K be a class of finite  $\tau$ -structures and  $P \subseteq K$ . If for every  $k, m \in \mathbb{N}$  with  $2k \leq m$  and every finite and nonempty set of primes  $\Omega$ , there is a pair of structures  $(\mathfrak{A}, \mathfrak{B})$ , such that  $\mathfrak{A} \in P$ ,  $\mathfrak{B} \notin P$ , and  $\mathfrak{A} \equiv_{\mathcal{M}}^{m,k,\Omega} \mathfrak{B}$ , then P is not  $IFP+R_{\mathbb{P}}$  definable, where  $\mathbb{P}$  is the set of all primes.

If we fix a finite set of primes  $\Omega$  in Lemma 3, then P is not IFP+R $_{\Omega}$  definable (see [13]) because a P-defining IFP+R $_{\Omega}$ -formula implies a winning strategy of Spoiler in the  $\mathcal{M}^{m,k,\Omega}$  game. Lemma 3 is proved in [13] for the  $(m,k,\Omega)$ -rank-pebble game, which induces the equivalence  $\equiv_{\mathcal{R}}^{m,k,\Omega}$ . Then the authors show that  $\equiv_{\mathcal{M}}^{m,k,\Omega}$  refines  $\equiv_{\mathcal{R}}^{m,k,\Omega}$ . It is an open problem whether the equivalence  $\equiv_{\mathcal{M}}^{m,k,\Omega}$  strictly refines  $\equiv_{\mathcal{R}}^{m,k,\Omega}$ .

### 5 CFI Structures

In this section we define a variant of the well-known CFI graphs. Starting from a so-called base graph, for every vertex in the base graph a gadget is constructed. In the seminal paper of Cai, Fürer, and Immerman [6] these gadgets consist of inner and outer vertices, where the latter are pairs of vertices. Each outer vertex pair induces the automorphism group  $\mathbb{Z}_2$  and the inner vertices realize the automorphism group  $\{\bar{a} \in \mathbb{Z}_2^d \mid \sum \bar{a} = 0\}$  of its d adjacent outer vertex pairs. Whenever two vertices in the base graph are adjacent, the two corresponding outer vertex pairs of the two gadgets are connected. Such a connection can either be "straight" or "twisted". This construction generalizes to other groups than  $\mathbb{Z}_2$ . In [28] a construction of gadgets for general abelian groups can be found. We are interested in cyclic groups  $\mathbb{Z}_{2^q}$ . The following construction only uses the inner vertices and directly connects the inner vertices of two gadgets. For  $\mathbb{Z}_2$ , this approach is given in [15].

A base graph is a simple, connected, and totally ordered graph. Let  $G = (V, E, \leq)$  be a base graph. Consider the additive group of  $\mathbb{Z}_{2^q}$ . For each vertex  $x \in V$  we define a gadget consisting of vertices  $A_x$  and two families of relations:

$$\begin{split} A_x &:= \left\{ \left. \bar{a} \in \mathbb{Z}_{2^q}^{N_G(x)} \; \middle| \; \sum \bar{a} = 0 \right. \right\}, & x \in V, \\ I_{x,y} &:= \left\{ \left. (\bar{a}, \bar{b}) \in A_x^2 \; \middle| \; \bar{a}(y) = \bar{b}(y) \right. \right\}, & x \in V, y \in N_G(x), \\ C_{x,y} &:= \left\{ \left. (\bar{a}, \bar{b}) \in A_x^2 \; \middle| \; \bar{a}(y) + 1 = \bar{b}(y) \right. \right\}, & x \in V, y \in N_G(x). \end{split}$$

Consider the sets  $A_{x,y,c} := \{\bar{a} \in A_x \mid \bar{a}(y) = c\}$  for  $y \in N_G(x)$  and  $c \in \mathbb{Z}_{2^q}$ . The relation  $I_{x,y}$  realizes these sets by disjoint cliques, one for each  $A_{x,y,c}$ . The relation  $C_{x,y}$  induces a directed cycle  $A_{x,y,c}, A_{x,y,c+1}, \ldots, A_{x,y,c+2^q-1}$  on these sets for a fixed y by adding directed complete bipartite graphs between subsequent cliques. In that way, the relation  $C_{x,y}$  realizes the group  $\mathbb{Z}_{2^q}$  on the sets  $A_{x,y,c}$ . By the condition  $\sum \bar{a} = 0$  on the vertices in  $A_x$ , a gadget thereby has an automorphism group isomorphic to  $\{\bar{a} \in \mathbb{Z}_{2^q}^d \mid \sum \bar{a} = 0\}$  where d is the degree of x.

Now we connect gadgets. We first extend the order  $\leq$  to the lexicographical order on tuples of vertices of G and further to sets of such tuples. Let  $g: E \to \mathbb{Z}_{2^q}$  be a function

defining the values by which the edges are twisted. For every edge  $\{x,y\} \in E$  we connect the gadgets of the incident vertices. We obtain the CFI structure

$$\mathsf{CFI}_{2^q}(G,g) := (A, R_I, R_C, R_{E,0}, \dots, R_{E,2^q-1}, \preceq)$$

as follows:

$$E_{\{x,y\},c} := \left\{ \{\bar{a},\bar{b}\} \mid \bar{a} \in A_{x}, \bar{b} \in A_{y}, \bar{a}(y) + \bar{b}(x) = c \right\}, \qquad \{x,y\} \in E, c \in \mathbb{Z}_{2^{q}},$$

$$\preceq := \left\{ (\bar{a},\bar{b}) \mid \bar{a} \in A_{x}, \bar{b} \in A_{y}, x \leq y \right\},$$

$$R_{I} := \left\{ (\bar{a},\bar{b},\bar{a}',\bar{b}') \mid \{ (x,y) \mid (\bar{a},\bar{b}) \in I_{x,y} \} \leq \{ (x',y') \mid (\bar{a}',\bar{b}') \in I_{x',y'} \} \right\},$$

$$R_{C} := \left\{ (\bar{a},\bar{b},\bar{a}',\bar{b}') \mid \{ (x,y) \mid (\bar{a},\bar{b}) \in C_{x,y} \} \leq \{ (x',y') \mid (\bar{a}',\bar{b}') \in C_{x',y'} \} \right\},$$

$$A := \bigcup_{x \in V} A_{x}, \qquad R_{E,c} := \bigcup_{e \in E} E_{e,c+g(e)}.$$

The unions above are meant to be disjoint. The relations  $I_{x,y}$  (and similarly  $C_{x,y}$ ) are encoded by  $R_I$  (and  $R_C$ ) as follows: All edges  $(\bar{a}, \bar{b}) \in A_x^2$  are partitioned according to the set of base vertices y such that  $(\bar{a}, \bar{b}) \in I_{x,y}$ . The partition is given by the equivalence classes of  $R_I$  (seen as equivalence on pairs). The relations  $I_{x,y}$  (respectively  $C_{x,y}$ ) are unions of  $R_I$ -equivalence classes (respectively  $R_C$ -equivalence classes).

**Definition 4** (Origin). We say that the vertices  $\bar{a} \in A_x$  originate from x or that their origin is x and write  $\operatorname{orig}(\bar{a}) := x$ . We extend this to tuples and define the origin of  $(\bar{a}_1, \ldots, \bar{a}_j) \in A^j$  as  $\operatorname{orig}((\bar{a}_1, \ldots, \bar{a}_j)) := (\operatorname{orig}(\bar{a}_1), \ldots, \operatorname{orig}(\bar{a}_j))$ . We will often view  $\operatorname{orig}((\bar{a}_1, \ldots, \bar{a}_j))$  as the set  $\{\operatorname{orig}(\bar{a}_1), \ldots, \operatorname{orig}(\bar{a}_j)\}$  and write  $x \in \operatorname{orig}((\bar{a}_1, \ldots, \bar{a}_j))$ . If M is a set of tuples of the same origin, we set  $\operatorname{orig}(M) := \operatorname{orig}(\bar{a})$  for some (and thus all)  $\bar{a} \in M$ . For a set  $W \subseteq V$ , we define the origin induced substructure

$$\mathsf{CFI}_{2^q}(G,g)[W] := \mathsf{CFI}_{2^q}(G,g)[\{\,\bar{a} \mid \mathsf{orig}(\bar{a}) \in W\,\}]$$

to be the substructure induced by all vertices whose origin is contained in W.

It will be always clear from the context whether we refer to the origin induced substructure (or just a standard induced substructure). In that case W and the universe of the CFI structure are disjoint.

For CFI structures it is well-known that  $\mathsf{CFI}_{2^q}(G,g) \cong \mathsf{CFI}_{2^q}(G,f)$  if and only if  $\sum_{e \in E} g(e) = \sum_{e \in E} f(e)$ . That is, there are up to isomorphism  $2^q$  many CFI structures of the base graph G.

**Lemma 5.** The automorphism group  $Aut(CFl_{2q}(G,g))$  of  $CFl_{2q}(G,g)$  is an abelian 2-group.

Proof. Every automorphism of  $\mathsf{CFI}_{2^q}(G,g)$  is origin-respecting, i.e., it maps a vertex to a vertex of the same origin, because the preorder  $\leq$  on the vertices is obtained from the total order  $\leq$  on G. It follows that  $\mathsf{Aut}(\mathsf{CFI}_{2^q}(G,g))$  is a subgroup of the direct product of the automorphism groups of every gadget. Because the automorphism group of each degree d gadget is the abelian 2-group  $\{\bar{a} \in \mathbb{Z}_{2^q}^d \mid \sum \bar{a} = 0\}$  as argued before, so is the direct product of them and in particular  $\mathsf{Aut}(\mathsf{CFI}_{2^q}(G,g))$ .

Other CFI Constructions. We compare our CFI construction to other version in the literature. The classical construction in [6] uses inner and outer vertices, while we only use inner ones. The sets  $A_{x,y,c}$  in our construction correspond to the outer vertices in the classical construction. We only use one type of vertices to avoid case distinctions between inner and outer vertices in the following.

Another approach to avoid this case distinction is to only use outer vertices and to replace the inner vertices by relations of higher arity (see, e.g., [22]). The arity of the relations corresponds to the degree of a vertex in the base graph. While this construction is more elegant, it is restricted to base graphs of bounded degree to obtain structures of a fixed signature. However, our argument separating rank logic and CPT requires base graphs of unbounded degree. Our construction always yields structures of arity 4, but the number of relations varies with the group  $\mathbb{Z}_{2^q}$ . Of course, we could use a single relation to encode the relations  $R_{E,c}$ . But in fact, it suffices only to use  $R_{E,0}$  to obtain a structure with the same automorphism group. Then all  $R_{E,c}$  are actually definable in 3-variable counting logic. We include all  $R_{E,c}$  in the structure for convenience.

In general, most properties of the structures transfer between the different constructions (with some quite obvious adaptations).

#### 5.1 Isomorphisms of CFI Structures

In this section we consider two classes of isomorphisms between CFI structures. They get important later in Section 9. Let  $q \in \mathbb{N}$  and  $G = (V, E, \leq)$  be a base graph. In the following, we denote for every  $f \colon E \to \mathbb{Z}_{2^q}$  by  $\mathfrak{A}_f$  the CFI structure  $\mathsf{CFI}_{\mathbb{Z}_{2^q}}(G, f)$ . These structures have by definition the same universe A for every  $f \colon E \to \mathbb{Z}_{2^q}$ .

**Definition 6** (Twisted Edge). Two functions  $f, g: E \to \mathbb{Z}_{2^q}$  twist an edge  $e \in E$  if  $f(e) \neq g(e)$ . We also say that e is twisted by f and g. For a set  $W \subseteq V$  we say that f and g do not twist W if no edge in G[W] is twisted by f and g.

We omit f and g if they are clear from the context. Let  $x \in V$  and  $\bar{a} \in \mathbb{Z}_{2^q}^{N_G(x)}$  satisfy  $\sum \bar{a} = 0$ . We identify  $\bar{a}$  with a permutation of vertices with origin x as follows: if u has origin x (in some CFI structure over G), then  $\bar{a}(u) := v$  such that  $v(y) = u(y) + \bar{a}(y)$  for all  $y \in N_G(x)$ . Because  $\sum \bar{a} = 0$ ,  $\bar{a}(u)$  is indeed a vertex with origin x, too.

**Definition 7** (Path Isomorphism). Let  $c \in \mathbb{Z}_{2^q}$  and  $\bar{s} = (x_1, \ldots, x_n)$  be a simple path in G. For every 1 < i < n, let  $\bar{a}_i \in \mathbb{Z}_{2^q}^{N_G(x_i)}$  such that  $\bar{a}_i(x_{i-1}) = c$ ,  $\bar{a}_i(x_{i+1}) = -c$ , and  $\bar{a}_i(y) = 0$  for all other  $y \in N_G(x_i)$ . The **path isomorphism**  $\vec{\pi}[c, \bar{s}]$  is defined by

$$\vec{\pi}[c, \bar{s}](u) := \begin{cases} \bar{a}_i(u) & \textit{if } \mathsf{orig}(u) = x_i \textit{ and } 1 < i < n \\ u & \textit{otherwise}. \end{cases}$$

**Lemma 8.** Let  $f, g: E \to \mathbb{Z}_{2^q}$ ,  $\bar{s} = (x_1, \ldots, x_n)$  be a simple path in G,  $e_1 = \{x_1, x_2\}$ , and  $e_2 = \{x_{n-1}, x_n\}$ . If no edge apart from  $e_1$  and  $e_2$  is twisted by f and g,  $g(e_1) = f(e_1) + c$ , and  $g(e_2) = f(e_2) - c$ , then  $\vec{\pi}[c, \bar{s}]$  is an isomorphism  $(\mathfrak{A}_f, \bar{p}) \to (\mathfrak{A}_g, \bar{p})$  for every tuple  $\bar{p} \in A^m$  satisfying  $\operatorname{dist}_G(\operatorname{orig}(\bar{p}), \{x_1, \ldots, x_n\}) > 1$ .

The proof of Lemma 8 is an obvious adaptation of the proof of Lemma 3.11 in [17]. This lemma uses a variant of CFI structures with outer vertices and relations, but the

arguments are similar. We additionally require that the tuple  $\bar{p}$  is fixed, but because its distance to the path  $\bar{s}$  is greater than 1, it is not affected by the path isomorphism at all, i.e.,  $\vec{\pi}[c,\bar{s}](\bar{p}) = \bar{p}$ . Isomorphisms between CFI structures satisfying  $\sum f = \sum g$ , in which more than two edges are twisted, can be composed out of multiple path isomorphisms. The following special case of such isomorphisms will play an important role later:

**Definition 9** (Star Isomorphism). Let  $z \in V$  be of degree d,  $\ell \leq d$ ,  $\bar{s}_1, \ldots, \bar{s}_\ell$  be simple paths,  $\bar{s}_i = (x_1^i, \ldots, x_{\ell_i}^i)$ ,  $x_{\ell_i}^i = z$  for all  $i \in [\ell]$ , and the  $\bar{s}_i$  be disjoint apart from z. We call  $\bar{s}_1, \ldots, \bar{s}_\ell$  a **star** and z the **center of the star**. For  $\bar{c} \in \mathbb{Z}_{2^q}^\ell$  satisfying  $\sum \bar{c} = 0$ , we define the **star-isomorphism**  $\pi^*[\bar{c}, \bar{s}_1, \ldots, \bar{s}_\ell]$  via

$$\pi^*[\bar{c}, \bar{s}_1, \dots, \bar{s}_\ell](u) := \begin{cases} \bar{c}'(u) & \text{if } \mathsf{orig}(u) = z, \\ \vec{\pi}[c_i, \bar{s}_i](u) & \text{if } \mathsf{orig}(u) \neq z \text{ and } \mathsf{orig}(u) \text{ is contained in } \bar{s}_i, \\ u & \text{otherwise,} \end{cases}$$

where  $\bar{c}' \in \mathbb{Z}_{2^q}^{N_G(z)}$  such that  $\bar{c}'(x_{\ell_i-1}^i) = c_i$  for all  $i \in [\ell]$  and  $\bar{c}'(y) = 0$  for all other  $y \in N_G(z)$ .

**Lemma 10.** Let  $f,g: E \to \mathbb{Z}_{2^q}$ ,  $\bar{s}_1, \ldots, \bar{s}_\ell$  be a star in G,  $\bar{s}_i = (x_1^i, \ldots, x_{\ell_i}^i)$  for all  $i \in [\ell]$ , and  $\bar{c} \in \mathbb{Z}_{2^q}^\ell$  such that  $\sum \bar{c} = 0$ . If no edge apart from the edges  $e_i = \{x_1^i, x_2^i\}$  for every  $i \in [\ell]$  is twisted by f and g and  $g(e_i) = f(e_i) + c_i$  for all  $i \in [\ell]$ , then  $\pi^*[\bar{c}, \bar{s}_1, \ldots, \bar{s}_\ell]$  is an isomorphism  $(\mathfrak{A}_f, \bar{p}) \to (\mathfrak{A}_g, \bar{p})$  for every tuple  $\bar{p} \in A^m$  satisfying  $\mathrm{dist}_G(\mathrm{orig}(\bar{p}), \{x_j^i \mid i \in [\ell], j \in [\ell_i]\}) > 1$ .

Proof. Let  $\bar{p} \in A^m$  satisfy  $\operatorname{dist}_G(\operatorname{orig}(\bar{p}), \{x_j^i \mid i \in [\ell], j \in [\ell_i]\}) > 1$  and let z be the center of the star  $\bar{s}_1, \ldots, \bar{s}_\ell$ . For every  $i \in [\ell-1]$ , let  $\bar{s}_i'$  be the  $x_1^i - x_1^{i+1}$ -path obtained by stitching  $\bar{s}_i$  and  $\bar{s}_{i+1}$  together at  $z := x_{\ell_i}^i$  (that is, the path  $\bar{s}_{i+1}$  is attached in reversed direction). Furthermore, for every  $i \in [\ell-1]$ , set  $\varphi_i := \vec{\pi}[\sum_{j \in [i]} c_j, \bar{s}_i']$ , and let  $f_i : E \to \mathbb{Z}_{2^q}$  be the function defined via  $f_i(e_j) = f(e_j) + c_j$  for every  $j \in [i]$ ,  $f_i(e_{i+1}) = f(e_{i+1}) - \sum_{j \in [i]} c_j$ , and  $f_i(e) = f(e)$  otherwise. Applying Lemma 8 inductively shows that  $\varphi_1 \circ \cdots \circ \varphi_i$  is an isomorphism  $(\mathfrak{A}_f, \bar{p}) \to (\mathfrak{A}_{f_i}, \bar{p})$ : For i = 1, the only twisted edges are  $e_1$  and  $e_2$  satisfying  $f_1(e_1) = f(e_1) + c_1$  and  $f_1(e_2) = f(e_2) - c_1$  and  $\varphi_1 : (\mathfrak{A}_f, \bar{p}) \to (\mathfrak{A}_{f_1}, \bar{p})$  is an isomorphism by Lemma 8. For every  $2 \le i \le \ell - 2$ , exactly the edges  $e_{i+1}$  and  $e_{i+2}$  are twisted by  $f_i$  and  $f_{i+1}$ . It holds that

$$f_{i+1}(e_{i+1}) = f(e_{i+1}) + c_{i+1} = f_i(e_{i+1}) + \sum_{j \in [i+1]} c_j,$$
  
$$f_{i+1}(e_{i+2}) = f(e_{i+2}) - \sum_{j \in [i+1]} c_i = f_i(e_{i+2}) - \sum_{j \in [i+1]} c_i.$$

Thus,  $\varphi_{i+1}$  is an isomorphism  $(\mathfrak{A}_{f_i}, \bar{p}) \to (\mathfrak{A}_{f_{i+1}}, \bar{p})$  by Lemma 8 and  $\varphi_1 \circ \cdots \circ \varphi_{i+1}$  is an isomorphism  $(\mathfrak{A}_f, \bar{p}) \to (\mathfrak{A}_{f_{i+1}}, \bar{p})$  by induction.

Now let  $\psi = \varphi_1 \circ \cdots \circ \varphi_\ell$ . To prove the claim it suffices to show that  $f_{\ell-1} = g$  and that  $\psi = \pi^*[\bar{c}, \bar{s}_1, \dots, \bar{s}_\ell]$ . The former holds because  $\sum \bar{c} = 0$ . To show the latter, first consider the path  $\bar{s}_1$ . On vertices with origin in  $\bar{s}_1$  different from z the action of  $\psi$  is equal to the action of  $\varphi_1$ . This exactly equals the definition of  $\pi^*[\bar{c}, \bar{s}_1, \dots, \bar{s}_\ell]$ . For vertices with origin in  $\bar{s}_\ell$  different from z the argument is similar and the action of  $\psi$  is equal to the

action of  $\varphi_{\ell-1}$ . The isomorphism  $\varphi_{\ell-1}$  twists the edge  $\{x_1^{\ell}, x_2^{\ell}\}$  by  $-\sum_{j \in [\ell-1]} c_j$ , which by assumption is equal to  $c_{\ell}$  because  $\sum \bar{c} = 0$ . Now consider vertices with origin in  $\bar{s}_i$  different from z for  $i \notin \{1, \ell\}$ . Here the action of  $\psi$  equals the action of  $\varphi_{i-1} \circ \varphi_i$ . The isomorphism  $\varphi_{i-1}$  twists the edge  $\{x_1^i, x_2^i\}$  by  $-\sum_{j \in [i-1]} c_j$  and  $\varphi_i$  twists the same edge by  $\sum_{j \in [i]} c_j$ . Note that  $\bar{s}'_i$  contains the vertices of  $\bar{s}_{i+1}$  in reversed order, so on all the vertices with origin different from z the action of  $\varphi_{i-1} \circ \varphi_i$  becomes equal to the action of the path isomorphism  $\vec{\pi}[c_i, \bar{s}_i]$ . Finally, by a similar argument, the action of  $\psi$  on vertices with origin z equals the action of  $\vec{c}'$  defined as in Definition 9.

#### 5.2 Orbits of CFI Structures

In this section we analyze the structure of k-orbits of CFI structures for highly connected base graphs. Let  $q, k, m \in \mathbb{N}$  and  $G = (V, E, \leq)$  be a (k+m+1)-connected base graph. We denote again for every  $f: E \to \mathbb{Z}_{2^q}$  by  $\mathfrak{A}_f$  the CFI structure  $\mathsf{CFI}_{\mathbb{Z}_{2^q}}(G, f)$  with universe A. Let  $\bar{p} \in A^m$  be arbitrary but fixed. We consider the k-orbits of pebbled structures  $(\mathfrak{A}_f, \bar{p})$ , i.e., orbits of k-tuples. Recall that  $\mathsf{Aut}((\mathfrak{A}_f, \bar{p}))$  is the automorphism group of  $(\mathfrak{A}_f, \bar{p})$  and that  $\mathsf{orbs}_k((\mathfrak{A}_f, \bar{p}))$  is the set of all k-orbits (cf. Section 2).

**Definition 11** (Type of a Tuple). The **isomorphism type** of a pebbled structure is the class of all isomorphic structures. For  $f: E \to \mathbb{Z}_{2^q}$  the **type** of a tuple  $\bar{u} \in A^k$  in  $(\mathfrak{A}_f, \bar{p})$  is the pair  $(\text{orig}(\bar{u}), T)$ , where T is the isomorphism type of  $(\mathfrak{A}_f[\text{orig}(\bar{p}\bar{u})], \bar{p}\bar{u})$ .

We omit the pebbled structure  $(\mathfrak{A}_f, \bar{p})$  if it is clear from the context. Including  $\operatorname{orig}(\bar{u})$  in the type is needed because the isomorphism type T respects the relative order of the gadgets in  $\leq$  only. If  $\mathfrak{A}_f$  was vertex-colored instead, this would not be a problem. We have to consider the origin induced substructure of  $\operatorname{orig}(\bar{p}\bar{u})$  and not of  $\bar{p}\bar{u}$  because only then the relations  $I_{x,v}$  and  $C_{x,v}$  can be recovered from  $R_I$  and  $R_C$ . Here, an edge coloring would resolve this issue.

**Lemma 12.** For every  $f: E \to \mathbb{Z}_{2^q}$  and every  $\bar{u}, \bar{v} \in A^k$  there is an automorphism  $\varphi \in \operatorname{Aut}(\mathfrak{A}_f, \bar{p})$  such that  $\varphi(\bar{u}) = \bar{v}$  if and only if  $\bar{u}$  and  $\bar{v}$  have the same type.

*Proof.* A similar argument to the following can be found in Lemma 3.15 in [17]. Let  $f: E \to \mathbb{Z}_{2^q}$  and  $\bar{u}, \bar{v} \in A^k$ . If  $\varphi(\bar{u}) = \bar{v}$  for some automorphism  $\varphi \in \mathsf{Aut}((\mathfrak{A}_f, \bar{p}))$ , then surely  $\bar{u}$  and  $\bar{v}$  have the same type.

For the other direction, assume that  $\bar{u}$  and  $\bar{v}$  have the same type. Then, by definition, there is an isomorphism  $\varphi \colon (\mathfrak{A}_f[\mathsf{orig}(\bar{p}\bar{u})], \bar{p}\bar{u}) \to (\mathfrak{A}_f[\mathsf{orig}(\bar{p}\bar{v})], \bar{p}\bar{v})$ . Because  $\bar{u}$  and  $\bar{v}$  have the same type, it follows that  $\mathsf{orig}(\bar{p}\bar{u}) = \mathsf{orig}(\bar{p}\bar{v})$  and in particular that  $\varphi$  is an automorphism of  $(\mathfrak{A}_f[\mathsf{orig}(\bar{p}\bar{u})], \bar{p})$ . We show that this local automorphism extends to an automorphism of  $(\mathfrak{A}_f, \bar{p})$ .

We extend  $\varphi$  by the identity map on all vertices with origin not in  $\operatorname{orig}(\bar{p}\bar{u})$ . Then  $\varphi$  is an isomorphism between  $(\mathfrak{A}_f,\bar{p})$  and another CFI structure, where all twisted edges  $e_1,\ldots,e_\ell$  leave  $\operatorname{orig}(\bar{u})$  and are not incident to  $\operatorname{orig}(\bar{p})$  (edges incident to  $\operatorname{orig}(\bar{p})$  cannot be twisted because  $\varphi$  fixes  $\bar{p}$ ). Let N be the neighborhood of  $\operatorname{orig}(\bar{u})$  (and thus of  $\operatorname{orig}(\bar{v})$ ). Because G is (k+m+1)-connected, there is an x-y-path not using  $\operatorname{orig}(\bar{p}\bar{u})$  for every  $x,y\in N$  because  $G\setminus\operatorname{orig}(\bar{p}\bar{u})$  is still connected when removing at most  $|\bar{p}\bar{u}|=k+m< k+m+1$  many vertices. Hence, we can use path isomorphisms to move the twists at every  $e_i$  all to  $e_1$ . But because  $\varphi$  was an automorphism of  $(\mathfrak{A}_f[\operatorname{orig}(\bar{p}\bar{u})],\bar{p})$ , the sum of

the twists is 0. Hence, composing  $\varphi$  and the mentioned path isomorphisms forms an automorphism  $\psi \in \operatorname{Aut}(\mathfrak{A}_f, \bar{p})$ . Because the selected paths do not use  $\operatorname{orig}(\bar{p}\bar{u})$ , we still have  $\psi(\bar{p}\bar{u}) = \bar{p}\bar{v}$ .

Corollary 13. For every  $f: E \to \mathbb{Z}_{2^q}$  and every  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$  there is a type such that P contains exactly the tuples of that type.

**Definition 14** (Type of an Orbit). For  $f: E \to \mathbb{Z}_{2^q}$  the **type** of a k-orbit in  $(\mathfrak{A}_f, \bar{p})$  is the type of its contained tuples.

Corollary 15. For every pair  $f, g: E \to \mathbb{Z}_{2^q}$  that does not twist  $\operatorname{orig}(\bar{p})$ , it holds that

$$\operatorname{orbs}_k((\mathfrak{A}_f, \bar{p})) = \operatorname{orbs}_k((\mathfrak{A}_g, \bar{p}))$$
 and  $\operatorname{Aut}((\mathfrak{A}_f, \bar{p})) = \operatorname{Aut}((\mathfrak{A}_g, \bar{p})).$ 

While the orbit partitions of  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$  are equal, it is in general not true that an orbit  $P \in \operatorname{\mathsf{orbs}}_k((\mathfrak{A}_f, \bar{p}))$  has the same type in  $(\mathfrak{A}_f, \bar{p})$  and in  $(\mathfrak{A}_g, \bar{p})$ .

**Lemma 16.** Suppose the functions  $f, g: E \to \mathbb{Z}_{2^q}$  do not twist  $\operatorname{orig}(\bar{p})$ . Then for every k-orbit  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$  there is a  $Q \in \operatorname{orbs}_k((\mathfrak{A}_q, \bar{p}))$  that has the same type.

*Proof.* It suffices to consider the case that exactly one edge  $e = \{x, y\}$  is twisted because isomorphisms preserve types and because no edge contained in  $\operatorname{orig}(\bar{p})$  is twisted, all twists can be moved to a single edge using isomorphisms.

Let  $P \in \operatorname{orbs}_k((\mathfrak{A}_f,\bar{p}))$ . If  $\{x,y\} \not\subseteq \operatorname{orig}(P)$ , then P has the same type in  $(\mathfrak{A}_f,\bar{p})$  and in  $(\mathfrak{A}_g,\bar{p})$ . Otherwise, let  $\{x,y\} \subseteq \operatorname{orig}(P)$  and assume w.l.o.g. that  $y \notin \operatorname{orig}(\bar{p})$  (if  $\{x,y\} \subseteq \operatorname{orig}(\bar{p})$ , then  $\{x,y\} \not\subseteq \operatorname{orig}(P)$  because the twisted edge is not contained in  $\operatorname{orig}(\bar{p})$ ). Furthermore, choose a path  $\bar{s} = (x,y,\ldots,z)$ , such that  $z \notin \operatorname{orig}(P)$  and the path, possibly apart from x, is disjoint from  $\operatorname{orig}(\bar{p})$ . Such a path exists, because  $G \setminus \operatorname{orig}(\bar{p}) \setminus \operatorname{orig}(P)$  is connected (at most m+k < m+k+1 many vertices are removed) and  $y \notin \operatorname{orig}(\bar{p})$  by assumption. So we can pick some vertex z not contained in  $\operatorname{orig}(P)$  and in  $\operatorname{orig}(\bar{p})$ . Now, we move the twist to an edge incident to z with the path isomorphism  $\varphi := \vec{\pi}[g(e) - f(e), \bar{s}]$ . Then P has the same type in  $(\mathfrak{A}_f, \bar{p})$  as in  $\varphi((\mathfrak{A}_g, \bar{p})) = (\varphi(\mathfrak{A}_g), \bar{p})$  because  $\mathfrak{A}_f[\operatorname{orig}(\bar{p}) \cup \operatorname{orig}(P)] = \varphi(\mathfrak{A}_g)[\operatorname{orig}(\bar{p}) \cup \operatorname{orig}(P)]$ . Because isomorphisms preserve types, there is an orbit  $Q \in \operatorname{orbs}_k((\mathfrak{A}_g, \bar{p}))$  with the same type in  $(\mathfrak{A}_g, \bar{p})$  as P has in  $(\mathfrak{A}_f, \bar{p})$ .

**Lemma 17.** Let  $f: E \to \mathbb{Z}_{2^q}$  and  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$ . Then the permutation group  $\Gamma$  on P induced by  $\operatorname{Aut}((\mathfrak{A}_f, \bar{p}))$  is a regular and abelian 2-group.

*Proof.* We first argue that the automorphism group of a gadget is a regular abelian 2-group. Recall that the vertices of a gadget for the vertex  $\bar{x} \in V$  are defined as  $A_x = \{a \in \mathbb{Z}_{2^q}^{N_G(x)} \mid \sum a = 0\}$ . So  $|A_x| = (2^q)^{d-1}$ , where d is the degree of x. We saw in Section 5.1 that the automorphism group of a gadget is transitive. We already argued that the automorphism group is isomorphic to  $\{\bar{a} \in \mathbb{Z}_{2^q}^d \mid \sum \bar{a} = 0\}$ . Thus, the automorphism group is a 2-group and has order  $(2^q)^{d-1}$ . Hence, it is a regular abelian 2-group.

The claim for k-orbits follows from the case of a gadget:  $\Gamma$  is a subgroup of the direct product of the automorphism groups of the gadgets of  $\operatorname{orig}(P)$ . That is,  $\Gamma$  is

an abelian 2-group. By definition of a k-orbit,  $\Gamma$  is transitive. For regularity, note that a gadget is partitioned into singleton orbits once one vertex of the gadget is fixed (cf. Lemma 3.13 in [17]). So if we fix a  $\bar{u} \in P$ , all gadgets in the origin of  $\bar{u}$  are fixed. So, if an automorphism  $\varphi$  maps  $\bar{u}$  to  $\bar{v}$ , then its action on P is fixed, i.e., there is exactly one permutation in  $\Gamma$  that maps  $\bar{u}$  to  $\bar{v}$ . Hence,  $|\Gamma| = |P|$  and  $\Gamma$  is regular.

#### 5.3 Composition of Orbits

Composing k-orbits out of k'-orbits for k' < k plays a special role later. We further analyze the structure of k-orbits and identify cases in which such a composition in possible. As in the previous section, let  $q, k, m \in \mathbb{N}$  and  $G = (V, E, \leq)$  be a (k + m + 1)-connected base graph, denote for  $f : E \to \mathbb{Z}_{2^q}$  by  $\mathfrak{A}_f$  the CFI structure  $\mathsf{CFI}_{\mathbb{Z}_{2^q}}(G, f)$  with universe A, and let  $\bar{p} \in A^m$ .

Let  $\bar{u} \in A^k$  and  $\text{orig}(\bar{u})$  (viewed as a set) be partitioned into M and N. We now introduce notation for splitting  $\bar{u}$  into its parts belonging to N and M and for recovering  $\bar{u}$  from these two parts again.

1. The tuple  $\bar{u}_N$  obtained from  $\bar{u}$  by deleting all entries whose origin is not in N (respectively for M), is

$$\bar{u}_N := \bar{u}_{\{i \in [k] \mid \operatorname{orig}(u_i) \in N\}}.$$

2. We define a concatenation operation for a permutation  $\sigma$  of [k] as follows:

$$\bar{u}_N \cdot_{\sigma} \bar{u}_M := \sigma(\bar{u}_N \bar{u}_M).$$

For a suitable  $\sigma$  we have  $\bar{u} = \bar{u}_N \cdot_{\sigma} \bar{u}_M$ . In this article we are only interested in permutations satisfying the former equation. Then  $\sigma$  is almost always fixed by the context and we use juxtaposition  $\bar{u}_N \bar{u}_M$ . It is never the case that we refer with  $\bar{u}_N \bar{u}_M$  to ordinary concatenation.

3. We define similar operations for orbits: For  $P \in \mathsf{orbs}_k((\mathfrak{A}, \bar{p}))$  we set

$$P|_{N} := \left\{ \left. \bar{u}_{N} \mid \bar{u} \in P \right. \right\},$$

$$P|_{N} \times_{\sigma} P|_{M} := \left\{ \left. \bar{u}_{N} \cdot_{\sigma} \bar{u}_{M} \mid \bar{u}_{N} \in P|_{N}, \bar{u}_{M} \in P|_{M} \right. \right\},$$

and leave out  $\sigma$  if clear from the context. This intentionally overloads notation. Because the tuples in P are indexed by [k],  $P|_N$  and  $P|_K$  for  $N \subseteq \mathsf{orig}(P) \subseteq V$  and  $K \subseteq [k]$  can always be distinguished.

We also use this notation if N and M are sets of sets, such that  $\operatorname{\sf orig}(\bar{u})$  is partitioned into  $\bigcup N$  and  $\bigcup M$ .

