# Weak Similarity in Higher-Order Mathematical Operational Semantics

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Abstract—Higher-order abstract GSOS is a recent extension of Turi and Plotkin's framework of Mathematical Operational Semantics to higher-order languages. The fundamental wellbehavedness property of all specifications within the framework is that coalgebraic strong (bi)similarity on their operational model is a congruence. In the present work, we establish a corresponding congruence theorem for *weak* similarity, which is shown to instantiate to well-known concepts such as Abramsky's applicative similarity for the  $\lambda$ -calculus. On the way, we develop several techniques of independent interest at the level of abstract categories, including relation liftings of mixed-variance bifunctors and higher-order GSOS laws, as well as Howe's method.

#### I. INTRODUCTION

Following the emergence of structural approaches to operational semantics (SOS), e.g. [28], [34], operational reasoning has developed into a widely used methodology in formal reasoning on higher-order languages. Numerous powerful operational techniques have been developed, tested, and refined, such as logical relations [36], [35], [33], [15] and Howe's method [24], [25], [14]. These methods have been found to be quite robust, being capable of providing solutions to challenging problems such as congruence proofs and reasoning about contextual equivalence, even in rather involved settings such as effectful, e.g. nondeterministic, higher-order languages.

Unfortunately, such power comes at a price. Operational methods are known to be both complex, requiring a daunting amount of machinery in order to be instantiated, and specialized, in the sense that they need to be developed on a per-case basis, and any small perturbation in the problem setting may break earlier machinery. A key ingredient that is needed to alleviate these issues is a sufficiently general rigorous notion of SOS specification of programming language semantics; without it, reasoning is inevitably bound to specific instances of SOS specifications, and the only 'free' mathematical principle is induction on the structure of terms. Capturing the essence of SOS in a single, precise definition in order to reason at a greater level of generality has thus been a topic of lasting interest. Rule formats such as GSOS [5] provide a handle to reason about classes of languages, as opposed to one language at a time. For instance, the property that

bisimilarity is a congruence holds for any language adhering to the GSOS format. On a more abstract and conceptual level, Turi and Plotkin's framework of Mathematical Operational Semantics [37], a.k.a. *abstract GSOS*, shows that rule formats such as GSOS are instances of a general principle, namely that operational rules amount to certain natural transformations, so-called *GSOS laws*. Abstract GSOS has been instantiated in quite diverse settings [3], [29], [17], [32], [19].

In recent work [20] we have reconciled Turi and Plotkin's ideas, originally applicable only in first-order settings, with higher-order languages. The main insight is that dinatural transformations are able to express higher-order operational rules in ways that the original approach based on naturality could not. Like a classical GSOS law, a higher-order GSOS law is a form of distributive law of a syntax functor  $\Sigma$ over a behaviour functor B, but in the context of higherorder languages, B in general needs to be a mixed-variance bifunctor, in the sense that it depends covariantly on the set of states or terms when these appear as results of functions, and contravariantly when they are used as arguments of functions. It is this phenomenon of mixed variance that necessitates the use of dinatural transformations. The main result of [20] is that the operational semantics of a higher-order GSOS law is *compositional*: for the initial (term) model  $\mu\Sigma$ , coalgebraic bisimilarity for the endofunctor  $B(\mu\Sigma, -)$  is a congruence. For instance, in the case where B(X, Y) is the behaviour bifunctor for the  $\lambda$ -calculus, this instantiates to a *strong* variant of Abramsky's applicative bisimilarity [1], which unlike applicative bisimilarity proper makes  $\beta$ -reductions observable.

The main contribution of the present paper is a generalization of our previous congruence result [20] from strong bisimilarity to weak (bi)similarity. It applies to higher-order GSOS laws whose initial model forms a higher-order lax bialgebra, extending the corresponding first-order concept [6]. When instantiated to the call-by-name  $\lambda$ -calculus, weak (bi)similarity amounts to standard applicative (bi)similarity. Hence we obtain a more useful general compositionality theorem, an instance of which is the classical result that applicative bisimilarity (rather than a previously unstudied notion of strong applicative bisimilarity as in [20]) in the call-by-name  $\lambda$ -calculus is a congruence [1]. Our approach is parameterized in such a way that strong similarity is an instance of weak similarity, so our main result subsumes that of [20].

The passage from strong to weak similarity comes with a number of technical challenges; most notably, simple and

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well-established proof techniques such as coinduction up to congruence now fail. To prove our main theorem, we develop an abstract categorical version of Howe's method (Proposition VIII.5). The abstraction depends centrally on new notions of bifunctorial graph and relation liftings (applied to liftings of the mixed-variance behaviour functor), which may in fact turn out to be of independent interest as generalizations of relation liftings of functors [22], [27] to higher-order behaviours.

#### For full proofs and additional details, see Appendix.

*Related Work:* Borthelle et al. [8] and Hirschowitz and Lafont [23] have recently developed a framework for congruence of applicative bisimilarity based on Howe's method. Their approach is conceptually quite different from ours: operational rules are given as endofunctors on a presheaf category of *transition systems* over models of a signature endofunctor, and the initial algebra for the rule endofunctor represents the induced transition system for the given semantics.

Dal Lago et al. [14] propose a generalization of Howe's method for call-by-value  $\lambda$ -calculi with algebraic effects, based on the theory of relators. Their notion of a *computational*  $\lambda$ -calculus is parametrized over a signature  $\Sigma$  and a monad T on sets, representing syntax and effects of the language. The operational semantics is given in big-step form.

Bonchi et al. [6] employ lax bialgebras to establish upto techniques for weak bisimulations in the context of (firstorder) abstract GSOS. Besides the differences in scope, two approaches diverge also in the way the are based on relation liftings: Bonchi et al. lift endofunctors from sets to preorders and further to up-closed relations, while we lift bifunctors from an abstract category  $\mathbb{C}$  to relations over  $\mathbb{C}$ , the upclosure being replaced with the abstract *good-for-simulations* condition (Definition IV.5).

#### **II. PRELIMINARIES**

#### A. Category Theory

We assume familiarity with basic category theory. In the following we recall some relevant terminology and notation.

*Products and coproducts:* Given objects  $X_1, X_2$  in a category  $\mathbb{C}$ , we write  $X_1 \times X_2$  for the product and  $\langle f_1, f_2 \rangle \colon X \to X_1 \times X_2$  for the pairing of morphisms  $f_i \colon X \to X_i, i = 1, 2$ . We let  $X_1 + X_2$  denote the coproduct, inl:  $X_1 \to X_1 + X_2$  and inr:  $X_2 \to X_1 + X_2$  the injections,  $[g_1, g_2] \colon X_1 + X_2 \to X$  the copairing of morphisms  $g_i \colon X_i \to X, i = 1, 2$ , and  $\nabla = [\operatorname{id}_X, \operatorname{id}_X] \colon X + X \to X$  the codiagonal.

Locally distributive categories: A category  $\mathbb{C}$  is distributive if it has finite products and coproducts, and for each  $X \in \mathbb{C}$  the endofunctor  $X \times (-)$  on  $\mathbb{C}$  preserves finite coproducts. It is *locally distributive* if for each  $X \in \mathbb{C}$  the slice category  $\mathbb{C}/X$  is distributive. Recall that  $\mathbb{C}/X$  has as objects all pairs  $(Y, p_Y)$  of an object  $Y \in \mathbb{C}$  and a morphism  $p_Y \colon Y \to X$ , and a morphism from  $(Y, p_Y)$  to  $(Z, p_Z)$  is a morphism  $f \colon Y \to Z$  of  $\mathbb{C}$  such that  $p_Y = p_Z \cdot f$ . The coslice category  $X/\mathbb{C}$  is defined dually. **Example II.1.** Examples of locally distributive categories include the category **Set** of sets and functions, the category **Set**<sup> $\mathbb{C}$ </sup> of presheaves on a small category  $\mathbb{C}$  and natural transformations, and the categories of posets and monotone maps, nominal sets and equivariant maps, metric spaces and non-expansive maps. In fact, they are all *lextensive* [11, Cor 4.9].

Algebras: Given an endofunctor F on a category  $\mathbb{C}$ , an F-algebra is a pair (A, a) which consists of an object A (the carrier of the algebra) and a morphism  $a: FA \to A$  (its structure). A morphism from (A, a) to an F-algebra (B, b) is a morphism  $h: A \to B$  of  $\mathbb{C}$  such that  $h \cdot a = b \cdot Fh$ . Algebras for F and their morphisms form a category  $\operatorname{Alg}(F)$ , and an initial F-algebra is simply an initial object in that category. We denote the initial F-algebra by  $\mu F$  if it exists, and its structure by  $\iota: F(\mu F) \to \mu F$ . If  $\mathbb{C}$  has binary products, initial algebras entail a useful definition principle known as primitive recursion: for every morphism  $a: F(\mu F \times A) \to A$  there exists a unique morphism pr a making the square below commute.

$$\begin{array}{ccc} F(\mu F) & \stackrel{\iota}{\longrightarrow} & \mu F \\ F\langle \operatorname{id}, \operatorname{pr} a \rangle & & & \downarrow \\ F(\mu F \times A) & \stackrel{a}{\longrightarrow} & A \end{array}$$
 (II.1)

More generally, a *free F*-algebra on an object *X* of  $\mathbb{C}$  is an *F*-algebra  $(F^*X, \iota_X)$  together with a morphism  $\eta_X : X \to F^*X$  of  $\mathbb{C}$  such that for every algebra (A, a) and every morphism  $h: X \to A$  in  $\mathbb{C}$ , there exists a unique *F*-algebra morphism  $h^* : (F^*X, \iota_X) \to (A, a)$  such that  $h = h^* \cdot \eta_X$ . If free algebras exist on every object, their formation induces a monad  $F^* : \mathbb{C} \to \mathbb{C}$ , the *free monad* generated by *F*. (Conversely, in complete and well-powered categories, existence of a free monad implies existence of free algebras [30, Thm. 4.2.15].) For every *F*-algebra (A, a), we obtain an Eilenberg-Moore algebra  $\hat{a}: F^*A \to A$  as the free extension of  $\mathrm{id}_A : A \to A$ .

