Expressivity of Quantitative Modal Logics
Categorical Foundations via Codensity and Approximation

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Abstract—A modal logic that is strong enough to fully characterize the behavior of a system is called expressive. Recently, with the growing diversity of systems to be reasoned about (probabilistic, cyber-physical, etc.), the focus shifted to quantitative settings which resulted in a number of expressivity results for quantitative logics and behavioral metrics. Each of these quantitative expressivity results uses a tailor-made argument; distilling the essence of these arguments is non-trivial, yet important to support the design of expressive modal logics for new quantitative settings. In this paper, we present the first categorical framework for deriving quantitative expressivity results, based on the new notion of approximating family. A key ingredient is the codensity lifting—a uniform observation-centric construction of various bisimilarity-like notions such as bisimulation metrics. We show that several recent quantitative expressivity results (e.g. by König et al. and by Fijalkow et al.) are accommodated in our framework; a new expressivity result is derived, too, for what we call bisimulation uniformity.

I. Introduction

a) (Quantitative) Modal Logics and Their Coalgebraic Unification: The role of different kinds of *modal logics* is pervasive in computer science. Their principal functionality is to specify and reason about behaviors of state-transition systems. With the growing diversity of target systems (probabilistic, cyber-physical, etc.), the use of *quantitative* modal logics—where truth values and logical connectives can involve real numbers—is increasingly common. For such logics, however, providing the necessary theoretical foundations takes a significant effort and is often done individually for each variant.

It is therefore desirable to establish unifying and abstract foundations once and for all, which readily instantiate to individual modal logics. This is the goal pursued by the study of *coalgebraic modal logic* [1]–[7], which builds on the general categorical modeling of state-transition systems as *coalgebras* [8], [9].

b) Expressivity of Modal Logics: When using a concrete modal logic, there are several important properties that we expect its metatheory to address, such as *soundness* and *completeness* of its proof system. In this paper, we are interested in the *adequacy* and *expressivity* properties of the logic. These

properties are about comparison between 1) the expressive power of the logic, and 2) some notion of *indistinguishability* that is inherent in the target state-transition systems.

A prototypical example of such notions of indistinguishability is *bisimilarity* [10]. Expressivity with respect to bisimilarity—that modal logic formulas can distinguish non-bisimilar states—is the classic result by Hennessy and Milner [11]. Adequacy, the opposite of expressivity, means that semantics of modal formulas is invariant under bisimilarity, and holds in most modal logics. In contrast, expressivity is a desired property but not always true. Expressivity, when it holds, relies on a delicate balance between the choice of modal operators, the underlying propositional connectives, and the "size" of (branching of) the target state-transition systems.

c) Quantitative Expressivity: The aforementioned interests in quantitative modal logics have sparked research efforts for *quantitative expressivity*. In quantitative settings, the inherent indistinguishability notion in target systems is quantitative, too, typically formulated in terms of a *bisimulation pseudometric* ("how much apart the two states are") that refines the quantitative notion of bisimilarity ("if the two states are indistinguishable or not") [12], [13]. In the expressivity problem, such an indistinguishability notion is compared against the quantitative truth values of logical formulas.

Recent works that study quantitative expressivity include [13]–[18]; they often involve coalgebraic generalization, too, since quantitative modal logics often have immediate variations. Their quantitative expressivity proofs are much more mathematically involved compared to qualitative expressivity proofs. This is because the aforementioned balance between syntax and semantic equivalences is much more delicate. Specifically, target systems are quantitative and thus exhibit *continuity* of behaviors, while logical syntax is inherently *disconnected*, in the sense that each logical formula is an inductively defined and thus finitary entity. Expressivity needs to bridge these two seemingly incompatible worlds.

In order to do so, each of the expressivity proofs in [14]–[18] uses some kind of "approximation." However, each of these arguments has a specialized, tailor-made flavor: Stone–Weierstrass-like arguments for metric spaces [17], the unique structure theorem for analytic spaces [15], and so on. It does not seem easy to distill the essence that is common to different quantitative expressivity proofs. Indeed, there has not been a

coalgebraic framework that unifies them.

d) Categorical Unification of Quantitative Expressivity via Codensity and Approximation: We present the first categorical framework that uniformly axiomatizes different approximation arguments—it uses a fibrational notion of *approximating family*—and unifies different quantitative expressivity results.

Our framework hinges on the construction called the *codensity lifting* [19], [20]; it is a general method for modeling a variety of bisimilarity-like notions (bisimilarity, probabilistic bisimilarity, bisimulation metric, etc.). The codensity lifting uses not only coalgebras (for unifying different state-transition systems) but also *fibrations* for different *observation modes*; the latter include Boolean predicates, quantitative/fuzzy predicates, equivalence relations, pseudometrics, topologies, etc. This use of fibrations provides flexibility to accommodate a variety of quantitative bisimilarity-like notions.

The codensity lifting, while defined in abstract categorical terms, has clear observational intuition (see §III-C). It also gives a class of *codensity bisimilarity games* that characterize a variety of (qualitative and quantitative) bisimilarity notions [20] (see also §III-C).

Our key contribution of a categorical formalization of approximation is enabled by the formalization of the codensity lifting. It has a similar observational intuition, too: see §IV-A, where we characterize an approximating family of observations as a "winnable" set of moves in a suitable sense.

On top of our fibrational notion of approximating family, we establish a general expressivity framework, which is the first to unify existing quantitative expressivity results including [15]–[17]. In our unified framework, we have two proof principles for expressivity—Knaster–Tarski (Thm. IV.5) and Kleene (Thm. IV.7)—that mirror two classic characterizations of greatest fixed points. Our general framework is presented in terms of predicate lifting [2], [3]. This is mostly for presentation purposes (showing concrete syntax is easier this way). A more abstract and fully fibrational recap of our framework— where a modal logic is formalized with a dual adjunction [4]– [7]—is found in §VIII.

We demonstrate our general framework with three examples: expressivity for the Kantorovich bisimulation metric (from [17], §V); that for Markov process bisimilarity (from [15], §VI); and that for the so-called *bisimulation uniformity* (§VII). Both the Knaster–Tarski and Kleene principles are used for proofs. See Table I. The last is a new expressivity result that is not previously found in the literature.

We note that the role of the notion of approximating family is as a useful axiomatization: it tells us what key lemma to prove in an expressivity proof, but it does not tell how to prove the key lemma. The proof of this key lemma is where the technical hardcore lies in existing expressivity proofs (by a Stone–Weierstrass-like result in [17], by the unique structure theorem in [15], etc.). For the new instance of bisimulation uniformity (§VII), the general axiomatization of approximating family allowed us to discover a result we need in a paper [21] that is seemingly unrelated to modal logic. The same result

guided us in the design of modal logic, too, especially in the choice of propositional connectives.

e) Contributions: We summarize our contributions.

- The notion of approximating family, whose instances occur in the key steps of existing quantitative expressivity proofs. It is built on top of the codensity lifting.
- We use it in a unified categorical expressivity framework. It offers two proof principles (Knaster–Tarski and Kleene) that have different applicability (Table I).
- The framework is instantiated to two known expressivity results [15], [17] and one new result (§VII).
- The framework is given an abstract and fully fibrational recap (§VIII) that exposes further fibrational structures.

f) Related Work: Here we list related work considering quantitative expressivity.

Our framework is parameterized both in the kind of coalgebra and in the observation mode. To our knowledge, the only existing work with this generality is [22] which combines coalgebras and fibrations to provide a general setting for proving expressivity. However, that approach does not accommodate approximation arguments, therefore failing to cover any of the aforementioned quantitative expressivity proofs [14]–[18]. Our §VIII can be seen as an extension of [22]; our main novelty is the accommodation of approximation arguments and thus quantitative expressivity results, as we already discussed.

The idea of behavioral metrics was first proposed in [12]. In the setting of category theory, the *behavioral pseudometric* is introduced in [14] in terms of coalgebras in the category $\mathbf{PMet}_1$ of 1-bounded pseudometric spaces, and a corresponding expressivity result is established. Many other formulations of quantitative bisimilarity are based on *fibrational coinduction* [23]. The work [24] discusses general behavioral metrics (but not modal logics); expressivity w.r.t. these metrics is studied in [17] for general $\mathbf{Set}$-coalgebras. The line of work on *codensity bisimilarity*—including [19], [20] and the current work—follows this fibrational tradition, too.

A recent work [18] uses a different formulation of bisimilarity-like notions: it does not use fibrations or functor liftings, but uses so-called *fuzzy lax extensions* of functors. This approach is a descendant of *relators* [25]; seeking the connection to these works is future work.

g) Organization: After recalling preliminaries in §II, we axiomatize the data under which we study expressivity—it is called an *expressivity situation*—in §III. In §IV we define our key notion of approximating family, from which we derive the Knaster–Tarski and Kleene proof principles for expressivity. §V–VII present instances of our framework: two known [15], [17] and one new (§VII). In §VIII, we recap our framework in more abstract terms, and identify expressivity as a problem of comparing coinductive predicates in two different fibrations.

Most proofs are deferred to the appendix found in the extended version [26].

II. PRELIMINARIES

We use coalgebras (§II-A) to accommodate different types of systems, and fibrations (§II-B) to accommodate different "ob-

TABLE I: Examples of expressivity situations. The non-shaded rows describe data in expressivity situations, and the shaded ones describe the resulting bisimilarity notions and modal logics, i.e. two constructs compared in the problem of expressivity.

parameter	[17] (§V) proved by Kleene	[15] (§VI) proved by Knaster–Tarski	§VII proved by Knaster–Tarski
$\mathbb{C}$ category of *spaces*	**Set** sets	**Meas** measurable sets	**Set** sets
B behavior functor	B (arbitrary)	$(\mathcal{G}_{\leq 1}\_)^A$ continuous-space Markov processes	B (arbitrary)
$p\colon \mathbb{E} \to \mathbb{C}$ observation mode	$\mathbf{PMet}_1 \to \mathbf{Set}$ 1-bounded pseudometrics	$\mathbf{EqRel}_{\mathbf{Meas}} \to \mathbf{Meas}$ equiv. relations	$\mathbf{Unif} \to \mathbf{Set}$ uniformity
$\Omega \in \mathbb{C}$ truth-value domain	$[0,1]$ the unit interval	$2 = \{0,1\}$ with the discrete σ-algebra	$[0,1]$ the unit interval
$\underline{\Omega} \in \mathbb{E}_\Omega$ observation predicate	$([0,1], d_e)$ Euclidean metric	$(2, =)$ equality	$([0,1], \mathcal{U}_e)$ metric uniformity
$(\tau_\lambda \colon B\Omega \to \Omega)_{\lambda \in \Lambda}$ observation modality	arbitrary but must satisfy Asm. V.3	$\Lambda = A \times (\mathbb{Q} \cap [0,1])$ $\tau_{a,r}((\mu_a)_{a \in A}) = 1$ iff $\mu_a(1) > r$	arbitrary but must satisfy Asm. VII.6
resulting **bisimilarity-like notion**	B-bisimulation metric [17]	probabilistic bisimilarity	bisimulation uniformity
Σ propositional connectives	$\top, \neg, \min,$ and $(\ominus q)$ for $q \in \mathbb{Q} \cap [0,1]$	$\top, \wedge$	$1, \min,$ and $(r+), (r\times)$ for $r \in \mathbb{R}$
$(f_\sigma)_{\sigma \in \Sigma}$ propositional structure	Zadeh logic connectives on $[0,1]$	meet-semilattice with $0 \sqsubseteq 1$	affine lattice structure on $\mathbb{R}$
resulting modal logic **(modal operators are $(\heartsuit_\lambda)_{\lambda \in \Lambda}$)**	The logic in [17], generalization of Zadeh fuzzy modal logic in [16]	$\mathrm{PML}_\wedge$ [15]	A new modal logic

servation modes." Then quantitative bisimilarity-like notions are formulated as coinductive predicates (§II-C).