**Definition 18** (Components of Tuples and Orbits). Let  $f: E \to \mathbb{Z}_{2^q}$ ,  $\bar{u} \in A^k$ , and  $N \subseteq \operatorname{orig}(\bar{u})$ . We call N a **component** of  $\bar{u}$  if N is a connected component of  $G[\operatorname{orig}(\bar{u})]$ . We call  $\bar{u}$  disconnected if it has more than one component.

Likewise, a k-orbit  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$  is disconnected if P contains some (and thus only) disconnected tuples. A set  $N \subseteq \operatorname{orig}(P)$  is a component of P if N is a connected component of  $G[\operatorname{orig}(P)]$ .

If a k-orbit P is disconnected, then we can split P into multiple k'-orbits for k' < k as follows.

**Lemma 19.** Let  $f: E \to \mathbb{Z}_{2^q}$ ,  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$ , and the components of P be partitioned into M and N. Then  $P = P|_M \times P|_N$ ,  $P|_M \in \operatorname{orbs}_{k_M}((\mathfrak{A}_f, \bar{p}))$ , and  $P|_N \in \operatorname{orbs}_{k_N}((\mathfrak{A}_f, \bar{p}))$  for suitable  $k_M$  and  $k_N$  such that  $k_M + k_N = k$ .

*Proof.* This can easily be seen by Corollary 13. Because M and N are sets of components, the type of  $\bar{u} \in P$  is given by the disjoint union of the types of  $\bar{u}_M$  and  $\bar{u}_N$  (even if  $\text{orig}(\bar{p})$  overlaps with M and N because  $\bar{p}$  has to be fixed by every automorphism).

Next, we show how to obtain k'-orbits from k-orbits with k' < k by fixing a vertex.

**Lemma 20.** Let  $f: E \to \mathbb{Z}_{2^q}$ ,  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$ ,  $K \subseteq [k]$ , and  $\operatorname{orig}(P|_K) = \{z\}$ . For every  $\bar{v} \in A^{|K|}$  and  $w \in A$  such that  $\operatorname{orig}(\bar{v}) = \{z\}$  and  $\operatorname{orig}(w) = z$ , the set

$$Q := \left\{ \bar{u}|_{[k]\setminus K} \mid \bar{u} \in P, \bar{u}|_K = \bar{v} \right\} \text{ satisfies}$$

$$Q \in \operatorname{orbs}_{k-|K|}((\mathfrak{A}_f, \bar{p}w)) \cup \{\emptyset\}.$$

If  $\bar{v}$  has the same type as  $\bar{u}|_K$  for some (and thus every)  $\bar{u} \in P$ , then  $Q \neq \emptyset$ .

Proof. We assume w.l.o.g. up to reordering that K = [|K|]. Let  $\bar{v} \in A^{|K|}$  such that  $\operatorname{orig}(\bar{v}) = \{z\}$ . Every vertex  $v_i$  forms a singleton orbit in  $\operatorname{orbs}_1((\mathfrak{A}_f, w))$  and in particular in  $\operatorname{orbs}_1((\mathfrak{A}_f, \bar{p}w))$  because  $v_i$  and w have the same origin z (all vertices with origin z can be distinguished by their distances to w in the  $C_{u,v}$  relation, cf. Lemma 3.13 in [17]). So it holds that  $\operatorname{Aut}((\mathfrak{A}_f, \bar{p}\bar{v})) = \operatorname{Aut}((\mathfrak{A}_f, \bar{p}w))$ . Assume that  $Q \neq \emptyset$ . Because P is an orbit, if  $\bar{v}\bar{u}, \bar{v}\bar{u}' \in P$ , then there is an automorphism  $\varphi \in \operatorname{Aut}((\mathfrak{A}_f, \bar{p}))$  such that  $\varphi(\bar{v}\bar{u}) = \bar{v}\bar{u}'$ . That is,  $\varphi \in \operatorname{Aut}((\mathfrak{A}_f, \bar{p}\bar{v})) = \operatorname{Aut}((\mathfrak{A}_f, \bar{p}w))$  and thus Q is a subset of an orbit in  $\operatorname{orbs}_{k-|K|}((\mathfrak{A}_f, \bar{p}w))$ . To show that Q is indeed an orbit, assume that  $\bar{u} \in Q$  and  $\varphi \in \operatorname{Aut}((\mathfrak{A}_f, \bar{p}w)) = \operatorname{Aut}((\mathfrak{A}_f, \bar{p}\bar{v}))$ . Because  $\bar{u} \in Q$ ,  $\bar{v}\bar{u} \in P$  and  $\varphi(\bar{v}\bar{u}) = \bar{v}\varphi(\bar{u}) \in P$ . Hence,  $\varphi(\bar{u}) \in Q$  and so  $Q \in \operatorname{orbs}_{k-|K|}((\mathfrak{A}_f, \bar{p}w))$ .

Now assume that there is some  $\bar{u} \in P$  such that  $\bar{u}|_K$  has the same type as  $\bar{v}$ . That is, there is an automorphism  $\varphi \in \operatorname{Aut}(\mathfrak{A}_f, \bar{p})$  such that  $\varphi(\bar{u})|_K = \bar{v}$  (Lemma 12). Hence,  $\varphi(\bar{u}) \in P$  and  $\varphi(\bar{u})|_{[k]\setminus K} \in Q$ .

Note that Q is independent of w, but not the type of Q in  $(\mathfrak{A}_f, \bar{p}w)$ .

Corollary 21. Let  $f: E \to \mathbb{Z}_{2^q}$ ,  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$ ,  $i \in [k]$ ,  $\operatorname{orig}(P|_{\{i\}}) = \{z\}$ , and let  $\operatorname{dist}_G(z, \operatorname{orig}(\bar{p})) > 1$ . For every  $v, w \in A$  such that  $\operatorname{orig}(v) = \operatorname{orig}(w) = z$  it holds that  $\{\bar{u}|_{[k]\setminus\{i\}} \mid \bar{u} \in P, u_i = v\} \in \operatorname{orbs}_{k-1}((\mathfrak{A}_f, \bar{p}w))$ .

*Proof.* We apply Lemma 20: Because  $\mathsf{dist}_G(z,\mathsf{orig}(\bar{p})) > 1$ , the type of w is the same as the type of every v with origin z, in particular the same as  $v_i$  for every  $\bar{v} \in P$ .

### 5.4 Rank Logic on CFI Structures

In this section we refine a result of [17] and show that on CFI structures over  $\mathbb{Z}_{2^q}$  the uniform rank logic IFP+R has the same expressiveness as the rank logic IFP+R<sub>{2}</sub> only with rank operators over  $\mathbb{F}_2$ .

**Definition 22.** For a class of base graphs K,

$$\mathsf{CFI}_{2^{\omega}}(\mathcal{K}) := \left\{ \left. \mathsf{CFI}_{2^q}(G, f) \; \right| \; q \in \mathbb{N}, G = (V, E, \leq) \in \mathcal{K}, f \colon E \to \mathbb{Z}_{2^q} \right\}$$

is the class of all CFI structures over K.

**Lemma 23.** Let K be a class of base graphs. For every IFP+R-formula  $\Phi$  there is an IFP+ $R_{\{2\}}$ -formula  $\Psi$  that is equivalent to  $\Phi$  on  $\mathsf{CFl}_{2^{\omega}}(K)$ .

*Proof.* Let **solvability logic** IFP+CS be the extension by IFP+C by the uniform solvability quantifier slv [17]. If  $s(\bar{x}, \bar{y})$  is a numeric term and t is a closed numeric term, then

$$\operatorname{slv}(\bar{x},\bar{y}).\ (s,t)$$

is a formula. Similar to the rank operator rk, the numeric term t defines a number p. If p is prime, then the solvability quantifier is satisfied if the linear system  $M_s^{\mathfrak{A}}x=1$  is solvable over  $\mathbb{F}_p$ . If otherwise p is not prime, then the operator is not satisfied. Let IFP+CS $_{\Omega}$  be the extension of IFP+C with solvability quantifiers  $\mathsf{slv}_p$  for each fixed field  $\mathbb{F}_p$  with  $p \in \Omega$  similar. We again left out parameters for readability.

Grädel and Pakusa [17] give a translation of IFP+R $_{\Omega}$ -formulas to IFP+C-formulas equivalent on CFI structures over  $\mathbb{F}_2$  for every set of primes  $\Omega$  satisfying  $2 \notin \Omega$ . The crucial point in their proofs is that the automorphism groups of these CFI structures are abelian 2-groups and that their k-orbits can be defined and ordered in IFP+C, that is, there is an IFP+C-definable total preorder on all k-tuples whose equivalence classes coincide with the k-orbits (their construction is not specific to  $\mathbb{F}_2$  but generally for  $\mathbb{F}_p$  whenever  $p \notin \Omega$  and p is the characteristic of the CFI structures). These assumptions are made explicit in Section 3.2 in [17]. Hence, the arguments work for CFI structures over  $\mathbb{Z}_{2^q}$  instead of  $\mathbb{F}_2$ , too. In [17], the authors use solvability logic as an intermediate step and first show that for all sets of primes  $\Omega$  (even with  $2 \in \Omega$ ) it holds that IFP+R $_{\Omega}$  = IFP+CS $_{\Omega}$  on CFI<sub>2 $\omega$ </sub>( $\mathcal{K}$ ) (Lemma 3.7 in [17]). This reduction works as well for the uniform case and shows IFP+R = IFP+CS on CFI<sub>2 $\omega$ </sub>( $\mathcal{K}$ ).

The second step in [17] is a recursive translation of IFP+R $_{\Omega}$ -formulas to IFP+C-formulas if  $2 \notin \Omega$  (Lemmas 3.4 to 3.6 in [17]). For every IFP+C-term s the solvability quantifier  $\Psi = \mathsf{slv}_p(\bar{x}, \bar{y})$ . s over  $\mathbb{F}_p$  can be simulated in IFP+C by computing the rank of the matrix  $M := M_s^{\mathfrak{A}}$  orbit-wise. This is expressible in IFP+C because the automorphism group is a 2-group and  $p \neq 2$ . This process works as follows: There is an IFP+C-formula that for every prime p and every term s exploits the orbits of the structure to define a matrix E such that Mx = 1 is solvable if and only if  $(M \cdot E)x = 1$  is solvable (Lemma 3.6 in [17]). Now, E is defined such that the columns of  $M \cdot E$  are totally ordered and thus the solution can be obtained in IFP+C.

Now, we translate an IFP+CS $_{\Omega}$ -formula (respectively term) with  $2 \in \Omega$  recursively into an IFP+CS $_{\{2\}}$ -formula (respectively term). Again consider a solvability quantifier  $\Psi = \mathsf{slv}_p(\bar{x}, \bar{y}).s.$  If p=2, then we recurse on s but do not replace the solvability quantifier. If otherwise  $p \neq 2$ , then we recurse on s and obtain an IFP+CS $_{\{2\}}$ -term equivalent to s, define the matrix E with the IFP+C-formula from above, and construct a formula defining whether  $M \cdot E = 1$  is solvable. Because this check can be done in IFP+C and M is defined by an IFP+CS $_{\{2\}}$ -term, we obtain an IFP+CS $_{\{2\}}$ -formula equivalent to  $\Psi$ .

We finally deal with the case of an IFP+CS formula, where the prime is defined by a numeric term t. Checking an ordered equation system for solvability is IFP+C-definable when the prime is given by a term, too. Let  $\Psi = \mathsf{slv}(\bar{x}, \bar{y})$ . (s, t) be a uniform solvability quantifier. Let  $\Psi_2$  be the formula obtained for  $\mathsf{slv}_2(\bar{x}, \bar{y})$ . s in the former case and  $\Psi_{\neq 2}$  be the formula for the case  $p \neq 2$ , where we already use t to obtain the prime. Indeed,  $\Psi_{\neq 2}$  is independent of p because defining the matrix E is independent of p and checking the linear equation system for consistency is already done using the prime-defining term t. Then the uniform solvability quantifier  $\Psi$  is equivalent to the IFP+CS<sub>{2}</sub>-formula  $(t=2 \to \Psi_2) \land (t \neq 2 \to \Psi_{\neq 2})$ . Obviously, an IFP+CS<sub>{2}</sub>-formula can be translated back into an IFP+R<sub>{2</sub>}-formula.

#### 6 Matrices over CFI Structures

In the invertible-map game, Duplicator has to partition the 2k-tuples of CFI structures and to provide a similarity matrix. For our arguments, we want that Duplicator plays with the 2k-orbit partitions. To construct the required similarity matrices, we develop a criterion for invertibility of matrices over  $\mathbb{F}_2$  and show that this criterion is preserved by matrix multiplication.

Let  $q, k, m \in \mathbb{N}$  and  $G = (V, E, \leq)$  be a (k + m + 1)-connected base graph. The connectivity is needed to apply the lemmas of Section 5.2. Again, we denote for a function  $f \colon E \to \mathbb{Z}_{2^q}$  by  $\mathfrak{A}_f$  the CFI structure  $\mathsf{CFI}_{\mathbb{Z}_{2^q}}(G, f)$  with universe A (which is equal for every  $f \colon E \to \mathbb{Z}_{2^q}$ ). Let  $\bar{p} \in A^m$  be arbitrary but fixed in this section.

**Definition 24** (Blurring the Twist). For  $f, g: E \to \mathbb{Z}_{2^q}$  not twisting  $\operatorname{orig}(\bar{p})$ , an  $A^k \times A^k$  matrix S over  $\mathbb{F}_2$  k-blurs the twist between  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$  if S is invertible and  $\chi^P \cdot S = S \cdot \chi^Q$  for every  $P \in \operatorname{orbs}_{2k}((\mathfrak{A}_f, \bar{p}))$  and  $Q \in \operatorname{orbs}_{2k}((\mathfrak{A}_g, \bar{p}))$  that are of the same type.

Note that by Corollary 13 two different orbits have different types and that by Lemma 16 for each  $P \in \operatorname{orbs}_{2k}(\mathfrak{A}_f, \bar{p})$  there is a  $Q \in \operatorname{orbs}_{2k}(\mathfrak{A}_g, \bar{p})$  of the same type. So we indeed get a bijection between the orbits and Duplicator can use the matrix S in the invertible-map game. Because S is invertible,  $\chi^P \cdot S = S \cdot \chi^Q$  is equivalent to  $\chi^P = S \cdot \chi^Q \cdot S^{-1}$ . Showing the former has the benefit that we do not need the inverse  $S^{-1}$ .

**Lemma 25.** Let  $f, g, h: E \to \mathbb{Z}_{2^q}$  pairwise not twist  $\operatorname{orig}(\bar{p})$  and S, T be  $A^k \times A^k$  matrices over  $\mathbb{F}_2$ . If S blurs the twist between  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$  and T blurs the twist between  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_h, \bar{p})$ , then  $S \cdot T$  blurs the twist between  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_h, \bar{p})$ .

Proof. Let  $P \in \operatorname{orbs}_{2k}((\mathfrak{A}_f, \bar{p}))$ ,  $Q \in \operatorname{orbs}_{2k}((\mathfrak{A}_g, \bar{p}))$ , and  $R \in \operatorname{orbs}_{2k}((\mathfrak{A}_h, \bar{p}))$  be of the same type. Recall that given P, the orbits Q and R are determined uniquely (Corollary 13). Then  $\chi^P \cdot S \cdot T = S \cdot \chi^Q \cdot T = S \cdot T \cdot \chi^R$ .

Now we want to develop combinatorial conditions for an  $A^k \times A^k$  matrix S over  $\mathbb{F}_2$ , which guarantee that S is invertible. The k-orbits (for given  $f,g \colon E \to \mathbb{Z}_{2^q}$ ) partition S into a block matrix. Each  $P \in \mathsf{orbs}_k((\mathfrak{A}_f,\bar{p}))$  corresponds to a subset of the rows of S and each  $Q \in \mathsf{orbs}_k((\mathfrak{A}_g,\bar{p}))$  corresponds to a subset of the columns of S. We denote by  $S_{P \times Q}$  the corresponding submatrix of S.

**Definition 26** (Orbit-Diagonal Matrix). For  $f, g: E \to \mathbb{F}_2$  not twisting  $\operatorname{orig}(\bar{p})$ , we call an  $A^k \times A^k$  matrix S over  $\mathbb{F}_2$  orbit-diagonal over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ , if for every  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$  and every  $Q \in \operatorname{orbs}_k((\mathfrak{A}_g, \bar{p}))$  it holds that if  $S_{P \times Q} \neq \emptyset$ , then P has the same type in  $(\mathfrak{A}_f, \bar{p})$  as Q has in  $(\mathfrak{A}_g, \bar{p})$ .

We have seen that for every  $P \in \operatorname{\sf orbs}_k((\mathfrak{A}_f, \bar{p}))$  there is exactly one  $Q \in \operatorname{\sf orbs}_k((\mathfrak{A}_g, \bar{p}))$  of the same type. So orbit-diagonal matrices are block-diagonal matrices, where orbits of the same type form the nonzero blocks. A permutation  $\sigma$  of  $A^k$  is applied to an  $A^k \times A^k$  matrix S in the natural way:  $(\sigma(S))(\bar{u}, \bar{v}) = S(\sigma(\bar{u}), \sigma(\bar{v}))$ . Of particular interest are automorphisms.

**Definition 27** (Orbit-Invariant Matrix). For  $f, g: E \to \mathbb{Z}_{2^q}$  that do not twist  $\operatorname{orig}(\bar{p})$ , an  $A^k \times A^k$  matrix S over  $\mathbb{F}_2$  is called **orbit-invariant** over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ , if for every  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$ ,  $Q \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$ , and  $\varphi \in \operatorname{Aut}((\mathfrak{A}_f, \bar{p})) = \operatorname{Aut}((\mathfrak{A}_g, \bar{p}))$  (cf. Corollary 15) the matrix S satisfies  $\varphi(S_{P \times Q}) = S_{P \times Q}$ .

**Lemma 28.** Let  $f, g, h \colon E \to \mathbb{Z}_{2^q}$  not twist  $\operatorname{orig}(\bar{p})$  and S, T be  $A^k \times A^k$  matrices over  $\mathbb{F}_2$ . If S is orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$  and T is orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_f, \bar{p})$ , then  $S \cdot T$  is orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_h, \bar{p})$ .

Proof. It is clear that  $S \cdot T$  is orbit-diagonal over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_h, \bar{p})$ . For k-orbits  $P \in \mathsf{orbs}_k((\mathfrak{A}_f, \bar{p})), \ Q \in \mathsf{orbs}_k((\mathfrak{A}_g, \bar{p})), \ \text{and} \ R \in \mathsf{orbs}_k((\mathfrak{A}_h, \bar{p})) \ \text{of the same type it holds}$  that  $(S \cdot T)_{P \times R}(\bar{u}, \bar{w}) = \sum_{\bar{v} \in Q} S_{P \times Q}(\bar{u}, \bar{v}) \cdot T_{Q \times R}(\bar{v}, \bar{w}).$  Let  $\varphi \in \mathsf{Aut}((\mathfrak{A}, \bar{p})).$  Then

$$(\varphi(S \cdot T))_{P \times R}(\bar{u}, \bar{w}) = (S \cdot T)_{P \times R}(\varphi(\bar{u}), \varphi(\bar{w}))$$

$$= \sum_{\bar{v} \in Q} S_{P \times Q}(\varphi(\bar{u}), \bar{v}) \cdot T_{Q \times R}(\bar{v}, \varphi(\bar{w}))$$

$$= \sum_{\bar{v} \in Q} S_{P \times Q}(\varphi(\bar{u}), \varphi(\bar{v})) \cdot T_{Q \times R}(\varphi(\bar{v}), \varphi(\bar{w}))$$

$$= \sum_{\bar{v} \in Q} S_{P \times Q}(\bar{u}, \bar{v}) \cdot T_{Q \times R}(\bar{v}, \bar{w})$$

$$= (S \cdot T)_{P \times R}(\bar{u}, \bar{w}).$$

Applying  $\varphi$  to  $\bar{v}$  is valid because  $\varphi$  is a permutation of Q and thus only permutes the summands. Then  $S_{P\times Q}(\varphi(\bar{u}), \varphi(\bar{v})) = S_{P\times Q}(\bar{u}, \bar{v})$  because S is orbit-invariant (and likewise for T).

**Definition 29** (Odd-Filled Matrix). A matrix over  $\mathbb{F}_2$  is called **odd-filled** if every row contains an odd number of ones.

**Lemma 30.** If two  $A^k \times A^k$  matrices S and T over  $\mathbb{F}_2$  are odd-filled, then so is  $S \cdot T$ .

*Proof.* Let  $R = S \cdot T$  and denote by  $r_{\bar{u}}$  and  $t_{\bar{v}}$  the rows of R and T indexed by  $\bar{u} \in A^k$  and  $\bar{v} \in A^k$ . Then

$$r_{\bar{u}} = \sum_{\bar{v} \in A^k} S(\bar{u}, \bar{v}) \cdot t_{\bar{v}}.$$

The number of ones modulo 2 is given by

$$\sum r_{\bar{u}} = \sum_{\bar{v} \in A^k} S(\bar{u}, \bar{v}) \cdot \sum t_{\bar{v}}.$$

Now  $S(\bar{u}, \bar{v}) = 1$  for an odd number of  $\bar{v} \in A^k$ , because S is odd-filled. Hence,  $\sum r_{\bar{u}}$  is the sum of an odd number of  $\sum t_{\bar{v}}$ , of which each is odd because T is odd-filled. So  $\sum r_{\bar{u}} = 1$  and  $r_{\bar{u}}$  contains an odd number of ones.

**Lemma 31.** Let  $f, g: E \to \mathbb{Z}_{2^q}$  not twist  $\operatorname{orig}(\bar{p})$  and S be an  $A^k \times A^k$  matrix over  $\mathbb{F}_2$ . If S is odd-filled and both orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ , then every column of S contains an odd number of ones.

Proof. Consider the block  $S_{P\times Q}$  for arbitrary  $P\in \operatorname{orbs}_k((\mathfrak{A}_f,\bar{p}))$  and  $Q\in \operatorname{orbs}_k((\mathfrak{A}_g,\bar{p}))$  of the same type. Let  $P=\{\bar{u}_1,\ldots,\bar{u}_n\}$  and  $Q=\{\bar{v}_1,\ldots,\bar{v}_n\}$ . Then consider automorphisms  $\varphi_i$  such that  $\varphi_i(\bar{u}_1)=\bar{u}_i$ . Because the induced action of  $\operatorname{Aut}((\mathfrak{A},\bar{p}))$  on P (and on Q) is regular (Lemma 17), the action of  $\varphi_i$  on P (and so Q) is uniquely determined. Now we consider w.l.o.g. the column indexed by  $\bar{v}_1$ :

$$S(\bar{u}_i, \bar{v}_1) = \varphi_i^{-1}(S)(\bar{u}_i, \bar{v}_1) = S(\bar{u}_1, \varphi_i^{-1}(\bar{v}_1))$$

because S is orbit-invariant. So the column indexed by  $\bar{v}_1$  contains exactly the entries of the row indexed by  $\bar{u}_1$ . That is, the number of ones in every column is odd.

**Lemma 32.** Let  $\bar{a} \in \mathbb{F}_2^N$  for some finite set N and  $\Gamma < \mathsf{Sym}(N)$  be a regular and abelian 2-group. If the number of ones in  $\bar{a}$  is odd, then the set  $B := \{\sigma(\bar{a}) \mid \sigma \in \Gamma\}$  is a basis of  $\mathbb{F}_2^N$ .

Proof. Assume w.l.o.g. that  $N = [\ell]$  and let  $W \subseteq \mathbb{F}_2^N$  be the linear space spanned by B. Because  $\Gamma$  is regular, it consists of  $\ell$  many permutations  $\Gamma = \{\sigma_1, \ldots, \sigma_\ell\}$  such that  $\sigma_i(1) = i$  for all  $i \in [\ell]$ . By definition, W is invariant under permutations of  $\Gamma$ . In coding theory, such a linear space is called an abelian code. It is known that W can be identified with an ideal of the group algebra  $\mathbb{F}_2[\Gamma]$  [4], which is the set of formal sums

$$\bigg\{ \sum_{g \in \Gamma} b_g g \ \bigg| \ b_g \in \mathbb{F}_2 \bigg\}.$$

This set is naturally an  $\mathbb{F}_2$ -vector space indexed by  $\Gamma$ . To turn it into a  $\mathbb{F}_2$ -algebra, multiplication is defined via

$$\left(\sum_{g\in\Gamma}b_gg\right)\cdot\left(\sum_{g\in\Gamma}c_gg\right):=\sum_{g,h\in\Gamma}(b_g\cdot c_h)(g\cdot h).$$

A set  $I \subseteq \mathbb{F}_2[\Gamma]$  is a (left) ideal of the algebra  $\mathbb{F}_2[\Gamma]$  if  $g \cdot h \in I$  for every  $g \in \mathbb{F}_2[\Gamma]$  and  $h \in I$ , i.e.,  $\mathbb{F}_2[\Gamma] \cdot I = I$ . The abelian code W is identified with an ideal of  $\mathbb{F}_2[\Gamma]$  via the bijection  $(b_1, \ldots, b_\ell) \mapsto \sum_{i=1}^\ell b_i \sigma_i$  for every  $\bar{b} \in W$ .

Let  $I \subseteq \mathbb{F}_2[\Gamma]$  be the corresponding ideal of W and let the number of ones of  $\bar{a} \in W$  be odd. Because  $\Gamma$  is a 2-group, there is a k such that  $\sigma_i^{(2^k)} = 1_{\Gamma}$  for all  $i \in [\ell]$ . Because  $\Gamma$  is abelian and we consider  $\mathbb{F}_2$ ,  $(b\sigma_i)(c\sigma_j) + (c\sigma_j)(b\sigma_i) = 2(b\sigma_i)(c\sigma_j) = 0$ . So  $(b\sigma_i + c\sigma_j)^2 = (b\sigma_i)^2 + (c\sigma_j)^2$  and  $(b\sigma_i + c\sigma_j)^{(2^k)} = (b\sigma_i)^{(2^k)} + (c\sigma_j)^{(2^k)}$ . It follows that

$$\left(\sum_{i=1}^{\ell} a_i \sigma_i\right)^{(2^k)} = \sum_{i=1}^{\ell} (a_i \sigma_i)^{(2^k)} = \sum_{i=1}^{\ell} a_i^{(2^k)} 1_{\Gamma} = \sum_{i=1}^{\ell} a_i 1_{\Gamma} = 1_{\Gamma}.$$

The last step holds because the number of ones in  $\bar{a}$  is odd. So  $\sum_{i=1}^{\ell} a_i \sigma_i$  is a unit with inverse  $(\sum_{i=1}^{\ell} a_i \sigma_i)^{2^k-1}$ . First,  $\sum_{i=1}^{\ell} a_i \sigma_i \in I$  because  $\bar{a} \in W$ . Second,  $1_{\Gamma} \in I$  because the inverse of  $\sum_{i=1}^{\ell} a_i \sigma_i \in I$  is clearly contained in  $\mathbb{F}_2[\Gamma]$  and  $\mathbb{F}_2[\Gamma] \cdot I = I$ . Thus,  $I = \mathbb{F}_2[\Gamma]$  and  $W = \mathbb{F}_2^N$ . Finally, B must be a basis of W because |B| = |N|.

**Lemma 33.** Let  $f, g: E \to \mathbb{Z}_{2^q}$  not twist  $\operatorname{orig}(\bar{p})$  and S be an  $A^k \times A^k$  matrix over  $\mathbb{F}_2$ . If S is odd-filled and both orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ , then S is invertible.

Proof. It suffices to show that each block on the diagonal of S is invertible because S is orbit-diagonal. Let  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$  and  $Q \in \operatorname{orbs}_k((\mathfrak{A}_g, \bar{p}))$  be of the same type. Because S is odd-filled and orbit-diagonal,  $S_{P\times Q}$  is also odd-filled. By Lemma 17, the action of  $\operatorname{Aut}((\mathfrak{A}_f, \bar{p}))$  on P induces a regular and abelian 2-group  $\Gamma$ . By Corollary 15, the action of  $\operatorname{Aut}((\mathfrak{A}_g, \bar{p}))$  on Q yields the same group  $\Gamma$ . Let  $n := |P|, P = \{\bar{u}_1, \ldots, \bar{u}_n\}$ , and  $s_i$  be the row of  $S_{P\times Q}$  indexed by  $\bar{u}_i$ . We want to show that  $s_i = \varphi_i(s_1)$  for a unique  $\varphi_i \in \Gamma$ . Each  $\varphi \in \Gamma$  acts as a permutation on the entries of each  $s_i$ , that is  $(\varphi(s_i))(\bar{v}) = s_i(\varphi(\bar{v}))$ . Let  $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$  such that  $\varphi_i^{-1}(\bar{u}_1) = \bar{u}_i$  for every  $i \in [n]$  (this is possible because  $\Gamma$  is regular). Then

$$(\varphi_i(s_1))(\bar{v}) = S_{P \times Q}(\bar{u}_1, \varphi_i(\bar{v})) = S_{P \times Q}(\varphi_i^{-1}(\bar{u}_1), \bar{v})$$

because S is orbit-invariant. Hence,

$$(\varphi_i(s_1))(\bar{v}) = S_{P \times Q}(\varphi_i^{-1}(\bar{u}_1), \bar{v}) = s_i(\bar{v}),$$

i.e.,  $\varphi_i(s_1) = s_i$ . Finally,  $\{\varphi_i(s_1) \mid i \in [n]\} = \{s_1, \dots, s_n\}$  forms a basis of  $\mathbb{F}_2^n$  by Lemma 32. That is,  $S_{P \times Q}$  has full rank and is invertible.

# 7 The Arity 1 Case

To separate rank logic from PTIME, we want to show that for every arity k and number of pebbles 2k+m, there are two non-isomorphic CFI structures over  $\mathbb{Z}_{2^q}$  for a suitable  $q \in \mathbb{N}$  for which Duplicator has a winning strategy in the invertible-map game  $\mathcal{M}^{2k+m,k,\{2\}}$ . This implies IFP+R<sub>{2}</sub>-undefinability of the CFI query by Lemma 3 and IFP+R-undefinability by Lemma 23. The most challenging part of constructing winning strategies for Duplicator in the invertible-map game is to provide similarity matrices. Indeed, our goal is to construct matrices blurring the twist. Once we achieve this, it suffices to ensure that the pebbled tuples in both structures always have the same type. This final step is made formal in Section 10. Constructing matrices blurring the twists for an arbitrary arity k turns out to be formally intricate and is in particular recursive on the arity. In this section, we start with constructing matrices for arity 1, which serve as a base case for the recursion. We introduce basic techniques that we generalize to higher arities later in Section 9.

Let  $q \geq 2$ ,  $m \in \mathbb{N}$ ,  $G = (V, E, \leq)$  be an (m+3)-connected base graph,  $z \in V$  be a vertex of degree d, and  $\{z, t\} \in E$ . Let  $f, g \colon E \to \mathbb{Z}_{2^q}$  such that  $\{z, t\}$  is the only twisted edge and  $g(\{z, t\}) = f(\{z, t\}) + 2^{q-1}$ . The number m is the number of pebbles remaining on the structure when Spoiler picks up the 2 = 2k many pebbles before Duplicator

needs to provide the similarity matrix (we consider arity k=1 in this section). From another perspective, m corresponds to the number of free variables of a rank operator. Set  $\mathfrak{A}_f := \mathsf{CFI}_{2^q}(G,f)$  and  $\mathfrak{A}_g := \mathsf{CFI}_{2^q}(G,g)$ , both with universe A. Let  $\bar{p} \in A^m$  such that  $\mathsf{dist}_G(z,\mathsf{orig}(\bar{p})) \geq 3$ , in particular g and f do not twist  $\mathsf{orig}(\bar{p})$ . The tuple  $\bar{p}$  is the tuple of vertices on which the pebbles remain. It suffices to consider only a single tuple  $\bar{p}$  for both structures because we will ensure that the pebbled tuples always have the same type in both structures. Whenever the pebbled tuples have the same type but are not equal, we can consider an isomorphic structure in which we moved the twist to an edge far apart from the pebbled tuples. Then the tuples are equal and we ensured that the distance between  $\mathsf{orig}(\bar{p})$  and the twisted edge is sufficiently large (details in Section 10).

For  $x \in V$ , let  $A_x$  be the set of vertices originating from x, i.e., the vertices of the gadget for x. The key idea is to "distribute" the twist among multiple edges, such that it cannot be detected by Spoiler. For this, we introduce blurrers, the key ingredient to define the desired similarity matrix.

**Definition 34.** Let  $\Xi \subseteq \mathbb{Z}_{2^q}^d$ . For  $b \in \mathbb{Z}_{2^q}$  and  $j \in [d]$  we define

$$\#_{j,b}(\Xi) := \left| \left\{ \bar{a} \in \Xi \mid a_j = b \right\} \right| \mod 2.$$

The set  $\Xi$  is called a (q, d)-blurrer if it satisfies

- 1.  $\sum \bar{a} = 0$  for all  $a \in \Xi$ ,
- 2.  $\#_{1,2^{q-1}}(\Xi) = 1$ ,
- 3.  $\#_{j,0}(\Xi) = 1 \text{ for all } 1 < j \leq d, \text{ and }$
- 4.  $\#_{j,b}(\Xi) = 0$  for all other pairs of  $b \in \mathbb{Z}_{2^q}$  and  $j \in [d]$ .

From now on, we use the letter  $\xi$  for elements of a blurrer  $\Xi$ . Note that  $\Xi$  consists solely of tuples satisfying  $\Sigma \xi = 0$ , i.e., we can later turn every  $\xi \in \Xi$  into an automorphism. But intuitively, when looking at a single index and summing over all  $\xi \in \Xi$ , it looks like there is a twist at index 1 and no twist at all other indices.

**Lemma 35.** The size  $|\Xi|$  of every (q, d)-blurrer is odd. For every  $d \geq 3$ , there is a (q, d)-blurrer.

*Proof.* By Conditions 2 and 4 it holds that

$$|\Xi| = \sum_{b \in \mathbb{Z}_{2^q}} \#_{1,b}(\Xi) = \#_{1,2^{q-1}}(\Xi) + \sum_{b \in \mathbb{Z}_{2^q} \setminus \{2^{q-1}\}} \#_{1,b}(\Xi) = 1 \mod 2.$$

For 
$$d \ge 3$$
, set  $\Xi := 2^{q-2} \cdot \{(3,0,1,0,\ldots,0), (3,1,0,0,\ldots,0), (2,1,1,0,\ldots,0)\}.$ 

Let  $\mathbf{P}_j = \mathsf{orbs}_j((\mathfrak{A}_f, \bar{p}))$  and  $\mathbf{Q}_j = \mathsf{orbs}_j((\mathfrak{A}_g, \bar{p}))$  for every  $j \in [2]$ . For  $P \in \mathbf{P}_2$  we set  $P_i := P|_i$  for every  $i \in [2]$  and likewise for a  $Q \in \mathbf{Q}_2$ . By Corollary 13,  $P_i$  satisfies  $P_i = A_x$  if  $x = \mathsf{orig}(P_i)$  and  $\mathsf{dist}_G(x, \mathsf{orig}(\bar{p})) > 1$ . Moreover, every  $P \in \mathbf{P}_1$  is also in  $\mathbf{Q}_1$  and has the same type in  $(\mathfrak{A}_f, \bar{p})$  as in  $(\mathfrak{A}_g, \bar{p})$ .