The most familiar example of functor algebras are algebras for a signature. An *algebraic signature* consists of a set  $\Sigma$  of operation symbols together with a map  $\operatorname{ar}: \Sigma \to \mathbb{N}$  associating to every  $f \in \Sigma$  its *arity* ar(f). Symbols of arity 0 are called *constants*. Every signature  $\Sigma$  induces the polynomial set functor  $\prod_{f \in \Sigma} (-)^{\operatorname{ar}(f)}$ , which we denote by the same letter  $\Sigma$ . An algebra for the functor  $\Sigma$  is precisely an algebra for the signature  $\Sigma$ , viz. a set A equipped with an operation  $f^A \colon A^n \to A$ for every *n*-ary operation symbol  $f \in \Sigma$ . Morphisms of  $\Sigma$ algebras are maps respecting the algebraic structure. Given a set X of variables, the free algebra  $\Sigma^{\star}X$  is the  $\Sigma$ -algebra of  $\Sigma$ -terms with variables from X. In particular, the free algebra on the empty set is the initial algebra  $\mu\Sigma$ ; it is formed by all closed terms of the signature. For every  $\Sigma$ -algebra (A, a), the induced Eilenberg-Moore algebra  $\widehat{a} \colon \Sigma^* A \to A$  is given by the map evaluating terms over A in the algebra.

A relation  $R \subseteq A \times A$  on a  $\Sigma$ -algebra A is called a *congruence* if for every *n*-ary  $f \in \Sigma$  and elements  $R(a_i, a'_i)$ ,  $i = 1, \ldots, n$ , one has  $R(f^A(a_1, \ldots, a_n), f^A(a'_1, \ldots, a'_n))$ . Note that we do not require R to be an equivalence relation.

*Coalgebras:* Dual to the notion of algebra, a *coalgebra* for an endofunctor F on  $\mathbb{C}$  is a pair (C, c) of an object C (the *carrier*) and a morphism  $c: C \to FC$  (its *structure*).

#### B. Higher-Order Abstract GSOS

We review the core principles behind *higher-order abstract GSOS* [20], a categorical framework modelling the operational semantics of higher-order languages. It is parametric in

- (1) a category  $\mathbb{C}$  with finite products and coproducts;
- (2) an object  $V \in \mathbb{C}$  of variables;

(3) two functors  $\Sigma \colon \mathbb{C} \to \mathbb{C}$  and  $B \colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$ , where  $\Sigma = V + \Sigma'$  for some functor  $\Sigma' \colon \mathbb{C} \to \mathbb{C}$ , and free  $\Sigma$ -algebras exist on every object (hence  $\Sigma$  generates a free monad  $\Sigma^*$ ).

Informally, the functors  $\Sigma$  and B represent the *syntax* and the *behaviour* of a higher-order language. The initial algebra  $\mu\Sigma$  is the object of programs, and the requirement that  $\Sigma = V + \Sigma'$  asserts that variables are programs. An object of  $V/\mathbb{C}$ , the coslice category of *V*-pointed objects, is thought of as a set X of programs with an embedding  $p_X : V \to X$  of the variables.

**Example II.2.** A simple instantiation is given by  $V = \emptyset$ , a polynomial functor  $\Sigma$  and the bifunctor  $B_0(X, Y) = Y + Y^X$  on Set. A map  $\gamma_0: \mu\Sigma \to \mu\Sigma + \mu\Sigma^{\mu\Sigma}$ , that is, a  $B_0(\mu\Sigma, -)$ -coalgebra with carrier  $\mu\Sigma$ , can be thought of as a description of the operational behaviour of deterministic higher-order programs: every program  $p \in \mu\Sigma$  either performs a silent computation step reducing p to  $\gamma(p) \in \mu\Sigma$ , or it acts as a function  $\gamma(p) \in \mu\Sigma^{\mu\Sigma}$  mapping programs to programs.

In order to actually construct coalgebras  $\gamma_0$  as in the above example, we use the following concept:

**Definition II.3.** A (V-pointed) higher-order GSOS law of  $\Sigma$  over B is a family of morphisms

$$\varrho_{(X,p_X),Y} \colon \Sigma(X \times B(X,Y)) \to B(X,\Sigma^*(X+Y)) \quad (\text{II.2})$$

dinatural in  $(X, p_X) \in V/\mathbb{C}$  and natural in  $Y \in \mathbb{C}$ .

**Notation II.4.** (1) We usually write  $\rho_{X,Y}$  for  $\rho_{(X,p_X),Y}$ , as the point  $p_X : V \to X$  will always be clear from the context. (2) For every  $\Sigma$ -algebra (A, a), we regard A as V-pointed by

$$p_A = \left( V \xrightarrow{\text{inl}} V + \Sigma' A = \Sigma A \xrightarrow{a} A \right).$$

**Definition II.5.** The *operational model* of a higher-order GSOS law  $\rho$  in (II.2) is the  $B(\mu\Sigma, -)$ -coalgebra

$$\gamma \colon \mu \Sigma \to B(\mu \Sigma, \mu \Sigma)$$

obtained via primitive recursion as the unique morphism making the diagram (II.3) in Figure 1 commute. Here we regard the initial algebra  $\mu\Sigma$  as V-pointed as in Notation II.4, and  $\hat{\iota}$  is the  $\Sigma^*$ -algebra corresponding to  $\iota: \Sigma(\mu\Sigma) \to \mu\Sigma$ .

**Remark II.6.** The commutative diagram (II.3) states that  $(\mu\Sigma, \iota, \gamma)$  forms a *bialgebra* for the higher-order GSOS law  $\varrho$ ; in fact, it is the initial such bialgebra [20, Prop. 4.20]. An important difference to first-order abstract GSOS [37] is that a final bialgebra usually does not exist even for

simple deterministic behaviour functors [20, Ex. 4.21]. This in part explains why higher-order compositionality results are technically involved and first-order proof methods fail.

Let us illustrate the above concepts in the setting of Example II.2. A higher-order GSOS law of a polynomial functor  $\Sigma$  over  $B_0(X, Y) = Y + Y^X$  is a family of maps  $\varrho^0_{X,Y} \colon \Sigma(X \times (Y + Y^X)) \to \Sigma^*(X + Y) + (\Sigma^*(X + Y))^X$ 

dinatural in  $X \in \mathbf{Set}$  and natural in  $Y \in \mathbf{Set}$ . Intuitively, on input  $f((p_1, b_1), \ldots, (p_n, b_n))$  for  $f \in \Sigma$ , the map  $\varrho_{X,Y}^0$ specifies the behaviour of the program  $f(p_1, \ldots, p_n)$  in terms of the behaviours  $b_1, \ldots, b_n \in Y + Y^X$  of its subprograms  $p_1, \ldots, p_n$ . (Di)naturality of  $\varrho^0$  ensures that the maps  $\varrho_{X,Y}^0$ are parametrically polymorphic, that is, they do not look into the structure of their arguments. This can be made formal via the following syntactic representation of higher-order GSOS laws. Fix metavariables  $x, x_i, y_i$  and  $y_i^z$  for  $i \in \mathbb{N}$  and  $z \in$  $\{x, x_1, x_2, x_3, \ldots\}$ . An  $\mathcal{HO}$  rule is an expression of the form (II.4) or (II.5), where  $f \in \Sigma$ ,  $n = \operatorname{ar}(f), W \subseteq \{1, \ldots, n\}, \overline{W} = \{1, \ldots, n\} \setminus W$ , and t is a  $\Sigma$ -term in the variables appearing in the premise, and additionally in x for (II.5).

$$\frac{(x_j \to y_j)_{j \in W} \quad (x_k \stackrel{z}{\longrightarrow} y_k^z)_{k \in \overline{W}, z \in \{x_1, \dots, x_n\}}}{\mathsf{f}(x_1, \dots, x_n) \to t} \quad (\text{II.4})$$

$$\frac{(x_j \to y_j)_{j \in W} \quad (x_k \xrightarrow{z} y_k^z)_{k \in \overline{W}, z \in \{x_1, \dots, x_n, x\}}}{\mathsf{f}(x_1, \dots, x_n) \xrightarrow{x} t}$$
(II.5)

An  $\mathcal{HO}$  specification is a complete set  $\mathcal{R}$  of  $\mathcal{HO}$  rules, that is, for each *n*-ary operation symbol  $f \in \Sigma$  and  $W \subseteq \{1, \ldots, n\}$  there is exactly one rule of the form (II.4) or (II.5) in  $\mathcal{R}$ .