A. Coalgebra

Coalgebras are commonly used as a categorical presentation of state-based transition systems [8], [9]. Let $\mathbb{C}$ be a category and $B\colon \mathbb{C} \to \mathbb{C}$ be a functor. A *B-coalgebra* is a pair (X, x) of an object $X \in \mathbb{C}$ and a $\mathbb{C}$-arrow $x\colon X \to BX$; this coalgebra is often denoted simply by $x\colon X \to BX$. A *morphism* of B-coalgebras from (X, x) to (Y, y) is a $\mathbb{C}$-arrow $f\colon X \to Y$ such that $y \circ f = Bf \circ x$.

The theory of coalgebras generalizes process theory and automata theory, where our interests are principally in the observational behaviors of transition systems that are insensitive to internal states. *Bisimilarity* by Park and Milner [10] is a prototype notion that captures such black-box behaviors. One of the early successes of coalgebras is a categorical characterization of bisimilarity that works for different B's, hence for different types of systems. In the theory, coalgebra morphisms are identified as "behavior-preserving maps."

A *final B-coalgebra* is $\zeta\colon Z \to BZ$ such that there is a unique morphism from each coalgebra (X, x) to (Z, ζ). It plays an important role as a fully abstract domain of B-behaviors.

This paper's use of coalgebras goes beyond what we described so far. We follow [23], [27]–[29] and use them in combination with *fibrations* (§II-B). In this case, a coalgebra in a fiber is understood as a predicate (or relation, pseudometric, etc.) with a suitable invariance property (§II-C). The importance of final coalgebras remains, since those in a fiber—called coinductive predicates in §II-C—characterize coinductively defined bisimilarity-like notions.

B. Fibration

Fibrations give a categorical way of organizing indexed entities. Roughly corresponding to indexed categories $\mathbb{C}^{\mathrm{op}} \to$ **Cat**, a fibration $p\colon \mathbb{E} \to \mathbb{C}$ can be thought of as a collection $(\mathbb{E}_C)_{C \in \mathbb{C}}$ of categories $\mathbb{E}_C$ given for each $C \in \mathbb{C}$, together with a suitable "pullback" action of arrows of $\mathbb{C}$. In a fibration, these *fiber categories* $\mathbb{E}_C$ are patched up to form a single *total category* $\mathbb{E}$, a formulation that accommodates many structural reasoning principles. See [30] for a comprehensive account.

The definition of fibration is simpler if restricted to posetal fibration; see e.g. [20]. We need the following general definition for the discussions in §VIII.

Definition II.1 (fibration). Let $p\colon \mathbb{E} \to \mathbb{C}$ be a functor.

- An $\mathbb{E}$-arrow $f\colon X \to Y$ is said to be *Cartesian* if it has the following universal property: for each $\mathbb{E}$-arrow $h\colon Z \to Y$, if ph (in $\mathbb{C}$) factors through pf (say $ph = pf \circ k$, see below middle), then there exists a unique $\mathbb{E}$-arrow $g\colon Z \to X$ such that $h = f \circ g$ and $pg = k$.

- The functor p is called a *fibration* if, for each $Y \in \mathbb{E}$ and each $\mathbb{C}$-arrow $i\colon C \to pY$, there exists an $\mathbb{E}$-arrow $\widetilde{i}Y\colon i^*Y \to Y$ such that $p(\widetilde{i}Y) = i$ and $\widetilde{i}Y$ is Cartesian. See above right. The object $i^*Y \in \mathbb{E}$ is called the *pullback* of Y along i (it is unique up-to isomorphism); the arrow $\widetilde{i}Y$ is called the *Cartesian lifting* of i with respect to Y.

We say that $X \in \mathbb{E}$ is *above* $C \in \mathbb{C}$ if $pX = C$; an $\mathbb{E}$-arrow *above* a $\mathbb{C}$-arrow is defined similarly. A fibration p gives rise to the *fiber category* $\mathbb{E}_C$ for each $C \in \mathbb{C}$: it consists of all the objects above C and all the arrows above id_C. The category $\mathbb{E}$ is called the *total category* of the fibration. Each $\mathbb{C}$-arrow $i\colon C \to C'$ gives rise to a *reindexing functor* $i^*\colon \mathbb{E}_{C'} \to \mathbb{E}_C$.

A fibration $p\colon \mathbb{E} \to \mathbb{C}$ is *posetal* if each fiber $\mathbb{E}_C$ is a poset. It is a $\mathbf{CLat}_{\sqcap}$-*fibration* if each fiber is a complete lattice, and each reindexing functor preserves arbitrary meets $\bigsqcap$.

$\mathbf{CLat}_{\sqcap}$-fibrations form a special class of *topological functors* [31]. We prefer the fibrational presentation, following works on coinductive predicates [19], [22], [23], [27]–[29].

Fibrations in general organize various indexed structures. In this paper, however, our examples share the following intuition.

- The base category $\mathbb{C}$ is that of sets (or spaces, structured sets, etc.) and functions between them (that preserve/respect those structures imposed on the objects). We often assume an endofunctor $B\colon \mathbb{C} \to \mathbb{C}$, for which coalgebras model state-based systems, as in §II-A.
- A fibration $p\colon \mathbb{E} \to \mathbb{C}$ specifies the *observation mode*, providing an additional reasoning structure for spaces in $\mathbb{C}$. Such a structure can be predicates, binary relations, pseudometrics, topologies, etc. (Example II.3). In particular, a bisimilarity-like notion over X is an object $P \in \mathbb{E}_X$.

For the sake of presentation, we fix the following terminology.

Definition II.2 ((fib.) predicate). In a fibration $p\colon \mathbb{E} \to \mathbb{C}$, an object $P \in \mathbb{E}_X$ is called a *(fibrational) predicate* over X.

Note that a (fibrational) predicate can be in fact a binary relation, a pseudometric, etc., depending on the choice of $p\colon \mathbb{E} \to \mathbb{C}$. This abuse of words will be useful in §II-C.

The coming examples are all $\mathbf{CLat}_{\sqcap}$-fibrations; arrows in fibers are denoted by $\sqsubseteq$. The intuition of $P \sqsubseteq Q$ is that the predicate P is more fine-grained and discriminating than Q.

Example II.3. ($\mathbf{Pred} \to \mathbf{Set}$) An object of $\mathbf{Pred}$ is a pair (X, P) of a set X and a predicate $P \subseteq X$. An arrow $f\colon (X, P) \to (Y, Q)$ in $\mathbf{Pred}$ is a function $f\colon X \to Y$ such that $f(x) \in Q$ whenever $x \in P$. The obvious forgetful functor $\mathbf{Pred} \to \mathbf{Set}$ is a fibration, with pullbacks given by $f^*(Y, Q) = (X, f^{-1}Q)$. Each fiber $\mathbf{Pred}_X$ is the powerset $\mathcal{P}X$ with $\sqsubseteq = \subseteq$ as arrows.

($\mathbf{ERel} \to \mathbf{Set}, \mathbf{EqRel} \to \mathbf{Set}, \mathbf{Pre} \to \mathbf{Set}$) $\mathbf{ERel} \to \mathbf{Set}$ is a binary variant of $\mathbf{Pred} \to \mathbf{Set}$: an object (X, R) of $\mathbf{ERel}$ is a set X with an endorelation $R \subseteq X \times X$. Pullbacks are given by inverse images, too. These relations are restricted to equivalence relations and preorders, in $\mathbf{EqRel} \to \mathbf{Set}$ and $\mathbf{Pre} \to \mathbf{Set}$, respectively.

($\mathbf{PMet}_1 \to \mathbf{Set}$) $\mathbf{PMet}_1$ consists of sets with 1-bounded pseudometrics and non-expansive maps. A pseudometric d is much like a metric but allows $d(x, y) = 0$ for $x \neq y$, a common setting where bisimulation metrics are formulated. 1-boundedness (that $d(x, y) \leq 1$ for all x, y) is assumed for technical convenience—any bound would do, such as ∞. The forgetful functor $\mathbf{PMet}_1 \to \mathbf{Set}$ is a fibration, in which pullbacks equip a set with an induced pseudometric: $f^*(Y, e) = \big(X, \lambda(x, x').\, e(f(x), f(x'))\big)$. A consequence is that the fiber $(\mathbf{PMet}_1)_X$ consists of all 1-bounded pseudometrics over X ordered by $\sqsubseteq\ =\ \geq$—this concurs with the above intuition that d is more discriminating if $d \sqsubseteq d'$.

($\mathbf{EqRel}_{\mathbf{Meas}} \to \mathbf{Meas}$) This is a measurable variant of $\mathbf{EqRel} \to \mathbf{Set}$; $\mathbf{Meas}$ is the category of measurable spaces and measurable maps. An object (X, R) of $\mathbf{EqRel}_{\mathbf{Meas}}$ is given by a measurable set X, together with $R \subseteq X \times X$.

C. Coinductive Predicate in a Fibration

The combination of coalgebras and fibrations has been actively studied, starting in [23] and more recently e.g. in [20], [22], [27]–[29]. Fibrations introduce additional reasoning structures (§II-B) which allow to accommodate bisimilarity-like notions beyond classical bisimilarity (including bisimilarity pseudometrics, see e.g. [20], [32]). In fact, these bisimilarity-like notions are defined coinductively, i.e., as suitable greatest fixed points, which are then identified with coalgebras in a fiber. Here we shall review this fibrational machinery for coinductive reasoning for coalgebras.

Definition II.4 (functor lifting). Let $p\colon \mathbb{E} \to \mathbb{C}$ be a fibration, and $B\colon \mathbb{C} \to \mathbb{C}$ be a functor. A functor $\overline{B}\colon \mathbb{E} \to \mathbb{E}$ is a *lifting* of B along p if $p \circ \overline{B} = B \circ p$. We say that a lifting $\overline{B}$ is *fibered* if it preserves Cartesian arrows.

A functor lifting determines the type of *coinductive predicates* through a *predicate transformer*.