Let  $\Xi$  be a (q, d)-blurrer (note that z is of degree  $d \geq 3$  because G is (m+3)-connected) and  $N_G(z) = \{t_1, \ldots, t_d\}$  such that  $t_1 = t$ . Then we can view  $\xi \in \Xi$  also as a tuple

 $\xi \in \mathbb{Z}_{2^q}^{N_G(z)}$ . Thus,  $\xi$  acts on vertices u originating from z and we denote this action by  $\xi(u)$  (cf. Section 5.1 for a definition of the action). Note that every  $\xi \in \Xi$  extends to an automorphism of  $(\mathfrak{A}_f, \bar{p})$  (and so of  $(\mathfrak{A}_g, \bar{p})$ ): By Corollary 13, the gadget of z consists of a single orbit because  $\operatorname{dist}_G(z, \bar{p}) \geq 3$ , i.e.,  $A_z \in \mathbf{P}_1$ . We define an  $A \times A$  matrix S over  $\mathbb{F}_2$ , which is orbit-diagonal over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ . We set  $S_P := S_{P \times P}$  and define

$$S_P(u,v) := \begin{cases} 1 & \text{if } \operatorname{\mathsf{orig}}(P) \neq z \text{ and } u = v, \\ 1 & \text{if } \operatorname{\mathsf{orig}}(P) = z \text{ and } \xi(u) = v \text{ for some } \xi \in \Xi, \\ 0 & \text{otherwise.} \end{cases}$$

Of particular interest is the unique 1-orbit  $P_z$  with origin z. We have already seen that  $P_z = A_z \in \mathbf{P}_1$ , because  $\mathsf{dist}_G(z,\mathsf{orig}(\bar{p})) \geq 3$  by assumption. For all other orbits  $P \in \mathbf{P}_1$ , it is easy to see that  $S_P = \mathbb{1}$ .

**Lemma 36.** The matrix S is orbit-invariant over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ .

Proof. Let  $P \in \mathbf{P}_1$ ,  $\varphi \in \operatorname{Aut}((\mathfrak{A}_f, \bar{p}))$ ,  $u \in P$ , and  $v \in Q = P \in \mathbf{Q}_1$ . If  $P \neq P_z$ , then clearly  $\varphi(S_P) = \varphi(\mathbb{1}) = \mathbb{1} = S_P$ . Otherwise,  $P = P_z$ . Because the automorphism group of  $\mathfrak{A}_f$  is abelian (Lemma 5) and every  $\xi \in \Xi$  extends to an automorphism, it holds that  $\xi(\varphi(u)) = \varphi(\xi(u))$ . So  $S_{P_z}(\varphi(u), \varphi(v)) = 1$  if and only if  $\xi(\varphi(u)) = \varphi(\xi(u)) = \varphi(v)$  for some  $\xi \in \Xi$  if and only if  $\xi(u) = v$  for some  $\xi \in \Xi$ , i.e,  $S_{P_z}(u, v) = 1$ .

**Lemma 37.** The matrix S is odd-filled.

Proof. Let  $P \in \mathbf{P}_1$ . For  $P \neq P_z$ , the number of ones in a row of  $S_P = \mathbb{1}$  is one and thus odd. In  $S_{P_z}$ , the number of ones in a row is  $|\Xi|$  because  $\xi(u) \neq \xi'(u)$  if  $\xi \neq \xi'$  (Lemma 17) and if  $u \in P_z$ , then  $\xi(u) \in P_z$  for every  $\xi \in \Xi$ . From Lemma 35 it follows that  $|\Xi|$  is odd.

Corollary 38. The matrix S is invertible.

*Proof.* Apply Lemmas 33, 36, and 37.

We want to define a function  $\lambda \colon \mathbf{P}_2 \to \mathbf{Q}_2$  such that it maps an orbit to another orbit of the same type. By Corollary 15, we know that  $\mathbf{P}_2 = \mathbf{Q}_2$  and by Lemma 16 that a type-preserving bijection exists. Let  $P \in \mathbf{P}_2$  with origin (x, y). If  $\{z, t\} \neq \{x, y\}$ , we set  $\lambda(P) := P$ . Otherwise if (t, z) = (x, y), then P has a different type in  $(\mathfrak{A}_f, \bar{p})$  than in  $(\mathfrak{A}_g, \bar{p})$ : Every vertex in  $P_1$  is related with every vertex in  $P_2$  via some  $R_{E,c}$ . By Corollary 13, we have that  $P = E_{\{z,t\},a}$  for some  $a \in \mathbb{Z}_{2^q}$  (recall our assumption  $\mathrm{dist}_G(z, \bar{p}) \geq 3$  and thus a determines the type of P). We set  $\lambda(P) := E_{\{z,t\},a+2^{q-1}}$ , which then has the same type in  $(\mathfrak{A}_g, \bar{p})$  because of the twist (cf. Figure 1). The case of (z, t) is analogous.

**Lemma 39.**  $\chi^P \cdot S = S \cdot \chi^{\lambda(P)}$  for every  $P \in \mathbf{P}_2$ .

Proof. Let  $P \in \mathbf{P}_2$  and  $\operatorname{orig}(P) = (x, y)$  and set  $Q := \lambda(P)$ . Clearly  $P \subseteq P_1 \times P_2$ . We also have  $P_1 = Q_1$  and  $P_2 = Q_2$  (as seen earlier by Corollary 13). Then the  $P_1 \times P_2$  block is the only nonzero block of  $\chi^P$ . Because S is orbit-diagonal,  $\chi^P \cdot S$  has only one nonzero block, namely the  $P_1 \times Q_2$  block, which satisfies  $(\chi^P \cdot S)_{P_1 \times Q_2} = \chi_{P_1 \times P_2}^P \cdot S_{P_2 \times Q_2}$ . Likewise,  $(S \cdot \chi^Q)_{P_1 \times Q_2} = S_{P_1 \times Q_1} \cdot \chi_{Q_1 \times Q_2}^Q$ . Recall that we have set  $S_{P_2} = S_{P_2 \times Q_2}$ . We identify  $\chi^P$  with  $\chi_{P_1 \times P_2}^P$  and likewise for  $\chi^Q$ . So we are left to show that  $\chi^P \cdot S_{P_2} = S_{Q_1} \cdot \chi^Q$ .

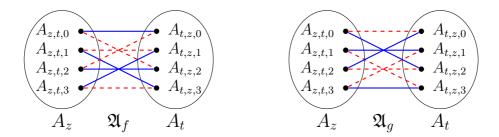


Figure 1: This figure shows the twist between  $\mathfrak{A}_f$  and  $\mathfrak{A}_g$ . It assumes that we consider  $\mathbb{Z}_4$  and that  $f(\{z,t\})=0$  and  $g(\{z,t\})=2$ . It shows the twisted connection for the edge  $\{z,t\}$ . On the left, there are the two gadgets for the vertices z and t in  $\mathfrak{A}_f$  and on the right there are the same gadgets in  $\mathfrak{A}_g$ . Every vertex represents a clique corresponding to the  $A_{z,t,c}$  and every edge a complete bipartite graph (cf. Section 5). The relation  $R_{E,0}$  is drawn in blue and  $R_{E,2}$  in red and dashed style. Restricted to the connection between  $\{z,t\}$ , we have in  $\mathfrak{A}_f$  that  $R_{E,0}=E_{\{z,t\},0}$  and  $R_{E,2}=E_{\{z,t\},2}$ . In  $\mathfrak{A}_g$  we have that  $R_{E,0}=E_{\{z,t\},2}$  and  $R_{E,2}=E_{\{z,t\},0}$ .

- Case  $z \notin \{x,y\}$ : Then  $Q = \lambda(P) = P$  and  $\chi^P \cdot S_{P_2} = \chi^P \cdot \mathbb{1} = \mathbb{1} \cdot \chi^Q = S_{Q_1} \cdot \chi^Q$ .
- Case x = y = z: Then  $Q = \lambda(P) = P$ . As already seen,  $P_1 = P_2 = Q_1 = Q_2 = P_z$ . So if  $u \in Q_2$ , then  $\xi^{-1}(u) \in Q_2$  for every  $\xi \in \Xi$ . We obtain

$$(\chi^P \cdot S_{P_2})(u, v) = \sum_{w \in P_2} \chi^P(u, w) \cdot S_{P_z}(w, v)$$
$$= \sum_{\xi \in \Xi} \chi^P(u, \xi^{-1}(v))$$
$$= \sum_{\xi \in \Xi} \chi^P(\xi(u), v).$$

The last step uses that  $\xi$  extends to an automorphism and thus  $\chi^P(u, \xi^{-1}(v)) = \xi(\chi^P)(u, \xi^{-1}(v)) = \chi^P(\xi(u), v)$ . The reverse direction is similar:

$$\sum_{\xi \in \Xi} \chi^{P}(\xi(u), v)$$

$$= \sum_{\xi \in \Xi} \chi^{Q}(\xi(u), v)$$

$$= \sum_{w \in Q_{1}} S_{Q_{1}}(u, w) \cdot \chi^{Q}(w, v)$$

$$= (S_{Q_{1}} \cdot \chi^{Q})(u, v).$$

• Case y = z and  $\{x, z\} \in E$  (the case x = z and  $\{z, y\} \in E$  is analogous): Again  $P_2 = P_z$ . We have

$$(\chi^{P} \cdot S_{P_{2}})(u, v)$$

$$= \sum_{w \in P_{z}} \chi^{P}(u, w) \cdot S_{P_{z}}(w, v)$$

$$= \sum_{\xi \in \Xi} \chi^{P}(u, \xi^{-1}(v))$$

$$= \begin{cases} 1 & \text{if } |\{\xi \in \Xi \mid (u, \xi^{-1}(v)) \in P\}| \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Let  $P = E_{\{x,z\},a}$  for some  $a \in \mathbb{Z}_{2^q}$  (cf. the definition of  $\lambda$ ) and  $(u,v) \in E_{\{x,z\},b}$  for some  $b \in \mathbb{Z}_{2^q}$ . Then, by definition of  $E_{\{x,z\},b}$ , it holds that u(z) + v(x) = b. Let  $i \in [d]$  such that  $x = t_i$  (recall that  $N_G(z) = \{t_1, \ldots, t_d\}$  and  $t_1 = t$ ). For every  $\xi \in \Xi$  it holds that  $(u, \xi^{-1}(v)) \in P = E_{(x,z),a}$  if and only if  $u(z) + \xi^{-1}(v)(x) = a$  if and only if  $\xi(i) = b - a$  because

$$u(z) + \xi^{-1}(v)(x) = u(z) + v(x) - \xi(i) = b - \xi(i).$$

We see that

$$\left| \left\{ \xi \in \Xi \mid (u, \xi^{-1}(v)) \in P \right\} \right| = \#_{i,b-a}(\Xi).$$

Set  $c := 2^{q-1}$  if i = 1 (and so x = t) and c := 0 otherwise. Then  $\#_{i,b-a}(\Xi) = 1$  if and only if b - a = c by the properties of a blurrer. It follows that

$$(\chi^P \cdot S_{P_2})(u, v) = \begin{cases} 1 & \text{if } b - a = c, \\ 0 & \text{otherwise.} \end{cases}$$

– If  $i \neq 1$  (so  $x \neq t'$ ), then c = 0 and  $(\chi^P \cdot S_{P_2})(u, v) = 1$  if and only if b = a, but that holds if and only if  $(u, v) \in P$ . So

$$\chi^P \cdot S_{P_2} = \chi^P = \mathbb{1} \cdot \chi^Q = S_{Q_1} \cdot \chi^Q$$

because  $Q = \lambda(P) = P$ .

– If i = 1 (so x = t), then  $(\chi^P \cdot S_{P_2})(u, v) = 1$  if and only if  $b - a = 2^{q-1}$ , i.e.,  $a + 2^{q-1} = b$ . But that holds by definition of  $\lambda$  if and only if

$$(u,v) \in Q = \lambda(P) = E_{(x,z),a+2^{q-1}}$$

and so

$$\chi^P \cdot S_{P_2} = \mathbb{1} \cdot \chi^Q = S_{Q_1} \cdot \chi^Q.$$

• Case y=z and  $\{x,z\} \notin E$  (the case x=z and  $\{z,y\} \notin E$  is analogous): By the assumption that  $\operatorname{dist}_G(z,\operatorname{orig}(\bar{p})) \geq 3$  the type of (u,v) and (u,v') for  $u \in A_x$  and  $v,v' \in A_z$  is equal. So  $(u,v) \in P$  if and only if  $(u,v') \in P$  by Corollary 13. In particular,  $(u,v) \in P$  if and only if  $(u,\xi^{-1}(v)) \in P$  for every  $\xi \in \Xi$ . Set

$$D:=\Big\{\,\xi\in\Xi\;\Big|\;(u,\xi^{\text{-}1}(v))\in P\;\Big\}.$$

Then we have

$$(\chi^P \cdot S_{P_2})(u, v) = \begin{cases} 1 & \text{if } |D| \text{ is odd,} \\ 0 & \text{otherwise} \end{cases}$$
$$= \chi^P(u, v).$$

The last step holds because if  $(u, v) \in P$ , then  $D = \Xi$  and  $|D| = |\Xi|$  is odd (Lemma 35), and if  $(u, v) \notin P$ , then  $D = \emptyset$  and |D| = 0. As seen before,

$$\chi^P \cdot S_{P_2} = \chi^P = \chi^Q = S_{Q_2} \cdot \chi^Q$$

because  $Q = \lambda(P) = P$ .

Corollary 40. The matrix S 1-blurs the twist between  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ .

We summarize the results of this section:

**Lemma 41.** For every  $q \geq 2$ ,  $m \in \mathbb{N}$ , every (m+3)-connected base graph  $G = (V, E, \leq)$ , every  $f, g \colon E \to \mathbb{Z}_{2^q}$  twisting exactly one edge  $\{z, t\}$  such that  $f(\{z, t\}) = g(\{z, t\}) + 2^{q-1}$ , and every m-tuple  $\bar{p} \in A^m$  of  $\mathfrak{A}_f := \mathsf{CFl}_{2^q}(G, f)$  and  $\mathfrak{A}_g := \mathsf{CFl}_{2^q}(G, f)$ , there is an odd-filled matrix S, both orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ , that 1-blurs the twist between  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$  and satisfies that  $S_{P,Q} = \mathbb{1}$  for every k-orbits  $P \in \mathsf{orbs}_1((\mathfrak{A}_f, \bar{p}))$  and  $Q \in \mathsf{orbs}_1((\mathfrak{A}_g, \bar{p}))$  of the same type with  $\mathsf{orig}(P) = \mathsf{orig}(Q) \neq z$ .

Constructing matrices blurring the twist for higher arities is more difficult: First, we have to generalize our notion of a blurrer to arity k. Second, we are faced with disconnected orbits, which do not pose a problem in the 1-ary case, but complicate matters in the general case. To deal with these orbits, we need to establish more technical lemmas for matrices over CFI structures.

## 8 The Active Region of a Matrix

In this section, we consider the part of a matrix S, where S "has a non-trivial effect". Intuitively, this means that S is locally not the identity matrix. We will call these parts the active region. Conversely, for parts where S is not active, S is locally the identity matrix. We now make this idea formal.

As in Section 6, let  $q, k, m \in \mathbb{N}$  and  $G = (V, E, \leq)$  be a (k + m + 1)-connected base graph. We again denote for every  $f \colon E \to \mathbb{Z}_{2^q}$  by  $\mathfrak{A}_f$  the CFI structure  $\mathsf{CFI}_{\mathbb{Z}_{2^q}}(G, f)$  and by A the universe of these CFI structures. Let  $\bar{p} \in A^m$  be arbitrary but fixed in this section. For  $N \subseteq V$ , the N-components  $C_N(P)$  of an orbit P is the set of components C of P satisfying  $C \subseteq N$ .

**Definition 42** (Active Region). Let  $f, g: E \to \mathbb{Z}_{2^q}$  not twist  $\operatorname{orig}(\bar{p})$ , S be an  $A^k \times A^k$  matrix over  $\mathbb{F}_2$ , and  $\mathbf{P}_k = \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$  and  $\mathbf{Q}_k = \operatorname{orbs}_k((\mathfrak{A}_g, \bar{p}))$ . For  $P \in \mathbf{P}_k$  and  $Q \in \mathbf{Q}_k$  of the same type, the matrix S is **active** (with respect to  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ ) on a component C of P (and so of Q), if there are  $\bar{u} \in P$  and  $\bar{v} \in Q$  such that  $\bar{u}_C \neq \bar{v}_C$  and  $S(\bar{u}, \bar{v}) = 1$ . We write  $\mathbf{A}^{f,g,\bar{p}}(S, P) = \mathbf{A}^{f,g,\bar{p}}(S, Q)$  for the set of components of P, on which S is active, and  $\mathbf{N}^{f,g,\bar{p}}(S, P) = \mathbf{N}^{f,g,\bar{p}}(S, Q)$  for the remaining components. The **active region**  $\mathbf{A}^{f,g,\bar{p}}(S) \subseteq V$  **of** S is the inclusion-wise smallest set satisfying the following:

- 1.  $C \subseteq \mathsf{A}^{f,g,\bar{p}}(S)$  for every  $C \in \mathsf{A}^{f,g,\bar{p}}(S,P)$  and every  $P \in \mathbf{P}_k$ .
- 2. For every  $P, P' \in \mathbf{P}_k$  and  $Q, Q' \in \mathbf{Q}_k$  such that

$$C_{\mathsf{A}^{f,g,\bar{p}}(S)}(P) = C_{\mathsf{A}^{f,g,\bar{p}}(S)}(P') = C_{\mathsf{A}^{f,g,\bar{p}}(S)}(Q) = C_{\mathsf{A}^{f,g,\bar{p}}(S)}(Q') =: \mathsf{A},$$

both P and Q (respectively P' and Q') have the same type, and thus  $\mathsf{N}^{f,g,\bar{p}}(S,P) = \mathsf{N}^{f,g,\bar{p}}(S,Q) =: \mathsf{N}$  (respectively  $\mathsf{N}^{f,g,\bar{p}}(S,P') = \mathsf{N}^{f,g,\bar{p}}(S,Q') =: \mathsf{N}'$ ), and every  $\bar{u} \in P$ ,  $\bar{u}' \in P'$ ,  $\bar{v} \in Q$ , and  $\bar{v}' \in Q'$ , it holds that if  $\bar{u}_{\mathsf{A}} = \bar{u}'_{\mathsf{A}}$ ,  $\bar{v}_{\mathsf{A}} = \bar{v}'_{\mathsf{A}}$ ,  $\bar{u}_{\mathsf{N}} = \bar{v}_{\mathsf{N}}$ , and  $\bar{u}'_{\mathsf{N}'} = \bar{v}'_{\mathsf{N}'}$ , then  $S(\bar{u},\bar{v}) = S(\bar{u}',\bar{v}')$ .

The active region is well-defined: Clearly V itself satisfies Conditions 1 and 2. If two sets  $X \subseteq V$  and  $Y \subseteq V$  satisfy the two conditions, then also  $X \cap Y$ . Note that  $C_{\mathsf{A}^{f,g,\bar{p}}(S)}(P)$  and  $\mathsf{N}^{f,g,\bar{p}}(S,P)$  are not necessarily disjoint, but  $\mathsf{N}^{f,g,\bar{p}}(S,P)$  contains all components of P not contained in  $C_{\mathsf{A}^{f,g,\bar{p}}(S)}(P)$ . Condition 2 equivalently can be stated only with the remaining components apart from  $C_{\mathsf{A}^{f,g,\bar{p}}(S)}(P)$  instead of  $\mathsf{N}^{f,g,\bar{p}}(S,P)$ .

Although Condition 2 is rather technical, it ensures that the "non-identity-part" of S only depends on the active region:  $S(\bar{u}, \bar{v})$  only depends on the components of  $\bar{u}$  and  $\bar{v}$ , on which S is active, as long as the entries for the other components are equal (otherwise  $S(\bar{u}, \bar{v}) = 0$  anyway by Condition 1). That is  $S(\bar{u}, \bar{v})$  only depends on whether  $\bar{u}_{N^{f,g,\bar{p}}(S,P)} = \bar{v}_{N^{f,g,\bar{p}}(S,P)}$  but not on e.g. the type of  $\bar{u}_{N^{f,g,\bar{p}}(S,P)}$ .

We first consider the matrix blurring the twist defined in Section 7:

**Lemma 43.** The matrix S given in the setting of Lemma 41 satisfies  $A^{f,g,\bar{p}}(S) = \{z\}$ .

Proof. Let  $P \in \operatorname{orbs}_1(\mathfrak{A}_f, \bar{p})$  and  $Q \in \operatorname{orbs}_1(\mathfrak{A}_g, \bar{p})$  be of the same type with origin  $x = \operatorname{orig}(P) = \operatorname{orig}(Q) \neq z$ . Then  $S_{P \times Q} = \mathbb{1}$  by Lemma 41, i.e., S is clearly not active on  $\{x\}$ . The matrix S has to be active on  $\{z\}$  because otherwise  $S = \mathbb{1}$  and the structures would be isomorphic. This proves Condition 1. In the 1-ary case, a 1-orbit can only have one component, so Condition 2 of the active region is trivially satisfied.

We now continue in the general case. The rest of this section establishes rather technical lemmas needed in Section 9. It is easy to see that if P and Q have the same type, whose origins contain no vertex of  $A^{f,g,\bar{p}}(S)$ , then  $S_{P\times Q}=\mathbb{1}$ . In the region of a twist, S has to be active:

**Lemma 44.** Let  $f, g: E \to \mathbb{Z}_{2^q}$  not twist  $\operatorname{orig}(\bar{p})$ , S be an  $A^k \times A^k$  matrix over  $\mathbb{F}_2$ , and  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$  and  $Q \in \operatorname{orbs}_k((\mathfrak{A}_g, \bar{p}))$  have the same type. If the block  $S_{P \times Q}$  is nonzero and C is a component of P (and thus of Q) such that  $P|_C \neq Q|_C$ , then  $C \in \mathsf{A}^{f,g,\bar{p}}(S,P)$ .

*Proof.* Let  $\bar{u} \in P$  and  $\bar{v} \in Q$  such that  $S(\bar{u}, \bar{v}) = 1$ . Such an entry must exist because  $S_{P \times Q}$  is nonzero. If  $\bar{u}_C = \bar{v}_C$ , then  $P|_C = Q|_C$  by Lemma 19 and Corollary 13, which contradicts our assumption. So  $\bar{u}_C \neq \bar{v}_C$  and  $C \in \mathsf{A}^{f,g,\bar{p}}(S,P)$ .

The next lemma shows that, as long as f and g agree on the edges in  $\text{orig}(\bar{p})$ , the actual values f and g assign to edges e are not important but only the difference f(e) - g(e) matters.

**Lemma 45.** Let  $f, g: E \to \mathbb{Z}_{2^q}$  not twist  $\operatorname{orig}(\bar{p})$  and S be an  $A^k \times A^k$  matrix over  $\mathbb{F}_2$ . Furthermore, let  $f', g': E \to \mathbb{Z}_{2^q}$  such that f'(e) = f(e) and g'(e) = g(e) for every  $e \in E$  with  $e \cap \operatorname{orig}(\bar{p}) \neq \emptyset$  and f'(e) - f(e) = g'(e) - g(e) for every other  $e \in E$ . Then  $A^{f,g,\bar{p}}(S) = A^{f',g',\bar{p}}(S)$  and if S is orbit-diagonal (respectively orbit-invariant) over  $(\mathfrak{A}_f,\bar{p})$  and  $(\mathfrak{A}_g,\bar{p})$ , then S is orbit-diagonal (respectively orbit-invariant) over  $(\mathfrak{A}_{f'},\bar{p})$  and  $(\mathfrak{A}_{g'},\bar{p})$ .

Proof. Note that if  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$  has the same type in  $(\mathfrak{A}_f, \bar{p})$  as  $Q \in \operatorname{orbs}_k((\mathfrak{A}_g, \bar{p}))$  has in  $(\mathfrak{A}_g, \bar{p})$ , then  $P \in \operatorname{orbs}_k((\mathfrak{A}_{f'}, \bar{p}))$  and  $Q \in \operatorname{orbs}_k((\mathfrak{A}_{g'}, \bar{p}))$  (Corollary 15) and P has the same type in  $(\mathfrak{A}_{f'}, \bar{p})$  as Q has in  $(\mathfrak{A}_{g'}, \bar{p})$ . So we only change the type of the

orbits, but not the correspondence between orbits of the same type. That is, if S is orbit-diagonal (respectively orbit-invariant) over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ , then S is orbit-diagonal (respectively orbit-invariant) over  $(\mathfrak{A}_{f'}, \bar{p})$  and  $(\mathfrak{A}_{g'}, \bar{p})$ . Furthermore, S is active on the same components with respect to  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$  as S is with respect to  $(\mathfrak{A}_{f'}, \bar{p})$  and  $(\mathfrak{A}_{g'}, \bar{p})$ . This implies that  $\mathsf{A}^{f,g,\bar{p}}(S) = \mathsf{A}^{f',g',\bar{p}}(S)$ .

We now show that the active region of products  $S \cdot T$  is bounded by the active regions of S and T. For two k-tuples  $\bar{u}, \bar{v} \in A^k$  we use the Kronecker delta  $\delta_{\bar{u},\bar{v}}$ , which is 1 if and only if  $\bar{u} = \bar{v}$  and 0 otherwise.

**Lemma 46.** Let  $f, g, h: E \to \mathbb{Z}_{2^q}$  pairwise not twist  $\operatorname{orig}(\bar{p})$  and S, T be  $A^k \times A^k$  matrices over  $\mathbb{F}_2$ . If S is orbit-diagonal over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$  and T is orbit-diagonal over  $(\mathfrak{A}_g, \bar{p})$  and  $(\mathfrak{A}_h, \bar{p})$ , then  $A^{f,h,\bar{p}}(S \cdot T) \subseteq A^{f,g,\bar{p}}(S) \cup A^{g,h,\bar{p}}(T)$ .

*Proof.* In this proof we omit the superscripts f, g, h, and  $\bar{p}$  for readability: for S we always refer to f and g, for T to g and h, and for  $S \cdot T$  to f and h. We show that  $\mathsf{A}(S) \cup \mathsf{A}(T)$  satisfies Conditions 1 and 2. Because the active region is the inclusion-wise minimal set satisfying the two conditions, it then follows that  $\mathsf{A}(S \cdot T) \subseteq \mathsf{A}(S) \cup \mathsf{A}(T)$ . Let  $\mathbf{P}_k = \mathsf{orbs}_k((\mathfrak{A}_f, \bar{p}))$ ,  $\mathbf{Q}_k = \mathsf{orbs}_k((\mathfrak{A}_g, \bar{p}))$ , and  $\mathbf{R}_k = \mathsf{orbs}_k((\mathfrak{A}_h, \bar{p}))$ .

We show Condition 1 by contraposition. Let  $P \in \mathbf{P}_k$  and C be a connected component of  $G[\mathsf{orig}(P)]$ . We show that if  $C \notin \mathsf{A}(S,P) \cup \mathsf{A}(T,P)$ , then  $C \notin \mathsf{A}(S \cdot T,P)$ . Let  $C \notin \mathsf{A}(S,P) \cup \mathsf{A}(T,P)$ ,  $Q \in \mathbf{Q}_k$  and  $R \in \mathbf{R}_k$  be of the same type as  $P, \bar{u} \in P$ , and  $\bar{w} \in R$ . Because S and T are orbit-diagonal,

$$(S \cdot T)(\bar{u}, \bar{w}) = \sum_{\bar{v} \in Q} S(\bar{u}, \bar{v}) \cdot T(\bar{v}, \bar{w}).$$

If  $S(\bar{u}, \bar{v}) = 1$  (i.e.,  $S(\bar{u}, \bar{v}) \neq 0$ ), then  $\bar{u}_C = \bar{v}_C$  because  $C \notin A(S, P)$ . Similarly,  $\bar{v}_C = \bar{w}_C$  if  $T(\bar{v}, \bar{w}) = 1$ . This implies  $\bar{u}_C = \bar{w}_C$  if  $(S \cdot T)(\bar{u}, \bar{w}) = 1$  (so there is at least one nonzero summand). Hence,  $C \notin A(S \cdot T, P)$ .

To show Condition 2, let  $P, P' \in \mathbf{P}_k$  and  $R, R' \in \mathbf{R}_k$  be arbitrary k-orbits, such that

$$A := C_{A(S) \cup A(T)}(P) = C_{A(S) \cup A(T)}(P') = C_{A(S) \cup A(T)}(R) = C_{A(S) \cup A(T)}(R'),$$

the orbits P and R have the same type, and P' and R' have the same type. Let N be the set of remaining components of P (and so of R) apart from A. Similarly, let N' be the set of remaining components of P' (and so of R') apart from A. Let  $\bar{u} \in P$ ,  $\bar{u}' \in P'$ ,  $\bar{w} \in R$ , and  $\bar{w}' \in R'$ , such that  $\bar{u}_A = \bar{u}'_A$ ,  $\bar{w}_A = \bar{w}'_A$ ,  $\bar{u}_N = \bar{w}_N$ , and  $\bar{u}'_{N'} = \bar{w}'_{N'}$ . We have to show that  $(S \cdot T)(\bar{u}, \bar{w}) = (S \cdot T)(\bar{u}', \bar{w}')$ .

By assumption,  $\bar{u}_A \in P|_A$ ,  $\bar{u}'_A \in P'|_A$ , and  $\bar{u}_A = \bar{u}'_A$ . So  $P|_A = P'|_A$  by Corollary 13 because they have the same type and contain the same tuple. Let  $Q \in \mathbf{Q}_k$  be of the same type as P and  $Q' \in \mathbf{Q}_k$  be of the same type as P'. Then  $Q|_A = Q'|_A$  and A and N (respectively N') are sets of components of Q (respectively Q'). We first assume that the blocks  $S_{P\times Q}$  and  $T_{Q\times R}$  are nonzero. We apply Lemma 19:  $Q = Q|_A \times Q|_N$ ,  $Q' = Q'|_A \times Q'|_{N'}$ , and likewise for P and P'.

$$(S \cdot T)(\bar{u}, \bar{w}) = \sum_{\bar{v} \in Q} S(\bar{u}, \bar{v}) \cdot T(\bar{v}, \bar{w})$$

$$= \sum_{\bar{v}_{\mathsf{A}} \in Q|_{\mathsf{A}}} \sum_{\bar{v}_{\mathsf{N}} \in Q|_{\mathsf{N}}} S(\bar{u}_{\mathsf{A}}\bar{u}_{\mathsf{N}}, \bar{v}_{\mathsf{A}}\bar{v}_{\mathsf{N}}) \cdot T(\bar{v}_{\mathsf{A}}\bar{v}_{\mathsf{N}}, \bar{w}_{\mathsf{A}}\bar{w}_{\mathsf{N}}). \tag{*}$$

From Lemma 44 it follows that  $P|_{\mathsf{N}} = Q|_{\mathsf{N}} = R|_{\mathsf{N}}$  (recall we assumed that the blocks  $S_{P\times Q}$  and  $T_{Q\times R}$  are nonzero), in particular  $\bar{u}_{\mathsf{N}}, \bar{w}_{\mathsf{N}} \in Q|_{\mathsf{N}}$ . We use again that the N-components are not in the active region of S and T. We continue the equation  $(\star)$ :

$$\begin{split} (\star) &= \sum_{\bar{v}_{\mathsf{A}} \in Q|_{\mathsf{A}}} \sum_{\bar{v}_{\mathsf{N}} \in Q|_{\mathsf{N}}} \delta_{\bar{u}_{\mathsf{N}},\bar{v}_{\mathsf{N}}} \cdot S(\bar{u}_{\mathsf{A}}\bar{u}_{\mathsf{N}},\bar{v}_{\mathsf{A}}\bar{u}_{\mathsf{N}}) \cdot \delta_{\bar{v}_{\mathsf{N}},\bar{w}_{\mathsf{N}}} \cdot T(\bar{v}_{\mathsf{A}}\bar{w}_{\mathsf{N}},\bar{w}_{\mathsf{A}}\bar{w}_{\mathsf{N}}) \\ &= \sum_{\bar{v}_{\mathsf{A}} \in Q|_{\mathsf{A}}} \delta_{\bar{u}_{\mathsf{N}},\bar{w}_{\mathsf{N}}} \cdot S(\bar{u}_{\mathsf{A}}\bar{u}_{\mathsf{N}},\bar{v}_{\mathsf{A}}\bar{u}_{\mathsf{N}}) \cdot T(\bar{v}_{\mathsf{A}}\bar{w}_{\mathsf{N}},\bar{w}_{\mathsf{A}}\bar{w}_{\mathsf{N}}) \\ &= \sum_{\bar{v}_{\mathsf{A}} \in Q'|_{\mathsf{A}}} \delta_{\bar{u}'_{\mathsf{N}'},\bar{w}'_{\mathsf{N}'}} \cdot S(\bar{u}_{\mathsf{A}}\bar{u}'_{\mathsf{N}'},\bar{v}_{\mathsf{A}}\bar{u}'_{\mathsf{N}'}) \cdot T(\bar{v}_{\mathsf{A}}\bar{w}'_{\mathsf{N}'},\bar{w}_{\mathsf{A}}\bar{w}'_{\mathsf{N}'}) \\ &= \sum_{\bar{v}'_{\mathsf{A}} \in Q'|_{\mathsf{A}}} \sum_{\bar{v}'_{\mathsf{N}'} \in Q'|_{\mathsf{N}'}} \delta_{\bar{u}'_{\mathsf{N}'},\bar{v}'_{\mathsf{N}'}} \cdot S(\bar{u}_{\mathsf{A}}\bar{u}'_{\mathsf{N}'},\bar{v}'_{\mathsf{A}}\bar{u}'_{\mathsf{N}'}) \cdot \delta_{\bar{v}'_{\mathsf{N}'},\bar{w}'_{\mathsf{N}'}} \cdot T(\bar{v}'_{\mathsf{A}}\bar{w}'_{\mathsf{N}'},\bar{w}_{\mathsf{A}}\bar{w}'_{\mathsf{N}'}). \end{split}$$

We used, as already seen,  $Q|_{A} = Q'|_{A}$ . We also used  $A(S \cdot T, P) \subseteq A(S, P) \cup A(T, P)$  as shown for Condition 1. So  $\bar{u}_{N}$  can be exchanged with  $\bar{u}'_{N'}$  and  $\bar{w}_{N}$  with  $\bar{w}'_{N'}$ . In the next step we use that  $\bar{u}_{A} = \bar{u}'_{A}$  and  $\bar{w}_{A} = \bar{w}'_{A}$  (by assumption) and again that S and T are not active on the N'-components.

$$\begin{split} (\star) &= \sum_{\bar{v}'_{\mathsf{A}} \in Q'|_{\mathsf{A}}} \sum_{\bar{v}'_{\mathsf{N}} \in Q'|_{\mathsf{N}'}} S(\bar{u}'_{\mathsf{A}}\bar{u}'_{\mathsf{N}'}, \bar{v}'_{\mathsf{A}}\bar{v}'_{\mathsf{N}'}) \cdot T(\bar{v}'_{\mathsf{A}}\bar{v}'_{\mathsf{N}'}, \bar{w}'_{\mathsf{A}}\bar{w}'_{\mathsf{N}'}) \\ &= \sum_{\bar{v}' \in Q'} S(\bar{u}', \bar{v}') \cdot T(\bar{v}', \bar{w}') \\ &= (S \cdot T)(\bar{u}', \bar{w}'). \end{split}$$

If  $S_{P\times Q}$  or  $T_{Q\times R}$  is zero, then  $S_{P'\times Q'}$  or  $T_{Q'\times R'}$  is zero because  $S(\bar{u},\bar{v})=S(\bar{u}',\bar{v}')=0$  and likewise for T. The claim follows because  $(S\cdot T)_{P\times Q}=0$  and  $(S\cdot T)_{P'\times Q'}=0$ .

We now consider products  $S \cdot T$  in the case that the active regions of S and T are disjoint. Intuitively, our goal is to prove that then  $S \cdot T$  is given by S on the active region of S and by T on the active region of T.