**Example II.7.** The *extended* SKI *calculus*, previously termed unary SKI *calculus* [20], is a combinatory logic expressively equivalent to Curry's SKI *calculus* [13], hence to the untyped  $\lambda$ -calculus. Its signature is given by  $\Sigma =$  $\{S/0, K/0, I/0, S'/1, K'/1, S''/2, \circ/2\}$  with arities as indicated. Informally, the operator  $-\circ -$  corresponds to function application (we write st for  $s \circ t$ ), and the constants S, K, Irepresent the functions  $(s, t, u) \mapsto (s u) (t u), (s, t) \mapsto s$ , and  $s \mapsto s$ . The operators S', S'', K' serve auxiliary purposes. The operational semantics is given by an  $\mathcal{HO}$  specification [20, Fig. 1]. For instance, the rules for application are

$$\frac{x_1 \to y_1}{x_1 x_2 \to y_1 x_2} \qquad \frac{x_1 \xrightarrow{x_2} x_1^{x_2}}{x_1 x_2 \to x_1^{x_2}} \qquad (II.6)$$

**Remark II.8.** By convention, a rule with incomplete premises represents the set of  $\mathcal{HO}$  rules obtained by adding missing premises in every feasible way. For example, in the first rule of (II.6) we can add  $x_2 \to y_2$ , or  $x_2 \xrightarrow{x_1} y_2^{x_1}$  and  $x_2 \xrightarrow{x_2} y_2^{x_2}$ .

**Proposition II.9** [20]. *Higher-order GSOS laws of*  $\Sigma$  *over*  $B_0$  *correspond bijectively to*  $\mathcal{HO}$  *specifications.* 

The bijection is based on the Yoneda lemma, and maps an  $\mathcal{HO}$  specification  $\mathcal{R}$  to the higher-order GSOS law  $\varrho^0$  defined as follows. Given  $X, Y \in \mathbf{Set}$  and

$$w = \mathsf{f}((p_1, b_1), \dots, (p_n, b_n)) \in \Sigma(X \times B_0(X, Y)),$$

Fig. 1. Operational model of a higher-order GSOS law

consider the unique rule in  $\mathcal{R}$  matching f and  $W = \{j \in \{1, \ldots, n\} : b_j \in Y\}$ . If the rule is of the form (II.4), then

$$\varrho^0_{X,Y}(w) \in \Sigma^*(X+Y) \subseteq B_0(X, \Sigma^*(X+Y))$$

is the term obtained by taking the term t in (II.4) and applying the following substitutions for  $i \in \{1, ..., n\}, j \in W, k \in \overline{W}$ :

$$x_i \mapsto p_i, \qquad y_j \mapsto b_j, \qquad y_k^{x_i} \mapsto b_k(p_i).$$

If the rule is of the form (II.5), then

$$\varrho^0_{X,Y}(w) \in (\Sigma^*(X+Y))^X \subseteq B_0(X, \Sigma^*(X+Y))$$

is the map  $e \mapsto t_e$ , where  $t_e$  is obtained by taking the term t in (II.5) and applying the above substitutions along with

$$x \mapsto e$$
 and  $y_k^x \mapsto b_k(e)$   $(k \in \overline{W})$ 

Instantiating Definition II.5, the operational model of a higherorder GSOS law  $\rho^0$  is the  $B_0(\mu\Sigma, -)$ -coalgebra

$$\gamma_0 \colon \mu \Sigma \to \mu \Sigma + \mu \Sigma^{\mu \Sigma} \tag{II.7}$$

that runs programs in  $\mu\Sigma$  according to the rules in the corresponding  $\mathcal{HO}$  specification.

#### III. Compositionality for $\mathcal{HO}$ Specifications

Our eventual goal is to reason about weak simulations and their congruence properties on operational models of higherorder GSOS laws. The required categorical machinery is developed from Section IV onwards. In the present section we motivate the categorical abstractions by again investigating the special case of  $\mathcal{HO}$  specifications, that is, we continue to work in the setting of Example II.2.

**Notation III.1.** (1) In addition to the polynomial functor  $\Sigma$  and  $B_0(X, Y) = Y + Y^X$ , we will also consider the bifunctor

$$B(X,Y) = \mathcal{P}B_0(X,Y) = \mathcal{P}(Y+Y^X)$$
: Set<sup>op</sup>×Set  $\rightarrow$  Set,

where  $\mathcal{P} \colon \mathbf{Set} \to \mathbf{Set}$  is the powerset functor.

(2) Given  $X \in$ **Set**, a coalgebra  $c: C \to C + C^X$  for the functor  $B_0(X, -)$  and  $p \in C$ , we write

$$p \to \overline{p} \quad \text{if} \quad c(p) \in C \text{ and } \overline{p} = c(p),$$

$$p \not\to \quad \text{if} \quad c(p) \notin C \text{ (that is, } c(p) \in C^X),$$

$$p \xrightarrow{x} p_x \quad \text{if} \quad c(p) \in C^X, x \in X, \text{ and } p_x = c(p)(x).$$

In the first case, we say that p reduces. Moreover, we put

$$p \Rightarrow \overline{p} \quad \text{if} \quad \exists k \ge 0, \exists p_0, \dots, p_k, p = p_0 \to \dots \to p_k = \overline{p}, \\ p \Downarrow \overline{p} \quad \text{if} \quad p \Rightarrow \overline{p} \text{ and } \overline{p} \not\to .$$

The weak transition system of  $c: C \to C + C^X$  is the coalgebra  $\tilde{c}: C \to \mathcal{P}(C+C^X)$  for the functor B(X, -) where

$$\widetilde{c}(p) = \{ \overline{p} \in C : p \Rightarrow \overline{p} \} \cup \{ c(\overline{p}) : p \Downarrow \overline{p} \}.$$

**Definition III.2.** A *weak simulation* on a  $B_0(X, -)$ -coalgebra  $c: C \to C + C^X$  is a relation  $R \subseteq C \times C$  such that for every R(p,q) and  $\overline{p} \in C$ , the following conditions hold:

$$\begin{array}{lll} p \Rightarrow \overline{p} & \Longrightarrow & \exists \overline{q} \in C. \ q \Rightarrow \overline{q} \land R(\overline{p}, \overline{q}); \\ p \Downarrow \overline{p} & \Longrightarrow & \exists \overline{q} \in C. \ q \Downarrow \overline{q} \land \forall x \in X. \ R(\overline{p}_x, \overline{q}_x) \end{array}$$

*Weak similarity* is the greatest weak simulation on (C, c), viz. the union of all weak simulations, denoted  $\leq_{(C,c)}$  or just  $\leq$ .

Note that dropping the first condition leads to the same weak similarity relation. We include it to match the abstract view on weak simulations in Remark III.3(2) below.

**Remark III.3.** We make some observations that will be key to our categorical generalization of weak simulations in Section VI and VIII.

(1) From a conceptual perspective, weak simulations can be understood in terms of *relation liftings* of the involved functors. Let **Rel** denote the category whose objects are pairs (X, R) of a set X and a binary relation  $R \subseteq X \times X$ , and whose morphisms  $h: (X, R) \to (Y, S)$  are maps  $h: X \to Y$ such that  $(h \times h)[R] \subseteq S$ . The functors  $\mathcal{P}$ ,  $B_0$  and  $B = \mathcal{P} \cdot B_0$ lift to functors  $\overline{\mathcal{P}}$ ,  $\overline{B}_0$  and  $\overline{B}$  on **Rel** making the diagram below commute, where |-| is the forgetful functor  $(X, R) \mapsto X$ .

$$\begin{array}{c} & \overline{B} \\ & & \overline{B} \\ & & & \overline{B} \\ \hline & & & & \overline{B} \\ |-|^{\mathsf{op}} \times |-| \\ & & & & \downarrow |-| \\ & & & \downarrow$$

(a) The lifting  $\overline{\mathcal{P}}$  of  $\mathcal{P}$  is given by

$$\overline{\mathcal{P}}(X,R) = (\mathcal{P}X,S_R), \qquad \overline{\mathcal{P}}h = \mathcal{P}h,$$

where  $S_R$  is the (one-sided) Egli-Milner relation on  $\mathcal{P}X$ :

$$S_R(U,V) \iff \forall u \in U. \exists v \in V. R(u,v).$$

(b) The lifting  $\overline{B}_0$  of  $B_0$  is given by

$$\overline{B}_0((X, R), (Y, S)) = (B_0(X, Y), E^0_{R,S})$$
$$\overline{B}_0(h, k) = B_0(h, k),$$

where  $E_{R,S}^0(u,v)$  holds for  $u,v \in B_0(X,Y) = Y + Y^X$ whenever either of the following conditions is satisfied: •  $u, v \in Y \land S(u, v);$ 

u, v ∈ Y<sup>X</sup> ∧ ∀x, x' ∈ X. (R(x, x') ⇒ S(u(x), v(x'))).
We note that ((X, R), (Y, S)) ↦ (Y<sup>X</sup>, E<sup>0</sup><sub>R,S</sub> ∩ Y<sup>X</sup> × Y<sup>X</sup>) is the internal hom-functor of the cartesian closed category Rel.
(c) Finally, we put B = P · B<sub>0</sub>. More explicitly,

$$\overline{B}((X,R),(Y,S)) = (B(X,Y), E_{R,S}), \quad \overline{B}(h,k) = B(h,k),$$

where  $E_{R,S}$  is the relation on  $\mathcal{P}(Y+Y^X)$  defined as follows:

$$E_{R,S}(U,V) \iff \forall u \in U. \ \exists v \in V. \ E^0_{R,S}(u,v).$$

(2) A relation  $R \subseteq C \times C$  forms a weak simulation on the coalgebra  $c: C \to C + C^X$  iff there exists a map  $\tilde{c}_R$  making the diagram (III.1) commute, where outl and outr are the left and right projections and  $\Delta_X \subseteq X \times X$  is the identity relation.