Definition II.5 ((fibrational) predicate transformer $x^* \circ \overline{B}$). In the setting of Def. II.4, let $\overline{B}$ be a lifting of B, and $x\colon X \to BX$ be a B-coalgebra. We then obtain an endofunctor

$$x^* \circ \overline{B}\colon \mathbb{E}_X \to \mathbb{E}_X \text{ by the composite } \mathbb{E}_X \xrightarrow{\overline{B}} \mathbb{E}_{BX} \xrightarrow{x^*} \mathbb{E}_X.$$

The functor $x^* \circ \overline{B}$ is called the *predicate transformer* induced by $\overline{B}$ over the B-coalgebra x.

It is standard to characterize bisimilarity as a suitable greatest fixed point (gfp). Accordingly, we are interested in the gfp of the predicate transformer $x^* \circ \overline{B}$. The latter amounts to the final $x^* \circ \overline{B}$-coalgebra, in our current setting where the fiber $\mathbb{E}_X$ is not necessarily a poset but a category.

Definition II.6 (coinductive predicate $\nu(x^* \circ \overline{B}) \in \mathbb{E}_X$, and invariant). In the setting of Def. II.5, the carrier of the final $x^* \circ \overline{B}$-coalgebra (if it exists) is called the $\overline{B}$-*coinductive predicate* over x. It is denoted by $\nu(x^* \circ \overline{B}) \in \mathbb{E}_X$.

An $x^* \circ \overline{B}$-coalgebra is called a $\overline{B}$-*invariant* over x.

The names in the above definition reflect the common reasoning principle for gfp specifications (such as safety), namely that an invariant underapproximates (and thus witnesses) the gfp specification. Each $\overline{B}$-invariant indeed witnesses the $\overline{B}$-coinductive predicate, in the sense that there is a unique morphism from the former to the latter.

Example II.7 ($\mathbf{Pred} \to \mathbf{Set}$). Liftings of a functor B along $\mathbf{Pred} \to \mathbf{Set}$ are well-studied—they correspond to the so-called *predicate liftings* [2], [3]. For example, for the powerset functor $B = \mathcal{P}$, two liftings $\overline{\mathcal{P}}_{\square}, \overline{\mathcal{P}}_{\Diamond}$ are given by

$$\overline{\mathcal{P}}_{\square}(X, P) = \big(\mathcal{P}X, \square P = \{U \subseteq X \mid U \subseteq P\}\big),$$
$$\overline{\mathcal{P}}_{\Diamond}(X, P) = \big(\mathcal{P}X, \Diamond P = \{U \subseteq X \mid U \cap P \neq \emptyset\}\big).$$

Thus a choice of lifting here amounts to a choice of modality.

Coinductive predicates in this setting represent various safety properties. On a Kripke frame $x: X \to \mathcal{P}X$, the coinductive predicate $\nu(x^* \circ \overline{\mathcal{P}_\Diamond}) \subseteq X$ designates those states from which there is an infinite path. A similar example appears in [27].

Example II.8 ($\mathbf{ERel} \to \mathbf{Set}, \mathbf{EqRel} \to \mathbf{Set}, \mathbf{Pre} \to \mathbf{Set}$). In these relational examples, a coinductive predicate embodies some bisimilarity-like relation—more specifically, the greatest among those relations which are preserved by one-step transitions. A class of examples is given by *coalgebraic bisimilarity*: see [23], where they assign a specific choice of lifting $\overline{B}$ to each functor B. Other examples include similarity [33] (see also [22, §4.3]), probabilistic bisimilarity [20] and the language equivalence between deterministic automata [20].

Example II.9 ($\mathbf{PMet}_1 \to \mathbf{Set}$). A prototypical example is given as follows. Let $B = \mathcal{D}_{\leq 1}$, the *subdistribution* functor that carries a set X to $\mathcal{D}_{\leq 1}X = \{\xi: X \to [0,1] \mid \sum_{x \in X} \xi(x) \leq 1\}$. Its action on arrows is given by push-forward distributions. One lifting $\overline{\mathcal{D}_{\leq 1}}$ of $\mathcal{D}_{\leq 1}$ along $\mathbf{PMet}_1 \to \mathbf{Set}$ is given by the *Kantorovich metric*: $\overline{\mathcal{D}_{\leq 1}}(X,d) = (\mathcal{D}_{\leq 1}X, \mathcal{K}d)$, where $\mathcal{K}d(\xi, \xi')$ is given by

$$\sup\nolimits_{f:\,(X,d)\to_{\mathrm{ne}}[0,1]} \left| \sum\nolimits_{x \in X} f(x) \cdot \xi(x) - \sum\nolimits_{x \in X} f(x) \cdot \xi'(x) \right|.$$

In the above supremum, f ranges over all *nonexpansive* functions of the designated type. The coinductive predicate for $\overline{\mathcal{D}_{\leq 1}}$ coincides with the *bisimulation metric* from [13].

The above construction of a lifting $\overline{B}$ along $\mathbf{PMet}_1 \to \mathbf{Set}$ has been generalized to an arbitrary functor B. This is called the *Kantorovich lifting* and is introduced in [24]. It uses a map $B[0,1] \to [0,1]$ as a parameter; the latter is much like a choice of a modality. We study this general setting in §V, following [17] but identifying the Kantorovich lifting as a special case of the *codensity lifting* (Def. III.8).

Here is an abstract account of coinductive predicates.

Proposition II.10 (from [28, Prop. 4.1 and 4.2]). *Each lifting $\overline{B}$ of B along $p: \mathbb{E} \to \mathbb{C}$ induces a functor $\mathbf{CoAlg}(p): \mathbf{CoAlg}(\overline{B}) \to \mathbf{CoAlg}(B)$; it carries a coalgebra $U \to \overline{B}U$ (in $\mathbb{E}$) to $pU \to B(pU)$ (in $\mathbb{C}$). Moreover, the functor $\mathbf{CoAlg}(p)$ is a fibration if $\overline{B}$ is fibered. The fiber $\mathbf{CoAlg}(\overline{B})_{(X,x)}$ over a coalgebra $x: X \to BX$ coincides with the category $\mathbf{CoAlg}(x^* \circ \overline{B})$ of $\overline{B}$-invariants over (X,x).* □

Lemma II.11 (from [20], [28]). *Let $\overline{B}$ be a lifting of B along p. If p is a $\mathbf{CLat}_\sqcap$-fibration, then $\nu(x^* \circ \overline{B})$ exists for each B-coalgebra $x: X \to BX$—it is the gfp of the monotone map $x^* \circ \overline{B}$ over the complete lattice $\mathbb{E}_X$. Moreover, if $\overline{B}$ is fibered, then these coinductive predicates are preserved by reindexing along coalgebra morphisms.* □

Notation II.12. Final objects shall be denoted with subscripts ($1_\mathbb{E}, 1_X$, etc.) to clarify in which category it is final, unless it is obvious. We will also write $\top$ for the maximum element of a poset, that is, the final object when the poset is thought of as a category. This typically happens with the final object $\top_X \in \mathbb{E}_X$ in a fiber $\mathbb{E}_X$ of a $\mathbf{CLat}_\sqcap$-fibration.

III. Expressivity Situation

On top of the above preliminaries, we fix the format of categorical data under which we study expressivity. It is called an expressivity situation. While it may seem overwhelming, we show that the data arises naturally, with clear intuition from the viewpoints of modal logics and observations (§III-B–III-C).

A. Definition

Definition III.1. An *expressivity situation* $\mathscr{S} = (p, B, \Omega, \underline{\Omega}, \Sigma, \Lambda, (f_\sigma)_{\sigma \in \Sigma}, (\tau_\lambda)_{\lambda \in \Lambda})$ is given by the following.

- A $\mathbf{CLat}_\sqcap$-fibration $p: \mathbb{E} \to \mathbb{C}$.
- A functor $B: \mathbb{C} \to \mathbb{C}$ (a *behavior functor*).
- An object $\Omega \in \mathbb{C}$ (a *truth-value object*) equipped with finite powers ($\Omega^n \in \mathbb{C}$ for $n \in \mathbb{N}$), and another object $\underline{\Omega}$ (an *observation predicate*) above it. It follows that $\underline{\Omega}$ also has finite powers [30, Prop. 9.2.1].
- A ranked alphabet Σ of *propositional connectives* and a family of arrows $(f_\sigma: \Omega^{\mathrm{rank}(\sigma)} \to \Omega)_{\sigma \in \Sigma}$ (a *propositional structure*). Moreover, we require that each $f_\sigma: \Omega^{\mathrm{rank}(\sigma)} \to \Omega$ has a *lifting* $g_\sigma: \underline{\Omega}^{\mathrm{rank}(\sigma)} \to \underline{\Omega}$ (in $\mathbb{E}$) such that $pg_\sigma = f_\sigma$.
- A set Λ of *modality indices* and a family of algebras $(\tau_\lambda: B\Omega \to \Omega)_{\lambda \in \Lambda}$ (*observation modalities*).

Roughly speaking, Σ and Λ are used for modal logic *syntax*, and $\mathbb{C}$, B, Ω, $(f_\sigma)_{\sigma \in \Sigma}$, and $(\tau_\lambda)_{\lambda \in \Lambda}$ are used for modal logic *semantics*. The other constructs (p and $\underline{\Omega}$) are there for defining a bisimilarity-like notion.

In what follows, we formulate the expressivity problem on top of Def. III.1, explaining the role of each piece of data in an expressivity situation $\mathscr{S}$. More specifically, we let $\mathscr{S}$ induce the following constructs: 1) the modal logic $L_\mathscr{S}$ (Def. III.2 and III.4); 2) the *fibrational logical equivalence* $\mathrm{LE}_\mathscr{S}(x)$ induced by $L_\mathscr{S}$ (Def. III.6); and 3) the bisimilarity-like notion $\mathrm{Bisim}^{\underline{\Omega},\tau}(x)$ as a *codensity bisimilarity* (Def. III.9). Comparison of the last two is the problem of expressivity. As an illustrating example, we use an expressivity situation $\mathscr{S}_{\mathrm{KMM}}$ that arises from the real-valued logic $\mathcal{M}(\Lambda)$ in [17] (see also §V).

B. Syntax and Semantics of Our Logic $L_\mathscr{S}$

The syntax of modal logic is specified by the propositional connectives in Σ and the modality indices in Λ.

Definition III.2 ($L_\mathscr{S}$). Let $\mathscr{S}$ be an expressivity situation in Def. III.1. The modal logic $L_\mathscr{S}$ has the following syntax.

$$\varphi, \varphi_1, \ldots, \varphi_n ::= \sigma(\varphi_1, \ldots, \varphi_{\mathrm{rank}(\sigma)}) \qquad (\sigma \in \Sigma)$$
$$\mid \heartsuit_\lambda \varphi \qquad\qquad\qquad\quad (\lambda \in \Lambda)$$

We also let $L_\mathscr{S}$ denote the set of all formulas.