**Lemma 47.** Let  $f, g, h : E \to \mathbb{Z}_{2^q}$  pairwise not twist  $\operatorname{orig}(\bar{p})$  and S, T be  $A^k \times A^k$  matrices over  $\mathbb{F}_2$ . Let S be orbit-diagonal over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ , T be orbit-diagonal over  $(\mathfrak{A}_g, \bar{p})$  and  $(\mathfrak{A}_h, \bar{p})$ , both be odd-filled,  $A^{f,g,\bar{p}}(S) \cap A^{g,h,\bar{p}}(T) = \emptyset$ ,  $P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$ ,  $Q \in \operatorname{orbs}_k((\mathfrak{A}_g, \bar{p}))$ , and  $R \in \operatorname{orbs}_k((\mathfrak{A}_h, \bar{p}))$  be of the same type, and the components of P (and thus the components of Q and Q) be partitioned into Q and Q0 such that  $C_{A^{g,h,\bar{p}}(S)}(P) \subseteq M$  and  $C_{A^{g,h,\bar{p}}(T)}(Q) \subseteq N$ .

(a) For every  $\bar{u} \in P$  and  $\bar{w} \in R$  it holds that

$$(S \cdot T)(\bar{u}, \bar{w}) = S(\bar{u}_M \bar{u}_N, \bar{w}_M \bar{u}_N) \cdot T(\bar{w}_M \bar{u}_N, \bar{w}_M \bar{w}_N).$$

(b) If S is orbit-invariant over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ , then for every  $\bar{u} \in P$  and  $\bar{w} \in R$  it holds that

$$\sum_{\bar{u}_M' \in P|_M} (S \cdot T)(\bar{u}_M' \bar{u}_N, \bar{w}_M \bar{w}_N) = T(\bar{w}_M \bar{u}_N, \bar{w}_M \bar{w}_N).$$

*Proof.* We first show Part (a). Let  $\bar{u} \in P$  and  $\bar{w} \in R$ .

$$(S \cdot T)(\bar{u}, \bar{w}) = \sum_{\bar{v} \in Q} S(\bar{u}_M \bar{u}_N, \bar{v}_M \bar{v}_N) \cdot T(\bar{v}_M \bar{v}_N, \bar{w}_M \bar{w}_N)$$

$$= \sum_{\bar{v} \in Q} \delta_{\bar{u}_N, \bar{v}_N} \cdot S(\bar{u}_M \bar{u}_N, \bar{v}_M \bar{u}_N) \cdot \delta_{\bar{v}_M, \bar{w}_M} \cdot T(\bar{w}_M \bar{v}_N, \bar{w}_M \bar{w}_N). \tag{*}$$

The last step uses that components of  $\bar{u}_N$  and  $\bar{w}_N$  consist only of components not contained in A(S) and likewise for  $\bar{v}_M$  and  $\bar{w}_M$ .

$$(\star) = \sum_{\substack{\bar{v} \in Q, \\ \bar{v} = \bar{w}_M \bar{u}_N}} S(\bar{u}_M \bar{u}_N, \bar{w}_M \bar{u}_N) \cdot T(\bar{w}_M \bar{u}_N, \bar{w}_M \bar{w}_N)$$
$$= S(\bar{u}_M \bar{u}_N, \bar{w}_M \bar{u}_N) \cdot T(\bar{w}_M \bar{u}_N, \bar{w}_M \bar{w}_N).$$

For the last step, we have to argue that  $\bar{w}_M \bar{u}_N \in Q$ . From Lemma 19 it follows that  $P = P|_M \times P|_N$ ,  $Q = Q|_M \times Q|_N$ , and  $R = R|_M \times R|_N$ . Because S is not active on the components in N and T is not active on the components in M, it follows from Lemma 44 that  $P|_N = Q|_N$  and that  $Q|_M = R|_M$  (the corresponding blocks of S and T are nonzero because S and T are odd-filled). Hence,  $\bar{w}_M \bar{u}_N \in Q$  because  $\bar{w}_M \in R|_M$  and  $\bar{u}_N \in P|_N$ .

We now show Part (b). We apply Part (a):

$$\begin{split} & \sum_{\bar{u}_{M}' \in P|_{M}} (S \cdot T)(\bar{u}_{M}' \bar{u}_{N}, \bar{w}_{M} \bar{w}_{N}) \\ &= \sum_{\bar{u}_{M}' \in P|_{M}} S(\bar{u}_{M}' \bar{u}_{N}, \bar{w}_{M} \bar{u}_{N}) \cdot T(\bar{w}_{M} \bar{u}_{N}, \bar{w}_{M} \bar{w}_{N}) \\ &= T(\bar{w}_{M} \bar{u}_{N}, \bar{w}_{M} \bar{w}_{N}) \cdot \sum_{\bar{u}_{M}' \in P|_{M}} S(\bar{u}_{M}' \bar{u}_{N}, \bar{w}_{M} \bar{u}_{N}). \end{split}$$

It suffices to show that the value of the sum is 1. We rewrite the sum using  $P = P|_M \times P|_N$  (Lemma 19):

$$\sum_{\bar{u}_{M}' \in P|_{M}} S(\bar{u}_{M}' \bar{u}_{N}, \bar{w}_{M} \bar{u}_{N}) = \sum_{\bar{u}_{M}' \bar{u}_{N}' \in P} S(\bar{u}_{M}' \bar{u}_{N}', \bar{w}_{M} \bar{u}_{N}) - \sum_{\substack{\bar{u}_{M}' \bar{u}_{N}' \in P, \\ \bar{u}_{N}' \neq \bar{u}_{N}}} S(\bar{u}_{M}' \bar{u}_{N}', \bar{w}_{M} \bar{u}_{N}).$$

In the right sum it always holds that  $S(\bar{u}'_M \bar{u}'_N, \bar{w}_M \bar{u}_N) = 0$  because  $\bar{u}'_N \neq \bar{u}_N$  and N is not in the active region of S. So the right sum is zero. Finally, the left sum  $\sum_{\bar{u}'_M \bar{u}'_N \in P} S(\bar{u}'_M \bar{u}'_N, \bar{w}_M \bar{u}_N)$  sums over a column of S because S is orbit-diagonal. Because S is orbit-invariant and odd-filled, this sum is 1 by Lemma 31.

Finally, we show the result of Lemma 47(b) for a product of three matrices  $S_1 \cdot S_2 \cdot S_3$ .

**Lemma 48.** Let  $g_i : E \to \mathbb{Z}_{2^q}$  pairwise not twist  $\operatorname{orig}(\bar{p})$  for every  $i \in [4]$ . Let  $S_i$  be an  $A^k \times A^k$  matrix over  $\mathbb{F}_2$  that is odd-filled and both orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_{g_i}, \bar{p})$  and  $(\mathfrak{A}_{g_{i+1}}, \bar{p})$  for every  $i \in [3]$ . If the active regions  $A^{g_i, g_{i+1}, \bar{p}}(S_i)$  are pairwise disjoint, then for all k-orbits  $P_i \in \operatorname{orbs}_k((\mathfrak{A}_{g_i}, \bar{p}))$  of the same type for all  $i \in [4]$ , every partition of the components of the  $P_i$  into  $M_1$ ,  $M_2$ , and  $M_3$  such that  $C_{A^{g_i, g_{i+1}, \bar{p}}(S_i)}(P_i) \subseteq M_i$  for every  $i \in [3]$ , and every  $\bar{u} \in P_1$  and  $\bar{w} \in P_4$  it holds that

$$\sum_{\bar{u}'_{M_2} \in P_1 \mid_{M_2}} (S_1 \cdot S_2 \cdot S_3)(\bar{u}_{M_1} \bar{u}'_{M_2} \bar{u}_{M_3}, \bar{w}_{M_1} \bar{w}_{M_2} \bar{w}_{M_3}) = (S_1 \cdot S_3)(\bar{u}_{M_1} \bar{w}_{M_2} \bar{u}_{M_3}, \bar{w}_{M_1} \bar{w}_{M_2} \bar{w}_{M_3}).$$

*Proof.* By Lemma 28, the matrix  $(S_1 \cdot S_2)$  is orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_{g_1}, \bar{p})$  and  $(\mathfrak{A}_{g_3}, \bar{p})$  and the matrix  $(S_2 \cdot S_3)$  is orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_{g_2}, \bar{p})$  and  $(\mathfrak{A}_{g_4}, \bar{p})$ . Both matrices are odd-filled by Lemma 30. We apply Lemma 47(a) for the partition of the components of P into  $M_1 \cup M_3$  and  $M_2$ :

$$\sum_{\bar{u}'_{M_2} \in P_1|_{M_2}} (S_1 \cdot S_2 \cdot S_3) (\bar{u}_{M_1} \bar{u}'_{M_2} \bar{u}_{M_3}, \bar{w}_{M_1} \bar{w}_{M_2} \bar{w}_{M_3})$$

$$= \sum_{\bar{u}'_{M_2} \in P_1|_{M_2}} S_1 (\bar{u}_{M_1} \bar{u}'_{M_2} \bar{u}_{M_3}, \bar{w}_{M_1} \bar{u}'_{M_2} \bar{u}_{M_3}) \cdot (S_2 \cdot S_3) (\bar{w}_{M_1} \bar{u}'_{M_2} \bar{u}_{M_3}, \bar{w}_{M_1} \bar{w}_{M_2} \bar{w}_{M_3})$$

$$= \sum_{\bar{u}'_{M_2} \in P_1|_{M_2}} S_1 (\bar{u}_{M_1} \bar{w}_{M_2} \bar{u}_{M_3}, \bar{w}_{M_1} \bar{w}_{M_2} \bar{u}_{M_3}) \cdot (S_2 \cdot S_3) (\bar{w}_{M_1} \bar{u}'_{M_2} \bar{u}_{M_3}, \bar{w}_{M_1} \bar{w}_{M_2} \bar{w}_{M_3}). \quad (\star)$$

The last step uses that  $M_2$  consists only of components not contained in  $A^{g_1,g_2,\bar{p}}(S_1)$ . We continue the equation by moving  $S_1$  out of the sum and applying Lemma 47(b) for the partition of the components of P into  $M_1 \cup M_3$  and  $M_2$ :

$$(\star) = S_{1}(\bar{u}_{M_{1}}\bar{w}_{M_{2}}\bar{u}_{M_{3}}, \bar{w}_{M_{1}}\bar{w}_{M_{2}}\bar{u}_{M_{3}}) \cdot \sum_{\bar{u}'_{M_{2}} \in P|_{M_{2}}} (S_{2} \cdot S_{3})(\bar{w}_{M_{1}}\bar{u}'_{M_{2}}\bar{u}_{M_{3}}, \bar{w}_{M_{1}}\bar{w}_{M_{2}}\bar{w}_{M_{3}})$$

$$= S_{1}(\bar{u}_{M_{1}}\bar{w}_{M_{2}}\bar{u}_{M_{3}}, \bar{w}_{M_{1}}\bar{w}_{M_{2}}\bar{u}_{M_{3}}) \cdot S_{3}(\bar{w}_{M_{1}}\bar{w}_{M_{2}}\bar{u}_{M_{3}}, \bar{w}_{M_{1}}\bar{w}_{M_{2}}\bar{w}_{M_{3}})$$

$$= (S_{1} \cdot S_{3})(\bar{u}_{M_{1}}\bar{w}_{M_{2}}\bar{u}_{M_{3}}, \bar{w}_{M_{1}}\bar{w}_{M_{2}}\bar{w}_{M_{3}}).$$

The last step follows from applying Lemmas 45 and 47(a) in the reverse direction by partitioning the components into  $M_1$  and  $M_2 \cup M_3$ .

### 9 The Arity k Case

We now construct a similarity matrix for the k-ary invertible-map game. Constructing this matrix and verifying its suitability will be quite technical and intricate. We first discuss the difficulties we have to overcome and why the approach for arity 1 cannot be generalized to arity k easily. In the following, we provide high-level intuition for constructing the similarity matrix for arity k. This prepares us for the lengthy formal definition of this matrix, which follows subsequently.

#### 9.1 Overview of the Construction

Orbits of the Same Type. Let  $\mathfrak{A}_f$  and  $\mathfrak{A}_g$  be two CFI structures, such that a single edge  $\{t,t'\}$  of the base graph G is twisted by f and g. Let  $\bar{p}$  be parameters, whose origin has sufficiently large distance to the twisted edge. We have seen in Section 7 that every 1-orbit has the same type in  $(\mathfrak{A}_f,\bar{p})$  as it has in  $(\mathfrak{A}_g,\bar{p})$ . For a k-orbit P, this is not the case whenever  $\{t,t'\}\subseteq \operatorname{orig}(P)$ . Ultimately, our goal is to construct an orbit-invariant, orbit-diagonal, and odd-filled similarity matrix S that k-blurs the twist. Because the blocks on the diagonal of S arise from orbits of the same type and because the characteristic matrices of orbits of the same type have to be simultaneously similar, we first want to define a bijection  $\operatorname{orbs}_{k'}((\mathfrak{A}_f,\bar{p})) \to \operatorname{orbs}_{k'}((\mathfrak{A}_g,\bar{p}))$  for every  $k' \leq 2k$  that preserves the orbit types. For this, we want to construct a function  $\tau \colon A^{\leq 2k} \to A^{\leq 2k}$  that preserves the type of tuples. Then  $\tau$  preserves orbit types, too. To do so, we pick

a vertex z satisfying  $\operatorname{dist}_G(t,z) > 2k$  and a path  $(z,\ldots,t',t)$ . We consider the path-isomorphism  $\varphi_{\tau}$  that twists the edge  $\{t,t'\}$  and the edge incident to z in the chosen path. That is, between  $\varphi_{\tau}(\mathfrak{A}_f)$  and  $\mathfrak{A}_g$  an edge incident to z is twisted but the edge  $\{t,t'\}$  is not. For the moment assume that we only consider connected tuples and thus only connected orbits. Let  $\tau$  be the function that applies the path-isomorphism  $\varphi_{\tau}$  to every tuple  $\bar{u}$  with  $\{t,t'\}\subseteq\operatorname{orig}(\bar{u})$  and is the identity function on all others. Let  $\bar{u}\subseteq A^{\leq 2k}$  be such a tuple with  $\{t,t'\}\subseteq\operatorname{orig}(\bar{u})$ . Because  $\operatorname{dist}_G(t,z)>2k$  and because we consider connected tuples, we have that  $z\notin\operatorname{orig}(\bar{u})$ . Hence,  $\mathfrak{A}_g[\operatorname{orig}(\bar{u})]=\varphi_{\tau}(\mathfrak{A}_f)[\operatorname{orig}(\bar{u})]$  and  $\bar{u}$  has the same type in  $(\mathfrak{A}_f,\bar{p})$  as  $\tau(\bar{u})$  has in  $(\mathfrak{A}_g,\bar{p})$ . Consequently, for every  $k'\leq 2k$  and  $P\in\operatorname{orbs}_{k'}((\mathfrak{A}_f,\bar{p}))$  it holds that  $\tau(P)\in\operatorname{orbs}_{k'}((\mathfrak{A}_g,\bar{p}))$  and  $\tau(P)$  has the same type in  $(\mathfrak{A}_g,\bar{p})$  as P has in  $(\mathfrak{A}_f,\bar{p})$ .

Generalized Blurrers. Next we transfer the concept of a blurrer to the k-ary case. Definition 34 of a (q, d)-blurrer requires that there seems to be a twist at index 1 but none at the others indices when considering only one of the d entries of the blurrer elements. Although, all tuples  $\xi$  in a blurrer satisfy  $\sum \xi = 0$ . We require the same property in the k-ary case, but now not only consider one index at a time but sets of k many indices. We will generalize (q, d)-blurrers to (k, q, a, d)-blurrers, where k is the arity, q specifies the ring  $\mathbb{Z}_{2^q}$ , d the length of the tuples in the blurrer, and  $a \in \mathbb{Z}_{2^q}$  the value of the twist (which was fixed to  $2^{q-1}$  before). Showing the existence of such blurrers will be more difficult, in particular we will have to use, for a given k, the ring  $\mathbb{Z}_{2^q}$  for a sufficiently large q = q(k).

In the 1-ary case, we identified a tuple  $\xi \in \Xi$  with a local automorphism of the gadget of z. We now describe the approach in the k-ary case. Assume we are given a generalized (k, q, a, d)-blurrer  $\Xi$  for arity k for some suitable  $q, a \in \mathbb{Z}_{2^q}$ , and d. We now require that the base graph G is regular of degree d. Recall that in Section 7 we blurred the twist between the edges incident to z, of which one was the twisted edge: We used multiple local automorphisms (one for each  $\xi \in \Xi$ ) to distribute the twist among these edges. When considering connected 2k-tuples, we want to ensure that the origin of every 2k-tuple contains at most one of the edges between which we blur the twist. So it is not possible to blur the twist between the incident edges of a single vertex. Instead, we will choose vertices  $t_1, \ldots, t_d$  and  $t'_1, \ldots, t'_d$ , such that  $t = t_1, t' = t'_1$ , and such that there are simple paths  $\bar{s}_i = (z, \dots, t'_i, t_i)$  of length at least 2k forming a star, i.e., the paths  $\bar{s}_i$ are disjoint apart from z (cf. Figure 2). Here it will be important to choose  $\bar{s}_1$  to be the path we used to define the tuple-type-preserving map  $\tau$  in the previous paragraph. We will ensure that such paths exist by requiring that the girth of G is large enough. We will blur the twist between the edges  $\{t_i, t_i'\}$ . In the 1-ary case, an element in a blurrer corresponded to an automorphism of the gadget of z, or equivalently to a starisomorphism, where the paths of the star have length 1. In the k-ary case we will identify a  $\xi \in \Xi$  with the star-isomorphism  $\varphi_{\xi} := \pi^*[\xi, \bar{s}_1, \dots, \bar{s}_d]$ . Again to preserve the type of tuples, we will only apply  $\xi$  to tuples  $\bar{u}$  satisfying  $\{t_i, t_i'\} \not\subseteq \text{orig}(\bar{u})$  for all  $i \in [d]$ . That is, on such a  $\bar{u}$ , the action of  $\xi$  could also be defined by an automorphism. This turns  $\xi$  into a "star-automorphism". Using a star in combination with the large girth ensures that the tips of the star, the edges  $\{t_i, t_i'\}$ , are sufficiently far apart. If we only had to deal with connected tuples, this approach would be sufficient to construct a similarity matrix (and in particular, we could even use easier blurrers). However, disconnected tuples complicate

matters.

Disconnected Tuples and Orbits. We have to consider disconnected tuples and orbits. While with connected tuples the approach just described is local (we only considered the 2k-neighborhood of z), there are disconnected tuples containing vertices scattered in the structure. But these vertices belong to different components of the tuple (cf. Definition 18). Lemma 19 tells us that the components of disconnected orbits are independent whenever the connectivity of G is sufficiently large. In a first step, we will salvage the previous approach by applying the path- and the star-isomorphism not to entire tuples, but instead to components of tuples. That is, if a component C contains the twisted edge  $\{t, t'\} = \{t_1, t'_1\}$ , then we apply the type-preserving map  $\tau$  to this component. If  $\{t_i, t'_i\} \not\subseteq C$  for all  $i \in [d]$ , i.e, C contains none of the edges between which we blur the twist, we apply the star-automorphisms  $\xi$  to this component. (Note that  $\xi$  is the identity map unless C intersects non-trivially with at least one path  $\bar{s}_i$ .)

This approach fails, when for a 2k-orbit P the two k-orbits  $P_1 := P|_{\{1,\dots,k\}}$  and  $P_2 := P|_{\{k+1,\dots,2k\}}$  contain the center z of the star and some of the edges  $\{t_i,t_i'\}$  in their origin. Because the edges  $\{t_i,t_i'\}$  are contained in the origin, we need to argue with the blurrer properties to show that we blur the twist. This is only possible if for two k-tuples  $\bar{u} \in P_1$  and  $\bar{v} \in P_2$  it only depends on up to k indices of a  $\xi \in \Xi$  whether  $\xi(\bar{u})\bar{v}$  is in the same orbit as  $\bar{u}\bar{v}$ . But because the center z is in the origin, this actually depends on all d entries of  $\xi$  and the blurrer properties do not apply. This is why we will have to distinguish two kinds of k-orbits. We call a k-orbit P blurrable, if  $z \notin \text{orig}(P)$ . For non-blurrable orbits, we need another technique as follows.

Recursive Blurring. Now consider a 2k-orbit P, such that both  $P_1 := P|_{\{1,\dots,k\}}$  and  $P_2 := P|_{\{k+1,\dots,2k\}}$  are non-blurrable k-orbits. Let us quickly recall the 1-ary case. It was possible to blur the twist in Lemma 39 because we summed over the tuples  $\xi(\bar{u})\bar{v}$  for all  $\xi \in \Xi$ . For a 2-orbit P, whose origin was the twisted edge  $\{z,t\}$ , w.l.o.g. the origin of  $P_2$  is z and for every  $v \in P$  it held that  $\xi(v) \neq \xi'(v)$  for every  $\xi \neq \xi'$  in the blurrer. But for  $P_1$  only one index of the blurrer was relevant. That is, for every  $u \in P_1$ ,  $v \in P_2$ , and  $\xi, \xi' \in \Xi$  such that  $\xi(1) = \xi'(1)$  we had that  $\xi(u) = \xi'(u)$  and that  $\xi(u)v$  is in the same orbit as  $\xi'(u)v$ . So were able to apply the properties of a blurrer, i.e., when summing over  $\xi(u)v$  for all  $\xi \in \Xi$  and if only one index matters, then the twist vanishes. The 2-orbits for which both  $P_1$  and  $P_2$  have origin  $\{z\}$  did not cover the twisted edge and so did not pose a problem in the 1-ary case.

Now consider the k-ary case again. Here of course there are orbits P such that both  $P_1$  and  $P_2$  are non-blurrable and they contain the twisted edge and the center z in their origins. Let  $\bar{u} \in P_1$  and  $\bar{v} \in P_2$ . Both  $\bar{u}$  and  $\bar{v}$  contain a vertex with origin z and the blurrer properties do not apply because the orbit of  $\xi(\bar{u})\bar{v}$  is different for every  $\xi \in \Xi$  (fixing one vertex of origin z separates the gadget of z into singleton orbits).

That is, when summing over all  $\xi \in \Xi$ , we map every  $\bar{u}\bar{v}$  to the tuple  $\xi(\bar{u})\bar{v}$ , whose type in  $(\varphi_{\xi}(\mathfrak{A}_g), \bar{p})$  is the same as the type of  $\bar{u}\bar{v}$  in  $(\mathfrak{A}_f, \bar{p})$ . But in  $(\mathfrak{A}_g, \bar{p})$  the tuple  $\xi(\bar{u})\bar{v}$  has a different type. Between  $(\mathfrak{A}_g, \bar{p})$  and  $(\varphi_{\xi}(\mathfrak{A}_g), \bar{p})$  the edges  $\{t_i, t_i'\}$  are twisted additionally (the values of the twists depend on  $\xi$ ). This, in some sense, introduces other twists, but only for said 2k-orbits P, where the origins of both  $P_1$  and  $P_2$  are not blurrable.

The idea is to fix an arbitrary vertex  $p_z$  with origin z and consider (k-1)-orbits of  $(\mathfrak{A}, \bar{p}p_z)$  (justified by Corollary 21). This can be done because all non-blurrable orbits contain z in its origin. For every  $\xi \in \Xi$ , we will recursively obtain a matrix  $S^{\xi}$  that (k-1)-blurs the twist between  $(\mathfrak{A}_f, \bar{p}p_z)$  and  $(\varphi_{\xi}^{-1}(\mathfrak{A}_g), \bar{p}p_z)$ . We use the inverse  $\varphi_{\xi}^{-1}$  of the star-isomorphism  $\varphi_{\xi}$  because we want to revert the twists introduced by  $\varphi_{\xi}$ . Here the need arises to blur a twist of value  $a \neq 2^{q-1}$ . Combining the blurrer  $\Xi$  with the matrices  $S^{\xi}$  to a matrix S that k-blurs the twist will become formally tedious. In particular, we will need to ensure that the  $S^{\xi}$  act "independently" on the  $\{t_i, t_i'\}$ , which we discuss next.

Active Region and Blurrers. The matrix S is defined for blocks of k-orbits. Blocks for blurrable k-orbits will be defined using  $\Xi$  and the matrices  $S^{\xi}$ . With this approach we will show that S is a similarity matrix for all orbits P, for which either  $P_1$  and  $P_2$  are both blurrable or  $P_1$  and  $P_2$  are both non-blurrable. In the former case, we will use the blurrer property, in the latter case, we will use induction. The case that  $P_1$  is blurrable and  $P_2$  is not or vice versa remains. We have to show that  $\chi^P \cdot S = S \cdot \chi^Q$  (for  $Q = \tau(P)$ , which has the same type as P). Assume that  $P_1$  is blurrable and  $P_2$  is not. For  $S \cdot \chi^Q$  solely the block  $S_{P_1 \times Q_1}$  of S is relevant. This block is defined using the blurrer  $\Xi$  because  $P_1$  is blurrable. Similarly, for  $\chi^P \cdot S$  solely the block  $S_{P_2 \times Q_2}$  is relevant. This block is defined using the blurrer  $\Xi$  and the matrices  $S^{\xi}$  because  $P_2$  is non-blurrable.

To use the blurrer properties also for  $P_2$ , we will define the matrices  $S^{\xi}$ , which blur multiple twists at the edges  $\{t_i, t_i'\}$ , as  $S^{\xi} := S^{\xi,1} \cdot \ldots \cdot S^{\xi,d}$ , where each  $S^{\xi,i}$  only blurs a single twist at the edge  $\{t_i, t_i'\}$ . We will ensure that the active region of  $S^{\xi,i}$  is bound by the r(k)-neighborhood of  $t_i$  for some suitable r(k). We then enlarge the star such that the paths  $\bar{s}_i$  have length greater than  $\max\{2k, r(k)\}$ . Now, the active regions of the  $S^{\xi,i}$  are disjoint and we can use Lemma 48 to show that indeed all except k many of the  $S^{\xi,i}$  cancel out. So finally, we can use the blurrer properties to show that S is a similarity matrix for orbits where  $P_1$  is blurrable and  $P_2$  is not. We now start with generalizing blurrers and then show the existence of the required similarity matrix.

#### 9.2 Blurrer

When dealing with arity k, the properties of a blurrer must be generalized from a single index to sets of indices of size at most k. Let  $q, d \in \mathbb{N}$  and  $\Xi \subseteq \mathbb{Z}_{2^q}^d$ . For  $K \subseteq [d]$  and  $\bar{b} \in \mathbb{Z}_{2^q}^{|K|}$  we count the tuples contained in  $\Xi$  whose restriction to K equals  $\bar{b}$ . We define

$$\#_{K,\bar{b}}(\Xi) := \left| \left\{ \bar{c} \in \Xi \mid \bar{c}|_K = \bar{b} \right\} \right| \mod 2.$$

**Definition 49** (Blurrer). Let  $d \geq k$ ,  $\Xi \subseteq \mathbb{Z}_{2^q}^d$ , and  $a \in \mathbb{Z}_{2^q}$ . The set  $\Xi \subseteq \mathbb{Z}_{2^q}^d$  is called a (k, q, a, d)-blurrer if it satisfies the following for all  $K \subseteq [d]$  with |K| = k:

- 1.  $\sum \xi = 0$  for all  $\xi \in \Xi$ .
- 2. If  $1 \in K$ , then  $\#_{K,(a,0,\ldots,0)}(\Xi) = 1$ .
- 3. If  $1 \notin K$ , then  $\#_{K,\bar{0}}(\Xi) = 1$ .
- 4.  $\#_{K,\bar{b}}(\Xi) = 0$  for all other pairs of K and  $\bar{b}$ .

The crucial property of a blurrer is the following:

**Lemma 50.** Let  $\Xi$  be a (k, q, a, d)-blurrer,  $K \subseteq [d]$  such that |K| = k, and define  $\xi_{\mathsf{twst}} := (a, 0, \dots, 0) \in \mathbb{Z}_{2^q}^d$ . Every function  $f : \Xi|_K \to \mathbb{F}_2$  satisfies

$$\sum_{\xi \in \Xi} f(\xi|_K) = f(\xi_{\mathsf{twst}}|_K).$$

In particular, there is a  $\xi_{\mathsf{tw}} \in \Xi$  such that  $\xi_{\mathsf{tw}}|_K = \xi_{\mathsf{twst}}|_K$ .

*Proof.* Because f takes k-tuples as input, it cannot distinguish whether it is applied to  $\xi|_K$  or to  $\xi_{\mathsf{twst}}|_K$  because by Conditions 2, and 3  $\xi_{\mathsf{twst}}|_K \in \Xi|_K$ . Conditions 2, 3, and 4 ensure that when summing over all  $\xi \in \Xi$ , all summands  $f(\xi|_K)$  apart from  $f(\xi_{\mathsf{twst}}|_K)$  cancel out (by Condition 4 and 2 if  $1 \in K$  or Condition 3 if  $1 \notin K$ ). The existence of a  $\xi_{\mathsf{tw}} \in \Xi$  as required follows from Conditions 2 and 3.

Note that while  $\xi$  only contains tuples satisfying  $\Sigma \xi = 0$ , we have that  $\Sigma \xi_{\mathsf{twst}} = a$ .

**Lemma 51.** Let  $\Xi$  be a (k, q, a, d)-blurrer. Then  $|\Xi|$  is odd.

*Proof.* Let  $K \subseteq [d]$  with |K| = k. We partition  $\Xi = M \cup N$  into

$$M := \left\{ \xi \in \Xi \mid \xi|_K = \xi_{\mathsf{twst}}|_K \right\},$$

$$N := \left\{ \xi \in \Xi \mid \xi|_K \neq \xi_{\mathsf{twst}}|_K \right\},$$

where  $\xi_{\mathsf{twst}} := (a, 0, \dots, 0)$  is the tuple from Lemma 50. The size of |M| is odd by Condition 2 if  $1 \in K$  and otherwise by Condition 3. By Condition 4, the size |N| is even. If it was odd, then some  $\bar{b}$  would violate Condition 4.

We now construct blurrers.

**Lemma 52.** If there is a (k, q, a, d)-blurrer  $\Xi$ , then

- 1. there is a (k, q, a, d')-blurrer for every  $d' \geq d$ ,
- 2.  $\Xi$  is a (k', q, a, d)-blurrer for every  $k' \leq k$ , and
- 3. there is a  $(k, q, c \cdot a, d)$ -blurrer for every  $c \in \mathbb{Z}_{2^q}$ .

*Proof.* To prove the first statement, we just fill up the tuples of  $\Xi$  with zeros to be of length d'. To prove the second statement, let  $K' \subseteq K \subseteq [d]$  such that |K| = k and let  $\bar{b}' \in \mathbb{Z}_{2^q}^{|K'|}$ . Then

$$\#_{K', \bar{b'}}(\Xi) = \sum_{\substack{\bar{b} \in \mathbb{Z}_{2q}^k, \ \bar{b}|_{K'} = \bar{b'}}} \#_{K, \bar{b}}(\Xi).$$

Assume  $1 \in K'$  and  $\bar{b}' = (a, 0, ..., 0)$ . Then for  $\bar{b} = (a, 0, ..., 0) \in \mathbb{Z}_{2^q}^k$  we have  $\bar{b}|_{K'} = \bar{b}'$  and  $\#_{K,\bar{b}}(\Xi) = 1$  by Condition 2. For all other  $\bar{b}$  we have  $\#_{K,\bar{b}}(\Xi) = 0$  by Condition 4. So the sum is 1. The case that  $1 \notin K'$  and  $\bar{b} = \bar{0}$  is similar using Condition 3. In the remaining case all summands are 0 by blurrer Condition 4.

To prove the last statement, let  $c \in \mathbb{Z}_{2^q}$  and set  $\Xi' := \{c \cdot \xi \mid \xi \in \Xi\}$ . If  $\sum \xi = 0$ , then clearly  $\sum c \cdot \xi = 0$ . We verify blurrer Condition 2, the others are similar. Let  $K \subseteq [d]$  of size k and  $\bar{b} = (a, 0, \dots, 0) \in \mathbb{Z}_{2^q}^k$ . From Conditions 2 and 3 it follows that

$$\#_{K,c.\bar{b}}(\Xi') = \#_{K,\bar{b}}(\Xi) + \sum_{\substack{\bar{b}' \in \mathbb{Z}_{2^q}^k, \\ \bar{b}' \neq \bar{b}, \\ c.\bar{b}' = c.\bar{b}}} \#_{K,\bar{b}'}(\Xi) = 1 + \sum_{\substack{\bar{b}' \in \mathbb{Z}_{2^q}^k, \\ \bar{b}' \neq \bar{b}, \\ c.\bar{b}' = c.\bar{b}}} 0 = 1.$$

**Lemma 53.** Let  $m, n \in \mathbb{N}$ . If  $0 < m < 2^n$ , then  $\binom{2^n}{m}$  is even. If  $m \leq 2^n - 1$ , then  $\binom{2^n - 1}{m}$  is odd.

*Proof.* Let  $k \in \mathbb{N}$  and consider  $\binom{k}{m}$ . We write k and m in base 2 representation

$$m = \sum_{i=0}^{j} m_i 2^i k = \sum_{i=0}^{j} k_i 2^i$$

for some suitable j and  $m_i, k_i \in \{0, 1\}$  for all  $i \in \{0, ..., j\}$ . We apply Lucas's Theorem [7]:

$$\binom{k}{m} \bmod 2 = \prod_{i=0}^{j} \binom{k_i}{m_i} \bmod 2,$$

where  $\binom{0}{1} = 0$  and  $\binom{0}{0} = \binom{1}{0} = \binom{1}{1} = 1$ . That is,  $\binom{k}{m} \mod 2 = 0$  if and only if there is an  $i \in \{0, \ldots, j\}$  such that  $k_i = 0$  and  $m_i = 1$ .

- (a) Let  $k=2^n$  and 0 < m < k. Then there is an i < n such that  $m_i=1$ . Because  $k=2^n, \ k_i=0$  and so  $\binom{k}{m}$  is even.
- (b) Let  $k = 2^n 1$  and  $m \le k$ . For every i < n we have  $k_i = 1$ . For every i such that  $m_i = 1$  it holds that i < n. That is,  $\binom{k}{m}$  is odd.

**Lemma 54.** For every  $i \in \mathbb{N}$ , there is a  $(2^{i-1} - 1, i, 2^{i-1}, 2^i - 1)$ -blurrer.

*Proof.* We set  $k := 2^{i-1} - 1$ ,  $d := 2^i - 1$ , and define  $\Xi$  as follows:

$$\Xi_{1} := \left\{ \left( 2^{i-1}, a_{2}, \dots, a_{2^{i}-1} \right) \middle| \sum_{j=2}^{2^{i}-1} a_{j} = 2^{i-1}, a_{j} \in \{0, 1\} \text{ for every } j \right\},$$

$$\Xi_{2} := \left\{ \left( 2^{i-1} + 1, a_{2}, \dots, a_{2^{i}-1} \right) \middle| \sum_{j=2}^{2^{i}-1} a_{j} = 2^{i-1} - 1, a_{j} \in \{0, 1\} \text{ for every } j \right\},$$

$$\Xi_{1} := \Xi_{1} \cup \Xi_{2}.$$

To verify that  $\Xi$  is indeed a  $(k, i, 2^{i-1}, d)$ -blurrer, let  $K \subseteq [d]$  be of size k and  $\bar{b} \in \mathbb{Z}_{2^i}^k$ . Set  $\overline{K} := [d] \setminus K$ .

• Let  $1 \in K$  and  $\bar{b} = (2^{i-1}, 0, \dots, 0)$ . Every  $\xi \in \Xi$  with  $\xi|_K = \bar{b}$  is contained in  $\Xi_1$ . Because every  $\xi \in \Xi_1$  contains  $2^{i-1}$  many 1-entries,  $\xi$  is of length  $d = 2^i - 1$ , and  $\bar{b}$  contains  $k - 1 = 2^{i-1} - 2$  many 0-entries, every  $\xi \in \Xi_1$  such that  $\xi|_K = \bar{b}$  satisfies  $\xi|_{\overline{K}} = (1, \dots, 1)$ . So there can be at most one such  $\xi \in \Xi_1$ . It exists by construction of  $\Xi_1$ . Hence,  $\#_{K,\bar{b}}(\Xi) = 1$ .