(3) For a relation  $R \subseteq C \times C$  to be a weak simulation, it suffices to restrict the premises of the two weak simulation conditions to strong transitions: for every R(p,q) and  $\overline{p} \in C$ ,

$$\begin{array}{rcl} p \to \overline{p} & \Longrightarrow & \exists \overline{q} \in C. \, q \Rightarrow \overline{q} \land R(\overline{p}, \overline{q}); \\ p \not\to & \Longrightarrow & \exists \overline{q} \in C. \, q \Downarrow \overline{q} \land \forall x \in X. \, R(p_x, \overline{q}_x). \end{array}$$

This amounts to the existence of a map  $\tilde{c}_R$  making the diagram (III.2) commute. Here we regard  $c: C \to C + C^X$  as a map  $c: C \to \mathcal{P}(C + C^X)$  by postcomposing with  $b \mapsto \{b\}$ .

The compositionality theorem for  $\mathcal{HO}$  specifications [20, Prop. 3.2] asserts that strong similarity on the operational model (II.7) is a congruence with respect to the operations from the signature  $\Sigma$ . (It is worth noting here that strong similarity coincides with strong bisimilarity because reductions are deterministic.) However, for weak similarity that result fails:

**Example III.4.** Consider the signature  $\Sigma = \{c, d, u\}$  where c, d are constants and u is unary, along with the  $\mathcal{HO}$  specification given by the following four rules:

$$\frac{x_1 \to y_1}{c \xrightarrow{x} c} \quad \frac{d \to c}{d \to c} \quad \frac{x_1 \to y_1}{u(x_1) \to u(x_1)} \quad \frac{x_1 \xrightarrow{x_1} y_1^{x_1}}{u(x_1) \to c}$$

Then  $c \leq d$  but  $u(c) \not\leq u(d)$ : we have  $u(c) \Downarrow c$  while  $u(d) \rightarrow u(d) \rightarrow u(d) \rightarrow \cdots$ , which means that  $u(d) \Downarrow t$  holds for no term t. Thus  $\leq$  is not a congruence on the initial algebra  $\mu\Sigma$ .

This example illustrates that unrestricted  $\mathcal{HO}$  rules are too liberal for our purposes: They allow operators to behave completely differently depending on whether a given subterm reduces or not, which is clearly against the spirit of weak similarity where individual reduction steps are meant to be unobservable. In the following we devise a natural condition on  $\mathcal{HO}$  specifications that avoids congruence-breaking behaviour.

**Definition III.5.** Suppose that  $\mathcal{R}$  is an  $\mathcal{HO}$  specification. We say that a rule (II.4)/(II.5) of  $\mathcal{R}$  is *sound for weak transitions* if its corresponding *weak rule* (III.3)/(III.4) shown below

$$\frac{(x_j \Rightarrow y_j)_{j \in W} \quad (x_j \stackrel{z}{\Rightarrow} y_k^z)_{k \in \overline{W}, \ z \in \{x_1, \dots, x_n\}}}{\mathsf{f}(x_1, \dots, x_n) \Rightarrow t} \quad (\text{III.3})$$

$$\frac{(x_j \Rightarrow y_j)_{j \in W} \quad (x_j \stackrel{z}{\Rightarrow} y_k^z)_{k \in \overline{W}, z \in \{x_1, \dots, x_n, x\}}}{\mathsf{f}(x_1, \dots, x_n) \stackrel{x}{\Rightarrow} t} \quad (\text{III.4})$$

is sound in the operational model (II.7). This means that for all  $p_1, \ldots, p_n, \overline{p}_1, \ldots, \overline{p}_n \in \mu\Sigma$  such that  $p_j \Rightarrow \overline{p}_j$  for  $j \in W$ and  $p_k \Downarrow \overline{p}_k$  for  $k \in \overline{W}$ , there exists  $\overline{p} \in \mu\Sigma$  where

for (III.3), one has f(p<sub>1</sub>,..., p<sub>n</sub>) ⇒ p̄ and the term p̄ ∈ μΣ emerges from t via the substitutions

$$x_i \mapsto p_i, \quad y_j \mapsto \overline{p}_j, \quad y_k^{x_i} \mapsto (\overline{p}_k)_{p_i},$$

for  $i \in \{1, \ldots, n\}, j \in W, k \in \overline{W};$ 

 for (III.4), one has f(p<sub>1</sub>,..., p<sub>n</sub>) ↓ p̄ and, for every e ∈ μΣ, the term p̄<sub>e</sub> ∈ μΣ emerges from t via the substitutions

$$x_i \mapsto p_i, \ x \mapsto e, \ y_j \mapsto \overline{p}_j, \ y_k^{x_i} \mapsto (\overline{p}_k)_{p_i}, \ y_k^x \mapsto (\overline{p}_k)_e$$
  
for  $i \in \{1, \dots, n\}, \ j \in W, \ k \in \overline{W}.$ 

**Example III.6.** In the specification of Example III.4 the last rule is unsound for weak transitions: we have  $d \Downarrow c$  but  $u(d) \not\Rightarrow c$  since u(d) reduces to itself. The remaining rules are sound. (For the first two note that all premise-free rules are sound.)

**Remark III.7.** The soundness condition can be expressed in terms of the higher-order GSOS law  $\rho^0$  of  $\Sigma$  over  $B_0(X, Y) =$  $Y + Y^X$  corresponding to the specification  $\mathcal{R}$ . First, we turn  $\rho^0$  into a higher-order GSOS law  $\rho$  of  $\Sigma$  over B(X, Y) = $\mathcal{P}(Y + Y^X)$ , where  $\rho_{X,Y}$  is defined to be the composite

$$\Sigma(X \times \mathcal{P}B_0(X, Y)) \xrightarrow{\Sigma \mathsf{st}_{X, B_0(X, Y)}} \Sigma \mathcal{P}(X \times B_0(X, Y))$$

$$\overbrace{\delta_{X \times B_0(X, Y)}}^{\delta_{X \times B_0(X, Y)}} (\text{III.5})$$

$$\mathcal{P}\Sigma(X \times B_0(X, Y)) \xrightarrow{\mathcal{P}\varrho^0_{X,Y}} \mathcal{P}B_0(X, \Sigma^{\star}(X+Y))$$

Here st is the *canonical strength* of the powerset functor  $\mathcal{P}$ ,

$$\mathsf{st}_{X,Z} \colon X \times \mathcal{P}Z \to \mathcal{P}(X \times Z), \ (x,U) \mapsto \{ (x,u) \colon u \in U \},\$$

and  $\delta: \Sigma \mathcal{P} \to \mathcal{P}\Sigma$  is the natural transformation whose component  $\delta_Z: \Sigma \mathcal{P}Z \to \mathcal{P}\Sigma Z$  is given by

$$f(Z_1,\ldots,Z_n)\mapsto \{f(z_1,\ldots,z_n): z_i\in Z_i \text{ for } i=1,\ldots,n\}.$$

The operational model of  $\rho$  is easily seen to be the composite

$$\mu\Sigma \xrightarrow{\gamma_0} B_0(\mu\Sigma, \mu\Sigma) \xrightarrow{\eta} \mathcal{P}B_0(\mu\Sigma, \mu\Sigma) = B(\mu\Sigma, \mu\Sigma),$$

where  $\gamma_0$  is the operational model of  $\rho^0$  and  $\eta(b) = \{b\}$ .

Then the rules of  $\mathcal{R}$  are sound for weak transitions iff the diagram (III.6) commutes laxly. Here  $\tilde{\gamma}$  is the weak transition system of  $\gamma$ , and the partial order  $\preceq$  on a hom-set  $\mathbf{Set}(X, \mathcal{P}Y)$  is given by  $f \preceq g$  iff  $f(x) \subseteq g(x)$  for all  $x \in X$ . In the terminology introduced later (Definition VIII.4),  $\iota$  and  $\tilde{\gamma}$  thus form a *lax bialgebra* for the higher-order GSOS law  $\varrho$ .