Example III.3. Let Λ be a set. To model the modal logic $\mathcal{M}(\Lambda)$ in [17], we let $\Sigma = \{\top^0, \min^2, \neg^1\} \cup \{(\ominus q)^1 \mid q \in \mathbb{Q} \cap [0,1]\}$. Then the syntax is given by

$$\varphi, \varphi_1, \ldots, \varphi_n ::= \top \mid \neg(\varphi) \mid \min(\varphi_1, \varphi_2)$$
$$\mid (\ominus q)\varphi \ (q \in \mathbb{Q} \cap [0,1]) \mid \heartsuit_\lambda \varphi \ (\lambda \in \Lambda).$$

Identifying $\heartsuit_\lambda$ with $[\lambda]$ in the original notation, this recovers the syntax of $\mathcal{M}(\Lambda)$ in [17].

Given a coalgebra $x\colon X \to BX$ of the behavior functor B, the semantics of each formula is a $\mathbb{C}$-arrow from the state space X to the truth-value object Ω, inductively defined as follows.

Definition III.4. Let $\mathscr{S}$ be an expressivity situation in Def. III.1 and let $x\colon X \to BX$ be a B-coalgebra. For each $\varphi \in L_\mathscr{S}$, the *interpretation* $\llbracket\varphi\rrbracket_x\colon X \to \Omega$ of φ with respect to x is defined inductively as follows:

$$\llbracket\sigma(\varphi_1,\ldots,\varphi_{\mathrm{rank}(\sigma)})\rrbracket = f_\sigma \circ \langle\llbracket\varphi_1\rrbracket,\ldots,\llbracket\varphi_{\mathrm{rank}(\sigma)}\rrbracket\rangle, \quad (\sigma \in \Sigma)$$
$$\llbracket\heartsuit_\lambda\varphi\rrbracket = \tau_\lambda \circ (B\llbracket\varphi\rrbracket) \circ x. \qquad (\lambda \in \Lambda)$$

Example III.5. Recall Ex. III.3. Let $B\colon \mathbf{Set} \to \mathbf{Set}$ be an endofunctor, and Ω be the unit interval $[0,1]$. We specify the propositional structure $(f_\sigma\colon [0,1]^{\mathrm{rank}(\sigma)} \to [0,1])_{\sigma\in\Sigma}$ by:

$$f_\top() = 1, \qquad f_{\min}(x,y) = \min(x,y),$$
$$f_\neg(x) = 1 - x, \qquad f_{\ominus q}(x) = \max(x - q, 0).$$

Here $\min$ plays the role of conjunction. Let $(\tau_\lambda\colon B[0,1] \to [0,1])$ be a family of observation modalities, and $x\colon X \to BX$ be a B-coalgebra. Then, the semantics $\llbracket\varphi\rrbracket_x$ of each formula φ in Def. III.4 coincides with the definition in [17, §3.2].

The following definition generalizes, in fibrational terms, the conventional definition that two states are logically equivalent if each formula's truth values coincide.

Definition III.6 (fibrational logical equivalence $\mathsf{LE}_\mathscr{S}(x)$). Let $\mathscr{S}$ be an expressivity situation in Def. III.1 and let $x\colon X \to BX$ be a B-coalgebra. The *fibrational logical equivalence* $\mathsf{LE}_\mathscr{S}(x)$ with respect to x is a predicate above X defined by

$$\mathsf{LE}_\mathscr{S}(x) = \textstyle\prod_{\varphi \in L_\mathscr{S}} \llbracket\varphi\rrbracket_x^* \underline{\Omega}, \quad \text{where} \quad \begin{array}{cc} \mathbb{E} & \llbracket\varphi\rrbracket_x^* \underline{\Omega} -\to \underline{\Omega} \\ {\scriptstyle\downarrow p} & \\ \mathbb{C} & X \xrightarrow{\llbracket\varphi\rrbracket_x} \Omega. \end{array}$$

Example III.7. Recall Ex. III.5. To define a logical distance, we let p be the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{PMet}_1 \to \mathbf{Set}$ (Ex. II.3), and $\underline{\Omega}$ be the usual Euclidean metric d_e on $[0,1]$.

Then, for each B-coalgebra $x\colon X \to BX$, the pseudometric $d_x^L := \mathsf{LE}_{\mathscr{S}_{\mathrm{KMM}}}(x)$ is equivalently described by

$$d_x^L(s,t) = \sup_\varphi d_e(\llbracket\varphi\rrbracket_x(s), \llbracket\varphi\rrbracket_x(t)),$$

where φ ranges over the modal formulas. Thus Def. III.6 coincides with the notion of logical distance in [17, Def. 25].

C. Codensity Lifting and Codensity Bisimilarity

We unify different quantitative bisimilarity notions—such as probabilistic bisimilarity and bisimulation metric—using *codensity bisimilarity*. This is what is compared with the fibrational logical equivalence (Def. III.6). Codensity bisimilarity arises natually from the notion of *codensity lifting* [19], [20].

The codensity lifting, although it is formulated in abstract terms (Def. III.8), has clear *observational* intuition. The codensity lifting $\overline{B}^{\underline{\Omega},\tau}\colon \mathbb{E} \to \mathbb{E}$ is defined for the following data, which is part of the data in an expressivity situation (Def. III.1).

$$\begin{array}{cc} \mathbb{E} & k^*\underline{\Omega} \to \underline{\Omega} \\ {\scriptstyle p\downarrow} & \\ \mathbb{C} & X \xrightarrow{k} \Omega \end{array}$$

- A fibration $p\colon \mathbb{E} \to \mathbb{C}$ for the observation mode (§II-B).
- An endofunctor $B\colon \mathbb{C} \to \mathbb{C}$; which is to be lifted (Def. II.4). Our target system is a B-coalgebra (§II-A).
- A truth value domain $\Omega \in \mathbb{C}$. Here we use "Ω-valued observations," that is, arrows $k\colon X \to \Omega$ in $\mathbb{C}$ (cf. the diagram above).
- An observation predicate $\underline{\Omega} \in \mathbb{E}$ above Ω. It is a "template of observations," whose pullback $k^*(\underline{\Omega})$ by an observation k defines an indistinguishability notion on X. See above. For $\mathbb{E} = \mathbf{EqRel}$, a common example is $\Omega = 2$ and $\underline{\Omega} = (2, \mathrm{Eq}_2 \subseteq 2 \times 2)$; it means we distinguish elements of X if they are mapped to different elements by an observation $k\colon X \to 2$.
- A family of observation modalities $(\tau_\lambda\colon B\Omega \to \Omega)_{\lambda\in\Lambda}$. An observation modality τ_λ specifies how observations interact with the behavior type B. Technically, it lifts
  - an observation $k\colon X \to \Omega$ of X
  - to an observation $BX \xrightarrow{Bk} B\Omega \xrightarrow{\tau_\lambda} \Omega$ of BX.

Given the above data with the observational intuition, the codensity lifting is defined as the "indistinguishability with respect to lifted Ω-valued observations," as below.

Definition III.8 (codensity lifting). Let $\mathscr{S}$ be an expressivity situation in Def. III.1. The *codensity lifting* of B with respect to $\underline{\Omega}$ and $(\tau_\lambda)_{\lambda\in\Lambda}$ is the functor $\overline{B}^{\underline{\Omega},\tau}\colon \mathbb{E} \to \mathbb{E}$, defined by

$$\overline{B}^{\underline{\Omega},\tau} P = \textstyle\prod_{\lambda\in\Lambda, h\in\mathbb{E}(P,\underline{\Omega})} (\tau_\lambda \circ B(ph))^* \underline{\Omega}. \qquad (1)$$

Some explanations are in order. An arrow $h\colon P \to \underline{\Omega}$ is a *legitimate* Ω-valued observation— it is a function $ph\colon X \to \Omega$ that respects indistinguishability predicates P and $\underline{\Omega}$. The latter legitimacy requirement instantiates to predicate- and relation-preservation, nonexpansiveness, continuity, etc., depending on the choice of the fibration p.

$$\begin{array}{cc} \mathbb{E} & \bullet\,-\,-\,-\,-\,-\to\underline{\Omega} \\ {\scriptstyle p\downarrow} & \\ \mathbb{C} & BX \xrightarrow{B(ph)} B\Omega \xrightarrow{\tau_\lambda} \Omega \end{array}$$

The predicate $(\tau_\lambda \circ B(ph))^* \underline{\Omega}$ in (1) is over BX and induced from "lifting h along τ_λ," as shown in the above diagram. In (1), in the end, $\overline{B}^{\underline{\Omega},\tau} P$ arises as the coarsest indistinguishability induced by such h's.

Definition III.9 (codensity bisimilarity $\mathsf{Bisim}^{\underline{\Omega},\tau}(x)$). Let $\mathscr{S}$ be an expressivity situation in Def. III.1 and let $x\colon X \to BX$ be a B-coalgebra. The *codensity bisimilarity* $\mathsf{Bisim}^{\underline{\Omega},\tau}(x)$ of x is the $\overline{B}^{\underline{\Omega},\tau}$-coinductive predicate (Def. II.6), i.e., the greatest fixed point of the map $x^* \circ \overline{B}^{\underline{\Omega},\tau}\colon \mathbb{E}_X \to \mathbb{E}_X$:

$$\mathsf{Bisim}^{\underline{\Omega},\tau}(x) = \nu(x^* \circ \overline{B}^{\underline{\Omega},\tau}) \in \mathbb{E}_X.$$

Example III.10. Recall Ex. III.7. In this case we can see that the codensity lifting coincides with the *Kantorovich lifting*; see §V for details. Thus the codensity bisimilarity coincides with the behavioral distance defined in [17, Def. 22].

Much like the codensity lifting explained in terms of "observations," codensity bisimilarity can be regarded as an outcome of "repeated observations." We recall here the following game-theoretic characterization [20], which is not used in the rest of the paper yet is useful for providing intuitions.

TABLE II: (Untrimmed) codensity bisimilarity game

position	player	possible moves
$P \in \mathbb{E}_X$	Spoiler	$k \colon X \to \Omega$ in $\mathbb{C}$ such that $\exists \lambda \in \Lambda.\, P \not\sqsubseteq x^*(Bk)^*\tau_\lambda^*\underline{\Omega}$
$k \in \mathbb{C}(X, \Omega)$	Duplicator	$P \in \mathbb{E}_X$ s.t. $P \not\sqsubseteq k^*\underline{\Omega}$

Fact III.11 (from [20, Cor. VI.4]). *In the setting of Def. III.9, we define a two-player infinite game called the* (untrimmed) codensity bisimilarity game *as shown in Table II. Its two players are called Spoiler and Duplicator; once a player gets stuck, the player loses; any infinite play is won by Duplicator.*

Then, for each $P \in \mathbb{E}_X$, P is below the codensity bisimilarity ($P \sqsubseteq \mathsf{Bisim}^{\Omega,\tau}(x)$) if and only if the position $P \in \mathbb{E}_X$ is winning for Duplicator.