• Let  $1 \in K$  and  $\bar{b} = (2^{i-1}, b_2, \dots, b_{2^{i-1}}), b_{\underline{j}} \in \{0, 1\}$  for all j, and not all  $b_j$  equal zero. Again, every  $\xi \in \Xi$  such that  $\xi|_K = b$  is contained in  $\Xi_1$ . Because

$$\sum \bar{b} \in \left\{ 2^{i-1} + 1, \dots, 2^{i-1} + k - 1 \right\} = \left\{ 2^{i-1} + 1, \dots, 2^{i} - 2 \right\}$$

it holds that

$$m := -\sum \bar{b} = \sum \xi|_{\overline{K}} \in \{2, \dots, 2^{i-1} - 1\}$$

for every  $\xi \in \Xi_1$ . By construction,  $\{\xi \in \Xi_1 \mid \xi \mid_K = \bar{b}\}|_{[2^i-1]\setminus K}$  is the set of all 0/1-tuples of length  $2^{i-1}$  that contain exactly m many ones. There are exactly  $\binom{2^{i-1}}{m}$  many 0/1-tuples of length  $2^{i-1}$ . For all possible values of m, the number  $\binom{2^{i-1}}{m}$  is even by Lemma 53. We conclude that  $\#_{K,\bar{b}}(\Xi) = 0$ .

• Let  $1 \in K$  and  $\bar{b} = (2^{i-1} + 1, b_2, \dots, b_{2^{i-1}})$ . Now, every  $\xi \in \Xi$  with  $\xi|_K = \bar{b}$  is contained in  $\Xi_2$ . Then

$$m := -\sum \bar{b} = \sum \xi|_{\overline{K}} \in \{1, \dots, 2^{i-1} - 1\}$$

for every  $\xi \in \Xi_2$  because

$$\sum \bar{b} \in \left\{ 2^{i-1} + 1, \dots, 2^{i-1} + 1 + (k-1) \right\} = \left\{ 2^{i-1} + 1, \dots, 2^{i} - 1 \right\}.$$

Again, there are  $\binom{2^{i-1}}{m}$  many 0/1-tuples extending  $\bar{b}$  to a  $\xi \in \Xi$ , which is an even number by Lemma 53 and thus  $\#_{K,\bar{b}}(\Xi) = 0$ .

- Let  $1 \in K$  and  $\bar{b}$  not be covered by two cases before. Then there is no  $\xi$  satisfying  $\xi|_K = \bar{b}$  and so  $\#_{K,\bar{b}}(\Xi) = 0$ .
- Now the case that  $1 \notin K$  remains. If there is no  $\xi \in \Xi$  satisfying  $\xi|_K = \bar{b}$ , then clearly  $\#_{K,\bar{b}}(\Xi) = 0$ . So assume that there is such a  $\xi \in \Xi$ .
  - Let us first consider  $\Xi_1$ . Let  $\xi \in \Xi_1$ . Then

$$0 = \sum \xi = \sum \bar{b} + \sum \xi|_{\overline{K}} = \sum \bar{b} + 2^{i-1} + \sum \xi|_{\overline{K} \backslash \{1\}}.$$

The tuple  $\xi|_{\overline{K}\setminus\{1\}}$  is a 0/1-tuple of length  $d-k-1=2^{i-1}-1$ . So

$$\sum \xi|_{\overline{K}\setminus\{1\}} \in \left\{0,\dots,2^{i-1}-1\right\}.$$

That is, if  $\sum \bar{b}=0$ , we obtain a contradiction because there is no  $\xi\in\Xi_1$  satisfying  $\xi|_K=\bar{b}$  and  $\#_{K,\bar{b}}(\Xi_1)=0$ . Otherwise, all  $\xi\in\Xi_1$  satisfying  $\xi|_K=\bar{b}$  extend  $\bar{b}$  by a 0/1-tuple of length  $2^{i-1}-1$  containing

$$m:=-\sum \bar{b}-2^{i-1}\in \left\{0,\dots,2^{i-1}-1\right\}$$

many ones. There are  $\binom{2^{i-1}-1}{m}$  many 0/1-tuples of length  $d-k-1=2^{i-1}-1$  and sum m, which is an odd number by Lemma 53. Hence,  $\#_{N,\bar{b}}(\Xi_1)=1$ .

- Now consider  $\Xi_2$ . For every  $\xi \in \Xi_2$  such that  $\xi|_K = \bar{b}$ , it similarly holds that

$$0 = \sum b + 2^{i-1} + 1 + \sum \xi|_{\overline{K} \setminus \{1\}}$$

and thus  $\sum \xi|_{\overline{K}\setminus\{1\}} \in \{0,\ldots,2^{i-1}\}$ . Every  $\xi \in \Xi_1$  satisfying  $\xi|_K = \bar{b}$  extends  $\bar{b}$  by a 0/1-tuple of length  $2^{i-1} - 1$  containing

$$m := -\sum \bar{b} - 2^{i-1} - 1 \in \{0, \dots, 2^{i-1} - 1\}$$

many ones. The number m is again odd by Lemma 53. Hence,  $\#_{N,\bar{b}}(\Xi_2) = 1$ .

Together, if  $\bar{b} = 0$ , then  $\#_{K,\bar{b}}(\Xi) = \#_{K,\bar{b}}(\Xi_1) + \#_{K,\bar{b}}(\Xi_2) = 0 + 1 = 1$ . Otherwise,  $\#_{K,\bar{b}}(\Xi_1) + \#_{K,\bar{b}}(\Xi_2) = 1 + 1$  which is 0 modulo 2.

**Remark 55.** Computer experiments suggest that for a given  $2^{i-2} \le k \le 2^{i-1} - 1$  our choice of q = i is minimal to construct a  $(k, q, 2^{i-1}, d)$ -blurrer and that  $d = 2^i - 1$  could be improved in the case that  $k \ne 2^{i-1} - 1$ , but is minimal in the case that  $k = 2^{i-1} - 1$ .

We now lift a (k, q, a, d)-blurrer from the ring  $\mathbb{Z}_{2^q}$  to the ring  $\mathbb{Z}_{2^{q+\ell}}$ . Here we have two choices, both of which we need later: the first is via the embedding of  $\mathbb{Z}_{2^q}$  in  $\mathbb{Z}_{2^{q+\ell}}$ , the second will not change the value a.

**Lemma 56.** Let  $q, \ell \in \mathbb{N}$  and  $\iota \colon \mathbb{Z}_{2^q} \to \mathbb{Z}_{2^{q+\ell}}$  be the embedding of  $\mathbb{Z}_{2^q}$  in  $\mathbb{Z}_{2^{q+\ell}}$  defined by  $a \mapsto 2^{\ell}a$ . If  $\Xi$  is a (k, q, a, d)-blurrer, then  $\iota(\Xi)$  is a  $(k, q + \ell, \iota(a), d)$ -blurrer.

*Proof.* This is straightforward from the definition.

**Lemma 57.** For every  $i, \ell \in \mathbb{N}$ , there is a  $(2^{i-1} - 1, i + \ell, 2^{i-1}, 2^i - 1)$ -blurrer.

*Proof.* Let  $\Xi$  be the  $(2^{i-1}-1,i,2^{i-1},2^i-1)$ -blurrer given by Lemma 54 and suppose  $c=2^{\ell+1}-1\in\mathbb{Z}_{2^{i+\ell}}$ . Let h be the following function that maps  $\Xi$  to  $\Xi':=\{h(\xi)\mid \xi\in\Xi\}$ :

$$\xi \mapsto \left(-c \cdot \sum_{j=2}^{d} \xi_j, c \cdot \xi_2, \dots, c \cdot \xi_d\right).$$

The operations are all in  $\mathbb{Z}_{2^{i+\ell}}$ . By definition  $\sum \xi' = 0$  for every  $\xi' \in \Xi'$ . Let  $K \subseteq [d]$  be of size k and  $\bar{b} \in \mathbb{Z}_{2^{i+\ell}}^k$ . Note that c is a unit because c is odd.

- Let  $1 \notin K$ . Because c is a unit, multiplication with c is a bijection and thus we have  $\#_{K,\bar{b}}(\Xi') = \#_{K,c^{-1}.\bar{b}}(\Xi)$ , which is 1 if and only if  $c^{-1} \cdot \bar{b} = \bar{b} = \bar{0}$ .
- Let  $1 \in K$ . We argue that also the action of h on the first position is a bijection. Because  $\sum \xi = 0$  for all  $\xi \in \Xi$ , the map  $\xi_1 \mapsto \sum_{j=2}^d \xi_j$  is a bijection and so is the action of h because c is a unit. So we have  $\#_{K,\bar{b}}(\Xi') = \#_{K,\bar{a}}(\Xi)$  for some  $\bar{a} \in \mathbb{Z}_{2^i}^k$ . It holds that  $-ca_1 = -(2^{\ell+1} 1)a_1 = a_1 2^{\ell+1}a_1$  (over  $\mathbb{Z}_{2^{i+\ell}}$ ). So  $a_1 = 2^{i-1}$  if and only if  $-ca_1 = 2^{i-1} 2^{\ell+1}2^{i-1} = 2^{i-1} 2^{\ell+i} = 2^{i-1} 0 = 2^{i-1}$ . Hence,

if and only if  $-ca_1 = 2^{i-1} - 2^{\ell+1}2^{i-1} = 2^{i-1} - 2^{\ell+i} = 2^{i-1} - 0 = 2^{i-1}$ . Hence,  $\bar{b} = (2^i, 0, \dots, 0)$  if and only if  $\bar{a} = (2^i, 0, \dots, 0)$ , which is the case if and only if  $\#_{K,\bar{b}}(\Xi') = 1$ .

### 9.3 Similarity Matrix for One Round

We now construct a similarity matrix k-blurring the twist. To be able to define this matrix, we need bounds on the degree, the girth, and the connectivity of the base graph as well as certain guarantees for the placement of the pebbles. Therefore, we define the following functions  $\mathbb{N}^+ \to \mathbb{N}^+$ , which will give us the needed bounds. In the definitions let  $i \in \mathbb{N}$  be the unique number that for the given k satisfies  $2^{i-1} - 1 < k \le 2^i - 1$ .

$$r(k) := \begin{cases} 1 & \text{if } k = 1, \\ \max\{4 \cdot r(k-1) + 2, 2k + 2\} & \text{otherwise,} \end{cases}$$
 
$$\theta(k) := \begin{cases} 1 & \text{if } k = 1, \\ i + \theta(k-1) & \text{otherwise,} \end{cases}$$
 
$$d(k, m) := \begin{cases} 3 + m & \text{if } k = 1, \\ \max\{2^{i+1} + m - 1, d(k-1, m+1)\} & \text{otherwise,} \end{cases}$$
 
$$q(k) := 1 + \theta(k).$$

**Lemma 58.** For every  $k, m \in \mathbb{N}$ ,

- every regular and (2k + m + 1)-connected base graph  $G = (V, E, \leq)$  of degree  $d \geq d(k, m)$  and girth at least 2r(k + 1),
- every edge  $\{t, t'\} \in E$ ,
- $every q \ge q(k)$ ,
- every  $\theta = a \cdot 2^{\theta(k)} \in \mathbb{Z}_{2^q}$  (for an arbitrary  $a \in \mathbb{Z}_{2^q}$ ),
- every  $f, g: E \to \mathbb{Z}_{2^q}$  such that f(e) = g(e) for all  $e \in E \setminus \{\{t, t'\}\}\}$  and  $g(\{t', t\}) = f(\{t', t\}) + \theta$ , and
- every m-tuple  $\bar{p} \in A^m$  of  $\mathfrak{A}_f := \mathsf{CFI}_{2^q}(G, f)$  and  $\mathfrak{A}_g := \mathsf{CFI}_{2^q}(G, g)$ , both with universe A, for which  $\mathsf{dist}_G(t, \mathsf{orig}(\bar{p})) > r(k+1)$

there is an odd-filled  $A^k \times A^k$  matrix S, both orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ , that k-blurs the twist between  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$  and those active region satisfies  $A^{f,g,\bar{p}}(S) \subseteq N_G^{r(k+1)}(t)$ .

The proof is by induction on k and spans the rest of this section. We already proved the case k=1 in Lemmas 41 and 43 for  $q \geq 2$  and  $\theta = 2^{q-1}$  using a  $(1,q,2^{q-1},d)$  blurrer. This can easily be adapted for the case that  $\theta = a \cdot 2^{\theta(1)} = 2a$  for some  $a \in \mathbb{Z}_{2^q}$ . We start with a (1,q,2,3)-blurrer given by Lemma 57 and turn it into a (1,q,2a,d)-blurrer using Lemma 52. Then the proof proceeds exactly the same.

So assume k > 1. Let  $m \in \mathbb{N}$ ,  $G = (V, E, \leq)$  be a regular and (2k + m + 1)-connected base graph of degree  $d \geq d(k, m)$  and girth at least 2r(k + 1), and  $\{t, t'\} \in E$ . The bound on the connectivity of G is needed in the following to apply the results from Sections 5.2, 6, and 8 and we will not further mention this when applying them. Let  $q \geq q(k)$ . As before, we denote for every  $f: E \to \mathbb{Z}_{2^q}$  by  $\mathfrak{A}_f$  the CFI structure  $\mathsf{CFI}_{2^q}(G, f)$ 

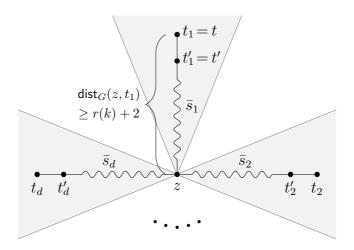


Figure 2: The star  $\bar{s}_1, \ldots, \bar{s}_d$ . Each path  $\bar{s}_i$  is contained in its own tree rooted at z (depicted in gray) because G has girth  $\geq 4r(k)$ .

with universe A (which is equal for all such f). Let  $f, g: E \to \mathbb{Z}_{2^q}$  such that f(e) = g(e) for all  $e \in E \setminus \{\{t, t'\}\}$  and  $g(\{t, t'\}) = f(\{t, t'\}) + \theta$ , where  $\theta = a \cdot 2^{\theta(k)}$  for some  $a \in \mathbb{Z}_{2^q}$ . Furthermore, let  $\bar{p} \in A^m$  such that  $\mathsf{dist}_G(t, \mathsf{orig}(\bar{p})) > r(k+1)$  be arbitrary but fixed. In particular, f, g do not twist  $\mathsf{orig}(\bar{p})$ .

Let z be a vertex with  $\operatorname{dist}_G(z,t') = r(k)+1$  and  $\operatorname{dist}_G(z,t) = r(k)+2$ . Choose vertices  $t=t_1,\ldots,t_d$  and  $t'=t'_1,\ldots,t'_d$  such that there are simple paths  $\bar{s}_i=(z,\ldots,t'_i,t_i)$  of length r(k)+2>2k+1 for all  $i\in[d]$  forming a star. Such paths exist because the girth of G is at least 2r(k+1)>2r(k)+4 and the degree is d (cf. Figure 2).

Claim 1. We have  $\operatorname{dist}_G(t'_i, \operatorname{orig}(\bar{p})) \geq 2r(k)$  for every  $i \in [d]$  and  $\operatorname{dist}_G(t'_i, t'_j) = 2r(k) + 2$  for every  $i \neq j$ .

Proof. By choice of z,  $\operatorname{dist}_G(z,t_i') = r(k) + 1$ . Let  $i,j \in [d]$  such that  $i \neq j$ . The  $t_i'$ - $t_j'$ -path obtained by joining  $\bar{s}_i$  and  $\bar{s}_j$  (after removing  $t_i$  and  $t_j$  respectively) has length  $2 \cdot (r(k) + 1)$  by construction. If this path was not a shortest path, then we would obtain a cycle of length less than  $4 \cdot (r(k) + 1)$ . This contradicts that G has girth at least  $2r(k+1) \geq 4 \cdot (r(k) + 1)$ . It follows  $\operatorname{dist}_G(t_i', t_j') = \operatorname{dist}_G(t_i', z) + \operatorname{dist}_G(z, t_j') = 2r(k) + 2$ .

Let  $y \in \operatorname{orig}(\bar{p})$ . Then  $\operatorname{dist}_G(y, t_1') \leq \operatorname{dist}_G(y, t_1') + \operatorname{dist}_G(t_1', t_1')$ . By assumption,  $\operatorname{dist}_G(y, t_1') \geq r(k+1) = 4r(k) + 2$  and by the former argument  $\operatorname{dist}_G(t_1', t_1') = 2r(k) + 2$ . It follows that

$${\sf dist}_G(y,t_i') \geq {\sf dist}_G(y,t_1') - {\sf dist}_G(t_1',t_i') = 4r(k) + 2 - 2r(k) - 2 \geq 2r(k). \qquad \exists$$

We call a set  $C \subseteq V$  such that  $|C| \leq 2k$  and G[C] is connected a 2k-component. Every 2k-component is a component of some 2k-orbit (cf. Definition 18). Because k is fixed in this proof, we just call them components in the following.

**Definition 59.** We call a component C

- an *i*-tip component if  $\{t_i, t_i'\} \subseteq C$  and  $i \in [d]$ ,
- a star component if C intersects non-trivially with the star  $\bar{s}_1, \ldots, \bar{s}_d$  and  $t_i \notin C$  for all  $i \in [d]$ ,

- an *i*-star component if C is a star component, C intersects non-trivially with  $\bar{s}_i$ , and for every other  $j \neq i$ , C intersects trivially with  $\bar{s}_j$ ,
- a star center component if C is a star component but not an i-star component for every  $i \in [d]$ ,
- and otherwise a **sky component**.

Claim 2. Let C be a component and  $C' \subseteq C$  be a component.

- 1. If C is an i-star component, then C' is an i-star or a sky component
- 2. If C is a star component, then C' is a star or a sky component.
- 3. If C is an i-tip component, then C' is an i-tip, an i-star, or a sky component.
- 4. If C is a star component, then C is an i-star component for some i if and only if  $z \notin C$ .

*Proof.* To show Case 1, let C be an i-star component. By definition, C only has a non-trivial intersection with  $\bar{s}_i$  and  $t_i \notin C$ . So every  $C' \subseteq C$  has either a trivial intersection with every  $\bar{s}_j$ , i.e., C' is a sky component, or a non-trivial intersection only with  $\bar{s}_i$  and  $t_i \notin C' \subseteq C$ , i.e., C' is an i-star component. The Case 2 where C is a star component in similar.

For Case 3, let C be an i-tip component. Because  $\operatorname{dist}_G(t_i', t_j') = 2r(k) + 2 > 4k + 4$  (Claim 1),  $\operatorname{dist}_G(t_i', z) = r(k) + 1 > 2k$ ,  $|C| \leq 2k$ , and because G[C] is connected, C has a trivial intersection with every  $\bar{s}_j$  for all  $j \neq i$ . Let  $C' \subseteq C$ . Now C' is an i-tip component if  $\{t_i, t_i'\} \subseteq C'$ , an i-star component if otherwise C' intersects non-trivially with  $\bar{s}_i$ , or otherwise a sky component.

Finally, to prove Case 4, let C be a star component. If C is also an i-star component, then  $z \notin C$  because otherwise C would intersect non-trivially with all  $\bar{s}_j$ . For the reverse direction, let  $z \notin C$  and C have a non-trivial intersection with  $\bar{s}_i$ . Because  $\text{dist}_G(t_i', t_j') = 2r(k) + 2 > 4k + 4$ ,  $|C| \leq 2k$ , G[C] is connected, and because  $z \notin C$ , the component C cannot have a non-trivial intersection with another  $\bar{s}_j$ . Hence, C is an i-star component.

To blur the twist, we want to distribute it among the edges  $\{t_i, t_i'\}$  (for all  $i \in [d]$ ) similar to the 1-ary case. Here we blurred the twist between all edges adjacent to z. In the 1-ary case, 1-tuples always had the same type in  $(\mathfrak{A}_f, \bar{p})$  and in  $(\mathfrak{A}_g, \bar{p})$ . However, for k-tuples, this is no longer the case. We now want to construct a function that maps a tuple  $\bar{u}$  to a tuple  $\bar{v}$  such that  $\bar{v}$  has the same type in  $(\mathfrak{A}_g, \bar{p})$  as  $\bar{u}$  has in  $(\mathfrak{A}_f, \bar{p})$ . To do so, we use a path isomorphism on the path  $\bar{s}_1$ . We generalize this and not only want to "repair" the types for a twist at the edge  $\{t_1, t_1'\}$  but possibly for multiple twists at all edges  $\{t_i, t_i'\}$ .

Let  $\bar{a} \in \mathbb{Z}_{2^q}^d$ . We define a function  $\tau_{\bar{a}} \colon A^{\leq 2k} \to A^{\leq 2k}$  that preserves the size of tuples. Set  $\varphi_{\tau}^{\bar{a},i} := \vec{\pi}[a_i, \bar{s}_i]$  (cf. Definition 7). The function  $\tau_{\bar{a}}$  applies  $\varphi_{\tau}^{\bar{a},i}$  to tuples, but only to those components containing some of the edges  $\{t_i, t_i'\}$ . These are precisely the *i*-tip

 $\dashv$ 

components:

$$\tau_{\bar{a}}(\bar{u}) := (v_1, \dots, v_{|\bar{u}|}), \text{ where } \bar{u} = (u_1, \dots, u_{|\bar{u}|}) \text{ and }$$

$$v_j := \begin{cases} \varphi_{\tau}^{\bar{a}, i}(u_j) & \text{if } \mathsf{orig}(u_j) \in C \text{ and } C \text{ is an } i\text{-tip component of } \bar{u}, \\ u_j & \text{otherwise.} \end{cases}$$

Given a function  $h: E \to \mathbb{Z}_{2^q}$ , we write  $h + \bar{a}$  for the function  $h': E \to \mathbb{Z}_{2^q}$  such that  $h'(\{t_i, t_i'\}) = h(\{t_i, t_i'\}) + a_i$  for all  $i \in [d]$ , and h'(e) = h(e) otherwise.

Claim 3. Suppose  $\bar{a} \in \mathbb{Z}_{2^q}^d$ ,  $h: E \to \mathbb{Z}_{2^q}$ , and  $k' \le 2k$ . If  $P \in \operatorname{orbs}_{k'}((\mathfrak{A}_h, \bar{p}))$ , then  $\tau_{\bar{a}}(P) \in \operatorname{orbs}_{k'}((\mathfrak{A}_{h+\bar{a}}, \bar{p}))$  and  $\tau_{\bar{a}}(P)$  has the same type in  $(\mathfrak{A}_{h+\bar{a}}, \bar{p})$  as P has in  $(\mathfrak{A}_h, \bar{p})$ .

Proof. Let  $P \in \operatorname{orbs}_{k'}(\mathfrak{A}_h, \bar{p})$ . It suffices to consider the case that P has a single component C because  $\tau_{\bar{a}}$  is defined component-wise and because by Lemma 19 the type of a disconnected orbit is given by the types of the restrictions to the components. If C does not contain  $t_i$  and  $t'_i$  for some  $i \in [d]$ , then  $\tau_{\bar{a}}$  is the identity function. Because P does not cover the twisted edge, it has the same type in  $(\mathfrak{A}_h, \bar{p})$  and in  $(\mathfrak{A}_{h+\bar{a}}, \bar{p})$ .

So assume  $\{t_i, t_i'\} \subseteq C$ . This is the case for exactly one  $i \in [d]$  by Claim 2. Let  $h_i$  be the function equal to h for all edges apart from  $h_i(\{t_i, t_i'\}) := h(\{t_i, t_i'\}) + a_i$  and  $h_i(\{z, z_i\}) := h(\{z, z_i\}) - a_i$ , where  $z_i$  is the neighbor of z used in the path  $\bar{s}_i$ . By Claim 1, the parameters  $\bar{p}$  have distance 2r(k) to  $t_i'$  and thus in particular are not contained in  $\bar{s}_i$ . So  $\varphi_{\tau}^{\bar{a},i}$  is an isomorphism between  $(\mathfrak{A}_h, \bar{p})$  and  $(\mathfrak{A}_{h_i}, \bar{p})$  by Lemma 8. Because P is a k'-orbit and  $k' \leq 2k$ , neither z nor its neighbors (in particular not  $z_i$ ) are contained in C, because  $2k < \operatorname{dist}_G(z, t_i') = r(k) + 1$ . So  $\tau_{\bar{a}}(P) = \varphi_{\tau}^{\bar{a},i}(P)$  has the same type in  $(\mathfrak{A}_{h_i}, \bar{p})$  as P has in  $(\mathfrak{A}_h, \bar{p})$ . Now, between  $(\mathfrak{A}_{h+\bar{a}}, \bar{p})$  and  $(\mathfrak{A}_{h_i}, \bar{p})$  all edges  $\{t_\ell, t_\ell'\}$  for  $\ell \neq i$  and the edge  $\{z, z_i\}$  are potentially twisted. Because  $z \notin C$  and  $\{t_\ell, t_\ell'\} \not\subseteq C$  for every  $\ell \neq i$ ,  $(\mathfrak{A}_{h_i}, \bar{p})[C] = (\mathfrak{A}_{h+\bar{a}}, \bar{p})[C]$ . Hence, the type of  $\varphi_{\tau}^{\bar{a},i}(P)$  in  $(\mathfrak{A}_{h_i}, \bar{p})$  is equal to the type of  $\tau_{\bar{a}}(P)$  in  $(\mathfrak{A}_{h+\bar{a}}, \bar{p})$ .

We now construct a blurrer for our setting. Let  $i \in \mathbb{N}$  such that  $2^{i-1} - 1 < k \leq 2^i - 1$ .

- 1. By Lemma 57, there is a  $(2^i 1, q \theta(k 1), 2^i, 2^{i+1} 1)$ -blurrer (note that  $q \theta(k 1) \ge q(k) \theta(k 1) = i + 1$ ).
- 2. We use Lemma 56 to turn it into a  $(2^i-1,q,2^{i+\theta(k-1)},2^{i+1}-1)$ -blurrer by embedding  $\mathbb{Z}_{2^q-\theta(k-1)}$  in  $\mathbb{Z}_{2^q}$ .
- 3. We use Lemma 52 to get a  $(k, q, 2^{i+\theta(k-1)}, 2^{i+1} 1)$ -blurrer because  $k \leq 2^i 1$ ,
- 4. then a  $(k, q, 2^{i+\theta(k-1)}, d)$ -blurrer because  $d \ge d(k, m) \ge 2^{i+1} 1$ , and finally
- 5. a  $(k, q, a \cdot 2^{i+\theta(k-1)}, d)$ -blurrer  $\Xi$ .

By this construction, for every  $\xi \in \Xi$  and every  $j \in [d]$  there is some  $b \in \mathbb{Z}_{2^q}$  such that  $\xi(j) = b \cdot 2^{\theta(k-1)}$  because we embedded  $\mathbb{Z}_{2^q - \theta(k-1)}$  in  $\mathbb{Z}_{2^q}$ . Let  $\xi_{\mathsf{twst}} = (a \cdot 2^{i + \theta(k-1)}, 0, \dots, 0)$  be the tuple given by Lemma 50 for  $\Xi$ . Then  $g = f + \xi_{\mathsf{twst}}$ . We set  $\tau := \tau_{\xi_{\mathsf{twst}}}$ .

Corollary 60. If  $k' \leq 2k$  and  $P \in \operatorname{orbs}_{k'}((\mathfrak{A}_f, \bar{p}))$ , then  $\tau(P) \in \operatorname{orbs}_{k'}((\mathfrak{A}_g, \bar{p}))$  and  $\tau(P)$  has the same type in  $(\mathfrak{A}_g, \bar{p})$  as P has in  $(\mathfrak{A}_f, \bar{p})$ .

We have seen that with  $\tau$  we can "repair" the types of the orbits, but  $\tau$  introduces inconsistencies between tuples along the path  $\bar{s}_1$ . This can easily be seen already for k=2: Consider a 2-tuple (u,v) with origin  $(t_1,t_1')$  and the 2-tuple (v,w) with origin  $(t_1',x)$ , where x is the next vertex in the path  $\bar{s}_1$ . Then  $\tau((u,v))=(u,v')$  for some  $v'\neq v$  but  $\tau((v,w))=(v,w)$ . So clearly the composed 4-tuple (u,v,v,w) has a different type than (u,v',v,w). For these inconsistencies, we use the blurrer  $\Xi$  as follows.

We now define another function which according to a  $\xi \in \Xi$  "distributes" the twists among the edges  $\{t_i, t_i'\}$  using a star isomorphism. We associate with each  $\xi \in \Xi$  a function  $\psi_{\xi} \colon A^{\leq k} \to A^{\leq k}$ , again preserving tuple sizes: Set  $\varphi_{\xi} := \pi^*[\xi, \bar{s}_1, \dots, \bar{s}_d]$  (cf. Definition 9). Then define  $\psi_{\xi} \colon A^{\leq k} \to A^{\leq k}$  as follows: Let  $\bar{u} \in A^{\leq k}$ . We set

$$\psi_{\xi}(\bar{u}) := (v_1, \dots, v_{|\bar{u}|}), \text{ where } \bar{u} = (u_1, \dots, u_{|\bar{u}|}) \text{ and}$$

$$v_i := \begin{cases} \varphi_{\xi}(u_i) & \text{if } \mathsf{orig}(u_i) \in C \text{ and } C \text{ is a star component of } \bar{u}, \\ u_i & \text{otherwise.} \end{cases}$$

That is, we apply  $\varphi_{\xi}$  to all components of  $\bar{u}$  that do not contain the tips of the star  $\bar{s}_1, \ldots, \bar{s}_d$ . On the sky components  $\varphi_{\xi}$  is the identity function anyway. From now on, we will identify  $\xi$  with  $\psi_{\xi}$  and write  $\xi(\bar{u})$ .

Claim 4. Let  $\bar{u} \in A^{\ell}$  and the components of  $\bar{u}$  be partitioned into D and D'. Then  $\tau_{\bar{a}}(\bar{u}) = \tau_{\bar{a}}(\bar{u}_D)\tau_{\bar{a}}(\bar{u}_{D'})$  and  $\xi(\bar{u}) = \xi(\bar{u}_D)\xi(\bar{u}_{D'})$  for every  $\bar{a} \in \mathbb{Z}_{2^q}^d$  and  $\xi \in \Xi$ .

*Proof.* The claim is immediate because  $\tau_{\bar{a}}$  and  $\psi_{\xi}$  are defined component-wise.

Claim 5. For every  $\bar{a} \in \mathbb{Z}_{2^q}^d$  and  $\xi \in \Xi$ , the functions  $\tau_{\bar{a}}$ ,  $\psi_{\xi}$ , and every automorphism  $\varphi \in \operatorname{Aut}((\mathfrak{A}_f, \bar{p})) = \operatorname{Aut}((\mathfrak{A}_g, \bar{p}))$  commute.

*Proof.* The functions  $\tau_{\bar{a}}$  and  $\psi_{\xi}$  are defined component-wise by isomorphisms. We saw in Section 5.1 that isomorphisms between CFI structures are composed of automorphisms of each gadget. Because the automorphism group of a gadget is abelian (Lemma 17), the said functions commute.

**Definition 61.** An orbit-automorphism is a function  $\zeta \colon A^{\leq 2k} \to A^{\leq 2k}$  that satisfies the following: For every  $P \in \operatorname{orbs}_{k'}((\mathfrak{A}_f, \bar{p})) = \operatorname{orbs}_{k'}((\mathfrak{A}_g, \bar{p}))$  with  $k' \leq 2k$  there is an automorphism  $\varphi_P \in \operatorname{Aut}((\mathfrak{A}_f, \bar{p})) = \operatorname{Aut}((\mathfrak{A}_g, \bar{p}))$  such that  $\zeta(\bar{u}) = \varphi_P(\bar{u})$  for all  $\bar{u} \in P$ .

That is, an orbit-automorphism is a function, whose action on a single orbit is the action of an automorphism. For different orbits, the corresponding automorphisms may be different. This matches the definition of an orbit-invariant matrix (cf. Definition 27), which is invariant under all orbit-automorphisms.

Claim 6. Every  $\xi \in \Xi$  is an orbit-automorphism.

*Proof.* By Lemma 19, it suffices to show the claim for connected orbits  $P \in \mathsf{orbs}_{k'}((\mathfrak{A}_f, \bar{p}))$  with  $k' \leq 2k$ . If the origin of P is not a star component, then  $\xi$  is the identity function on P and so clearly an orbit-automorphism.

Otherwise, the origin  $C := \mathsf{orig}(P)$  is a star component. Then in particular  $t_i \notin C$  for all  $i \in [d]$ . We show that there are paths  $\bar{s}'_i = (t'_i, t_i, \dots, t_1, t'_1)$  that are completely disjoint

from  $\operatorname{orig}(\bar{p})$  and possibly apart from  $t_i', t_1'$  disjoint from C. Set  $C' := C \setminus \{t_i' \mid i \in [d]\}$ . Consider the graph  $G \setminus C' \setminus \operatorname{orig}(\bar{p})$ . We removed at most  $|\operatorname{orig}(\bar{p})| + k' \leq 2k + m$  many vertices. Because G is (2k+m+1)-connected, the claimed paths exists in  $G \setminus C' \setminus \operatorname{orig}(\bar{p})$ . We then use path isomorphisms  $\varphi_i := \vec{\pi}[\xi(i), \bar{s}_i']$  for all  $i \in [d] \setminus \{1\}$  to move the twist introduced by  $\psi_{\xi}$  at the  $\{t_i, t_i'\}$  to  $\{t_1, t_1'\}$ . We set  $\psi := \psi_{\xi} \circ \varphi_2 \circ \cdots \circ \varphi_d$ . Now, we have that  $(\mathfrak{A}_f, \bar{p}) \cong \psi((\mathfrak{A}_f, \bar{p})) = (\mathfrak{A}_f, \bar{p})$  by Lemmas 8 and 10. Let  $\bar{u} \in P$ . The isomorphisms  $\varphi_i$  are the identity function on vertices in  $\operatorname{orig}(P)$  because  $\varphi_i$  is the identity on  $t_i'$  and  $t_1'$  (cf. Definition 7) and other vertices in  $\operatorname{orig}(P)$  are not contained in  $\bar{s}_i'$  for every  $i \in [d]$ . That is,  $P = \psi(P) = \psi_{\xi}(P) = \xi(P)$  and  $\xi$  is an orbit-automorphism.

We show how  $\xi$  and  $\tau_{\xi}$  work together. That is, for  $\bar{u}\bar{v}$  of length at most 2k, the tuple  $\tau_{\xi}(\xi(\bar{u}))\tau_{\xi}(\xi(\bar{v}))$  has the same type as the tuple  $\tau_{\xi}(\bar{u}\bar{v})$ . This is not true for applying only  $\tau_{\xi}$  to  $\bar{u}$  and  $\bar{v}$  because their origins may overlap.

Claim 7. Let  $h: E \to \mathbb{Z}_{2^q}$ ,  $k' \le 2k$ ,  $P \in \operatorname{orbs}_{k'}((\mathfrak{A}_h, \bar{p}))$ , and  $\xi \in \Xi$ . Every  $\bar{u}\bar{v} \in A^{k'}$  satisfies  $\bar{u}\bar{v} \in P$  if and only if  $\tau_{\xi}(\xi(\bar{u}))\tau_{\xi}(\xi(\bar{v})) \in \tau_{\xi}(P)$ .

Proof. Set  $P_1 := P|_{\{1,\dots,|\bar{u}|\}}$  and  $P_2 := P|_{\{|\bar{u}|+1,\dots,k'\}}$ . Let  $C_1^i,\dots,C_{\ell_i}^i$  be the components of  $P_i$  for every  $i \in [2]$ . Because  $\tau_{\xi}$  and  $\xi$  are defined component-wise, it suffices by Lemma 19 to assume that P has a single component C. We need to verify that  $\bar{u}\bar{v} \in P$  if and only if  $\tau_{\xi}(\xi(\bar{u}))\tau_{\xi}(\xi(\bar{v})) \in \tau_{\xi}(P)$ . The component C is the union of the  $C_j^i$ . We perform the following case distinction:

• Assume C is an n-tip component. For every  $i \in [2]$ , let  $D_i^T$  be the set of the n-tip components  $C_j^i$ ,  $D_i^S$  be the set of the n-star components  $C_j^i$ , and  $D_i^R$  be the set of sky components  $C_i^i$ . This yields a partition of all  $C_i^i$  by Claim 2.