**Theorem III.8.** For every  $\mathcal{HO}$  specification whose rules are sound for weak transitions, the weak similarity relation  $\leq$  on the canonical model  $\gamma: \mu\Sigma \to \mu\Sigma + \mu\Sigma^{\mu\Sigma}$  is a congruence.

The proof uses Howe's method [25], a standard technique for establishing higher-order congruence results.

**Notation III.9.** The *Howe closure* of a relation  $R \subseteq \mu \Sigma \times \mu \Sigma$  is the relation

$$\widehat{R} = \bigcup_{m \in \mathbb{N}} \widehat{R}_m$$

on  $\mu\Sigma$  where  $\widehat{R}_0 \subseteq \widehat{R}_1 \subseteq \widehat{R}_2 \subseteq \cdots$  are defined inductively:  $\widehat{R}_0 = R$  and for every  $m \in \mathbb{N}$  and  $p, r \in \mu\Sigma$ , one has  $\widehat{R}_{m+1}(p,r)$  whenever  $\widehat{R}_m(p,r)$  or

$$\exists \mathsf{f} \in \Sigma, \vec{p}, \vec{q} \in (\mu\Sigma)^{\mathsf{ar}(\mathsf{f})}, \, p = \mathsf{f}(\vec{p}) \, \land \, \widehat{R}_m(\vec{p}, \vec{q}) \, \land \, R(\mathsf{f}(\vec{q}), r).$$

Here  $\widehat{R}_m(\vec{p}, \vec{q})$  means  $\widehat{R}_m(p_i, q_i)$  for  $i = 1, \dots, \mathsf{ar}(\mathsf{f})$ .

**Remark III.10.** (1) If R is reflexive, then the Howe closure  $\hat{R}$  is a congruence: put  $r = f(\vec{q})$  in the definition of  $\hat{R}_{m+1}$ . (2) If R is transitive, then  $\hat{R}$  satisfies a weak transitivity property:  $\hat{R}(p,r)$  and R(r,r') implies  $\hat{R}(p,r')$  for all  $p, r, r' \in \mu \Sigma$ . This follows by induction on the least m such that  $\hat{R}_m(p,r)$ . (3) Thus, if R is both reflexive and transitive (in particular, if it is some weak similarity relation), then  $\hat{R}$  is the least weakly transitive congruence containing R.

*Proof of Theorem III.8.* Form the Howe closure  $\widehat{\leq}$  of  $\lesssim$ . Since  $\widehat{\leq}$  is a congruence, it suffices to prove  $\widehat{\leq} = \lesssim$ . The inclusion  $\lesssim \subseteq \widehat{\leq}$  is clear. For the inclusion  $\widehat{\leq} \subseteq \lesssim$  we show that  $\widehat{\leq}$  is a weak simulation; then the inclusion holds because  $\lesssim$  is the greatest weak simulation. By Remark III.3(3), we need to establish the following for every  $p \widehat{\leq} r$  and  $\overline{p} \in \mu \Sigma$ :

$$p \to \overline{p} \implies \exists \overline{r} \in \mu \Sigma. r \Rightarrow \overline{r} \land \overline{p} \lesssim \overline{r};$$
 (III.7)

$$p \not\rightarrow \implies \exists \overline{r} \in \mu \Sigma. \, r \Downarrow \overline{r} \land \forall e \in \mu \Sigma. \, p_e \stackrel{<}{\lesssim} \overline{r}_e. \quad \text{(III.8)}$$

In lieu of (III.8) we will actually prove a stronger statement:

$$p \not\rightarrow \implies \exists \overline{r} \in \mu \Sigma. \, r \Downarrow \overline{r} \land \forall d \stackrel{\widehat{\lesssim}}{\underset{\sim}{\sim}} e. \, p_d \stackrel{\widehat{\lesssim}}{\underset{\sim}{\sim}} \overline{r}_e.$$
(III.9)

The proof is by induction on the least m such that  $p \widehat{\leq}_m r$ .

**Induction base** (m = 0). Suppose that  $p \lesssim_0 r$ , that is,  $p \lesssim r$ .

*Proof of* (III.7). If  $p \to \overline{p}$ , since  $\lesssim$  is a weak simulation, there exists  $\overline{r} \in \mu\Sigma$  such that  $r \Rightarrow \overline{r}$  and  $\overline{p} \lesssim \overline{r}$ , hence also  $\overline{p} \lesssim \overline{r}$ .

*Proof of* (III.9). If  $p \not\rightarrow$ , since  $\lesssim$  is a weak simulation, there exists  $\overline{r} \in \mu\Sigma$  such that  $r \Downarrow \overline{r}$  and  $p_e \lesssim \overline{r}_e$  for  $e \in \mu\Sigma$ . By definition of the  $\mathcal{HO}$  format, there exists a term  $t_p(x)$  in a

single variable x such that  $p_e = t_p(e)$  for  $e \in \mu\Sigma$ . Since  $\widehat{\lesssim}$  is a congruence, it follows that, for  $d \widehat{\lesssim} e$ ,

$$p_d = t_p(d) \lesssim t_p(e) = p_e \lesssim \overline{r}_e$$

Thus  $p_d \widehat{\lesssim} \overline{r}_e$  by weak transitivity of the relation  $\widehat{\lesssim}$ .

**Induction step**  $(m \to m + 1)$ . Suppose that  $p \lesssim_{m+1} r$ . We only verify condition (III.9), the argument for (III.7) is analogous. Thus suppose that  $p \not\rightarrow$ . If  $p \lesssim_m r$ , we are done by induction. Otherwise, by definition of  $\lesssim_{m+1}$ , there exists an *n*-ary operation symbol  $f \in \Sigma$  and  $\vec{p}, \vec{q} \in (\mu \Sigma)^n$  such that

$$p = f(\vec{p}), \qquad \vec{p} \lesssim_m \vec{q}, \qquad q := f(\vec{q}) \lesssim r.$$

To avoid bulky notation, we consider the representative case of a binary operator f where  $p_1$  reduces (say  $p_1 \rightarrow \overline{p}_1$ ) and  $p_2 \not\rightarrow$ . Then we know by induction that

• 
$$\exists \overline{q}_1 \in \mu \Sigma. q_1 \Rightarrow \overline{q}_1 \land \overline{p}_1 \stackrel{<}{\underset{\sim}{\sim}} \overline{q}_1;$$
  
•  $\exists \overline{q}_2 \in \mu \Sigma. q_2 \Downarrow \overline{q}_2 \land \forall d \stackrel{<}{\underset{\sim}{\sim}} e. (p_2)_d \stackrel{<}{\underset{\sim}{\sim}} (\overline{q}_2)_e.$   
Since  $p = f(p_1, p_2) \not\rightarrow$ , the rule applying to  $p$  has the form

Thus, for every  $d \in \mu \Sigma$ ,

$$p \xrightarrow{a} p_d = t(p_1, p_2, d, \overline{p}_1, (p_2)_{p_1}, (p_2)_{p_2}, (p_2)_d).$$

The above rule is sound for weak transitions, and we have  $q_1 \Rightarrow \overline{q}_1$  and  $q_2 \Downarrow \overline{q}_2$ , so there exists  $\overline{q} \in \mu \Sigma$  such that

$$q \Downarrow \overline{q} \quad \text{and} \quad \overline{q} \xrightarrow{e} \overline{q}_e = t(q_1, q_2, e, \overline{q}_1, (\overline{q}_2)_{q_1}, (\overline{q}_2)_{q_2}, (\overline{q}_2)_e)$$

for all  $e \in \mu\Sigma$ . Thus for  $d \lesssim e$  we have  $p_d \lesssim \overline{q}_e$  because  $\lesssim$  is a congruence and the terms substituted in t for the variables are related by  $\lesssim$ . Moreover, since  $\lesssim$  is a weak simulation and  $q \lesssim r$ , there exists  $\overline{r} \in \mu\Sigma$  such that  $r \Downarrow \overline{r}$  and  $\overline{q}_e \lesssim \overline{r}_e$ for all  $e \in \mu\Sigma$ . It follows that  $p_d \lesssim \overline{r}_e$  for  $d \lesssim e$  because  $p_d \lesssim \overline{q}_e \lesssim \overline{r}_e$  and the relation  $\lesssim$  is weakly transitive.

**Remark III.11.** (1) The strengthening (III.9) of the induction hypothesis is required, for otherwise the proof gets stuck: the argument in the induction step showing  $p_d \lesssim \overline{q}_e$  for  $d \lesssim e$  (or even  $p_e \lesssim \overline{q}_e$  for  $e \in \mu\Sigma$ ) relies on relations such as  $(p_2)_{p_1} \lesssim (\overline{q_2})_{q_1}$ , which only hold by (III.9), not by (III.8).