In the codensity bisimilarity game, Spoiler repeatedly carries out observations $k \colon X \to \Omega$, trying to show that the previous move $P \in \mathbb{E}_X$ by Duplicator was in fact not below the codensity bisimilarity. Duplicator responds with a counter-argument that Spoiler's $k \colon X \to \Omega$ is illegitimate, not respecting the indistinguishability predicates P (on X) and $\underline{\Omega}$ (on Ω).

D. Adequacy and Expressivity

We are ready to formulate adequacy and expressivity. Recall that $P \sqsubseteq Q$ in a fiber means that P is more discriminating.

Definition III.12. Let $\mathscr{S}$ be an expressivity situation (Def. III.1) and $x \colon X \to BX$ be a B-coalgebra.
- $\mathscr{S}$ is *expressive* for x if $\mathsf{Bisim}^{\Omega,\tau}(x) \sqsupseteq \mathsf{LE}_{\mathscr{S}}(x)$ holds.
- $\mathscr{S}$ is *adequate* for x if $\mathsf{Bisim}^{\Omega,\tau}(x) \sqsubseteq \mathsf{LE}_{\mathscr{S}}(x)$ holds.

$\mathscr{S}$ is *expressive* (or *adequate*) if it is expressive (or adequate, respectively) for any B-coalgebra x.

The following result justifies our axiomatization in Def. III.1: adequacy, a property that is a prerequisite in most usage scenarios of modal logics, follows easily from the axiomatization itself.

Proposition III.13. *Any expressivity situation $\mathscr{S}$ in Def. III.1 is adequate.* $\square$

Example III.14. Recall Ex. III.10. Expressivity of the expressivity situation $\mathscr{S}_{\mathrm{KMM}}$ means that, for each $x \colon X \to BX$ and each pair $(s,t) \in X^2$ of states, the inequality $d_x(s,t) \leq d_x^L(s,t)$ holds between the behavioral and logical distances ("d_x^L is more discriminating"). Adequacy means that, for each x and (s,t), $d_x(s,t) \geq d_x^L(s,t)$ holds.

IV. Approximation in Quantitative Expressivity

In this section, based on the axiomatization in §III, we present a fibrational notion of *approximating family of observations*. The notion axiomatizes and unifies the "approximation" properties that are key steps in many recent quantitative expressivity proofs, such as in [15]–[17], [34].

We then proceed to present two proof principles for expressivity—*Knaster–Tarski* and *Kleene*—that mirror two classic characterizations of greatest fixed points. These proof principles make a large part of an expressivity proof routine.

The remaining technical challenges are 1) choosing a suitable propositional signature and 2) identifying suitable approximating families; our general framework singles out these technical challenges and thus eases the efforts for addressing them.

We recall the two characterizations of gfps.

Theorem IV.1. *Let $(L, \sqsubseteq)$ be a complete lattice, and $f \colon L \to L$ be a monotone function.*
1) *(**Knaster–Tarski**) The set $\{l \in L \mid l \sqsubseteq f(l)\}$ of post-fixed points is a complete lattice. Its maximum z satisfies $z = f(z)$, hence z is the greatest fixed point νf.*
2) *(**Kleene**) Consider the following ω^{op}-chain in L.*

$$\top \sqsupseteq f(\top) \sqsupseteq f^2(\top) \sqsupseteq \cdots \quad (2)$$

If f preserves the meet $\bigsqcap_{i \in \omega^{\mathrm{op}}} f^i(\top)$, then $\bigsqcap_{i \in \omega^{\mathrm{op}}} f^i(\top)$ is the greatest fixed point νf.

A. Approximating Family of Observations

Our categorical notion of *approximating family of observations* designates a "good" subset $S \subseteq \mathbb{C}(X, \Omega)$ of Ω-valued observations in a suitable sense. We will be asking if the set $\{[\![\varphi]\!]_x \colon X \to \Omega \mid \varphi \in L'\}$ of "logical observations" is approximating or not, where L' is some set of modal formulas.

Definition IV.2 (approximating family). Let $\mathscr{S}$ be an expressivity situation in Def. III.1 and X be an object of $\mathbb{C}$. A subset $S \subseteq \mathbb{C}(X, \Omega)$ is an *approximating family of observations*, or simply *approximating*, if, for every morphism

$$h \colon \left(\bigsqcap_{k \in S} k^*\underline{\Omega}\right) \longrightarrow \underline{\Omega} \quad (3)$$

of $\mathbb{E}$ and every $\lambda \in \Lambda$, the following inequality holds:

$$\bigsqcap_{k' \in S, \lambda' \in \Lambda} (\tau_{\lambda'} \circ Bk')^*\underline{\Omega} \sqsubseteq (\tau_\lambda \circ B(ph))^*\underline{\Omega}. \quad (4)$$

Note that $k \colon X \to \Omega$ is a $\mathbb{C}$-arrow while h is an $\mathbb{E}$-arrow.

Some explanation is in order. Intuitively, in the definition above, the set S is a set of "logical" observations. Each h as in (3) is a "non-logical" legitimate observation. For such h, the r.h.s. of (4) is the information obtained from the observation h. (Note the way h is used: it is not $h^*\underline{\Omega}$, but $(\tau_\lambda \circ B(ph))^*\underline{\Omega}$. This corresponds to (1).) On the other hand, the l.h.s. of (4) is the information from "logical" observations. Thus, an intuitive meaning of the definition above is that no "non-logical" observation gives any additional information. In many cases, the "logical" observations in S approximate each "non-logical" ones h. See Rem. IV.3 for details.

Another intuition is given in terms of the codensity bisimilarity game (Fact III.11). Roughly, S being an approximating family says that Spoiler may restrict its moves to $S \subseteq \mathbb{C}(X, \Omega)$.

Remark IV.3. In many examples, S being an approximating family is proved in the following two steps: 1) showing that ph can be approximated by observations in S; and 2) this approximation is preserved along the lifting $k \mapsto \tau_\lambda \circ Bk$ of observations over X to those over BX. The former step is usually the harder one, and proved via arguments specific to the current situation (pseudometric spaces, measurable spaces, etc.).

Example IV.4. Recall Ex. III.14. Let X be a set and $S \subseteq$ **Set**$(X, [0,1])$. In this case, $\bigsqcap_{k \in S} k^* \underline{\Omega} \in (\mathbf{PMet}_1)_X$ is the pseudometric d_S given by $d_S(x, y) = \sup_{k \in S} d_e(k(x), k(y))$. Therefore, in order to show S being an approximating family, we have to recover the pseudometric induced by $h \colon (X, d_S) \to ([0,1], d_e)$ from observations in S, for each h.

In Prop. V.6 later, it will turn out that S is approximating if the following hold (under Asm. V.3):

- S is closed under the four operations $\top$, min, $\neg$, and $\ominus q$ for every $q \in \mathbb{Q} \cap [0,1]$.
- (X, d_S) is totally bounded.

In this case, any $h \colon (X, d_S) \to ([0,1], d_e)$ can be uniformly approximated by a countable sequence of arrows in S. Moreover, this approximation is preserved by the lifting $k \mapsto \tau_\lambda \circ Bk$ (this is what we require in Asm. V.3). These two facts establish that S is approximating (cf. Rem. IV.3). See Prop. V.6 for details.

B. The Knaster–Tarski Proof Principle for Expressivity

From the Knaster–Tarski theorem, we can derive the following simple expressivity proof principle. Its proof is by showing that the logical equivalence $\mathsf{LE}_{\mathscr{S}}(x)$ is a suitable invariant and thus underapproximates the codensity bisimilarity (Thm. IV.1).

Theorem IV.5 (the Knaster–Tarski proof principle). *Let $\mathscr{S}$ be an expressivity situation in Def. III.1 and $x \colon X \to BX$ be a B-coalgebra. If $\{\llbracket \varphi \rrbracket_x \mid \varphi \in L_{\mathscr{S}}\} \subseteq \mathbb{C}(X, \Omega)$ is approximating, then $\mathscr{S}$ is expressive for x.* $\qquad\square$

The theorem's applicability hinges on whether we can show that the set $\{\llbracket \varphi \rrbracket_x \mid \varphi \in L_{\mathscr{S}}\} \subseteq \mathbb{C}(X, \Omega)$ is an approximating family (where φ ranges over all formulas). We use the theorem for the examples in §VI & VII.

C. The Kleene Proof Principle for Expressivity

To make use of Kleene theorem, we have to consider

$$\top \;\sqsupseteq\; (x^* \circ \overline{B}^{\Omega, \tau})(\top) \;\sqsupseteq\; (x^* \circ \overline{B}^{\Omega, \tau})^2(\top) \;\sqsupseteq\; \cdots \quad (5)$$

where the functor $x^* \circ \overline{B}^{\Omega, \tau}$ is from Def. III.9. We also have to assume that this sequence *stabilizes after ω steps*, i.e., $\bigsqcap_{i < \omega} (x^* \circ \overline{B}^{\Omega, \tau})^i(\top)$ is a fixed point of $x^* \circ \overline{B}^{\Omega, \tau}$.

We stratify $L_{\mathscr{S}}$ corresponding to the sequence (5).

Definition IV.6 (depth). Let $\mathscr{S}$ be an expressivity situation in Def. III.1. For each $\varphi \in L_{\mathscr{S}}$, the *depth of φ* $\mathrm{depth}(\varphi)$ is a natural number defined inductively as follows:

$$\mathrm{depth}(\sigma(\varphi_1, \ldots, \varphi_{\mathrm{rank}(\sigma)}))$$
$$= \max(\mathrm{depth}(\varphi_1), \ldots, \mathrm{depth}(\varphi_{\mathrm{rank}(\sigma)})) \quad (\sigma \in \Sigma)$$
$$\mathrm{depth}(\heartsuit_\lambda \varphi) = \mathrm{depth}(\varphi) + 1 \quad (\lambda \in \Lambda)$$

For $\sigma \in \Sigma$ with $\mathrm{rank}(\sigma) = 0$, $\mathrm{depth}(\sigma())$ is defined to be 0.

We formulate the following proof principle. Unlike Knaster–Tarski (Thm. IV.1), it uses an explicit induction on the depth i. Its proof is therefore more involved but not much more.

Theorem IV.7 (the Kleene proof principle). *Let $\mathscr{S}$ be an expressivity situation as in Def. III.1 and $x \colon X \to BX$ be a*

B-coalgebra. Assume that the chain (5) in $\mathbb{E}_X$ stabilizes after ω steps. If the set $\{\llbracket \varphi \rrbracket_x \mid \varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq i\} \subseteq \mathbb{C}(X, \Omega)$ is approximating for each i, then $\mathscr{S}$ is expressive for x. $\qquad\square$

In Thm. IV.7, we require that $\{\llbracket \varphi \rrbracket_x \mid \varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq i\}$ is approximating for each depth i; this is often easier than the case where φ ranges over all formulas (as in Thm. IV.5). We use the theorem for the example in §V.