Then we have by the definitions of  $\tau_{\xi}$  and  $\xi$  that  $\xi$  is the identity function on  $D_i^T$ ,  $\tau_{\xi}$  is the identity on  $D_i^S$ , and both are the identity on  $D_i^R$ . That is,

$$\begin{split} &\tau_{\xi}(\xi(\bar{u}))\tau_{\xi}(\xi(\bar{v}))\\ &=\tau_{\xi}(\xi(\bar{u}_{D_{1}^{T}}\bar{u}_{D_{1}^{S}}\bar{u}_{D_{1}^{R}}))\tau_{\xi}(\xi(\bar{v}_{D_{2}^{T}}\bar{v}_{D_{2}^{S}}\bar{v}_{D_{2}^{R}}))\\ &=\tau_{\xi}(\bar{u}_{D_{1}^{T}})\xi(\bar{u}_{D_{1}^{S}})\tau_{\xi}(\bar{u}_{D_{1}^{R}})\tau_{\xi}(\bar{v}_{D_{2}^{T}})\xi(\bar{v}_{D_{2}^{S}})\tau_{\xi}(\bar{v}_{D_{2}^{S}}). \end{split} \tag{$\star$}$$

When working on the whole component C,  $\tau_{\xi}$  applies  $\varphi_{\tau}^{\xi,n}$  to vertices in components of  $D_1^S$  and  $D_2^S$  because C is an n-tip component. We see that  $\varphi_{\xi}|_{D_i^S} = \varphi_{\tau}^{\xi,n}|_{D_i^S}$  because  $D_i^S$  is an n-star-component and so does not contain z (cf. the Definitions 7 and 9 and the definitions of  $\varphi_{\xi}$  and  $\varphi_{\tau}^{\xi,n}$ ). It follows that

$$(\star) = \tau_{\xi}(\bar{u}_{D_{1}^{T}}\bar{u}_{D_{1}^{S}}\bar{u}_{D_{1}^{R}}\bar{v}_{D_{2}^{T}}\bar{v}_{D_{2}^{S}}\bar{v}_{D_{2}^{R}}) = \tau_{\xi}(\bar{u}\bar{v}).$$

So  $\bar{u}\bar{v} \in P$  if and only if  $\tau_{\xi}(\xi(\bar{u}))\tau_{\xi}(\xi(\bar{v})) = \tau_{\xi}(\bar{u}\bar{v}) \in \tau_{\xi}(P)$ .

- Otherwise, C is not a tip component. We distinguish two more cases:
  - If C is a star component, let  $D_i^S$  be the set of star components  $C_j^i$  and  $D_i^R$  be the set of sky components  $C_j^i$  for all  $i \in [2]$ . There are no tip components of the  $C_j^i$  by Claim 2. So we again partitioned all components  $C_j^i$ . Now  $\tau_{\xi}$  is the identity function on all  $D_i^S$  and  $D_i^R$  and  $\xi$  is the identity on all  $D_i^R$ . So we have  $\bar{u}\bar{v} \in P$  if and only if  $\tau_{\xi}(\xi(\bar{u}))\tau_{\xi}(\xi(\bar{v})) = \xi(\bar{u})\xi(\bar{v}) = \xi(\bar{u}\bar{v}) \in \tau_{\xi}(P) = P$  by Claim 6.

– Otherwise, C is a sky component and both  $\tau_{\xi}$  and  $\xi$  are the identity function on C and all  $C_i^i$  and the claim follows immediately.

Using the blurrer  $\Xi$ , we will be able to blur the twist in many cases, but not in all. The problem is the following: If we only look at k many of the  $\{t_i, t_i'\}$  edges, then the blurrer properties will ensure that we cannot see the twist, i.e., "summing" over all elements in the blurrer maps a 2k-orbit of  $(\mathfrak{A}_f, \bar{p})$  to a 2k-orbit of the same type in  $(\mathfrak{A}_g, \bar{p})$  similar to the 1-ary case. Let us briefly recall the arguments to prove Lemma 39, which shows that summing over blurrer elements indeed yields a matrix bluring the twist for arity 1. There are two cases: First, for a tuple (u, v) with origin (z, z) the action of every  $\xi \in \Xi$  is the action of an automorphism, so  $(\xi(u), v)$  and  $(u, \xi^{-1}(v))$  are in the same orbit. Second, for a tuple (u, v) with origin  $(z, t_1)$  only one index (namely the first for  $t_1$ ) was relevant:  $\xi(u)v$  and  $\xi'(u)v$  are in the same orbit if  $\xi(1) = \xi'(1)$ . So whenever  $\xi(1) = \xi'(1)$ , the terms for  $\xi(u)v$  and  $\xi'(u)v$  canceled out in the summation. The blurrer properties ensured that only one term for  $\xi(u)v$  of the same type in  $(\mathfrak{A}_g, \bar{p})$  as uv in  $(\mathfrak{A}_f, \bar{p})$  remained.

For arity k the two cases (automorphism or blurrer properties) can be mixed. Consider k=2 and a 4-tuple  $\bar{u}$  with origin  $(z,t_1',z,t_1)$ . Then for every  $\xi,\xi'\in\Xi$ ,  $\xi(u_1u_2)u_3u_4$  and  $\xi'(u_1u_2)u_3u_4$  are in the same orbit if and only if  $\xi=\xi'$  (because fixing one vertex with origin z splits the gadget of z into singleton orbits). That is, we cannot argue solely with blurrer properties. However, the two tuples  $\xi(u_1u_2)u_3u_4$  and  $u_1u_2\xi^{-1}(u_3u_4)$  are not in the same orbit in general because every automorphism mapping  $u_1u_2$  to  $\xi(u_1u_2)$  cannot be the identity for  $u_4$  but which  $\xi^{-1}(u_3u_4)$  is. So we also cannot argue solely with automorphisms. The techniques of the 1-ary case can only be applied if z is not in the origin of at least one of  $u_1u_2$  and  $u_3u_4$ .

In general, let  $P \in \operatorname{orbs}_{2k}((\mathfrak{A}_f, \bar{p}))$  and  $\bar{u} \in P$ . Then in  $\chi^P$  the first k positions of  $\bar{u}$  will serve as row index and the remaining k positions as column index. The problem with the blurrer only occurs if both the first and second half of  $\bar{u}$  contain z in its origin. So we make a case distinction on whether a k-orbit contains z in its origin.

### **Definition 62.** We call $a P \in \operatorname{orbs}_k((\mathfrak{A}_f, \bar{p}))$ blurrable if $z \notin \operatorname{orig}(P)$ .

To also be able to blur the twist for non-blurrable orbits, we use a recursive approach. Because  $\sum \xi = 0$ ,  $\mathfrak{A}_{g-\xi}$  is isomorphic to  $\mathfrak{A}_g$  for every  $\xi \in \Xi$ . Let  $p_z$  be an arbitrary vertex with origin z. Our goal now is to blur the twist between  $(\mathfrak{A}_f, \bar{p}p_z)$  and  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$ . This will exactly undo the action of a  $\xi \in \Xi$ , when we consider an orbit that fixes a vertex with origin z. We exploit the high girth of G to blur the twists at each  $\{t_i, t_i'\}$  independently. Before we start to blur twists between  $(\mathfrak{A}_f, \bar{p}p_z)$  and  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$ , we first have to show that  $\tau_{\xi}$  and  $\xi$  are compatible with orbits when fixing the additional vertex  $p_z$ . The following two claims are in some sense refinements of Claims 3 and 7.

Claim 8. For every  $\bar{a} \in \mathbb{Z}_{2^q}$  and  $k' \leq 2k$ , every orbit  $P \in \operatorname{orbs}_{k'}((\mathfrak{A}_{f+\bar{a}}, \bar{p}p_z))$  satisfying  $\operatorname{orig}(P) \cap N^1_G(z) = \emptyset$  is contained in  $\operatorname{orbs}_{k'}((\mathfrak{A}_{f+\bar{a}}, \bar{p}))$ .

Proof. Let  $P \in \operatorname{orbs}_{k'}((\mathfrak{A}_{f+\bar{a}}, \bar{p}p_z))$  such that  $\operatorname{orig}(P) \cap N_G^1(z) = \emptyset$ . By Lemma 12, it suffices to show that two tuples  $\bar{u}, \bar{v} \in A^{k'}$ , such that  $\operatorname{orig}(\bar{u}) = \operatorname{orig}(\bar{v})$  is disjoint from  $N_G^1(z)$ , have the same type in  $(\mathfrak{A}_{f+\bar{a}}, \bar{p})$  if and only if they have the same type in  $(\mathfrak{A}_{f+\bar{a}}, \bar{p}p_z)$ . Because  $\operatorname{orig}(P) \cap N_G^1(z) = \emptyset$ , the components of  $\bar{u}\bar{p}p_z$  are the components of  $\bar{u}\bar{p}$  and the one of  $p_z$ . Hence, if  $\bar{u}$  and  $\bar{v}$  have the same type in  $(\mathfrak{A}_{f+\bar{a}}, \bar{p})$ , then they also have the same type in  $(\mathfrak{A}_{f+\bar{a}}, \bar{p}p_z)$ . The other direction is trivial.

Claim 9. Let  $\bar{a} \in \mathbb{Z}_{2^q}^d$ ,  $h: E \to \mathbb{Z}_{2^q}$ , and  $k' \le 2k$ . If  $P \in \operatorname{orbs}_{k'}((\mathfrak{A}_h, \bar{p}p_z))$ , then  $\tau_{\bar{a}}(P) \in \operatorname{orbs}_{k'}((\mathfrak{A}_{h+\bar{a}}, \bar{p}p_z))$  and  $\tau_{\bar{a}}(P)$  has the same type in  $(\mathfrak{A}_{h+\bar{a}}, \bar{p}p_z)$  as P has in  $(\mathfrak{A}_h, \bar{p}p_z)$ .

Proof. Let  $P \in \operatorname{orbs}_{k'}((\mathfrak{A}_h, \bar{p}p_z))$ , R be the set of components C of P with  $C \cap N_G^1(z) \neq \emptyset$ , and D be the set of all remaining components of P. Then  $P = P|_R \times P|_D$  by Claim 19. Similarly,  $\tau_{\bar{a}}(P) = \tau_{\bar{a}}(P)|_R \times \tau_{\bar{a}}(P)|_D$ . Every component C in R does not contain the edges  $\{t_i, t_i'\}$  for all  $i \in [d]$  because  $|C| \leq k'$  but every path  $\bar{s}_i$  has length  $r(k) + 2 > 2k + 1 \geq k' + 1$ . That is,  $\tau_{\bar{a}}$  is the identity on  $P|_R$  and so  $P|_R = \tau_{\bar{a}}(P)|_R$  and  $P|_R$  has the same type in  $(\mathfrak{A}_{h+\bar{a}}, \bar{p}p_z)$  as it has in  $(\mathfrak{A}_h, \bar{p}p_z)$ . By Claim 8, the orbit  $\tau_{\bar{a}}(P)|_D$  is an orbit of  $(\mathfrak{A}_h, \bar{p})$  and has by Claim 3 the same type in  $(\mathfrak{A}_h, \bar{p})$  as  $\tau_{\bar{a}}(P)|_D$  has in  $(\mathfrak{A}_{h+\bar{a}}, \bar{p})$ . It follows that P has the same type in  $(\mathfrak{A}_h, \bar{p})$  as  $\tau_{\bar{a}}(P)$  has in  $(\mathfrak{A}_{h+\bar{a}}, \bar{p})$ .

Claim 10. Suppose  $Q \in \operatorname{orbs}_{k'}((\mathfrak{A}_{g-\xi}, \bar{p}p_z))$  for some  $\xi \in \Xi$  and  $k' \leq 2k-2$ . Then  $\tau_{\xi}(Q) \in \operatorname{orbs}_{k'}((\mathfrak{A}_g, \bar{p}p_z))$  and has the same type as Q. Let C be the connected component of  $G[\operatorname{orig}(Q) \cup \{z\}]$  containing z, R be the set of components of Q contained in C, D be the set of all other components of Q, and let  $\bar{w}\bar{v} \in A^{k'}$ . Then  $\bar{w}\bar{v} \in Q$  if and only if  $\bar{w}_R\xi(\tau_{\xi}(\bar{w}_D))\bar{v}_R\xi(\tau_{\xi}(\bar{v}_D)) \in \tau_{\xi}(Q)$ .

Proof. We split Q using Lemma 19 in  $Q = Q|_R \times Q|_D$ . Because  $k' \leq 2k - 2$ , components in R cannot be tip components. Hence,  $\tau_{\xi}(Q) = Q|_R \times \tau_{\xi}(Q|_D)$ . By Claim 8,  $Q|_D$  is also an orbit of  $(\mathfrak{A}_{g-\xi}, \bar{p})$  because its origin has distance greater than 1 to  $\operatorname{orig}(p_z)$ . Then  $\tau_{\xi}(Q|_D)$  has the same type in  $(\mathfrak{A}_g, \bar{p})$  and is also an orbit of  $(\mathfrak{A}_g, \bar{p}p_z)$  of the same type, too, by Claim 3. It follows that  $\tau_{\xi}(Q)$  has the same type as Q. Let  $\bar{w}\bar{v} \in A^{k'}$ . Using the splitting above, from Claim 7 it follows that  $\bar{w}_D\bar{v}_D \in Q|_D$  if and only if  $\xi(\tau_{\xi}(\bar{v}_D))\xi(\tau_{\xi}(\bar{v}_D)) \in \tau_{\xi}(Q|_D)$ . The claim follows because  $Q = Q|_R \times Q|_D$ .

Now we construct matrices (k-1)-bluring the twist between  $(\mathfrak{A}_f, \bar{p}p)$  and  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$ . For every  $\xi \in \Xi$  and  $j \in [d+1]$  we define  $g^{\xi,j} \colon E \to \mathbb{Z}_{2^q}$  to be the following function:

$$g^{\xi,j}(e) := \begin{cases} f(\{t_i, t_i'\}) & \text{if } e = \{t_i, t_i'\} \text{ for some } i \ge j, \\ (g - \xi)(e) & \text{otherwise.} \end{cases}$$

Note that  $f(e) = g(e) = g^{\xi,j}(e)$  for all e different from the edges  $\{t_i, t_i'\}$ ,  $g^{\xi,1} = f$ ,  $g^{\xi,d+1} = g - \xi$ , and the only possibly twisted edge by  $g^{\xi,j}$  and  $g^{\xi,j+1}$  is  $\{t_j, t_j'\}$  for every  $j \in [d]$ . Define  $N_j := N_G^{r(k)}(t_j)$  for every  $j \in [d]$ .

Claim 11. For every  $\xi \in \Xi$  and  $j \in [d]$ , there is an  $A^{k-1} \times A^{k-1}$  matrix  $S^{\xi,j}$ , which is odd-filled and both orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_{g^{\xi,j}}, \bar{p}p_z)$  and  $(\mathfrak{A}_{g^{\xi,j+1}}, \bar{p}p_z)$ , which (k-1)-blurs the twist between  $(\mathfrak{A}_{g^{\xi,j}}, \bar{p}p_z)$  and  $(\mathfrak{A}_{g^{\xi,j+1}}, \bar{p}p_z)$ , and those active region satisfies  $\mathsf{A}^{g^{\xi,j},g^{\xi,j+1},\bar{p}p_z}(S^{\xi,j}) \subseteq N_j$ . In particular,  $S^{\xi,j} = \mathbb{1}$  if  $\xi(j) = 0$ .

Proof. Let  $\xi \in \Xi$  and  $j \in [d]$ . If  $\xi(j) = 0$ , then  $g^{\xi,j} = g^{\xi,j+1}$  and  $S^{\xi,j} := \mathbb{1}$  trivially satisfies the claim. Otherwise, the matrix  $S^{\xi,j}$  is obtained from the induction hypothesis: The number of parameters increased by one, but we consider tuples of length k-1. We continue to consider  $\mathbb{Z}_{2^q}$ .

• Clearly,  $q \ge q(k) \ge q(k-1)$ , the degree of G is  $d \ge d(k,m) \ge d(k-1,m+1)$ , and the girth of G is at least 4r(k) + 2 > 2r(k).

- We have  $2k + m + 1 \ge 2(k 1) + (m + 1) + 1$  and so G satisfies the connectivity condition.
- By construction, we have that  $g^{\xi,j}(e) = g^{\xi,j+1}(e)$  for all  $e \in E \setminus \{\{t_j,t_j'\}\}$  for every  $j \in [d]$ .
- We consider the value of the twist: Let  $j \in [d]$ . Then  $\xi(j) = b \cdot 2^{\theta(k-1)}$  for some  $b \in \mathbb{Z}_{2^q}$  (as shown before when constructing the blurrer  $\Xi$ ). If  $j \neq 1$ , then it holds that  $g^{\xi,j+1}(\{t_j,t_j'\}) = g^{\xi,j}(\{t_j,t_j'\}) b \cdot 2^{\theta(k-1)}$ . If otherwise j=1, then we have  $g^{\xi,2}(\{t_1,t_1'\}) = g(\{t_1,t_1'\}) \xi(1) = g^{\xi,1}(\{t_1,t_1'\}) \xi(1) + \theta$  because  $g^{\xi,1} = f$  and  $g(\{t_1,t_1'\}) = f(\{t_1,t_1'\}) + \theta$ . By assumption, we have that  $\theta = a \cdot 2^{\theta(k)} = a \cdot 2^{i+\theta(k-1)}$  for some  $a \in \mathbb{Z}_{2^q}$ . Clearly  $-\xi(1) + \theta = -b \cdot 2^{\theta(k-1)} + a \cdot 2^{i+\theta(k-1)} = (a \cdot 2^i b) \cdot 2^{\theta(k-1)}$ . So the value of the twist at the  $\{t_j,t_j'\}$  is in all cases  $c \cdot 2^{\theta(k-1)}$  for some  $c \in \mathbb{Z}_{2^q}$ .
- By Claim 1, it holds that  $\mathsf{dist}_G(t_i',\mathsf{orig}(\bar{p})) \geq 2r(k)$  for every  $i \in [d]$ , in particular  $\mathsf{dist}_G(t_i,\mathsf{orig}(\bar{p})) > r(k)$ . By construction, it holds that  $\mathsf{dist}_G(t_i',\mathsf{orig}(p_z)) = \mathsf{dist}_G(t_i',z) = r(k) + 1$ . So  $\mathsf{dist}_G(t_i,\mathsf{orig}(\bar{p}p_z)) > r(k)$  for every  $i \in [d]$ .

We now define for every  $\xi \in \Xi$  the  $A^{k-1} \times A^{k-1}$  matrix  $S^{\xi}$  as follows:

$$S^{\xi} := S^{\xi,1} \cdot \ldots \cdot S^{\xi,d},$$

where  $S^{\xi,j}$  is the matrix given by Claim 11 for  $\xi$  and j.

Claim 12. For every  $\xi \in \Xi$ , the matrix  $S^{\xi}$  is odd-filled and both orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_f, \bar{p}p_z)$  and  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$ , (k-1)-blurs the twist between  $(\mathfrak{A}_f, \bar{p}p_z)$  and  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$ , and satisfies  $\mathsf{A}^{f,g-\xi,\bar{p}p_z}(S^{\xi}) \subseteq \bigcup_{i=1}^d N_i$ .

*Proof.* For every  $j \in [d]$ , the matrix  $S^{\xi,j}$  is odd-filled and both orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_{g^{\xi,j}}, \bar{p}p_z)$  and  $(\mathfrak{A}_{g^{\xi,j+1}}, \bar{p}p_z)$ , (k-1)-blurs the twist between  $(\mathfrak{A}_{g^{\xi,j}}, \bar{p}p_z)$  and  $(\mathfrak{A}_{g^{\xi,j+1}}, \bar{p}p_z)$ , and satisfies  $\mathsf{A}^{g^{\xi,j},g^{\xi,j+1},\bar{p}p_z}(S^{\xi,j}) \subseteq N_j$  by Claim 11. Because  $f = g^{\xi,1}$  and  $g - \xi = g^{\xi,d+1}$ ,  $S^{\xi} = S^{\xi,1} \cdot \ldots \cdot S^{\xi,d}$  is orbit-diagonal and

Because  $f = g^{\xi,1}$  and  $g - \xi = g^{\xi,d+1}$ ,  $S^{\xi} = S^{\xi,1} \cdot \ldots \cdot S^{\xi,d}$  is orbit-diagonal and orbit-invariant over  $(\mathfrak{A}_f, \bar{p}p_z)$  and  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$  by Lemmas 47 and 28. By Lemma 30, the matrix  $S^{\xi}$  is odd-filled. It (k-1)-blurs the twist between  $(\mathfrak{A}_f, \bar{p}p_z)$  and  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$  by Lemma 25. Finally,  $\mathsf{A}^{f,g-\xi,\bar{p}p_z}(S^{\xi}) \subseteq \bigcup_{i=1}^d \mathsf{A}^{g^{\xi,j},g^{\xi,j+1},\bar{p}p_z}(S^{\xi,j}) \subseteq \bigcup_{i=1}^d N_i$  by Lemma 46.  $\dashv$ 

Claim 13. For every pair of distinct  $i, j \in [d]$  it holds that  $N_i \cap N_j = \emptyset$ .

Proof. Let  $i \neq j$ . Assume that there is an  $x \in N_i \cap N_j$ . Then there is a path from  $t_i$  to  $t_j$  of length at most 2r(k). By construction  $\mathsf{dist}_G(t_i, z) = \mathsf{dist}_G(t_j, z) = r(k) + 2$  and so  $z \notin N_i$  and  $z \notin N_j$ . But that means that there is a cycle of length at most  $\mathsf{dist}_G(t_i, z) + \mathsf{dist}_G(t_j, z) + \mathsf{dist}_G(t_i, x) + \mathsf{dist}_G(t_j, x) \leq 4r(k) + 4$  contradicting that G has girth at least  $2r(k+1) \geq 8r(k) + 4$ .

Claim 14. For every  $\xi \in \Xi$ , every  $P \in \operatorname{orbs}_{k-1}((\mathfrak{A}_f, \bar{p}p_z))$ ,  $Q \in \operatorname{orbs}_{k-1}((\mathfrak{A}_{g-\xi}, \bar{p}p_z))$  of the same type in  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$  as P in  $(\mathfrak{A}_f, \bar{p}p_z)$ ,  $\bar{u} \in P$ ,  $\bar{v} \in Q$ , and  $\varphi \in \operatorname{Aut}((\mathfrak{A}_f, \bar{p}))$  the matrix  $S^{\xi}$  satisfies  $S^{\xi}(\bar{u}, \bar{v}) = S^{\xi}(\varphi(\bar{u}), \varphi(\bar{v}))$ .

*Proof.* Let  $\xi \in \Xi$ . By Claim 12, the matrix  $S^{\xi}$  is orbit-invariant over  $(\mathfrak{A}_f, \bar{p}p_z)$  and  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$  and thus satisfies the claim for all  $\varphi \in \mathsf{Aut}((\mathfrak{A}_f, \bar{p}p_z))$ . But now we also want to consider automorphisms not fixing  $p_z$ .

So let  $P \in \mathsf{orbs}_{k-1}((\mathfrak{A}_f,\bar{p}p_z)), \ Q \in \mathsf{orbs}_{k-1}((\mathfrak{A}_{g-\xi},\bar{p}p_z))$  be of the same type in  $(\mathfrak{A}_{g-\xi},\bar{p}p_z)$  as P in  $(\mathfrak{A}_f,\bar{p}p_z), \ \bar{u} \in P, \ \bar{v} \in Q, \ \text{and} \ \varphi \in \mathsf{Aut}((\mathfrak{A}_f,\bar{p})).$  Let R be the set of components of  $\bar{u}$  (and thus of  $\bar{v}$ ) containing a vertex of  $N_G^1(z)$  (so in particular z itself). Let D be the set of the remaining components.

Because  $\bar{u}_D\bar{v}_D$  and  $\varphi(\bar{u}_D\bar{v}_D)$  are in the same orbit in  $(\mathfrak{A}_{f+\bar{a}},\bar{p})$ , they are also in the same orbit in  $(\mathfrak{A}_{f+\bar{a}},\bar{p}p_z)$  by Claim 8. Hence, there is a  $\psi \in \operatorname{Aut}((\mathfrak{A}_{f+\bar{a}},\bar{p}p_z))$  satisfying  $\varphi(\bar{u}_D\bar{v}_D) = \psi(\bar{u}_D\bar{v}_D)$ . We now apply that  $S^{\xi}$  is orbit-invariant over  $(\mathfrak{A}_f,\bar{p}p_z)$  and  $(\mathfrak{A}_{g-\xi},\bar{p}p_z)$ :

$$S^{\xi}(\bar{u}, \bar{v}) = S^{\xi}(\bar{u}_R \bar{u}_D, \bar{v}_R \bar{v}_D)$$

$$= S^{\xi}(\bar{u}_R \psi(\bar{u}_D), \bar{v}_R \psi(\bar{v}_D))$$

$$= S^{\xi}(\bar{u}_R \varphi(\bar{u}_D), \bar{v}_R \varphi(\bar{v}_D)). \qquad (\star)$$

For every  $C \in R$  it holds that  $C \not\subseteq \mathsf{A}^{f,g-\xi,\bar{p}p_z}(S^\xi) \subseteq \bigcup_{i=1}^d N_i$  because  $N_G^1(z) \cap \bigcup_{i=1}^d N_i = \emptyset$ , which follows from  $N_i = N_G^{r(k)}(t_i)$  and  $\mathsf{dist}_G(z,t_i) = r(k)+2$ . So we can apply Condition 2 of the active region because  $\bar{u}_R = \bar{v}_R$  if and only if  $\varphi(\bar{u}_R) = \varphi(\bar{v}_R)$ :

$$(\star) = S^{\xi}(\varphi(\bar{u}_{D_z})\bar{u}_{D_R}, \varphi(\bar{v}_{D_z})\bar{v}_{D_R})$$
$$= S^{\xi}(\varphi(\bar{v}), \varphi(\bar{v})). \qquad \exists$$

Claim 15. Let  $k' \leq k-1$ ,  $\xi \in \Xi$ ,  $P' \in \operatorname{orbs}_{k'}((\mathfrak{A}_f, \bar{p}p_z))$ ,  $K \subseteq [d]$ , D be the set of all components C of P' satisfying  $C \subseteq N_i$  for some  $i \in K$ , and R be the set of remaining components. Let  $Q' = \tau(P') \in \operatorname{orbs}_{k'}((\mathfrak{A}_g, \bar{p}p_z))$ ,  $\bar{u} \in P'$ , and  $\bar{v} \in Q'$ . Then

$$\sum_{\bar{w}_D \in P'|_D} S^{\xi}(\bar{w}_D \bar{u}_R, \bar{v}_D \bar{v}_R) = \left(\prod_{i \in [d] \setminus K} S^{\xi, i}\right) (\bar{v}_D \bar{u}_R, \bar{v}_D \bar{v}_R)$$

and  $\mathsf{A}^{f,f+\bar{a},\bar{p}p_z}(\prod_{i\in[d]\setminus K}S^{\xi,i})\subseteq\bigcup_{i\in[d]\setminus K}N_i$ , where  $\bar{a}\in\mathbb{Z}_{2^q}^d$  satisfies  $a_i=(g-\xi)(\{t_i,t_i'\})-f(\{t_i,t_i'\})$  if  $i\notin K$  and  $a_i=0$  otherwise for every  $i\in[d]$ .

*Proof.* Recall that  $S^{\xi} = S^{\xi,1} \cdot \ldots \cdot S^{\xi,d}$  and that  $\mathsf{A}^{g^{\xi,j},g^{\xi,j+1},\bar{p}p_z}(S^{\xi,j}) \subseteq N_j$  for every  $j \in [d]$  by Claim 11. The first part of the claim follows from repeated application of Lemma 48 using that the sets  $N_j$  are disjoint (Claim 13). The second part follows from repeated application of Lemmas 45 and 46.

We introduce more notation. Let  $\bar{u} \in A^{k'}$  such that  $z \in \text{orig}(\bar{u})$ . Then  $\bar{u}^{-z} \in A^{k'-1}$  is the tuple obtained from  $\bar{u}$  by deleting the first entry with origin z. This first entry is denoted by  $\bar{u}_z$ . Similarly to our convention for  $\bar{u}_C$  for a component C in Section 5.2, we write  $\bar{u}_z\bar{u}^{-z}$  not for concatenation but for inserting  $\bar{u}_z$  at the correct position such that  $\bar{u}_z\bar{u}^{-z} = \bar{u}$ . Now we are ready to define the  $A^k \times A^k$  matrix S. For  $j \in \{k, 2k\}$  we set  $\mathbf{P}_j := \text{orbs}_j((\mathfrak{A}_f, \bar{p}))$ . We define the  $P \times \tau(P)$  block of S for every  $P \in \mathbf{P}_k$ . All other blocks are zero.

$$S_{P\times\tau(P)}(\bar{u},\bar{v}) := \begin{cases} \sum\limits_{\xi\in\Xi,\\\tau(\xi(\bar{u}))=\bar{v}} 1 & \text{if } P \text{ is blurrable,} \\ \sum\limits_{\xi\in\Xi,\\\tau_{\xi}(\xi(\bar{u}_{z}))=\bar{v}_{z}} S^{\xi}(\xi(\bar{u}^{-z}),\tau_{\xi}^{-1}(\bar{v}^{-z})) & \text{if } P \text{ is not blurrable.} \end{cases}$$

We first check that we make reasonable use of the matrices  $S^{\xi}$ :

Claim 16. Let  $P \in \mathbf{P}_k$  be non-blurrable,  $\bar{u} \in P$ , and  $\bar{v} \in \tau(P)$ . If  $\bar{u}$  has the same type in  $(\mathfrak{A}_f, \bar{p})$  as  $\bar{v}$  has in  $(\mathfrak{A}_g, \bar{p})$ , then for every  $\xi \in \Xi$  such that  $\tau_{\xi}(\xi(\bar{u}_z)) = \bar{v}_z$ , the tuple  $\xi(\bar{u}^{-z})$  has the same type in  $(\mathfrak{A}_f, \bar{p}p_z)$  as  $\tau_{\xi}^{-1}(\bar{v}^{-z})$  has in  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$ .

Proof. Let  $\xi \in \Xi$  such that  $\tau_{\xi}(\xi(\bar{u}_z)) = \bar{v}_z$ . Let R be the star component of  $\operatorname{orig}(P)$  containing z and let D be the set of remaining components of  $\operatorname{orig}(P)$ . Then  $P = P|_R \times P|_D$  by Lemma 19. In particular  $\tau(P) = P|_R \times \tau(P|_D)$  because  $\tau$  is the identity on star components. The orbit  $\tau(P|_D)$  has the same type in  $(\mathfrak{A}_g, \bar{p})$  as  $P|_D$  has in  $(\mathfrak{A}_f, \bar{p})$  (Corollary 60) and  $\tau_{\xi}(\tau(P|_D))$  has thus the same type in  $(\mathfrak{A}_{g-\xi}, \bar{p})$  as  $P|_D$  has in  $(\mathfrak{A}_f, \bar{p})$  (Claim 3). Because  $z \in R$  (P is not blurrable), all components in D have distance at least 2 to z. Thus,  $P|_D$  is also an orbit of  $(\mathfrak{A}_f, \bar{p}p_z)$  (Claim 8) and has the same type in  $(\mathfrak{A}_f, \bar{p}p_z)$  as  $\tau_{\xi}(\tau(P|_D))$  has in  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$ . Because  $\xi$  is an orbit-automorphism (Claim 6),  $\xi(P|_D) = P|_D$ . It follows that  $\xi(\bar{u}_D) \in P|_D$  and  $\tau_{\xi}(\bar{v}_D) \in \tau_{\xi}(\tau(P|_D))$  have the same type.

It suffices to show that  $\xi(\bar{u}_R)$  has the same type in  $(\mathfrak{A}_f,\bar{p}p_z)$  as  $\tau_{\xi}(\bar{v}_R)$  has in  $(\mathfrak{A}_{g-\xi},\bar{p}p_z)$ . Because R contains star components,  $\tau_{\xi}$  is the identity on R and so  $\bar{v}_R \in P|_R$ . Further,  $\xi(\bar{u}_R) \in P|_R$  because  $\xi$  is an orbit-automorphism. That is, there is an automorphism  $\psi \in \operatorname{Aut}((\mathfrak{A}_f,\bar{p}))$  such that  $\psi(\xi(\bar{u}_R)) = \bar{v}_R$ . Because by assumption  $\tau_{\xi}(\xi(\bar{u}_z)) = \xi(\bar{u}_z) = \bar{v}_z$  ( $\tau_{\xi}$  is the identity on vertices with origin z),  $\psi$  is the identity on vertices with origin z and thus  $\psi \in \operatorname{Aut}((\mathfrak{A}_f,\bar{p}p_z))$ . That is,  $\xi(\bar{u}_R)$  and  $\bar{v}_R$  are in the same orbit of  $(\mathfrak{A}_f,\bar{p}p_z)$ . Because R contains star components, it does not contain any edge twisted by f and  $g-\xi$  and thus  $\xi(\bar{u}_R)$  and  $\bar{v}_R$  have the same type in  $(\mathfrak{A}_f,\bar{p}p_z)$  and in  $(\mathfrak{A}_{g-\xi},\bar{p}p_z)$ .

Claim 17. The matrix S is orbit-diagonal over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ .

*Proof.* By definition, the only nonzero blocks of S are the  $P \times \tau(P)$  blocks. By Corollary 60, P and  $\tau(P)$  have the same type for every  $P \in \mathbf{P}_k$ .

Claim 18. The matrix S is orbit-invariant over  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ .

Proof. Let  $\varphi \in \operatorname{Aut}((\mathfrak{A}_f, \bar{p})) = \operatorname{Aut}((\mathfrak{A}_g, \bar{p}))$ ,  $P \in \mathbf{P}_k$ ,  $\bar{u} \in P$ , and  $\bar{v} \in Q := \tau(P)$ . We perform a case distinction: Assume that P is blurrable. The functions  $\tau$ ,  $\xi$ , and  $\varphi$  commute (Claim 5) and thus  $\tau(\xi(\varphi(\bar{u}))) = \varphi(\tau(\xi(\bar{u})))$  for every  $\xi \in \Xi$ . Because  $\varphi$  is a bijection,  $\varphi(\tau(\xi(\bar{u}))) = \varphi(\bar{v})$  if and only if  $\tau(\xi(\bar{u})) = \bar{v}$ . So

$$S(\varphi(\bar{u}), \varphi(\bar{v})) = \sum_{\substack{\xi \in \Xi, \\ \tau(\xi(\varphi(\bar{u}))) = \varphi(\bar{v})}} 1 = \sum_{\substack{\xi \in \Xi, \\ \varphi(\tau(\xi(\bar{u}))) = \varphi(\bar{v})}} 1 = \sum_{\substack{\xi \in \Xi, \\ \tau(\xi(\bar{u})) = \bar{v}}} 1 = S(\bar{u}, \bar{v}).$$

Otherwise, assume that P is non-blurrable. Then for every  $\xi \in \Xi$  the following holds because  $\tau_{\xi}$ ,  $\xi$ , and  $\varphi$  commute by Claim 5 and because  $S^{\xi}$  is invariant under  $\varphi$  by Claim 14:

$$\begin{split} S^{\xi}(\xi(\varphi(\bar{u}^{-z})),\tau_{\xi}^{-1}(\varphi(\bar{v}^{-z}))) \\ &= S^{\xi}(\varphi(\xi(\bar{u}^{-z})),\varphi(\tau_{\xi}^{-1}(\bar{v}^{-z}))) \\ &= S^{\xi}(\xi(\bar{u}^{-z}),\tau_{\xi}^{-1}(\bar{v}^{-z})). \end{split}$$

Because  $\tau_{\xi}$ ,  $\xi$ , and  $\varphi$  commute and because  $\varphi$  is a bijection, it holds that  $\tau_{\xi}(\xi(\varphi(\bar{u}_z))) = \varphi(\bar{v}_z)$  if and only if  $\tau_{\xi}(\xi(\bar{u}_z)) = \bar{v}_z$ . Hence,

$$\begin{split} S(\varphi(\bar{u}), \varphi(\bar{v})) &= \sum_{\substack{\xi \in \Xi, \\ \tau_{\xi}(\xi(\varphi(\bar{u}_{z}))) = \varphi(\bar{v}_{z}) \\ = \sum_{\substack{\xi \in \Xi, \\ \tau_{\xi}(\xi(\bar{u}_{z})) = \bar{v}_{z}}} S^{\xi}(\xi(\varphi(\bar{u}^{-z})), \tau_{\xi}^{-1}(\bar{v}^{-z})) = S(\bar{u}, \bar{v}). \end{split}$$

#### Claim 19. The matrix S is odd-filled.

*Proof.* Let  $P \in \mathbf{P}_k$  and  $\bar{u} \in P$ . Then  $\tau(\bar{u}) \in \tau(P)$  and because every  $\xi \in \Xi$  is an orbit-automorphism (Claim 6), it holds that  $\xi(\bar{u}) \in P$  and  $\tau(\xi(\bar{u})) \in \tau(P)$  for every  $\xi \in \Xi$ , too.