(2) The strengthened induction hypothesis (III.7) + (III.9) can be expressed via the relation lifting of the bifunctor  $\overline{B}$ , see Remark III.3(1): It amounts to the existence of a map  $\delta$  making the diagram below commute.

$$\begin{array}{c} \mu\Sigma \xleftarrow{\operatorname{outl}_{\widehat{\lesssim}}} \widehat{\lesssim} \xrightarrow{\operatorname{outl}_{\widehat{\lesssim}}} \mu\Sigma \\ \gamma \downarrow & \downarrow^{\delta} & \downarrow^{\widetilde{\gamma}} \\ \mathcal{P}(\mu\Sigma + \mu\Sigma^{\mu\Sigma}) \xleftarrow{\operatorname{outl}_{\widehat{\lesssim},\widehat{\lesssim}}} E_{\widehat{\lesssim},\widehat{\lesssim}} \xrightarrow{\operatorname{outl}_{\widehat{\lesssim},\widehat{\lesssim}}} \mathcal{P}(\mu\Sigma + \mu\Sigma^{\mu\Sigma}) \end{array}$$

(3) The induction does not go through when the Howe closure  $\hat{\lesssim}$  is replaced with more obvious candidates. If  $\hat{\lesssim}$  is taken to be the least congruence containing  $\lesssim$ , then already the

induction base fails, as the argument requires weak transitivity of  $\widehat{\leq}$ . If  $\widehat{\leq}$  is taken to be the least transitive congruence containing  $\leq$ , it is no longer clear how to construct  $\widehat{\leq}$  as a union of inductively defined relations  $\widehat{\leq}_m$  in a way that makes the induction step work. It thus appears that Howe's method is the simplest and most natural approach to the present result.

We conclude this section by identifying a natural class of  $\mathcal{HO}$  specifications, the *cool*  $\mathcal{HO}$  *specifications*, whose rules are sound for weak transitions. It resembles first-order formats such as *cool GSOS* [4], [38] for labelled transition systems, and *cool stateful SOS* [19] for stateful computations.

**Definition III.12.** (1) An *n*-ary operator  $f \in \Sigma$  is *passive* if it is specified by a premise-free rule (cf. Remark II.8)

$$f(x_1, \dots, x_n) \to t$$
 or  $f(x_1, \dots, x_n) \xrightarrow{x} t$  (III.10)

where t is a term in the variables  $x_1, \ldots, x_n$  or  $x_1, \ldots, x_n, x$ , resp. Thus the behaviour of f does not depend on the behaviour of its subterms. An *active* operator is one which is not passive.

(2) An  $\mathcal{HO}$  specification is *cool* if for every active *n*-ary operator f there exists  $j \in \{1, ..., n\}$  (called the *receiving position of* f) such that all rules for f are of the form

$$\frac{x_j \to y_j}{\mathsf{f}(x_1, \dots, x_j, \dots, x_n) \to \mathsf{f}(x_1, \dots, y_j, \dots, x_n)} (\text{III.11})$$

$$\frac{(x_j \xrightarrow{z} y_j^z)_{z \in \{x_1, \dots, x_n\}}}{\mathsf{f}(x_1, \dots, x_n) \to t} \text{ or } \frac{(x_j \xrightarrow{z} y_j^z)_{z \in \{x_1, \dots, x_n, x\}}}{\mathsf{f}(x_1, \dots, x_n) \xrightarrow{x} t}$$

where t is a term in the variables  $x_i$  and  $y_j^{x_i}$   $(i \in \{1, \ldots, n\} \setminus \{j\})$ , and moreover in x and  $y_j^x$  for the third rule in (III.11).

Coolness thus asserts that for active f, a program  $p = f(p_1, \ldots, p_n)$  must run its *j*-th subprogram  $p_j$  (for some fixed *j* depending only on f) until it does not further reduce, correctly propagate all reduction steps of  $p_j$  to *p*, and continue the computation as a program *t* that no longer refers to  $p_j$ .

**Proposition III.13.** For cool HO specifications, all rules are sound for weak transitions.

Thus, we obtain as an instance of Theorem III.8:

**Corollary III.14.** For cool HO specifications, the weak similarity relation on the operational model is a congruence.

This generalizes corresponding congruence results for cool first-order specifications [4], [38], [19].

**Example III.15.** The extended SKI calculus (Example II.7) has application  $-\circ -$  as its only active operator, whose rules

(II.6) are cool. Therefore weak similarity on the operational model is a congruence. This means that, for instance,  $p \leq q$  implies  $pr \leq qr$  and  $rp \leq rq$  for all  $r \in \mu\Sigma$ .

The aim of the following sections is to generalize the congruence result of Theorem III.8 to the level of abstract higherorder GSOS laws. The technical key lies in the construction of relation liftings of bifunctors (Section V), along with a suitable categorification of Howe's method (Section VII).

#### IV. GRAPHS, RELATIONS, AND PREORDERS

For our categorical account of weak similarity we will need to restrict to base categories where operations on relations, such as union or composition, are well-behaved and interact with each other in a way familiar from the category of sets. Therefore, we work under the following global assumptions:

Assumptions IV.1. From now on, fix a category  $\ensuremath{\mathbb{C}}$  such that

(1)  $\mathbb{C}$  is complete, cocomplete, and well-powered;

(2) strong epimorphisms in  $\mathbb{C}$  are pullback-stable: for every pullback as shown below, if e is strongly epic then so is  $\overline{e}$ ;



(3)  $\mathbb{C}$  is locally distributive.

All categories of Example II.1 satisfy these assumptions. Since  $\mathbb{C}$  is complete and well-powered, the subobjects of a fixed object form a complete lattice, and every morphism has a (strong epi, mono)-factorization [7, Prop. 4.4.3]. All our results easily generalize to arbitrary proper factorization systems.

#### A. Graphs and Relations

We review some terminology for the categorical version of graphs (more precisely, directed multigraphs) and relations.

1) Graphs in a category: A graph in  $\mathbb{C}$  is a quadruple  $(X, R, \operatorname{outl}_R, \operatorname{outr}_R)$  given by two objects  $X, R \in \mathbb{C}$  and a parallel pair of morphisms  $\operatorname{outl}_R, \operatorname{outr}_R \colon R \to X$ . A graph is usually denoted by its pair (X, R) of objects. A morphism from (X, R) to a graph (Y, S) is a pair  $h = (h_0, h_1)$  of  $\mathbb{C}$ -morphisms making the diagram below commute:

$$\begin{array}{c|c} X & \xleftarrow{\operatorname{outl}_R} & R \xrightarrow{\operatorname{outr}_R} X \\ h_0 & & \downarrow_{h_1} & \downarrow_{h_0} \\ Y & \xleftarrow{\operatorname{outl}_S} S \xrightarrow{\operatorname{outr}_S} Y \end{array}$$
(IV.1)

We let  $\mathbf{Gra}(\mathbb{C})$  denote the category of graphs in  $\mathbb{C}$  and their morphisms. For every  $X \in \mathbb{C}$  we write  $\mathbf{Gra}_X(\mathbb{C}) \hookrightarrow \mathbf{Gra}(\mathbb{C})$ 

for the non-full subcategory consisting of all graphs of the form (X, R) and graph morphisms h such that  $h_0 = id_X$ .

2) Relations in a category: A graph  $(X, R) \in \mathbf{Gra}(\mathbb{C})$  is a relation if  $\operatorname{outl}_R$  and  $\operatorname{outr}_R$  are jointly monic, or equivalently if the morphism  $\langle \operatorname{outl}_R, \operatorname{outr}_R \rangle \colon R \to X \times X$  is monic. We let  $\mathbf{Rel}(\mathbb{C}) \hookrightarrow \mathbf{Gra}(\mathbb{C})$  and  $\mathbf{Rel}_X(\mathbb{C}) \hookrightarrow \mathbf{Gra}_X(\mathbb{C})$  denote the full subcategories given by relations; note that  $\mathbf{Rel}_X(\mathbb{C})$  is thin, i.e. an ordered set, and a complete lattice when isomorphic relations are identified. Both subcategories are reflective: The reflection of a graph (X, R) is given by  $(\mathrm{id}_X, e_R) \colon (X, R) \twoheadrightarrow (X, R^{\dagger})$  where  $e_R$  and  $R^{\dagger}$  are obtained via the (strong epi, mono)-factorization of  $\langle \mathrm{outl}_R, \mathrm{outr}_R \rangle$ :

$$\begin{array}{c} & & & & \\ & & & & \\ R & & & & \\ \hline & & & & \\ R & & & & \\ \end{array} \xrightarrow{e_R} & & & & \\ R^{\dagger} & & & & \\ & & & & \\ \end{array} \xrightarrow{\langle \text{outl}_{R^{\dagger}}, \, \text{outr}_{R^{\dagger}} \rangle} X \times X \quad (\text{IV.2}) \end{array}$$

The various categories are connected by the functors

where  $(-)^{\dagger}$  denotes the reflector and |-| is the projection functor given by  $(X, R) \mapsto X$  and  $h \mapsto h_0$ . We regard  $\mathbb{C}$ as a full subcategory of  $\operatorname{Rel}(\mathbb{C})$  by identifying  $X \in \mathbb{C}$  with the *identity relation*  $(X, X, \operatorname{id}_X, \operatorname{id}_X) \in \operatorname{Rel}(\mathbb{C})$ , which we simply denote by (X, X).