Example IV.8. Sufficient conditions for being an approximating family were given in Ex. IV.4. Combined with Thm. IV.7, it yields expressivity (Cor. V.9), one of the main results of [17].

Remark IV.9. In Thm. IV.7, we assumed the stabilization of the chain (5) at length ω. This assumption turns out to be benign, essentially because our modal formulas all have a finite depth (§III-B). Specifically we can show the following: if the logic $L_{\mathscr{S}}$ is expressive for $x \colon X \to BX$, then the chain (5) stabilizes after ω steps.

V. EXPRESSIVITY FOR THE KANTOROVICH BISIMULATION METRICS

This section shows how one of the main results in [17], expressivity of a real-valued logic w.r.t. bisimulation metric, is proved by our Kleene proof principle (Thm. IV.7). See also Ex. III.3, III.5 and III.7.

Definition V.1. Define an expressivity situation $\mathscr{S}_{\mathrm{KMM}}$ by:

- Its fibration is $\mathbf{PMet}_1 \to \mathbf{Set}$ (Ex. II.3).
- Its truth-value object is $[0,1]$ and its observation predicate is d_e, the usual Euclidean metric on $[0,1]$.
- The ranked alphabet of its propositional connectives is $\Sigma = \{\top^0, \min^2, \neg^1\} \cup \{(\ominus q)^1 \mid q \in \mathbb{Q} \cap [0,1]\}$. Its propositional structure $(f_\sigma \colon [0,1]^{\mathrm{rank}(\sigma)} \to [0,1])_{\sigma \in \Sigma}$ is specified by:

$$f_\top() = 1 \qquad\qquad f_{\min}(x, y) = \min(x, y)$$
$$f_\neg(x) = 1 - x \qquad\quad f_{\ominus q}(x) = \max(x - q, 0)$$

- The behavior functor $B \colon \mathbf{Set} \to \mathbf{Set}$, the set of its modality indices Λ, and its observation modalities $(\tau_\lambda \colon B[0,1] \to [0,1])_{\lambda \in \Lambda}$ are arbitrary.

The modal logic $L_{\mathscr{S}_{\mathrm{KMM}}}$ is the same as the logic $\mathcal{M}(\Lambda)$ in [17, Table 1]. What they call an *evaluation map* $\gamma \in \Gamma$ corresponds to an observation modality $\tau_\lambda(\lambda \in \Lambda)$ in our framework. Thus, the fibrational logical equivalence $\mathsf{LE}_{\mathscr{S}_{\mathrm{KMM}}}(\alpha)$ (Def. III.6) coincides with the *logical distance* d_α^L [17, Def. 25] for a coalgebra $\alpha \colon X \to BX$.

Moreover, the codensity lifting $\overline{B}^{d_e, \tau}$ specializes to the *Kantorovich lifting* by [24]:

$$\overline{B}^{d_e, \tau}(X, d) = (BX, d_B) \quad \text{where}$$
$$d_B(t_1, t_2) = \sup_{\lambda, h} d_e(\tau_\lambda((Bh)(t_1)), \tau_\lambda((Bh)(t_2))).$$

In the above sup, λ, h ranges over Λ and $\mathbf{PMet}_1((X, d), ([0,1], d_e))$, respectively. Thus, the codensity bisimilarity $\mathsf{Bisim}^{d_e, \tau}(\alpha)$ (Def. III.9) recovers the definition of the *behavioral distance* d_α [17, Def. 22] for a coalgebra $\alpha \colon X \to BX$.

From Prop. III.13 we obtain:

Corollary V.2. *For $\alpha\colon X \to BX$, $d_\alpha \geq d_\alpha^L$ holds.* □

As mentioned in [17], it is harder to prove expressivity. We use the Kleene proof principle (Thm. IV.7) here. For this argument to work, we have to make further assumptions.

Assumption V.3. *For $\mathscr{S}_{\mathrm{KMM}}$, assume the following:*

1) Λ *is finite.*
2) *If a sequence k_i of functions of type $X \to [0,1]$ uniformly converges into l, then $\tau_\lambda \circ Bk_i\colon BX \to [0,1]$ uniformly converges into $\tau_\lambda \circ Bl$ for each $\lambda \in \Lambda$.*

In particular, condition 2 above is satisfied if each τ_λ induces a non-expansive predicate lifting [17, Def. 17].

The notion of *total boundedness* below is pivotal in [17].

Definition V.4 (from [17, Def. 28]). *$(X,d) \in \mathbf{PMet}_1$ is totally bounded if, for any $\varepsilon > 0$, there is a finite set $F_\varepsilon \subseteq X$ satisfying the following: for each $x \in X$, there is $y \in F_\varepsilon$ such that $d(x,y) < \varepsilon$.*

A critical step in their proof used a Stone–Weierstrass-like property of totally bounded spaces.

Proposition V.5. *Let (X,d) be a totally bounded pseudometric space. A subset $S \subseteq \mathbf{PMet}_1((X,d),([0,1],d_e))$ is dense in the topology of uniform convergence if the following are satisfied:*

1) *S is closed under the four operations $\top$, $\min$, $\neg$ and $\ominus q$ for every $q \in \mathbb{Q} \cap [0,1]$;*
2) *for every $h \in \mathbf{PMet}_1((X,d),([0,1],d_e))$, and every $x,y \in X$, we have*

$$d_e(h(x),h(y)) \leq \sup_{g \in S} d_e(g(x),g(y)).$$ □

In our framework, this can be stated in the following form:

Proposition V.6. *Assume the setting of Def. V.1. Let $X \in \mathbf{Set}$. Under Asm. V.3, a subset $S \subseteq \mathbf{Set}(X,[0,1])$ is approximating if the following hold:*

- *S is closed under the four operations $\top$, $\min$, $\neg$ and $\ominus q$ for every $q \in \mathbb{Q} \cap [0,1]$.*
- *(X,d_S) is totally bounded, where $d_S(x,y) = \sup_{k \in S} d_e(k(x),k(y))$.* □

From now we use some facts on totally bounded space. Using the variation of Arzelà–Ascoli theorem [16, Lemma 5.6] for totally bounded spaces, we can show the following:

Fact V.7. *Under Asm. V.3, if $(X,d) \in \mathbf{PMet}_1$ is totally bounded, then*

- *$\overline{B}^{d_e,\tau}(X,d)$ is also totally bounded. [1]*
- *If $(X,d) \sqsubseteq (X,d')$, (X,d') is also totally bounded.*

These enable us to use Thm. IV.7:

---

[1]Here the finiteness of the number of modalities is crucial. When Λ is infinite, the Kantorovich lifting does not preserve total boundedness.

Proposition V.8. *Let $x\colon X \to BX$ be a coalgebra. Under Asm. V.3, for each i, $\{[\![\varphi]\!] \mid \varphi \in L_{\mathscr{S}_{\mathrm{KMM}}}, \mathrm{depth}(\varphi) \leq i\} \subseteq \mathbf{Set}(X,[0,1])$ is approximating.* □

Corollary V.9 (from [17, Thm. 32]). *Let $\alpha\colon X \to BX$ be a coalgebra. Assume that the sequence $\top \sqsupseteq (x^*\overline{B}^{d_e,\tau})(\top) \sqsupseteq (x^*\overline{B}^{d_e,\tau})^2(\top) \sqsupseteq \cdots$ stabilizes after ω steps (as in Thm. IV.7). Then, under Asm. V.3, $d_\alpha \leq d_\alpha^L$ holds. In particular, d_α is characterized as the greatest pseudometric that makes all $[\![\varphi]\!]_\alpha$ nonexpansive.* □

VI. Expressivity for Markov Process Bisimilarity

This section shows how one of the main results in [15], expressivity of probabilistic modal logic w.r.t. bisimilarity of labelled Markov process, is proved by our Knaster–Tarski proof principle (Thm. IV.5).

Throughout this section, fix a set A of labels.

Definition VI.1. Define an expressivity situation $\mathscr{S}_{\mathrm{CFKP}}$ by:

- Its fibration is $\mathbf{EqRel}_{\mathbf{Meas}} \to \mathbf{Meas}$ (Ex. II.3).
- Its behavior functor $B\colon \mathbf{Meas} \to \mathbf{Meas}$ is $BX = (\mathcal{G}_{\leq 1}X)^A$, where $\mathcal{G}_{\leq 1}$ is the variation of Giry functor, which sends each measurable space to its space of sub-distributions.
- Its truth-value object is $2 = \{0,1\}$ with all subsets measurable and its observation predicate is the equality relation Eq_2 on 2.
- The ranked alphabet of its propositional connectives is $\Sigma = \{\top^0, \wedge^2\}$. Its propositional structure $(f_\top, f_\wedge)$ is specified as the usual boolean operations.
- The set of its modality indices is $A \times (\mathbb{Q} \cap [0,1])$. For each $(a,r) \in A \times (\mathbb{Q} \cap [0,1])$, the observation modality $\tau_{a,r}\colon (\mathcal{G}_{\leq 1}2)^A \to 2$ is defined by

$$\tau_{a,r}((\mu_a)_{a \in A}) = \mathrm{thr}_r(\mu_a(\{1\})),$$

where $\mathrm{thr}_r(s) = 1$ if and only if $s > r$.

Note that a *labelled Markov process (LMP)* with label set A [15, Definition 5.1] is the same as B-coalgebra. The modal logic $L_{\mathscr{S}_{\mathrm{CFKP}}}$ (Def. III.2) has the following syntax:

$$\varphi_1, \varphi_2 ::= \top \mid \wedge(\varphi_1,\varphi_2) \mid \heartsuit_{a,r}\varphi_1 \; ((a,r) \in A \times (\mathbb{Q} \cap [0,1]))$$

So if we identify $\heartsuit_{a,r}$ with $\langle a \rangle_r$ in the original notation, this recovers the syntax of $\mathrm{PML}_\wedge$ defined in [15, Def. 2.3]. Under this identification, the semantics (Def. III.4) is also essentially the same as the original logic: $[\![\varphi]\!]_x(s) = 1 \iff s \vDash \varphi$ holds for any LMP $x\colon X \to (\mathcal{G}_{\leq 1}X)^A$, any point $s \in X$, and any formula φ. The fibrational logical equivalence (Def. III.6) can be concretely represented as

$$\mathsf{LE}_{\mathscr{S}_{\mathrm{CFKP}}}(x) = \{(s,t) \mid \forall \varphi \in L_{\mathscr{S}_{\mathrm{CFKP}}}, s \vDash \varphi \iff t \vDash \varphi\}.$$

By expanding the definition of the codensity lifting (Def. III.8) of $(\mathcal{G}_{\leq 1\_})^A$, we can see that it coincides with the one used to define *probabilistic bisimulation*:

Proposition VI.2. *The codensity lifting $\overline{(\mathcal{G}_{\leq 1\_})^A}^{\mathrm{Eq}_2,\tau}$ satisfies the following: for each $(\mu_a)_{a \in A}, (\nu_a)_{a \in A} \in (\mathcal{G}_{\leq 1}X)^A$, they*

are equivalent in $\overline{(\mathcal{G}_{\leq 1\_})^A}^{\mathrm{Eq}_2,\tau}(X,R)$ if and only if, for each $a \in A$ and each R-closed measurable set $S \subseteq X$, $\mu_a(S) = \nu_a(S)$ holds. □

Thus the codensity bisimilarity $\mathrm{Bisim}^{\mathrm{Eq}_2,\tau}(x)$ (Def. III.9) coincides with the probabilistic bisimilarity used in [15].