Assume first that P is blurrable. Now, we sum up (over  $\mathbb{F}_2$ ) the entries in the row indexed by  $\bar{u}$ :

$$\sum_{\bar{v}\in\tau(P)} S(\bar{u},\bar{v}) = \sum_{\bar{v}\in\tau(P)} \sum_{\substack{\xi\in\Xi,\\\tau(\xi(\bar{u}))=\bar{v}}} 1 = \sum_{\substack{\xi\in\Xi,\\\tau(\xi(\bar{u}))\in\tau(P)}} 1 = |\Xi| \bmod 2.$$

The last step holds, because – as seen –  $\tau(\xi(\bar{u})) \in \tau(P)$  for every  $\xi \in \Xi$ . Finally,  $|\Xi|$  is odd by Lemma 51 and so the number of ones in the row indexed by  $\bar{u}$  is odd, too.

Assume otherwise that P is not blurrable and set  $Q := \tau(P)$ , which is of the same type as P (Corollary 60). For every  $\xi \in \Xi$  we set  $Q_{\xi} := \{\bar{v}^{-z} \mid \bar{v} \in Q, \tau_{\xi}(\xi(\bar{u}_z)) = \bar{v}_z\}$ . Then  $Q_{\xi} \in \mathsf{orbs}_{k-1}((\mathfrak{A}_g, \bar{p}p_z)) = \mathsf{orbs}_{k-1}((\mathfrak{A}_{g-\xi}, \bar{p}p_z))$  by Corollaries 21 and 15 (the center z has distance greater than 1 to  $\mathsf{orig}(\bar{p})$ ).

$$\sum_{\bar{v} \in Q} S(\bar{u}, \bar{v}) = \sum_{\bar{v} \in Q} \sum_{\substack{\xi \in \Xi, \\ \tau_{\xi}(\xi(\bar{u}_{z})) = \bar{v}_{z}}} S^{\xi}(\xi(\bar{u}^{-z}), \tau_{\xi}^{-1}(\bar{v}^{-z}))$$

$$= \sum_{\xi \in \Xi} \sum_{\substack{\bar{v} \in Q, \\ \tau_{\xi}(\xi(\bar{u}_{z})) = \bar{v}_{z}}} S^{\xi}(\xi(\bar{u}^{-z}), \tau_{\xi}^{-1}(\bar{v}^{-z}))$$

$$= \sum_{\xi \in \Xi} \sum_{\bar{v} \in Q_{\xi}} S^{\xi}(\xi(\bar{u}^{-z}), \tau_{\xi}^{-1}(\bar{v}))$$

$$= \sum_{\xi \in \Xi} \sum_{\bar{v} \in Q_{\xi}} S^{\xi}(\xi(\bar{u}^{-z}), \bar{v}). \qquad (\star)$$

 $\dashv$ 

In the first line of the equation,  $\xi(\bar{u}^{-z})$  has the same type in  $(\mathfrak{A}_f, \bar{p}p_z)$  as  $\tau_{\xi}^{-1}(\bar{v}^{-z})$  has in  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$  for every  $\xi \in \Xi$  such that  $\tau_{\xi}(\xi(\bar{u}_z)) = \bar{v}_z$  (Claim 16). One sees that we always sum over the same column indices of  $S^{\xi}$  (for a fixed  $\xi$ ) and we only manipulate the way in which we express the sum. Hence, in the last line,  $\xi(\bar{u}^{-z})$  has the same type in  $(\mathfrak{A}_f, \bar{p}p_z)$  as  $\bar{v}$  has in  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$ . By Claim 9,  $\tau_{\xi}^{-1}(Q_{\xi}) \in \operatorname{orbs}_{k-1}((\mathfrak{A}_{g-\xi}, \bar{p}p_z))$ , so  $\sum_{\bar{v} \in \tau_{\xi}^{-1}(Q_{\xi})} S^{\xi}(\xi(\bar{u}^{-z}), \bar{v}) = 1$  because we sum over all entries in one row of a block on the diagonal of  $S^{\xi}$  and  $S^{\xi}$  is odd-filled (Claim 12). It follows that

$$(\star) = \sum_{\xi \in \Xi} 1 = |\Xi| \mod 2 = 1.$$

For the last step we used again that  $|\Xi|$  is odd by Lemma 51.

Now it follows from Lemma 33 that S is invertible.

Claim 20.  $A^{f,g,\bar{p}}(S) \subseteq N_G^{r(k+1)}(t)$ .

*Proof.* We show that  $N_G^{r(k+1)}(t)$  satisfies the conditions of the active region. This implies that  $\mathsf{A}^{f,g,\bar{p}}(S)\subseteq N_G^{r(k+1)}(t)$ . To show Condition 1 of the active region, we have to show that  $C\subseteq N_G^{r(k+1)}(t)$  for every component  $C\in \mathsf{A}^{f,g,\bar{p}}(S,P)$  for every  $P\in \mathbf{P}_k$ .

Let  $P \in \mathbf{P}_k$  be blurrable and C be a component of P. By definition of  $\xi$  and  $\tau$ , the matrix S is only active on C if C is a star or a tip component. But this means that  $C \subseteq N_G^{r(k)+2+k}(z)$  because C is connected, of size at most k, and contains a vertex x of some  $\bar{s}_i$  (which have length r(k) + 2). Because  $\mathrm{dist}_G(z,t) = r(k) + 2$ , every vertex with distance at most r(k) + 2 + k to z has distance at most  $2r(k) + 4 + k \le 4r(k) + 2 = r(k+1)$  to t (one immediately sees that  $r(k) \ge k \ge 2$ ). Thus,

$$C \subseteq N_G^{r(k)+2+k}(z) \subseteq N_G^{r(k+1)}(t).$$

Let otherwise  $P \in \mathbf{P}_k$  be non-blurrable and C be a component of P. Then S is possibly active on C if  $C \subseteq N_G^{r(k)+2+k}(t)$  (as seen in the blurrable case) or

$$C \subseteq \mathsf{A}^{f,g-\xi,\bar{p}p_z}(S^\xi) \subseteq \bigcup_{i \in d} N_i$$

for some  $\xi \in \Xi$  (Claim 12). Recall that  $N_i = N_G^{r(k)}(t_i)$  and that  $\operatorname{dist}_G(t_i, z) = r(k) + 2$  by construction. If follows that

$$\bigcup_{i \in d} N_i \subseteq N_G^{4r(k)+2}(z) = N_G^{r(k+1)}(t)$$

because every vertex with distance r(k) to some  $t_i$  has distance at most  $3r(k) + 4 \le 4r(k) + 2$  to t (we again use that  $r(k) \ge 2$  for  $k \ge 2$ ).

To prove Condition 2 of the active region, we see that  $\xi$ ,  $\tau$ , and  $\tau_{\xi}$  are defined component-wise (for every  $\xi \in \Xi$ ) and that  $A^{f,g-\xi,\bar{p}p_z}(S^{\xi}) \subseteq \bigcup_{i\in d} N_i$ .

Now, we want to show that S actually k-blurs the twist. For  $P \in \mathbf{P}_{2k}$  we set  $P_1 := P|_{\{1,\dots,k\}} \in \mathbf{P}_k$  and  $P_2 := P|_{\{k+1,\dots,2k\}} \in \mathbf{P}_k$  to be the unique k-orbits such that  $P \subseteq P_1 \times P_2$  (and similar for a  $Q \in \mathbf{Q}_{2k}$ ). Our aim is to prove that  $\chi^P \cdot S = S \cdot \chi^Q$  for every  $P \in \mathbf{P}_{2k}$  and  $Q := \tau(P)$ . Because S is orbit-diagonal and  $\chi^P_{P_1 \times P_2}$  and  $\chi^Q_{Q_1 \times Q_2}$  are the only nonzero blocks of  $\chi^P$  and  $\chi^Q$ , it suffices to show that  $\chi^P_{P_1 \times P_2} S_{P_2 \times Q_2} = S_{P_1 \times Q_1} \chi^Q_{Q_1 \times Q_2}$ . We begin with blurrable orbits and define the set of indices  $i \in [d]$  of the blurrer  $\Xi$  which are relevant for a blurrable orbit.

**Definition 63.** The occupied indices Occ(P) of a blurrable orbit  $P \in \mathbf{P}_k$  is the set of indices  $i \in [d]$ , such that there is a component C of P that satisfies  $C \subseteq N_i$  or C is an i-star component.

Note that the definition also covers *i*-tip components because every *i*-tip component is contained in  $N_i$ . Also note that  $|\mathsf{Occ}(P)| \leq k$  because P is a k-orbit and thus we can use the blurrer properties later. The following lemma states that if one of  $P_1$  and  $P_2$  is blurrable, say  $P_1$ , then it does not matter whether we apply  $\xi$  or  $\xi'$  to  $P_2$  as long as  $\xi$  and  $\xi'$  agree on  $\mathsf{Occ}(P_1)$ .

Claim 21. Suppose  $P \in \mathbf{P}_{2k}$ ,  $\xi, \xi' \in \Xi$ , and  $\bar{u}, \bar{v} \in A^k$ . If  $P_1 \in \mathbf{P}_k$  is blurrable and  $\xi|_{\mathsf{Occ}(P_1)} = \xi'|_{\mathsf{Occ}(P_1)}$ , then  $\bar{u}\xi(\bar{v}) \in P$  if and only if  $\bar{u}\xi'(\bar{v}) \in P$ . Likewise, if  $P_2$  is blurrable and  $\xi|_{\mathsf{Occ}(P_2)} = \xi'|_{\mathsf{Occ}(P_2)}$ , then  $\xi(\bar{u})\bar{v} \in P$  if and only if  $\xi'(\bar{u})\bar{v} \in P$ . The same holds for  $Q := \tau(P)$ .

*Proof.* Consider the case that  $P_1$  is blurrable. Assume that  $\bar{u}\xi(\bar{v}) \in P$ . Let  $C_1^i, \ldots, C_{\ell_i}^i$  be the components of  $P_i$  for every  $i \in [2]$ . Because  $\tau$  and  $\xi$  are defined component-wise, it suffices by Lemma 19 to assume that P is a k'-orbit of  $(\mathfrak{A}_f, \bar{p})$  for some  $k' \leq k$  and has a single component C. The component C is the union of all  $C_i^i$ .

First consider the case that C is a star component. Because  $P_1$  is blurrable, we can partition all components  $C_j^1$  by Claim 2 into  $D_1^1, \ldots, D_d^1$  and  $D_R^1$  such that  $D_i^1$  contains all i-star components for all  $i \in [d]$  and  $D_R^1$  the remaining sky components. Set  $\xi'' := \xi' - \xi$ . Then  $\xi''|_{\mathsf{Occ}(P_1)} = 0$ ,  $\xi''(\xi(\bar{v})) = \xi'(\bar{v})$ , and  $\xi''$  is the identity function on the components in all  $D_i^1$  because every  $D_i^1$  only contains i-star components and is nonempty only if  $i \in \mathsf{Occ}(P_1)$ . Additionally,  $\xi''$  is the identity on the sky components  $D_R^1$ . Hence  $\varphi_{\xi''}(\bar{u}) = \bar{u}$  by Definition 9. Because C is a star component, we have that

$$\xi''(\bar{u}\xi(\bar{v})) = \varphi_{\xi''}(\bar{u}\xi(\bar{v})) = \varphi_{\xi''}(\bar{u})\varphi_{\xi''}(\xi(\bar{v})) = \bar{u}\xi''(\xi(\bar{v})) = \bar{u}\xi'(\bar{v}).$$

Because  $\xi''$  is an orbit-automorphism by Claim 6, it holds that  $\bar{u}\xi(\bar{v}) \in P$  if and only if  $\xi''(\bar{u}\xi(\bar{v})) = \bar{u}\xi'(\bar{v}) \in P$ .

If otherwise C is not a star component, then none of the  $C_j^i$  is a star component and  $\xi$  and  $\xi'$  are the identity function on  $\bar{v}$  and the claim follows immediately. The cases for  $P_2$  and Q are analogous.

Claim 22. For every  $P \in \mathbf{P}_{2k}$ ,  $\xi \in \Xi$  such that  $\xi|_{\mathsf{Occ}(P)} = \xi_{\mathsf{twst}}|_{\mathsf{Occ}(P)}$ , and  $\bar{u}, \bar{v} \in A^k$  it holds that  $\bar{u}\bar{v} \in P$  if and only if  $\tau(\xi(\bar{u}))\tau(\xi(\bar{v})) \in \tau(P)$ .

Proof. Let  $P \in \mathbf{P}_{2k}$ ,  $\xi \in \Xi$  such that  $\xi|_{\mathsf{Occ}(P)} = \xi_{\mathsf{twst}}|_{\mathsf{Occ}(P)}$ , and  $\bar{u}, \bar{v} \in A^k$ . Using Claim 7 we obtain  $\bar{u}\bar{v} \in P$  if and only if  $\tau_{\xi}(\xi(\bar{u}))\tau_{\xi}(\xi(\bar{v})) \in \tau_{\xi}(P)$ . Because  $\xi|_{\mathsf{Occ}(P)} = \xi_{\mathsf{twst}}|_{\mathsf{Occ}(P)}$ , the action of  $\tau_{\xi}$  and  $\tau$  is equal on the *i*-tip components of P for  $i \in \mathsf{Occ}(P)$ . Thus,  $\tau_{\xi}(P) = \tau(P)$ . Similarly,  $\tau_{\xi}(\xi(\bar{u})) = \tau(\xi(\bar{u}))$  and  $\tau_{\xi}(\xi(\bar{v})) = \tau(\xi(\bar{v}))$ .

For every blurrable orbit  $P \in \mathbf{P}_k$  it holds that  $|\mathsf{Occ}(P)| \leq k$  and so we can apply the blurrer properties as follows:

Claim 23. Let  $P \in \mathbf{P}_{2k}$  and  $Q = \tau(P)$ . If  $P_1$  (and so  $Q_1$ ) and  $P_2$  (and so  $Q_2$ ) are blurrable, then  $\chi^P \cdot S = S \cdot \chi^Q$ .

*Proof.* With the definition of S on blurrable orbits we obtain for  $\bar{u} \in P_1$  and  $\bar{v} \in Q_2$  that

$$(\chi^{P} \cdot S)(\bar{u}, \bar{v}) = \sum_{\bar{w} \in P_{2}} \chi^{P}(\bar{u}, \bar{w}) \cdot S_{P_{2} \times Q_{2}}(\bar{w}, \bar{v})$$

$$= \sum_{\bar{w} \in P_{2}} \chi^{P}(\bar{u}, \bar{w}) \cdot \sum_{\substack{\xi \in \Xi, \\ \tau(\xi(\bar{w})) = \bar{v}}} 1$$

$$= \sum_{\xi \in \Xi} \chi^{P}(\bar{u}, \xi^{-1}(\tau^{-1}(\bar{v}))). \tag{*}$$

Now for every  $\xi \in \Xi$ ,  $\chi^P(\bar{u}, \xi^{-1}(\tau^{-1}(\bar{v})))$  depends only on  $\xi|_{\mathsf{Occ}(P_1)}$  by Claim 21, i.e.,  $\xi \mapsto \chi^P(\bar{u}, \xi^{-1}(\tau^{-1}(\bar{v})))$  is actually a function  $\Xi|_{\mathsf{Occ}(P_1)} \to \mathbb{F}_2$ . Because  $P_1$  is a blurrable

k-orbit, it holds that  $|\operatorname{Occ}(P_1)| \leq k$ . Then, by Lemma 50, it follows that for some  $\xi_{\mathsf{tw}} \in \Xi$  with  $\xi_{\mathsf{tw}}|_{\operatorname{Occ}(P_1)} = \xi_{\mathsf{twst}}|_{\operatorname{Occ}(P_1)}$  we have that

$$\begin{split} (\star) &= \chi^P(\bar{u}, \xi_{\mathsf{tw}}^{-1}(\tau^{-1}(\bar{v}))) \\ &= \chi^Q(\tau(\xi_{\mathsf{tw}}(\bar{u})), \bar{v}) \\ &= \sum_{\xi \in \Xi} \chi^Q(\tau(\xi(\bar{u})), \bar{v}) \\ &= \sum_{\bar{w} \in Q_1} S_{P_1 \times Q_1}(\bar{u}, \bar{w}) \cdot \chi^Q(\bar{w}, \bar{v}) \\ &= (S \cdot \chi^Q)(\bar{u}, \bar{v}), \end{split}$$

where the transition from P to Q is by Claim 22. The last step is the inverse reasoning as for P using Claim 21.

Next, we want to consider the case that  $P_1$  is blurrable and  $P_2$  is not (or vice versa, which is symmetric). We would like to use Claim 22 as in the case that both  $P_1$  and  $P_2$  are blurrable. But this is not sufficient because  $S_{P_2 \times \tau(P_2)}$  is defined using the matrices  $S^{\xi}$ . We show that the matrices  $S^{\xi}$  cancel out in this case. Intuitively, the idea is to use that the active regions of the  $S^{\xi,i}$  are disjoint (Claim 13). As a consequence the matrices  $S^{\xi,i}$  for all  $i \notin \operatorname{Occ}(P_1)$  cancel out. For the remaining  $S^{\xi,i}$  we use the blurrer properties to show that they vanish.

Claim 24. Suppose  $P \in \mathbf{P}_{2k}$  and  $Q = \tau(P)$ . If  $P_1$  is blurrable and  $P_2$  is not, then  $\chi^P \cdot S = S \cdot \chi^Q$ .

*Proof.* We first unfold the definition of S. Let  $\bar{u} \in P_1$  and  $\bar{v} \in Q_2$ .

$$(\chi^{P} \cdot S)(\bar{u}, \bar{v}) = \sum_{\bar{w} \in P_{2}} \chi^{P}(\bar{u}, \bar{w}) \cdot S_{P_{2} \times Q_{2}}(\bar{w}, \bar{v})$$

$$= \sum_{\bar{w} \in P_{2}} \chi^{P}(\bar{u}, \bar{w}) \cdot \sum_{\substack{\xi \in \Xi, \\ \xi(\bar{w}_{z}) = \tau_{\xi}^{-1}(\bar{v}_{z})}} S^{\xi}(\xi(\bar{w}^{-z}), \tau_{\xi}^{-1}(\bar{v}^{-z}))$$

$$= \sum_{\xi \in \Xi} \sum_{\bar{w} \in P_{2}, \\ \xi(\bar{w}_{z}) = \tau_{\xi}^{-1}(\bar{v}_{z})} \chi^{P}(\bar{u}, \bar{w}) \cdot S^{\xi}(\xi(\bar{w}^{-z}), \tau_{\xi}^{-1}(\bar{v}^{-z})). \tag{*}$$

For every  $\xi \in \Xi$ , we set  $P'_{\xi,2} := \{\bar{w}^{-z} \mid \bar{w} \in P_2, \bar{w}_z = \xi(\bar{v}_z)\}$  (note that  $\tau_{\xi}^{-1}(\bar{v}_z) = \bar{v}_z$  because  $\operatorname{orig}(\bar{v}_z) = z$ ). Here  $\bar{w}\xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_z))$  denotes the tuple  $\bar{w}'$  such that  $\bar{w}'^{-z} = \bar{w}$  and  $\bar{w}'_z = \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_z))$ . It holds that  $P'_{\xi,2} \in \operatorname{orbs}_{k-1}((\mathfrak{A}_f, \bar{p}p_z))$  by Corollary 21. Then

$$(\star) = \sum_{\xi \in \Xi} \sum_{\bar{w} \in P'_{\xi,2}} \chi^{P}(\bar{u}, \bar{w}\xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_{z}))) \cdot S^{\xi}(\xi(\bar{w}), \tau_{\xi}^{-1}(\bar{v}^{-z})).$$

Let  $K \subseteq [d]$  be the maximal set of indices, such that there is no component C of  $P_1$  satisfying  $C \subseteq N_i$ . So in particular  $Occ(P_1) \subseteq [d] \setminus K$ . We partition the components of  $P_2$  as follows:

• Let D be the set of components C of  $P_2$  that are also components of P and satisfy  $C \subseteq N_i$  for some  $i \in K$ .

- Let E be the set of C of  $P_2$  that are not contained in D and satisfy  $C \subseteq N_i$  for some  $i \in [d]$ .
- Let R be the set of all remaining components of  $P_2$ .

We split  $\bar{w} = \bar{w}_D \bar{w}_E \bar{w}_R$  into the components belonging to D, E, and R. We split  $\bar{v} = \bar{v}_D \bar{v}_E \bar{v}_R \bar{v}_z$  likewise, where, for simplicity, we set  $\bar{v}_D := \bar{v}_D^{-z}$  and similar for  $\bar{v}_E$  and  $\bar{v}_R$ . From Lemma 19 it follows that  $P'_{\xi,2} = P'_{\xi,2}|_D \times P'_{\xi,2}|_{E \cup R}$  and  $P = P|_D \times P|_{R'}$ , where R' are the components of P not contained in D. By the definition of K and D, we have that  $P|_D = P'_{\xi,2}|_D$  because the components in D are components of P, are disjoint with  $\operatorname{orig}(P_1)$ , and do not contain z ( $z \notin N_i$  for all  $i \in [d]$ ). We obtain that

$$\begin{split} &(\star) \\ &= \sum_{\xi \in \Xi} \sum_{\bar{w}_D \in P'_{\xi,2}|_D} \sum_{\bar{w}_E \bar{w}_R \in P'_{\xi,2}|_{E \cup R}} \chi^P(\bar{u}, \bar{w}_D \bar{w}_E \bar{w}_R \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_z))) \cdot S^{\xi}(\xi(\bar{w}_D \bar{w}_E \bar{w}_R), \tau_{\xi}^{-1}(\bar{v}_D \bar{v}_E \bar{v}_R)) \\ &= \sum_{\xi \in \Xi} \sum_{\bar{w}_E \bar{w}_R \in P'_{\xi,2}|_{E \cup R}} \chi^{P|_{R'}}(\bar{u}, \bar{w}_E \bar{w}_R \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_z))) \cdot \sum_{\bar{w}_D \in P'_{\xi,2}|_D} S^{\xi}(\xi(\bar{w}_D \bar{w}_E \bar{w}_R), \tau_{\xi}^{-1}(\bar{v}_D \bar{v}_E \bar{v}_R)). \end{split}$$

By Claim 4, the functions  $\xi$  and  $\tau_{\xi}$  can be applied component-wise. For every  $\xi$ , the matrix  $S^{\xi}$  is not active on the components in R (by definition of R and Claim 12). Thus,  $\xi(\bar{w}_R) = \tau_{\xi}^{-1}(\bar{v}_R)$  unless  $S^{\xi}(\xi(\bar{w}_D\bar{w}_E\bar{w}_R), \tau_{\xi}^{-1}(\bar{v}_D\bar{v}_E\bar{v}_R)) = 0$ . We again use Lemma 19 to split  $P'_{\xi,2}|_{E\cup R} = P'_{\xi,2}|_E \times P'_{\xi,2}|_R$ . Such a split of P is not possible because the components of  $P'_{\xi,2}$  in E and R might not be components of P. Here we have  $\xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_R))\xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_z)) = \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_R\bar{v}_z))$  because if  $C \cup \{z\}$  is a component, then  $C \cup \{z\}$  is a star component and C is a union of star and sky components. Thus,

$$= \sum_{\xi \in \Xi} \sum_{\bar{w}_E \in P'_{\xi,2}|_E} \chi^{P|_{R'}}(\bar{u}, \bar{w}_E \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_R \bar{v}_z))) \cdot \sum_{\bar{w}_D \in P'_{\xi,2}|_D} S^{\xi}(\xi(\bar{w}_D \bar{w}_E) \tau_{\xi}^{-1}(\bar{v}_R), \tau_{\xi}^{-1}(\bar{v}_D \bar{v}_E \bar{v}_R)).$$

We know that  $P'_{\xi,2}|_D \in \operatorname{orbs}_{k'}((\mathfrak{A}_f, \bar{p}p_z))$  for some  $k' \leq k-1$  by Lemma 19. Because  $\xi$  is an orbit-automorphism for every  $\xi \in \Xi$  (Claim 6) and D contains no star center components, every  $\xi \in \Xi$  permutes  $P'_{\xi,2}|_D$ . Hence, we can sum over  $\xi(\bar{w}_D) \in P'_{\xi,2}|_D$  instead over  $\bar{w}_D \in P'_{\xi,2}|_D$ . Then we can apply Claim 15:

$$(\star) = \sum_{\xi \in \Xi} \sum_{\bar{w}_E \in P'_{\xi,2}|_E} \chi^{P|_{R'}}(\bar{u}, \bar{w}_E \xi^{-1}(\tau_\xi^{-1}(\bar{v}_R \bar{v}_z))) \cdot \\ \left( \prod_{i \in [d] \backslash K} S^{\xi,i} \right) (\tau_\xi^{-1}(\bar{v}_D) \xi(\bar{w}_E) \tau_\xi^{-1}(\bar{v}_R), \tau_\xi^{-1}(\bar{v}_D \bar{v}_E \bar{v}_R)).$$

We show that this term only depends on  $\xi|_{[d]\setminus K}$  as follows: Let  $\xi, \xi' \in \Xi$  such that  $\xi|_{[d]\setminus K} = \xi'|_{[d]\setminus K}$ .

(a) Consider the right term  $(\prod_{i \in [d] \setminus K} S^{\xi,i})(\cdot, \cdot)$  in the equation. First,  $S^{\xi,i} = S^{\xi',i}$  for every  $i \in [d] \setminus K$  by construction of the matrices  $S^{\xi,i}$  because  $\xi(i) = \xi'(i)$ . Because R does not contain tip components, it holds that  $\tau_{\xi}(\bar{v}_R) = \bar{v}_R$ . Second, all components in E are contained in some  $N_i$  for  $i \in [d] \setminus K$  by definition of E. That is,  $\xi(\bar{w}_E) = \bar{v}_R$ .

 $\xi'(\bar{w}_E)$  and  $\tau_{\xi}(\bar{v}_E) = \tau_{\xi'}(\bar{v}_E)$ . Third, the active region of  $\prod_{i \in [d] \setminus K} S^{\xi,i}$  is bounded by  $\bigcup_{i \in [d] \setminus K} N_i$  by Claim 15. Because components in  $D \cup R$  are not contained in  $\bigcup_{i \in [d] \setminus K} N_i$  by definition of D and R, we can exploit Condition 2 of the active region and apply  $\tau_{\xi}$  to  $\bar{v}_D$  and  $\bar{v}_R$  on both sides (because  $\tau_{\xi}$  is a bijection), that is,

$$\left(\prod_{i \in [d] \setminus K} S^{\xi,i}\right) \left(\tau_{\xi}^{-1}(\bar{v}_{D})\xi(\bar{w}_{E})\tau_{\xi}^{-1}(\bar{v}_{R}), \tau_{\xi}^{-1}(\bar{v}_{D}\bar{v}_{E}\bar{v}_{R})\right) 
= \left(\prod_{i \in [d] \setminus K} S^{\xi,i}\right) \left(\bar{v}_{D}\xi(\bar{w}_{E})\bar{v}_{R}, \bar{v}_{D}\tau_{\xi}^{-1}(\bar{v}_{E})\bar{v}_{R}\right) 
= \left(\prod_{i \in [d] \setminus K} S^{\xi,i}\right) \left(\bar{v}_{D}\xi'(\bar{w}_{E})\bar{v}_{R}, \bar{v}_{D}\tau_{\xi'}^{-1}(\bar{v}_{E})\bar{v}_{R}\right) 
= \left(\prod_{i \in [d] \setminus K} S^{\xi',i}\right) \left(\tau_{\xi'}^{-1}(\bar{v}_{D})\xi'(\bar{w}_{E})\tau_{\xi'}^{-1}(\bar{v}_{R}), \tau_{\xi'}^{-1}(\bar{v}_{D}\bar{v}_{E}\bar{v}_{R})\right).$$

(b) Now consider the left term  $\chi^{P|_{R'}}(\cdot,\cdot)$ . First, note that R does not contain tip components and thus  $\tau_{\xi}^{-1}(\bar{v}_R\bar{v}_z) = \bar{v}_R\bar{v}_z$ . Second,  $\bar{u}\bar{w}_E\xi^{-1}(\bar{v}_R\bar{v}_z) \in P|_{R'}$  if and only if  $\bar{u}\bar{w}_E\xi'^{-1}(\bar{v}_R\bar{v}_z) \in P|_{R'}$  by Claim 21: By repeating entries, one sees that Claim 21 also holds for k'-orbits with  $k' \leq 2k$  and a partition of [k'] into two parts each of size at most k. Hence,

$$\chi^{P|_{R'}}(\bar{u},\bar{w}_E\xi^{\text{-}1}(\tau_\xi^{\text{-}1}(\bar{v}_R\bar{v}_z))) = \chi^{P|_{R'}}(\bar{u},\bar{w}_E\xi'^{\text{-}1}(\tau_{\xi'}^{\text{-}1}(\bar{v}_R\bar{v}_z))).$$

Hence, the Equation  $(\star)$  is of the form  $\sum_{\xi \in \Xi} h(\xi|_{[d] \setminus K})$  for some function  $h \colon \Xi|_{[d] \setminus K} \to \mathbb{F}_2$ . Because  $P_1$  contains k-tuples, it holds that  $|K| \geq d - k$  and hence that  $|[d] \setminus K| \leq k$ . That is, we can apply Lemma 50 and obtain for some  $\xi_{\mathsf{tw}} \in \Xi$ , which satisfies that  $\xi_{\mathsf{tw}}|_{[d] \setminus K} = \xi_{\mathsf{twst}}|_{[d] \setminus K}$ , the following:

$$(\star) = \sum_{\bar{w}_E \in P'_{\xi_{\mathsf{tw}},2}|_E} \chi^{P|_{R'}} (\bar{u}, \bar{w}_E \xi_{\mathsf{tw}}^{-1} (\tau_{\xi_{\mathsf{tw}}}^{-1} (\bar{v}_R \bar{v}_z))) \cdot \\ \\ \left( \prod_{i \in [d] \backslash K} S^{\xi_{\mathsf{tw}},i} \right) (\tau_{\xi_{\mathsf{tw}}}^{-1} (\bar{v}_D) \xi_{\mathsf{tw}} (\bar{w}_E) \tau_{\xi_{\mathsf{tw}}}^{-1} (\bar{v}_R), \tau_{\xi_{\mathsf{tw}}}^{-1} (\bar{v}_D \bar{v}_E \bar{v}_R)).$$

Now, the matrices  $S^{\xi_{\mathsf{tw}},i}$  in the equation blur a twist of value 0 for every  $i \in [d] \setminus K$ :  $S^{\xi_{\mathsf{tw}},i}$  blurs a twist of value  $(g - \xi_{\mathsf{tw}} - f)(\{t_i, t_i'\}) = (\xi_{\mathsf{twst}} - \xi_{\mathsf{tw}})(i)$ , which is 0 for every  $i \in [d] \setminus K$  because  $\xi_{\mathsf{tw}}|_{[d]\setminus K} = \xi_{\mathsf{twst}}|_{[d]\setminus K}$ . That is,  $S^{\xi_{\mathsf{tw}},i} = 1$  by definition (cf. Claim 11) and  $\xi_{\mathsf{tw}}(\bar{w}_E) = \tau_{\xi_{\mathsf{tw}}}^{-1}(\bar{v}_E)$  unless the right factor is zero. Hence,

$$(\star) = \chi^{P|_{R'}}(\bar{u}, \xi_{\rm tw}^{-1}(\tau_{\xi_{\rm tw}}^{-1}(\bar{v}_E \bar{v}_R \bar{v}_z))).$$

On the components of E the functions  $\tau$  and  $\tau_{\xi_{tw}}$  act equally because  $\xi_{tw}|_{[d]\setminus K} = \xi_{twst}|_{[d]\setminus K}$ . On R, both are the identity. Hence,  $\tau_{\xi_{tw}}^{-1}(\bar{v}_E\bar{v}_R\bar{v}_z) = \tau^{-1}(\bar{v}_E\bar{v}_R\bar{v}_z)$  and

$$(\star) = \chi^{P|_{R'}}(\bar{u}, \xi_{\text{tw}}^{-1}(\tau^{-1}(\bar{v}_E \bar{v}_R \bar{v}_z))).$$

We show that  $\bar{u}\xi_{\mathsf{tw}}^{-1}(\tau^{-1}(\bar{v}_E\bar{v}_R\bar{v}_z)) \in P|_{R'}$  if and only if  $\bar{u}\xi_{\mathsf{tw}}^{-1}(\tau^{-1}(\bar{v}_D\bar{v}_E\bar{v}_R\bar{v}_z)) \in P$ . This holds because D is a set of component of P,  $\bar{v}_D \in Q|_D$ , and  $\tau^{-1}(\bar{v}_D) \in P|_D$  if and only if  $\bar{v}_D \in Q|_D$  by Claim 22 and  $\tau^{-1}(\bar{v}_D) \in P|_D$  if and only if  $\xi_{\mathsf{tw}}^{-1}(\tau^{-1}(\bar{v}_D)) \in P|_D$  by Claim 6. It follows that

$$(\star) = \chi^P(\bar{u}, \xi_{\mathsf{tw}}^{-1}(\tau^{-1}(\bar{v}_D \bar{v}_E \bar{v}_R \bar{v}_z))) = \chi^P(\bar{u}, \xi_{\mathsf{tw}}^{-1}(\tau^{-1}(\bar{v}))).$$

We finish the proof similar to Claim 23 because  $P_1$  (and thus  $Q_1$ ) is blurrable:

$$(\star) = \chi^{Q}(\tau(\xi_{\mathsf{tw}}(\bar{u})), \bar{v})$$

$$= \sum_{\xi \in \Xi} \chi^{Q}(\tau(\xi(\bar{u})), \bar{v})$$

$$= \sum_{\bar{w} \in Q_{1}} S_{P_{1} \times Q_{1}}(\bar{u}, \bar{w}) \cdot \chi^{Q}(\bar{w}, \bar{v})$$

$$= (S \cdot \chi^{Q})(\bar{u}, \bar{v}).$$

The case when  $P_2$  is blurrable and  $P_1$  is not is analogous. Finally, to solve the case where both  $P_1$  and  $P_2$  are non-blurrable, we argue with the induction hypothesis for the matrices  $S^{\xi}$  (Claim 12). It formally becomes elaborate for two reasons. First, we need to argue precisely that the types of occurring orbits are the same. Second, we need to treat the components containing z differently (Claim 10) because in the recursive step we have  $p_z$  as an additional parameter and thus cannot apply the orbit-automorphisms  $\xi \in \Xi$  freely (because they are not the identity function on  $p_z$ ). These components can be treated specially because they are not contained in the active region of the matrices  $S^{\xi}$ .