3) Limits and colimits: The categories  $\mathbf{Gra}(\mathbb{C})$ ,  $\mathbf{Gra}_X(\mathbb{C})$ ,  $\mathbf{Rel}(\mathbb{C})$ ,  $\mathbf{Rel}_X(\mathbb{C})$  are complete and cocomplete. Coproducts in  $\mathbf{Gra}(\mathbb{C})$  and  $\mathbf{Gra}_X(\mathbb{C})$ , denoted by (X, R) + (Y, S) and  $(X, R) +_X(X, S)$ , are formed using  $\mathbb{C}$ -coproducts. Coproducts in  $\mathbf{Rel}(\mathbb{C})$  are given by  $(X, R) \lor (Y, S) = ((X, R) + (Y, S))^{\dagger}$ and in  $\mathbf{Rel}_X(\mathbb{C})$  by  $(X, R) \lor_X(X, S) = ((X, R) +_X(X, S))^{\dagger}$ .

Products  $(X, R) \times (Y, S)$  in both  $\mathbf{Gra}(\mathbb{C})$  and  $\mathbf{Rel}(\mathbb{C})$  are formed in  $\mathbb{C}$ . The product  $(X, R) \times_X (X, S)$  in  $\mathbf{Gra}_X(\mathbb{C})$  and  $\mathbf{Rel}_X(\mathbb{C})$  is the pullback of  $\langle \mathsf{outl}_R, \mathsf{outr}_R \rangle$  and  $\langle \mathsf{outl}_S, \mathsf{outr}_S \rangle$ .

4) Composition of graphs and relations: The composite (X, R); (X, R') of two graphs (X, R) and (X, R') is the graph (X, R; R') defined via the following pullback:



The *composite* of two relations (X, R), (X, R'), given by

$$(X, R) \bullet (X, R') = ((X, R); (X, R'))^{\dagger},$$

defines a bifunctor  $(-)\bullet(-)$  on  $\operatorname{Rel}_X(\mathbb{C})$  (that is, composition is a monotone map on the ordered set of relations). Using Assumptions IV.1(2),(3), relation composition can be shown to distribute over coproducts. This is the key property of relations needed for our account of Howe's method in Section VII.

5) Reflexive and transitive relations: Given graphs (X, R)and (X, R') in  $\mathbf{Gra}_X(\mathbb{C})$ , we put  $(X, R) \leq (X, R')$  if there exists a  $\mathbf{Gra}_X(\mathbb{C})$ -morphism from (X, R) to (X, R'). For relations,  $(X, R) \leq (X, R') \leq (X, R)$  implies  $(X, R) \cong$ (X, R'). A relation (X, R) is *reflexive* if  $(X, X) \leq (X, R)$ , and *transitive* if  $(X, R) \bullet (X, R) \leq (X, R)$ .

6) Reindexing: Every morphism  $f: X \to Y$  in  $\mathbb{C}$  induces a functor  $f_*: \operatorname{\mathbf{Gra}}_X(\mathbb{C}) \to \operatorname{\mathbf{Gra}}_Y(\mathbb{C})$  given by

$$(X, R, \mathsf{outl}_R, \mathsf{outr}_R) \mapsto (Y, R, f \cdot \mathsf{outl}_R, f \cdot \mathsf{outr}_R).$$

Readers familiar with the language of fibrations may note that |-|: **Gra**( $\mathbb{C}$ )  $\to \mathbb{C}$  is a bifibration with fibres **Gra**<sub>X</sub>( $\mathbb{C}$ ), and  $f_{\star}$  is the reindexing functor induced by opcartesian lifts.

#### B. Preorders

We extend some of the above terminology to graphs over preordered objects. Recall that a *preorder* on a set X is a reflexive and transitive relation  $\preceq \subseteq X \times X$ . Replacing elements  $1 \to X$  with "generalized elements"  $Y \to X$ , one obtains a categorical notion of preorder.

1) Preorders in a category: A preordered object in  $\mathbb{C}$  is a pair  $(X, \preceq)$  of an object  $X \in \mathbb{C}$  and a family  $\preceq = (\preceq_Y)_{Y \in \mathbb{C}}$  where  $\preceq_Y$  is a preorder on the hom-set  $\mathbb{C}(Y, X)$  satisfying

$$f \preceq_Y g \implies f \cdot h \preceq_Z g \cdot h \text{ for all } h \colon Z \to Y.$$

We usually drop subscripts and write  $\leq$  for  $\leq_Y$ ,  $\leq_Z$ , etc.

**Example IV.2.** (1) Every preordered set  $(X, \preceq)$  in the usual order-theoretic sense can be regarded as a preordered object in Set by taking the pointwise preorder on  $\mathbf{Set}(Y, X)$ :

$$f \preceq g \quad \iff \quad \forall y \in Y. \ f(y) \preceq g(y).$$

(2) On every  $X \in \mathbb{C}$ , one has the *discrete* preordered object (X, =), where = is the equality preorder.

2) Preordered functors: A preordered functor is a functor  $F: \mathbb{D} \to \mathbb{C}$  equipped with a preorder  $(FD, \preceq)$  for all  $D \in \mathbb{D}$ .

**Example IV.3.** The powerset functor  $\mathcal{P} \colon \mathbf{Set} \to \mathbf{Set}$  is preordered by taking the inclusion preorder  $\subseteq$  on  $\mathcal{P}X$ .

3) Right-lax morphisms: Given a preordered object  $(Y, \preceq)$ , a right-lax morphism from a graph (X, R) to a graph (Y, S) is a pair  $h = (h_0, h_1)$  of  $\mathbb{C}$ -morphisms such that

$$\begin{array}{c|c} X & \xleftarrow{\mathsf{outl}_R} & R \xrightarrow{\mathsf{outr}_R} X \\ h_0 & \searrow & \downarrow h_1 \not\succ & \downarrow h_0 \\ Y & \xleftarrow{\mathsf{outl}_S} & S \xrightarrow{\mathsf{outr}_S} Y \end{array}$$

For preordered  $(X, \preceq)$  we put  $(X, R) \preceq (X, S)$  if there exists a right-lax morphism  $h: (X, R) \rightarrow (X, S)$  where  $h_0 = id_X$ .

**Example IV.4.** Given relations (X, R), (X, S) on a preordered set  $(X, \preceq)$ , regarded as a preordered object as in Example IV.2(1), we have  $(X, R) \preceq (X, S)$  iff for  $x, y \in X$ ,

$$R(x,y) \quad \Longrightarrow \quad \exists z \in X. \ S(x,z) \land z \preceq y.$$

Graphs over a fixed preordered object  $(X, \preceq)$  and rightlax morphisms h satisfying  $h_0 = id_X$  form a category, but in contrast to the unordered case, the full subcategory of relations is usually not reflective. This turns out to be the main technical challenge for our preorder-based approach to simulations. The key concept to overcome this issue is as follows:

**Definition IV.5.** Let  $(X, \preceq)$  be a preordered object. A relation (X, S) is *good for simulations* if, for all  $(X, R) \in \mathbf{Gra}_X(\mathbb{C})$ ,

$$(X,R) \preceq (X,S) \implies (X,R) \le (X,S).$$

Note that (X, R) ranges over graphs, not just relations, and that the implication " $\Leftarrow$ " also holds trivially. The good-for-simulations condition thus ensures that right-lax graph morphisms into (X, S) can be turned into strict ones.

**Example IV.6.** For every relation (X, R), the relation  $(\mathcal{P}X, S_R)$  is good for simulations, where  $\mathcal{P}X$  is equipped with the inclusion preorder and  $S_R$  is the Egli-Milner relation (Remark III.3(1)). This follows from the observation that  $S_R$  is up-closed:  $S_R(A, B)$  and  $B \subseteq B'$  implies  $S_R(A, B')$ .

## V. LIFTING (BI-)FUNCTORS AND HIGHER-ORDER GSOS LAWS

As pointed out in Remark III.11, the compositionality proof for  $\mathcal{HO}$  specifications implicitly relies on the fact that the behaviour bifunctor  $B(X, Y) = \mathcal{P}(Y + Y^X)$  admits a lifting to the category of relations. We next study liftings of endofunctors, mixed-variance bifunctors, and higher-order GSOS laws on  $\mathbb{C}$  to the categories  $\mathbf{Gra}(\mathbb{C})$  and  $\mathbf{Rel}(\mathbb{C})$  of graphs and relations. We start with the case of endofunctors, which is straightforward and well-known:

**Definition V.1.** Let  $\Sigma \colon \mathbb{C} \to \mathbb{C}$  be an endofunctor.

(1) A graph lifting of  $\Sigma$  is a functor  $\overline{\Sigma}$ :  $\mathbf{Gra}(\mathbb{C}) \to \mathbf{Gra}(\mathbb{C})$  making the diagram on the left below commute.