From Prop. III.13, we readily obtain the following:

Corollary VI.3. *Let $x\colon X \to (\mathcal{G}_{\leq 1}X)^A$ be an LMP. If $s,t \in X$ are probabilistically bisimilar, for any $\varphi \in L_{\mathscr{S}_{\mathrm{CFKP}}}$, $s \vDash \varphi \iff t \vDash \varphi$ holds.* □

To show expressivity, we first have to review some mathematical key facts. In the rest of this section, we write $\sigma(\mathscr{E})$ for the σ-algebra generated by a family of sets $\mathscr{E}$.

Definition VI.4. A *Polish space* is a separable topological space which is metrizable by a complete metric. For any continuous map $f\colon X \to Y$ between Polish spaces X and Y, the image of f is called an *analytic topological space*. For an analytic topological space $(X, \mathcal{O}_X)$, the measurable space $(X, \sigma(\mathcal{O}_X))$ is called an *analytic measurable space*.

Let us review the two key facts they used in [15]. The first one is the following "elegant Borel space analogue of the Stone–Weierstrass theorem" [35].

Fact VI.5 (Unique Structure Theorem [35, Thm. 3.3.5]). *Let $X \in \mathbf{Meas}$ be an analytic measurable space and $\mathscr{E}$ be an (at most) countable family of measurable subsets of X such that $X \in \mathscr{E}$. Define an equivalence relation $\equiv_{\mathscr{E}}$ by*

$$x \equiv_{\mathscr{E}} y \iff \forall S \in \mathscr{E}, (x \in S \iff y \in S).$$

If $S \subseteq X$ is measurable and $\equiv_{\mathscr{E}}$-closed, then $S \in \sigma(\mathscr{E})$.

In the fact above, we use the operations of σ-algebras to construct S. The second key fact is about "decomposing" those operations into two parts.

Definition VI.6. Let X be a set. A family of subsets of X is called a *π-system* if it is closed under finite intersections. A family of subsets of X is a *λ-system* if it is closed under complement and countable disjoint unions.

Intuitively, π-systems correspond to the propositional connectives of $\mathscr{S}_{\mathrm{CFKP}}$ and λ-systems correspond to "approximation." These two operations are enough to recover all σ-algebra operations:

Fact VI.7 (π-λ Theorem [36]). *If Π is a π-system, Λ is a λ-system, and $\Pi \subseteq \Lambda$, then $\sigma(\Pi) \subseteq \Lambda$.*

Using Fact VI.5 and VI.7, we obtain a sufficient condition for being an approximating family. The proof follows the two steps outlined in Rem. IV.3: 1) we can approximate a given $h\colon X \to 2$ by σ-algebra operations (Fact VI.5), which can be reduced to λ-system operations (Fact VI.7); and 2) λ-system operations are in some sense "preserved" by the modalities (since measures are σ-additive).

Proposition VI.8. *Assume the setting of Def. VI.1. Let $X \in \mathbf{Meas}$. A subset $S \subseteq \mathbf{Meas}(X, 2)$ is approximating if the following hold:*

- *X is an analytic measurable space.*
- *S is at most countable.*
- *For $k, l \in S$, $\top$ and $k \wedge l$ are also included in S.* □

From this proposition and Thm. IV.5, we obtain the following expressivity result:

Corollary VI.9. *Let $x\colon X \to (\mathcal{G}_{\leq 1}X)^A$ be an LMP and $s, t \in X$ its states. Assume that the label set A is at most countable and that X is an analytic measurable space.*

Then $\mathscr{S}_{\mathrm{CFKP}}$ is expressive for x (Def. III.12): that is, If $s \vDash \varphi \iff t \vDash \varphi$ holds for every $\varphi \in L_{\mathscr{S}_{\mathrm{CFKP}}}$, then s and t are probabilistically bisimilar. □

VII. EXPRESSIVITY FOR THE BISIMULATION UNIFORMITY

In this section, we introduce *bisimulation uniformity* as a coinductive predicate in a fibration and a logic for it. By using our main results and a known mathematical result analogous to the Stone–Weierstrass theorem, the logic is readily proved to be adequate and expressive w.r.t. bisimulation uniformity. This example shows how our abstract framework can help to explore new bisimilarity-like notions.

A. Uniform Structure as Fibrational Predicate

Topological space can be regarded as an abstraction of (pseudo-)metric spaces w.r.t. continuous maps. In much the same way, *uniform space* [37] is an abstraction of (pseudo-)metric spaces w.r.t. uniformly continuous maps.

Definition VII.1 (from [37, Def. 1]). A *uniform structure*, or *uniformity*, on a set X is a nonempty family $\mathscr{U} \subseteq \mathcal{P}(X \times X)$ of subsets of $X \times X$ satisfying the following:

- If $V \in \mathscr{U}$ and $V \subseteq V' \subseteq X \times X$, then $V' \in \mathscr{U}$.
- If $V, W \in \mathscr{U}$, then $V \cap W \in \mathscr{U}$.
- If $V \in \mathscr{U}$, then $\{(x, x) \mid x \in X\} \subseteq V$.
- If $V \in \mathscr{U}$, then $\{(y, x) \mid (x, y) \in V\} \in \mathscr{U}$.
- If $V \in \mathscr{U}$, then there exists $W \in \mathscr{U}$ such that $\{(x, z) \mid \exists y\ (x, y) \in W \wedge (y, z) \in W\} \subseteq V$.

Here each element $V \in \mathscr{U}$ is called an *entourage*. A pair $(X, \mathscr{U})$ of a set and a uniformity on it is called a *uniform space*.

A function $f\colon X \to Y$ is a *uniformly continuous map* from $(X, \mathscr{U}_X)$ to $(Y, \mathscr{U}_Y)$ if, for each entourage $V \in \mathscr{U}_Y$, $\{(x, x') \mid (f(x), f(x')) \in V\} \subseteq X \times X$ is an enrourage of $(X, \mathscr{U}_X)$. The category of uniform spaces and uniformly continuous maps is denoted $\mathbf{Unif}$.

Each entourage represents some degree of "closeness." The following example is an archetypal one:

Example VII.2. Let (X, d) be a pseudometric space. Define a family $\mathscr{U} \subseteq \mathcal{P}(X \times X)$ as the set of all relations of the form $\{(x, x') \mid d(x, x') < \varepsilon\}$ for $\varepsilon > 0$ and their supersets. Then $(X, \mathscr{U})$ is a uniform space.

Some of the concepts considered for metric spaces, like completion, total boundedness, and characterization of compactness, can be lifted to uniform spaces. For us, the most important fact is that they form a $\mathbf{CLat}_\sqcap$-fibration:

Proposition VII.3 (from [37, Propositions 4 and 5]). *The forgetful functor* $\mathbf{Unif} \to \mathbf{Set}$ *is a* $\mathbf{CLat}_\sqcap$-*fibration.* $\square$

Thus we can use uniform structures as a sort of indistinguishability structure. A uniform structure on a finite set is essentially the same as an equivalence relation. For infinite sets, however, it can be a helpful way to analyze coalgebras that is more quantitative than an equivalence relation and more robust than a pseudometric.

B. Expressivity Situation for Bisimulation Uniformity

Definition VII.4. Define an expressivity situation $\mathscr{S}_{\mathrm{BU}}$ by:

- Its fibration is $\mathbf{Unif} \to \mathbf{Set}$ (Prop. VII.3).
- Its truth-value object is $\mathbb{R}$ and its observation predicate is $\mathscr{U}_e$, the uniformity defined using the usual Euclidean metric as in Ex. VII.2.
- The ranked alphabet of its propositional connectives is $\Sigma = \{1^0, \min^2\} \cup \{(r+)^1, (r\times)^1 \mid r \in \mathbb{R}\}$. Its propositional structure $(f_\sigma : \mathbb{R}^{\mathrm{rank}(\sigma)} \to \mathbb{R})_{\sigma \in \Sigma}$ is specified by:

$$f_1() = 1 \qquad\qquad f_{\min}(x,y) = \min(x,y)$$
$$f_{r+}(x) = r + x \qquad\qquad f_{r\times}(x) = rx$$

- The behavior functor $B : \mathbf{Set} \to \mathbf{Set}$, the set of its modality indices Λ, and its observation modalities $(\tau_\lambda : B\mathbb{R} \to \mathbb{R})_{\lambda \in \Lambda}$ are arbitrary.

For a B-coalgebra $x : X \to BX$, the codensity lifting $\overline{B}^{\mathscr{U}_e, \tau}$ (Def. III.8) yields the codensity bisimilarity $\mathrm{Bisim}^{\mathscr{U}_e, \tau}(x) \in \mathbf{Unif}_X$ (Def. III.9), which is a uniformity on the set X. We call it the *bisimulation uniformity of x*.

On the other hand, the logic $L_{\mathscr{S}_{\mathrm{BU}}}$ induces the fibrational logical equivalence $\mathsf{LE}_{\mathscr{S}_{\mathrm{BU}}}(x)$ (Def. III.6) for each $x : X \to BX$. We call this the *logical uniformity of x*.

By Prop. III.13, we obtain the following:

Proposition VII.5. *Assume the setting of Def. VII.4. Let $x : X \to BX$ be a B-coalgebra. Any entourage of the logical uniformity is also an entourage of the bisimulation uniformity. In particular, for any $\varphi \in L_{\mathscr{S}_{\mathrm{BU}}}$, $[\![\varphi]\!]_x : X \to \mathbb{R}$ is uniformly continuous w.r.t. the bisimulation uniformity.* $\square$

To prove expressivity, we have to make further assumptions.