Claim 25. Suppose  $P \in \mathbf{P}_{2k}$  and  $Q = \tau(P)$ . If  $P_1$  and  $P_2$  are non-blurrable, then  $\chi^P \cdot S = S \cdot \chi^Q$ .

*Proof.* Let  $\bar{u} \in P_1$  and  $\bar{v} \in Q_2$ . We expand the definition of S:

$$(\chi^{P} \cdot S)(\bar{u}, \bar{v}) = \sum_{\bar{w} \in P_{2}} \chi^{P}(\bar{u}, \bar{w}) \cdot S_{P_{2} \times Q_{2}}(\bar{w}, \bar{v})$$

$$= \sum_{\bar{w} \in P_{2}} \chi^{P}(\bar{u}, \bar{w}) \cdot \sum_{\substack{\xi \in \Xi, \\ \xi(\bar{w}_{z}) = \tau_{\xi}^{-1}(\bar{v}_{z})}} S^{\xi}(\xi(\bar{w}^{-z}), \tau_{\xi}^{-1}(\bar{v}^{-z}))$$

$$= \sum_{\xi \in \Xi} \sum_{\substack{\bar{w} \in P_{2}, \\ \bar{w}_{z} = \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_{z}))}} \chi^{P}(\bar{u}, \bar{w}) \cdot S^{\xi}(\xi(\bar{w}^{-z}), \tau_{\xi}^{-1}(\bar{v}^{-z})). \tag{*}$$

We define for every  $\xi \in \Xi$ 

$$\begin{split} P_{\xi} &:= \Big\{ \left. \bar{u}'^{-z} \bar{w}'^{-z} \; \middle| \; \bar{u}' \in P_1, \bar{w}' \in P_2, \bar{u}' \bar{w}' \in P, \bar{u}_z' = \bar{u}_z, \bar{w}_z' = \xi^{-1} (\tau_{\xi}^{-1} (\bar{v}_z)) \; \Big\}, \\ P_{\xi,2} &:= \Big\{ \left. \bar{w}'^{-z} \; \middle| \; \bar{w}' \in P_2, \bar{w}_z' = \xi^{-1} (\tau_{\xi}^{-1} (\bar{v}_z)) \; \right\}. \end{split}$$

By Lemma 20, we have that  $P_{\xi} \in \mathsf{orbs}_{2k-2}((\mathfrak{A}_f, \bar{p}p_z)) \cup \{\emptyset\}$  and, by Corollary 21, that  $P_{\xi,2} \in \mathsf{orbs}_{k-1}((\mathfrak{A}_f, \bar{p}p_z))$ . It depends on  $\xi \in \Xi$  whether the set  $P_{\xi}$  is empty. We continue the equation:

$$(\star) = \sum_{\xi \in \Xi} \sum_{\bar{w} \in P_{\xi,2}} \chi^{P_{\xi}}(\bar{u}^{-z}, \bar{w}) \cdot S^{\xi}(\xi(\bar{w}), \tau_{\xi}^{-1}(\bar{v}^{-z})).$$

We use that  $S^{\xi}$  is invariant under automorphism of  $(\mathfrak{A}_f, \bar{p})$  (Claim 14) and that  $\xi$  is an orbit-automorphism (Claim 6) for very  $\xi \in \Xi$ .

$$\begin{split} (\star) &= \sum_{\xi \in \Xi} \sum_{\bar{w} \in P_{\xi,2}} \chi^{P_{\xi}}(\bar{u}^{-z}, \bar{w}) \cdot S^{\xi}(\bar{w}, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}^{-z}))) \\ &= \sum_{\xi \in \Xi} (\chi^{P_{\xi}} \cdot S^{\xi})(\bar{u}^{-z}, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}^{-z}))) \\ &= \sum_{\xi \in \Xi} (S^{\xi} \cdot \chi^{Q_{\xi}})(\bar{u}^{-z}, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}^{-z}))) \\ &= \sum_{\xi \in \Xi} \sum_{\bar{w} \in Q_{\xi,1}} S^{\xi}(\bar{u}^{-z}, \bar{w}) \cdot \chi^{Q_{\xi}}(\bar{w}, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}^{-z}))), \end{split}$$

where  $Q_{\xi} \in \mathsf{orbs}_{2k-2}((\mathfrak{A}_{g-\xi}, \bar{p}p_z))$  has the same type in  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$  as  $P_{\xi}$  has in  $(\mathfrak{A}_f, \bar{p}p_z)$  and  $Q_{\xi,1} := Q_{\xi}|_{[k-1]}$  for every  $\xi \in \Xi$  (or  $Q_{\xi} = Q_{\xi,1} = \emptyset$  if  $P_{\xi} = \emptyset$ ). The step from  $P_{\xi}$  to  $Q_{\xi}$  is possible because  $S^{\xi}$  blurs the twist between  $(\mathfrak{A}_f, \bar{p}p_z)$  and  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$  for every  $\xi \in \Xi$  (Claim 12).

We analyze the structures of these orbits. Let R be the star center component of P (and so of Q). (Note that R may be split into multiple components for  $P_1$ ,  $P_2$ ,  $P_{\xi}$ , etc.). We apply Lemma 19 to split  $P = P|_R \times P|_D$ ,  $P_{\xi} = P_{\xi}|_R \times P_{\xi}|_D$ , and  $Q_{\xi} = Q_{\xi}|_R \times Q_{\xi}|_D$  for every  $\xi \in \Xi$ , where D is the set of components of P apart from R. The components in D have distance greater than 1 to z because  $z \in R$ . Hence,  $P|_D = P_{\xi}|_D$  and

$$P = P|_R \times P_{\varepsilon}|_D$$

for every  $\xi \in \Xi$ . Because  $P_{\xi}|_R$  has the same type in  $(\mathfrak{A}_f, \bar{p}p_z)$  as  $Q_{\xi}|_R$  has in  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$  and their origins do not contain any of the vertices  $t_i'$ , it even follows that  $P_{\xi}|_R = Q_{\xi}|_R$  for every  $\xi \in \Xi$  because  $(\mathfrak{A}_f, \bar{p}p_z)[R] = (\mathfrak{A}_{g-\xi}, \bar{p}p_z)[R]$ . So

$$Q_{\xi} = P_{\xi}|_{R} \times Q_{\xi}|_{D}$$

for every  $\xi \in \Xi$ . For readability, we set  $\bar{u}_D := \bar{u}_D^{-z}$  and  $\bar{u}_R := \bar{u}_R^{-z}$ . Then  $\bar{u}^{-z} = \bar{u}_R \bar{u}_D$ . We perform the same for  $\bar{v}^{-z} = \bar{v}_R \bar{v}_D$  and  $\bar{w} = \bar{w}_R \bar{w}_D$ . So we obtain in the next step that

$$(\star) = \sum_{\xi \in \Xi} \sum_{\bar{w}_R \bar{w}_D \in Q_{\xi,1}} S^{\xi}(\bar{u}_R \bar{u}_D, \bar{w}_R \bar{w}_D) \cdot \chi^{Q_{\xi}}(\bar{w}_R \bar{w}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_R \bar{v}_D))).$$

We set

$$Q_D := \tau_{\xi}(Q_{\xi}|_D)$$

for some  $\xi \in \Xi$ . Claim 10 states that  $Q_D$  is an orbit of  $(\mathfrak{A}_g, \bar{p}p_z)$  and has the same type in  $(\mathfrak{A}_g, \bar{p}p_z)$  as  $Q_{\xi}|_D$  has in  $(\mathfrak{A}_{g-\xi}, \bar{p}p_z)$ . As seen before, this is the same type as  $P_{\xi}|_D$  has in  $(\mathfrak{A}_f, \bar{p}p_z)$ . Because, as already seen,  $P_{\xi}|_D = P|_D$  for every  $\xi \in \Xi$ , the type of  $Q_D$  is independent of  $\xi$  and  $Q_D$  is well-defined. So  $Q_D$  is also an orbit of  $(\mathfrak{A}_g, \bar{p})$  and  $\xi(Q_D) = Q_D$  for every  $\xi \in \Xi$  because  $\xi$  is an orbit-automorphism (Claim 6). Thus,  $Q_D = \xi(\tau_{\xi}(Q_{\xi}|_D))$  for every  $\xi \in \Xi$ . We set for every  $\xi \in \Xi$ 

$$Q'_{\xi} := Q_{\xi}|_{R} \times Q_{D} = P_{\xi}|_{R} \times Q_{D}.$$

By Claim 10, for every  $\xi \in \Xi$  it holds that

$$\chi^{Q_{\xi}}(\bar{w}_{R}\bar{w}_{D}, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_{R}\bar{v}_{D})))$$

$$= \chi^{Q'_{\xi}}(\bar{w}_{R}\xi(\tau_{\xi}(\bar{w}_{D})), \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_{R}))\xi(\tau_{\xi}(\xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_{D})))))$$

$$= \chi^{Q'_{\xi}}(\bar{w}_{R}\xi(\tau_{\xi}(\bar{w}_{D})), \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_{R}))\bar{v}_{D}).$$

We used that  $\xi$  and  $\tau_{\xi}$  commute (Claim 5) and can be applied component-wise (Claim 4). With  $Q'_{\xi,1} := Q'_{\xi}|_{[k-1]}$  for every  $\xi \in \Xi$ , we obtain that

$$(\star) = \sum_{\xi \in \Xi} \sum_{\bar{w}_R \bar{w}_D \in Q_{\xi,1}} S^{\xi}(\bar{u}_R \bar{u}_D, \bar{w}_R \bar{w}_D) \cdot \chi^{Q'_{\xi}}(\bar{w}_R \xi(\tau_{\xi}(\bar{w}_D)), \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_R))\bar{v}_D)$$

$$= \sum_{\xi \in \Xi} \sum_{\bar{w}_R \bar{w}_D \in Q'_{\xi,1}} S^{\xi}(\bar{u}_R \bar{u}_D, \bar{w}_R \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_D))) \cdot \chi^{Q'_{\xi}}(\bar{w}_R \bar{w}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_R))\bar{v}_D).$$

We claim that

$$Q = P|_R \times Q_D.$$

First,  $P|_R \times Q_D \in \operatorname{orbs}_{2k}((\mathfrak{A}_g, \bar{p}))$  by Lemma 19. Both parts are orbits and their origins are not connected. Second, because  $Q_D$  has the same type in  $(\mathfrak{A}_g, \bar{p}p_z)$  as  $P_{\xi}|_D = P|_D$  in  $(\mathfrak{A}_f, \bar{p}p_z)$ ,  $Q_D$  has also the same type in  $(\mathfrak{A}_g, \bar{p})$  as  $P_{\xi}|_D = P|_D$  in  $(\mathfrak{A}_f, \bar{p})$ . Because the components in R do not contain a twisted edge,  $P|_R = Q|_R$  and so indeed  $Q = P|_R \times Q_D$ . Because  $Q'_{\xi}|_R = P_{\xi}|_R$  and by the definition of  $P_{\xi}|_R$ , it holds that  $\bar{w}_R\bar{w}_D\xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_R)) \in Q'_{\xi}$  if and only if  $\bar{w}_z\bar{w}_R\bar{w}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_z\bar{v}_R)) \in Q$ . We continue as follows:

$$(\star) = \sum_{\xi \in \Xi} \sum_{\bar{w}_R \bar{w}_D \in Q'_{\xi,1}} S^{\xi}(\bar{u}_R \bar{u}_D, \bar{w}_R \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_D))) \cdot \chi^Q(\bar{u}_z \bar{w}_R \bar{w}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_z \bar{v}_R)) \bar{v}_D)$$

$$= \sum_{\xi \in \Xi} \sum_{\bar{w}_z \bar{w}_R \bar{w}_D \in Q_1, \atop \bar{w}_z = \bar{u}_z} S^{\xi}(\bar{u}_R \bar{u}_D, \bar{w}_R \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_D))) \cdot \chi^Q(\bar{w}_z \bar{w}_R \bar{w}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_z \bar{v}_R)) \bar{v}_D).$$

For every  $\xi \in \Xi$ , it holds that  $\xi^{-1}(\tau_{\xi}^{-1}(Q|_R)) = Q|_R$  because  $\tau_{\xi}$  is the identity function on R-vertices (R is a star center component) and because  $\xi$  is an orbit-automorphism (Claim 6). So by Lemma 19,  $\xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_R))\bar{w}_D \in Q$  if and only if  $\bar{w}_R\bar{w}_D \in Q$  for every  $\xi \in \Xi$ . We substitute  $\bar{w}_R\bar{w}_D \mapsto \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_R))\bar{w}_D$  (this is a bijection).

$$\begin{split} &(\star) \\ &= \sum_{\xi \in \Xi} \sum_{\substack{\bar{w}_z \bar{w}_R \bar{w}_D \in Q_1, \\ \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_z)) = \bar{u}_z}} S^{\xi}(\bar{u}_R \bar{u}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_R)) \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_D))) \cdot \\ &= \sum_{\xi \in \Xi} \sum_{\substack{\bar{w}_z \bar{w}_R \bar{w}_D \in Q_1, \\ \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_z)) = \bar{u}_z}} S^{\xi}(\bar{u}_R \bar{u}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_R \bar{w}_D))) \cdot \chi^Q(\xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_z \bar{w}_R)) \bar{w}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_z \bar{v}_R)) \bar{v}_D) \\ &= \sum_{\xi \in \Xi} \sum_{\substack{\bar{w}_z \bar{w}_R \bar{w}_D \in Q_1, \\ \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_z)) = \bar{u}_z}} S^{\xi}(\bar{u}_R \bar{u}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_R \bar{w}_D))) \cdot \chi^Q(\xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_z \bar{w}_R)) \bar{w}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{v}_z \bar{v}_R)) \bar{v}_D) \\ &= \sum_{\xi \in \Xi} \sum_{\substack{\bar{w}_z \bar{w}_R \bar{w}_D \in Q_1, \\ \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_z)) = \bar{u}_z}} S^{\xi}(\bar{u}_R \bar{u}_D, \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_R \bar{w}_D))) \cdot \chi^Q(\bar{w}_z \bar{w}_R \bar{w}_D, \bar{v}_z \bar{v}_R \bar{v}_D). \end{split}$$

In the last step we again used that  $\tau_{\xi}$  is the identity on the R-component and that  $\xi$  is an orbit-automorphism. We finish the proof with the reverse steps as to how we started

it.

$$\begin{split} (\star) &= \sum_{\xi \in \Xi} \sum_{\substack{\bar{w} \in Q_1, \\ \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}_z)) = \bar{u}_z}} S^{\xi}(\bar{u}^{-z}, \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}^{-z}))) \cdot \chi^Q(\bar{w}, \bar{v}) \\ &= \sum_{\bar{w} \in Q_1} \sum_{\substack{\xi \in \Xi, \\ \xi(\bar{u}_z) = \tau_{\xi}^{-1}(\bar{w}_z)}} S^{\xi}(\bar{u}^{-z}, \xi^{-1}(\tau_{\xi}^{-1}(\bar{w}^{-z}))) \cdot \chi^Q(\bar{w}, \bar{v}) \\ &= \sum_{\bar{w} \in Q_1} S(\bar{u}, \bar{w}) \cdot \chi^Q(\bar{w}, \bar{v}) \\ &= (S \cdot \chi^Q)(\bar{u}, \bar{v}). \end{split}$$

Claims 23, 24, and 25 show that S indeed k-blurs the twist between  $(\mathfrak{A}_f, \bar{p})$  and  $(\mathfrak{A}_g, \bar{p})$ . This finishes the proof of Lemma 58.

**Remark 64.** We checked our construction in the proof for  $k \leq 2$  on the computer. For larger k it was computationally not tractable.

# 10 Separating Rank Logic from Choiceless Polynomial Time

In this section we finally separate rank logic from CPT. To apply Lemma 58, we need to construct a suitable class of base graphs.

**Lemma 65.** For every  $n \in \mathbb{N}$ , there is a regular graph that has degree at least n, girth at least n, and is n-connected.

*Proof.* A (d, g)-cage is a d-regular graph with girth g of minimum order. For every odd  $g \geq 7$ , every (d, g)-cage is  $\lceil \frac{d}{2} \rceil$ -connected [2]. So it suffices to show that for every n there is a  $d \geq 2n$  and an odd  $g \geq n$  such that there is a (d, g)-cage. For every  $d \geq 2$  and  $g \geq 3$  there is a d-regular graph of girth g [32] and so in particular a (d, g)-cage.

**Lemma 66.** Let G = (V, E) be a d-regular graph of girth at least  $2(\ell + 2) + 1$  for some  $\ell \in \mathbb{N}$ . Then for every set  $V' \subseteq V$  of size |V'| < d, there is a vertex  $x \in V$  such that  $\operatorname{dist}_G(V', x) > \ell$ .

Proof. For every  $y \in V$  and  $i \leq \ell + 2$  it holds that  $|N_G^i(y)| = 1 + d \sum_{j=0}^{i-1} (d-1)^j$  because G is d-regular and has girth at least  $2(\ell+2)+1$  and thus  $G[N_G^i(y)]$  is a tree, in which the root y has d many subtrees, which are all complete (d-1)-ary trees of height i-1. Let  $y \in V'$ . Then

$$\begin{split} \left| N_G^{\ell+2}(y) \setminus \bigcup_{z \in V'} N_G^{\ell}(z) \right| &\geq 1 + d \sum_{j=0}^{\ell+1} (d-1)^j - |V'| \cdot \left( 1 + d \sum_{j=0}^{\ell-1} (d-1)^j \right) \\ &\geq d \sum_{j=0}^{\ell+1} (d-1)^j - (d-1) - d \sum_{j=0}^{\ell} (d-1)^j \\ &\geq d (d-1)^{\ell+1} - (d-1) > 0. \end{split}$$

Hence, there is at least one  $x \in V$  satisfying the claim.

**Theorem 67.** There is a class of base graphs K, such that for every  $k, m \in \mathbb{N}$ , there is a graph  $G = (V, E, \leq) \in K$  and a  $q \in \mathbb{N}$  such that  $\mathsf{CFl}_{2^q}(G, f) \equiv_{\mathcal{M}}^{2k+m,k,\{2\}} \mathsf{CFl}_{2^q}(G, g)$  for every  $f, g \colon E \to \mathbb{Z}_{2^q}$  satisfying  $\sum g = \sum f + 2^{q-1}$ .

Proof. Let  $\mathcal{K}$  be the class of graphs given by Lemma 65 for every  $n \in \mathbb{N}$  equipped with some total order. Let  $k, m \in \mathbb{N}$  and  $G \in \mathcal{K}$  such that it has degree  $d \geq d(k, m) > m$ , girth at least 2(2r(k+1)+2)+1, and G is at least (2k+m+1)-connected. Let q=q(k),  $e=\{x,y\}\in E$ , and  $g,f\colon E\to \mathbb{Z}_{2^q}$  such that  $\sum g=\sum f+2^{q-1}$ . Up to isomorphism of the CFI structures, we can assume that e is the only edge twisted by f and g and that  $g(e)=f(e)+2^{q-1}$ . Let  $\mathfrak{A}_f=\mathsf{CFI}_{2^q}(G,f)$  and  $\mathfrak{A}_g=\mathsf{CFI}_{2^q}(G,g)$  both with universe f. We show that Duplicator has a winning strategy in the invertible-map game f and f an

We consider the case where m many pebbles are placed on the structures. Starting with fewer pebbles makes the proof more elaborate, without providing any additional insights. Let  $\bar{u} \in A^{2k+m}$  and  $\bar{v} \in A^{2k+m}$  such that the type of  $\bar{u}$  in  $\mathfrak{A}_f$  is the same as of  $\bar{v}$  in  $\mathfrak{A}_g$  and  $e \not\subseteq \operatorname{orig}(\bar{u})$ , i.e., there exists a local isomorphism mapping  $\bar{u}$  to  $\bar{v}$ . Assume we play on  $(\mathfrak{A}_f, \bar{u})$  and  $(\mathfrak{A}_g, \bar{v})$ . Let  $P \in \operatorname{orbs}_{2k+m}((\mathfrak{A}_f, \bar{u}))$  contain  $\bar{u}$  and  $Q \in \operatorname{orbs}_{2k+m}((\mathfrak{A}_g, \bar{v}))$  contain  $\bar{v}$ . Because  $\bar{u}$  and  $\bar{v}$  have the same type, P and Q have the same type. Because  $e \not\subseteq \operatorname{orig}(\bar{u})$ , we have that  $\mathfrak{A}_f[\operatorname{orig}(P)] = \mathfrak{A}_g[\operatorname{orig}(Q)]$  and thus P = Q. That is, there is an automorphism  $\varphi \in \operatorname{Aut}(\mathfrak{A}_g)$  such that  $\varphi(\bar{v}) = \bar{u}$  (Corollary 13). Up to isomorphism, we can continue the game on  $(\mathfrak{A}_f, \bar{u})$  and  $\varphi((\mathfrak{A}_g, \bar{v})) = (\mathfrak{A}_g, \bar{u})$ . Spoiler picks up 2k pebbles on each graph leaving us with the structures  $(\mathfrak{A}_f, \bar{w})$  and  $(\mathfrak{A}_g, \bar{w})$  for some  $\bar{w} \in A^m$ , where  $\bar{w}$  has the same type in  $\mathfrak{A}_f$  and in  $\mathfrak{A}_g$ .

There is a vertex  $z \in V$  such that  $\operatorname{dist}_G(\operatorname{orig}(\bar{w}), z) > 2r(k+1)$  by Lemma 66. We can assume up to exchanging x and y that  $e \cap \operatorname{orig}(\bar{w}) \subseteq \{x\}$  because  $e \not\subseteq \operatorname{orig}(\bar{w})$ . Because G is (2k+m+1)-connected, there is a path  $\bar{s}=(x,y,\ldots,z',z)$  such that  $\bar{s}$  and  $\operatorname{orig}(\bar{w})$  are disjoint apart from x. Then we apply the path isomorphism  $\psi:=\vec{\pi}[2^{q-1},\bar{s}]$  and w.l.o.g. can continue the game on  $(\mathfrak{A}_f,\bar{w})$  and  $\psi((\mathfrak{A}_g,\bar{w}))=(\mathfrak{A}_h,\bar{w})$ , where f and h only twist the edge  $\{z,z'\}$ . Now between  $\mathfrak{A}_f$  and  $\mathfrak{A}_h$  the twist is moved sufficiently far away from  $\operatorname{orig}(\bar{w})$ .

Duplicator chooses the partitions  $\mathbf{P} := \mathsf{orbs}_{2k}((\mathfrak{A}_f, \bar{w}))$  and  $\mathbf{Q} := \mathsf{orbs}_{2k}((\mathfrak{A}_h, \bar{w}))$  of  $A^k \times A^k$  and the invertible matrix S that k-blurs the twist between  $(\mathfrak{A}_f, \bar{w})$  and  $(\mathfrak{A}_h, \bar{w})$  given by Lemma 58. By construction, the conditions of the lemma are satisfied. The matrix S induces a map  $\mathbf{P} \to \mathbf{Q}$  mapping  $P \mapsto Q$  if and only if P and Q have the same type in  $(\mathfrak{A}_f, \bar{w})$  respectively  $(\mathfrak{A}_h, \bar{w})$  (Definition 24). Spoiler chooses orbits  $P \in \mathbf{P}$  and  $Q \in \mathbf{Q}$  of the same type and  $\bar{u}' \in P$  and  $\bar{v}' \in Q$ . Then  $\bar{u}'$  has the same type in  $(\mathfrak{A}, \bar{w})$  as  $\bar{v}'$  in  $(\mathfrak{A}_h, \bar{w})$ . So  $\bar{w}\bar{u}'$  and  $\bar{w}\bar{v}'$  induce a local isomorphism and the next round starts.

As we can see, we are in the same situation as before, that is  $\bar{w}\bar{u}'$  and  $\bar{w}\bar{v}'$  have the same type in  $\mathfrak{A}_f$  and  $\mathfrak{A}_h$  respectively and we can apply Duplicator's strategy again. So Duplicator has winning strategy in the  $\mathcal{M}^{2k+m,k,\{2\}}$ -game.

**Theorem 68.** There is a class of  $\tau$ -structures K, such that IFP+R < PTIME on K and CPT = PTIME on K.

*Proof.* Let  $\mathcal{K}'$  be the graph class from Theorem 67, and set  $\mathcal{K} := \mathsf{CFI}_{2^{\omega}}(\mathcal{K}')$  (recall Definition 22). We want to show that the CFI query for  $\mathcal{K}$  is not IFP+R-definable. This is the task of deciding whether for a given  $\mathsf{CFI}_{2^q}(G,f) \in \mathcal{K}$  it holds that  $\sum f = 0$ . By

Lemma 23, it suffices to show that the CFI query is not IFP+ $R_{\{2\}}$ -definable. The claim follows from Lemma 3 and Theorem 67.

We now argue that CPT captures PTIME on  $\mathcal{K}$ . By Theorem 2, it suffices to show that  $\mathcal{K}$  is a class of structures with abelian and ordered colors (cf. Section 3). By Lemma 5, the colors are abelian and it remains to define for every color class an ordered and transitive permutation group in CPT. Every gadget forms a color class. As seen in Section 5, every gadget forms a 1-orbit and has a regular, so in particular transitive, automorphism group (Lemmas 12 and 17). Consider a degree d gadget. Then every automorphism  $\varphi$  of the gadget can be identified with a tuple  $\bar{a} \in \mathbb{Z}_{2^q}^d$  satisfying  $\sum \bar{a} = 0$  such that  $\varphi(u) = v$  if and only if  $\bar{a}(u) = v$ . Indeed, every such tuple represents an automorphism. So the automorphism group of the gadget can be represented by  $\{\bar{a} \in \mathbb{Z}_{2^q}^d \mid \sum \bar{a} = 0\}$ , which is a CPT-definable set (the tuples, although of unbounded length, can be represented by standard set encoding of tuples). This set can be ordered using the lexicographical order on tuples. Because the base graph is ordered, the automorphism corresponding to the tuple  $\bar{a}$  is CPT-definable: The i-th entry defines the action on the i-th outgoing edge, which is definable using the relation  $R_I$ .

## 11 Discussion

We showed that rank logic does not capture CPT and in particular not PTIME on the class of CFI structures over rings  $\mathbb{Z}_{2^i}$ , even if the base graph is totally ordered. To do so, we used combinatorial objects called blurrers and a recursive approach over the arity. The non-locality of k-tuples for k > 1 increased the difficulty of k-ary rank operators dramatically compared to the arity 1 case. It was suggested in [17] that CFI graphs over  $\mathbb{Z}_4$  could be a separating example for rank logic and PTIME. Our concepts for blurrers required rings  $\mathbb{Z}_{2^i}$  with i > 2 for higher arities. Actually, our computer experiments to check Lemma 58 indicate that CFI graphs over  $\mathbb{Z}_4$  could possibly be distinguished using the k-ary invertible map game for k > 1. It might also be possible that the CFI query over  $\mathbb{Z}_4$  is definable in rank logic using rank operators of higher arities, but this remains an open question.

There are various definitions of rank logic, which slightly differ in the way the matrices in the rank operator are defined. In particular, there is an extension, in which rank operators not only bind universe variables, but also numeric variables [17, 25, 30]. It is not clear whether this extension is more expressive or not. However, for a suitable adaptation of the invertible-map game, which also supports numeric variables to construct matrices, we strongly believe that our arguments work exactly the same. In fact, we think that at least in the invertible-map game numeric variables do not increase the expressiveness and thus our arguments directly apply.

A natural question is how rank logic can be extended such that it can define the CFI query. We have shown that it is not sufficient to compute ranks over finite fields only. Even more, our construction applies to arbitrary linear-algebraic operators over finite fields [12]. However, it is not clear how rank logic can be extended to rings  $\mathbb{Z}_i$ . Over rings, there are several non-equivalent notions of the rank of a matrix. For a discussion see [9, 30]. As opposed to rank logic, solvability logic can easily be extended to rings and thus should be able to define the CFI query over all  $\mathbb{Z}_i$ . Notably, such an extension

would also capture PTIME on structures with bounded and abelian colors [30].

### Acknowledgments

I thank Jendrik Brachter for his help with Lemma 33. I also thank the anonymous reviewers for their detailed and helpful comments.

# References

- [1] Albert Atserias, Andrei A. Bulatov, and Anuj Dawar. Affine systems of equations and counting infinitary logic. *Theor. Comput. Sci.*, 410(18):1666–1683, 2009.
- [2] Camino Balbuena and Julián Salas. A new bound for the connectivity of cages. *Appl. Math. Lett.*, 25(11):1676–1680, 2012.
- [3] Christoph Berkholz and Martin Grohe. Linear diophantine equations, group csps, and graph isomorphism. In Philip N. Klein, editor, *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19*, pages 327–339. SIAM, 2017.
- [4] Samuil D. Berman. On the theory of group codes. *Kibernetika*, 3(1):31–39, 1967.
- [5] Andreas Blass, Yuri Gurevich, and Saharon Shelah. Choiceless polynomial time. *Ann. Pure Appl. Logic*, 100(1-3):141–187, 1999.
- [6] Jin-yi Cai, Martin Fürer, and Neil Immerman. An optimal lower bound on the number of variables for graph identification. *Combinatorica*, 12(4):389–410, 1992.
- [7] Peter J. Cameron. Combinatorics: Topics, Techniques, Algorithms. Cambridge University Press, 1994.
- [8] Ashok K. Chandra and David Harel. Structure and complexity of relational queries. J. Comput. Syst. Sci., 25(1):99–128, 1982.
- [9] Anuj Dawar, Erich Grädel, Bjarki Holm, Eryk Kopczynski, and Wied Pakusa. Definability of linear equation systems over groups and rings. *Log. Methods Comput. Sci.*, 9(4), 2013.
- [10] Anuj Dawar, Erich Grädel, and Wied Pakusa. Approximations of isomorphism and logics with linear-algebraic operators. In 46th International Colloquium on Automata, Languages, and Programming, ICALP 2019., pages 112:1–112:14, 2019.
- [11] Anuj Dawar, Martin Grohe, Bjarki Holm, and Bastian Laubner. Logics with rank operators. In *Proceedings of the 24th Annual IEEE Symposium on Logic in Computer Science*, *LICS 2009*, 11-14 August 2009, Los Angeles, CA, USA, pages 113–122. IEEE Computer Society, 2009.
- [12] Anuj Dawar, Erich Grädel, and Moritz Lichter. Limitations of the invertible-map equivalences. *Journal of Logic and Computation*, 09 2022.

- [13] Anuj Dawar and Bjarki Holm. Pebble games with algebraic rules. In Artur Czumaj, Kurt Mehlhorn, Andrew M. Pitts, and Roger Wattenhofer, editors, Automata, Languages, and Programming 39th International Colloquium, ICALP 2012, Warwick, UK, July 9-13, 2012, Proceedings, Part II, volume 7392 of Lecture Notes in Computer Science, pages 251–262. Springer, 2012.
- [14] Anuj Dawar, David Richerby, and Benjamin Rossman. Choiceless polynomial time, counting and the Cai-Fürer-Immerman graphs. *Ann. Pure Appl. Logic*, 152(1-3):31–50, 2008.
- [15] Martin Fürer. Weisfeiler-Lehman refinement requires at least a linear number of iterations. In Automata, Languages and Programming, 28th International Colloquium, ICALP 2001, Crete, Greece, July 8-12, 2001, Proceedings, volume 2076 of Lecture Notes in Computer Science, pages 322–333. Springer, 2001.
- [16] Erich Grädel and Martin Grohe. Is polynomial time choiceless? In Lev D. Beklemishev, Andreas Blass, Nachum Dershowitz, Bernd Finkbeiner, and Wolfram Schulte, editors, Fields of Logic and Computation II Essays Dedicated to Yuri Gurevich on the Occasion of His 75th Birthday, volume 9300 of Lecture Notes in Computer Science, pages 193–209. Springer, 2015.
- [17] Erich Grädel and Wied Pakusa. Rank logic is dead, long live rank logic! J. Symb. Log., 84(1):54–87, 2019.
- [18] Martin Grohe. The quest for a logic capturing PTIME. In Proceedings of the Twenty-Third Annual IEEE Symposium on Logic in Computer Science, LICS 2008, 24-27 June 2008, Pittsburgh, PA, USA, pages 267–271. IEEE Computer Society, 2008.
- [19] Martin Grohe. Fixed-point definability and polynomial time on graphs with excluded minors. In *Proceedings of the 25th Annual IEEE Symposium on Logic in Computer Science, LICS 2010, 11-14 July 2010, Edinburgh, United Kingdom*, pages 179–188. IEEE Computer Society, 2010.
- [20] Martin Grohe and Daniel Neuen. Canonisation and definability for graphs of bounded rank width. In 34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27, 2019, pages 1–13. IEEE, 2019.
- [21] Yuri Gurevich and Saharon Shelah. On finite rigid structures. J. Symb. Log., 61(2):549–562, 1996.
- [22] Lauri Hella. Logical hierarchies in PTIME. Inf. Comput., 129(1):1–19, 1996.
- [23] Bjarki Holm. Descriptive complexity of linear algebra. PhD thesis, University of Cambridge, UK, 2011.
- [24] Neil Immerman. Expressibility as a complexity measure: results and directions. In Proceedings of the Second Annual Conference on Structure in Complexity Theory, Cornell University, Ithaca, New York, USA, June 16-19, 1987. IEEE Computer Society, 1987.

- [25] Bastian Laubner. The structure of graphs and new logics for the characterization of Polynomial Time. PhD thesis, Humboldt University of Berlin, 2011.
- [26] Moritz Lichter. Separating rank logic from polynomial time. In 36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 July 2, 2021, pages 1–13. IEEE, 2021.
- [27] Moritz Lichter and Pascal Schweitzer. Canonization for bounded and dihedral color classes in choiceless polynomial time. In Christel Baier and Jean Goubault-Larrecq, editors, 29th EACSL Annual Conference on Computer Science Logic, CSL 2021, January 25-28, 2021, Ljubljana, Slovenia (Virtual Conference), volume 183 of LIPIcs, pages 31:1–31:18. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2021.
- [28] Daniel Neuen and Pascal Schweitzer. Benchmark graphs for practical graph isomorphism. In Kirk Pruhs and Christian Sohler, editors, 25th Annual European Symposium on Algorithms, ESA 2017, September 4-6, 2017, Vienna, Austria, volume 87 of LIPIcs, pages 60:1–60:14. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2017.
- [29] Martin Otto. Bounded variable logics and counting a study in finite models, volume 9 of Lecture Notes in Logic. Springer, 1997.
- [30] Wied Pakusa. Linear Equation Systems and the Search for a Logical Characterisation of Polynomial Time. PhD thesis, RWTH Aachen University, 2016.
- [31] Wied Pakusa, Svenja Schalthöfer, and Erkal Selman. Definability of Cai-Fürer-Immerman problems in choiceless polynomial time. In Jean-Marc Talbot and Laurent Regnier, editors, 25th EACSL Annual Conference on Computer Science Logic, CSL 2016, August 29 September 1, 2016, Marseille, France, volume 62 of LIPIcs, pages 19:1–19:17. Schloss Dagstuhl Leibniz-Zentrum fuer Informatik, 2016.
- [32] Horst Sachs. Regular graphs with given girth and restricted circuits. *J. London Math. Soc.*, s1-38(1):423–429, 1963.
- [33] Faried Abu Zaid, Erich Grädel, Martin Grohe, and Wied Pakusa. Choiceless polynomial time on structures with small abelian colour classes. In Erzsébet Csuhaj-Varjú, Martin Dietzfelbinger, and Zoltán Ésik, editors, Mathematical Foundations of Computer Science 2014 39th International Symposium, MFCS 2014, Budapest, Hungary, August 25-29, 2014. Proceedings, Part I, volume 8634 of Lecture Notes in Computer Science, pages 50-62. Springer, 2014.