Assumption VII.6. *For $\mathscr{S}_{\mathrm{BU}}$, assume the following:*

1) *If $k : X \to \mathbb{R}$ is bounded, then $\tau_\lambda \circ Bk : BX \to \mathbb{R}$ is also bounded for each $\lambda \in \Lambda$.*
2) *If a sequence k_i of functions of type $X \to \mathbb{R}$ uniformly converges into h, then $\tau_\lambda \circ Bk_i : BX \to \mathbb{R}$ uniformly converges into $\tau_\lambda \circ Bh$ for each $\lambda \in \Lambda$.*

The key in the expressivity proof is the following known Stone–Weierstrass-like result:

Fact VII.7 (from [21, Thm. 1]). *Let X be a set and $\Gamma \subseteq \mathbb{R}$ a set of real numbers unbounded both from above and below. Assume a family Φ of bounded real-valued function satisfies the following:*

1) *Every constant is in Φ.*
2) *For $f \in \Phi$ and $r \in \Gamma$, $rf \in \Phi$ holds.*
3) *For $f \in \Phi$ and $r \in \mathbb{R}$, $r + f \in \Phi$ holds.*
4) *For $f, g \in \Phi$, $\min(f,g), \max(f,g) \in \Phi$ holds.*

Let $\mathscr{U}_\Phi$ be the coarsest uniformity on X that makes every function in Φ uniformly continuous. Then any real-valued function uniformly continuous w.r.t. $\mathscr{U}_\Phi$ is the limit of a uniformly convergent sequence of elements of Φ.

By using this, we can show that a suitable set is approximating. In its proof, we follow the two steps discussed in Rem. IV.3.

Proposition VII.8. *Assume the setting of Def. VII.4. Let $X \in \mathbf{Set}$. Under Asm. VII.6, a subset $S \subseteq \mathbf{Set}(X, \mathbb{R})$ is approximating if the following hold:*

- *Every function in S is bounded.*
- *$1 \in S$.*
- *S is closed under the three operations $\min$, $(r+)$, and $(r\times)$ for every $r \in \mathbb{R}$.* $\square$

From this, we can obtain expressivity:

Corollary VII.9. *Assume the setting of Def. VII.4. Let $x : X \to BX$ be a B-coalgebra. Under Asm. VII.6, the bisimulation uniformity coincides with the logical uniformity, i.e., the former is characterized as the coarsest uniformity making every $[\![\varphi]\!]_x : X \to \mathbb{R}$ uniformly continuous.* $\square$

VIII. AN ABSTRACT LOOK AT EXPRESSIVITY AND APPROXIMATION

Lastly, we take an abstract approach to the concept of expressivity and approximation by combining two studies on coalgebraic modal logic: fibrational formulation of adequacy and expressivity [22], and Klin's reformulation of duality-based modal logic using comma categories [38]. The combination leads us to a new look at adequacy and expressivity as a comparison problem of final coalgebras through a functor. An approximating family is then a key construct to solve this comparison problem.

A. Fibrational Theory of Adequacy and Expressivity

In [22], the third and fourth authors integrated duality-based modal logic [6] and fibrational theory of bisimulation [23], and formulated adequacy and expressivity of modal logic. Their formulation is built upon the following categorical situation [22, Asm. 14, Def. 15], which we tentatively call a *modal logic with coinductive predicates* $\mathscr{K}$:

$$
\begin{array}{l}
\overline{B} \circlearrowright \mathbb{E} \\
\quad p \downarrow \quad \overline{Q} \\
B \circlearrowright \mathbb{C} \underset{Q}{\overset{P}{\underset{\perp}{\rightleftarrows}}} \mathbb{D}^{\mathrm{op}} \circlearrowright L^{\mathrm{op}}
\end{array}
\quad +
\begin{pmatrix} \alpha : L\Phi \to \Phi \text{ initial} \\ \delta : B \circ Q \to Q \circ L^{\mathrm{op}} \end{pmatrix}
\;\mathrm{s.t.}\;
\begin{pmatrix} p : \mathbf{CLat}_\sqcap\text{-fibration} \\ \overline{B} : \text{lifting of } B \\ p \circ \overline{Q} = Q \end{pmatrix}
\quad (6)
$$

The tuple $\mathscr{L} \triangleq (B, P \dashv Q, L, \delta, \alpha)$ is called a duality-based modal logic, while the tuple $(p, B, \overline{B})$ determines a setting for fibrational bisimulation (§II-C). Now let $x : X \to BX$ be a coalgebra. Following [22, Def. 15], $\mathscr{K}$ is said to be

- *adequate* for x if $\nu(x^* \circ \overline{B}) \sqsubseteq \mathrm{th}_x^*(Q\Phi)$, and
- *expressive* for x if $\nu(x^* \circ \overline{B}) \sqsupseteq \mathrm{th}_x^*(Q\Phi)$.

Here, $\mathrm{th}_x : X \to Q\Phi$ is the theory morphism induced by x. It corresponds to the interpretation function $[\![\_]\!]_x$ in Def. III.4; see [6] for detail.

One might wonder how the above definition of adequacy and expressivity is related to ours in Def. III.12. The following construction establishes a formal connection:

Theorem VIII.1. *For any expressivity situation $\mathscr{S} = (p, B, \Omega, \underline{\Omega}, \cdots, \tau)$ whose base category $\mathbb{C}$ has small powers and equalizers, there is a modal logic with coinductive predicates $\mathrm{MLB}(\mathscr{S})$ such that for any coalgebra $x : X \to BX$, $\nu(x^* \circ \overline{B}) = \mathsf{Bisim}^{\Omega,\tau}(x)$ and $\mathrm{th}_x^*(\overline{Q}\Phi) = \mathsf{LE}_{\mathscr{S}}(x)$ holds.* $\square$

B. Fibration from Duality-Based Modal Logic

In [38], Klin studied duality-based modal logic using a comma category (see [39, Sect. II.6] for the definition). We quickly review his study, reusing the duality-based modal logic $\mathscr{L}$ in (6). We consider the comma category $\mathrm{Id}_{\mathbb{C}} \downarrow Q^2$ with the evident first projection functor $\pi_1 : \mathrm{Id}_{\mathbb{C}} \downarrow Q \to \mathbb{C}$. He showed the following results that are relevant to us:

- [38, Sect. 4] The natural transformation δ determines a *lifting* Δ of B along π_1, whose object part is given by $\Delta(X, Y, f) = (BX, LY, \delta_Y \circ Bf)$.
- [38, Corollary 4.4] The initial L-algebra α induces a right adjoint right inverse R of $\mathbf{CoAlg}(\pi_1) : \mathbf{CoAlg}(\Delta) \to \mathbf{CoAlg}(B)$ (see Prop. II.10). Its object part sends a coalgebra $x : X \to BX$ to $Rx = (X, \Phi, \mathrm{th}_x)$.

We re-interpret these results in terms of the fibrational theory of bisimulation. First, one can easily verify that $\pi_1 : \mathrm{Id}_{\mathbb{C}} \downarrow Q \to \mathbb{C}$ is a fibration and Δ is a fibred lifting. Second, for any coalgebra $x : X \to BX$, Rx is a final object in the fiber category $\mathbf{CoAlg}(\pi_1)_x$, which corresponds to a Δ-coinductive predicate $\nu(x^* \circ \Delta)$ over x by Prop. II.10. To summarize, a duality-based modal logic induces a setting for fibrational bisimulation admitting coinductive predicates.

C. Another Abstract Look at Adequacy and Expressivity

We combine fibrational theory of adequacy and expressivity, and a fibrational presentation of the duality-based modal logic based on Klin's study. The key factor connecting these two studies is the fibred functor $H : \pi_1 \to p$ given by $H(X, Y, f) = f^*(\overline{Q}Y)$.

$$
\overline{B} \,\circlearrowright\, \mathbb{E} \xleftarrow{\quad H \quad} \mathrm{Id}_{\mathbb{C}} \downarrow Q \,\circlearrowright\, \Delta \qquad (7)
$$
$$
\begin{array}{c} \downarrow{\scriptstyle p} \qquad\qquad\qquad {\scriptstyle \pi_1} \downarrow \\ B \,\circlearrowright\, \mathbb{C} \xleftarrow{\qquad} \end{array}
$$

Using H, the definitions of adequacy and expressivity are equivalently rewritten as follows. Let $x : X \to BX$ be a

---

coalgebra. Then the modal logic with coinductive predicates $\mathscr{K}$ is

- *adequate* for x if $\nu(x^* \circ \overline{B}) \sqsubseteq H(\nu(x^* \circ \Delta))$, and
- *expressive* for x if $\nu(x^* \circ \overline{B}) \sqsupseteq H(\nu(x^* \circ \Delta))$.

That is, establishing adequacy and expressivity can be viewed as a familiar problem of comparing final coalgebras in two (fibre) categories connected by a functor.

D. Another Abstract Look at Kleene Proof Principle

From the above reformulation of expressivity (and adequacy), we easily notice the following sound proof method. Below we impose the following conditions (*) on

Theorem VIII.2. *Suppose that in the modal logic with coinductive predicates $\mathscr{K}$ (6), $\mathbb{D}$ has an initial object, L-initial sequence stabilizes, and $\overline{Q}$ is a right adjoint. Let $x : X \to BX$ be a coalgebra such that $x^* \circ \overline{B}$-final sequence stabilizes after ω-steps. Then each of (8) and (9) implies that $\mathscr{K}$ is expressive for x.*

$$
\forall i \in \omega \ . \ H((x^* \circ \Delta)^i(\top)) \sqsubseteq (x^* \circ \overline{B})^i(\top) \qquad (8)
$$
$$
\forall i \in \omega \ . \ H \circ \Delta \circ (x^* \circ \Delta)^i(\top) \sqsubseteq \overline{B} \circ H \circ (x^* \circ \Delta)^i(\top) \quad (9)
$$
$$
\square
$$

In other words, (8) says that the H-image of the $x^* \circ \Delta$-final sequence is bound by the $x^* \circ \overline{B}$-final sequence. The above theorem is working behind the proof of Kleene proof principle for expressivity (Thm. IV.7); in the situation (7) arising from $\mathrm{MLB}(\mathscr{S})$, Thm. IV.7 first shows (8) using the assumption of approximating family of observations, then concludes the expressivity by invoking Thm. VIII.2. On the other hand, (9) is an analogy of approximating family of observations appeared in Thm. IV.7 in the abstract set-up (7).

IX. CONCLUSIONS AND FUTURE WORK

We introduced a categorical framework to study expressivity of quantitative modal logics, based on the novel notion of approximating family. This enabled us to cover not only existing examples (§V and §VI) but also a new one (§VII). We conclude with some future research directions.

a) Making Use of Size Restrictions on Functors: Many existing expressivity results make use of size restriction condition on the behavior functor B [2], [11], [18]. Importing these size restrictions is future work. A starting point can be [28].

b) Study of Bisimulation Uniformity: We defined bisimulation uniformity in §VII, but there are many topics left to study. One primary subject is the connection to bisimilarity and bisimulation metric. It is also important to see if it is robust under parameter changes of the target system.

c) Seeking Stone–Weierstrass-like Theorems: To use our framework to show expressivity, one has to obtain a sufficient condition for being an approximating family. In many cases, this is reduced to finding an appropriate "Stone–Weierstrass-like" theorem. Concretely find ones and apply them to modal logics (other than those we have mentioned) is future work. Another research direction is to a seek connection to [40], where "Stone–Weierstrass-like" theorems are formulated in another way.

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[2] This is isomorphic to the category $P \downarrow \mathrm{Id}_{\mathbf{Set}}\Sigma$ employed by Klin in [38].

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