(2) A relation lifting of  $\Sigma$  is a functor  $\overline{\Sigma} \colon \mathbf{Rel}(\mathbb{C}) \to \mathbf{Rel}(\mathbb{C})$  making the diagram on the right below commute.

$$\begin{array}{lll} \mathbf{Gra}(\mathbb{C}) & \xrightarrow{\Sigma} & \mathbf{Gra}(\mathbb{C}) & & \mathbf{Rel}(\mathbb{C}) & \xrightarrow{\Sigma} & \mathbf{Rel}(\mathbb{C}) \\ |-| & & & \downarrow |-| & & |-| & & \downarrow |-| \\ \mathbb{C} & \xrightarrow{\Sigma} & \mathbb{C} & & & \mathbb{C} & \xrightarrow{\Sigma} & \mathbb{C} \end{array}$$

**Construction V.2.** Every functor  $\Sigma \colon \mathbb{C} \to \mathbb{C}$  admits a *canonical graph lifting*  $\overline{\Sigma}_{\mathbf{Gra}} \colon \mathbf{Gra}(\mathbb{C}) \to \mathbf{Gra}(\mathbb{C})$  and a *canonical relation lifting*  $\overline{\Sigma}_{\mathbf{Rel}} \colon \mathbf{Rel}(\mathbb{C}) \to \mathbf{Rel}(\mathbb{C})$  defined as follows: (1) The functor  $\overline{\Sigma}_{\mathbf{Gra}}$  is given on objects and morphisms by

$$(X, R) \mapsto (\Sigma X, \Sigma R, \Sigma \mathsf{outl}_R, \Sigma \mathsf{outr}_R), \quad h \mapsto (\Sigma h_0, \Sigma h_1).$$

(2) The functor  $\overline{\Sigma}_{Rel}$  is the composite

$$\operatorname{\mathbf{Rel}}(\mathbb{C}) \longrightarrow \operatorname{\mathbf{Gra}}(\mathbb{C}) \xrightarrow{\overline{\Sigma}_{\operatorname{\mathbf{Gra}}}} \operatorname{\mathbf{Gra}}(\mathbb{C}) \xrightarrow{(-)^{\dagger}} \operatorname{\mathbf{Rel}}(\mathbb{C}).$$

(This is similar to the usual Barr extension [2], except that relations are treated as objects rather than as morphisms.)

**Example V.3.** For a polynomial functor  $\Sigma$  on Set, the canonical relation lifting is the restriction of the canonical

graph lifting to **Rel**. Thus  $\overline{\Sigma}_{\mathbf{Rel}}(X, R) = (\Sigma X, \Sigma R)$  where  $\Sigma R(\mathfrak{f}(x_1, \ldots, x_n), \mathfrak{f}(x'_1, \ldots, x'_n))$  iff  $R(x_i, x'_i)$  for all *i*.

**Proposition V.4.** Suppose that  $\Sigma \colon \mathbb{C} \to \mathbb{C}$  preserves strong epimorphisms and generates a free monad  $\Sigma^*$ . Then  $\overline{\Sigma}_{\mathbf{Gra}}$  and  $\overline{\Sigma}_{\mathbf{Rel}}$  generate free monads satisfying

$$(\overline{\Sigma}_{\mathbf{Gra}})^{\star} = (\overline{\Sigma}^{\star})_{\mathbf{Gra}} \quad and \quad (\overline{\Sigma}_{\mathbf{Rel}})^{\star} = (\overline{\Sigma}^{\star})_{\mathbf{Rel}}.$$

Next we turn to liftings of mixed-variance bifunctors.

**Definition V.5.** A *relation lifting* of a functor  $B \colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$  is a functor  $\overline{B}$  such that the diagram below commutes.

$$\begin{array}{ccc} \mathbf{Rel}(\mathbb{C})^{\mathsf{op}} \times \mathbf{Rel}(\mathbb{C}) & \stackrel{B}{\longrightarrow} \mathbf{Rel}(\mathbb{C}) \\ |-|^{\mathsf{op}} \times |-| & & & \downarrow |-| \\ & & \mathbb{C}^{\mathsf{op}} \times \mathbb{C} & \stackrel{B}{\longrightarrow} & \mathbb{C} \end{array}$$

Every bifunctor admits a canonical relation lifting, generalizing the lifting  $\overline{B}_0$  of Remark III.3(1). Since the construction is more involved than for endofunctors, and our compositionality result works with any lifting, we refer to the Appendix (Section C). Finally, we lift higher-order GSOS laws:

**Definition V.6.** Let  $\Sigma: \mathbb{C} \to \mathbb{C}$  and  $B: \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$  be functors with relation liftings  $\overline{\Sigma}$  and  $\overline{B}$ , respectively, where  $\Sigma$ preserves strong epimorphisms and  $\overline{\Sigma} = \overline{\Sigma}_{Rel}$  is the canonical lifting. Given a V-pointed higher-order GSOS law

$$\varrho_{X,Y} \colon \Sigma(X \times B(X,Y)) \to B(X,\Sigma^{\star}(X+Y))$$

of  $\Sigma$  over B, a *relation lifting* of  $\rho$  is a (V, V)-pointed higherorder GSOS law

$$\Sigma((X, R) \times B((X, R), (Y, S)))$$

$$\downarrow^{\overline{\varrho}_{(X,R),(Y,S)}}$$

$$\overline{B}((X, R), \overline{\Sigma}^{\star}((X, R) \lor (Y, S)))$$

of  $\overline{\Sigma}$  over  $\overline{B}$  such that

$$(\overline{\varrho}_{(X,R),(Y,S)})_0 = \varrho_{X,Y}$$

for  $((X, R), p_{(X,R)}) \in (V, V)/\operatorname{\mathbf{Rel}}(\mathbb{C})$  and  $(Y, S) \in \operatorname{\mathbf{Rel}}(\mathbb{C})$ . Here we regard X as V-pointed by  $p_X = (p_{(X,R)})_0 \colon V \to X$ .

**Remark V.7.** (1) Recall that for a V-pointed higher-order GSOS law  $\rho$  we assume the functor  $\Sigma$  to be of the form  $V + \Sigma'$ . This implies  $\overline{\Sigma} = (V, V) \vee \overline{\Sigma'}$ , as required.

(2) The product  $\times$  and coproduct  $\vee$  in  $\operatorname{Rel}(\mathbb{C})$  are formed as explained in Section IV-A3, and we have  $\overline{\Sigma}^* = \overline{\Sigma}^*$ by Proposition V.4. It follows that  $(\overline{\varrho}_{(X,R),(Y,S)})_0$  is a  $\mathbb{C}$ morphism of type  $\Sigma(X \times B(X,Y)) \to B(X, \Sigma^*(X+Y))$ .

(3) Since  $\operatorname{Rel}(\mathbb{C})$ -morphisms are uniquely determined by their  $(-)_0$ -component, a higher-order GSOS law  $\rho$  admits at most one lifting  $\overline{\rho}$ . The requirement that the morphisms  $\overline{\varrho}_{(X,R),(Y,S)}$  form a higher-order GSOS law of  $\overline{\Sigma}$  over  $\overline{B}$  is thus vacuous: the (di-)naturality of  $\overline{\rho}$  is implied by that of  $\rho$ .

For the canonical relation lifting  $\overline{B}$  of B, every higher-order GSOS law admits a relation lifting (see Appendix, Section D).

#### VI. WEAK SIMULATIONS

We next introduce the notion of weak simulation featuring in our abstract congruence result.

**Notation VI.1.** Fix a functor  $F \colon \mathbb{C} \to \mathbb{C}$  and a relation lifting  $\overline{F}$ . We denote the relation  $\overline{F}(X, R)$  by  $(FX, E_R)$ .

We recall the notion of *(lifting) bisimulation* [27] for coalgebras. We use the term *simulation* instead, as this is what the concept amounts to in our applications, due to the use of asymmetric liftings such as the one-sided Egli-Milner lifting. An alternative approach to simulations uses lax liftings [26].

**Definition VI.2.** Let (C, c) be an *F*-coalgebra. A relation (C, R) is a *simulation* on (C, c) if  $c_{\star}(C, R) \leq \overline{F}(C, c)$ , that is, there exists a morphism  $c_R$  making (VI.1) commute.

$$C \xleftarrow{\operatorname{outl}_R} R \xrightarrow{\operatorname{outr}_R} C$$

$$c \downarrow \qquad \qquad \downarrow^{c_R} \qquad \downarrow^c \qquad (VI.1)$$

$$FC \xleftarrow{\operatorname{outl}_{E_R}} E_R \xrightarrow{\operatorname{outr}_{E_R}} FC$$

If it exists, the greatest simulation with respect to the partial order  $\leq$  on  $\operatorname{\mathbf{Rel}}_{C}(\mathbb{C})$  is called the *similarity relation* on (C, c).

**Lemma VI.3.** Suppose that the functor  $\overline{F}$  satisfies the following conditions for all  $X \in \mathbb{C}$  and  $(X, R), (X, S) \in \mathbf{Rel}_X(\mathbb{C})$ :

(1) the relation  $\overline{F}(X, X)$  is reflexive;

(2)  $\overline{F}(X,R) \bullet \overline{F}(X,S) \le \overline{F}((X,R) \bullet (X,S)).$ 

Then for every F-coalgebra (C, c) the similarity relation exists, and it is reflexive and transitive.

The conditions in the above lemma are similar to ones occurring in work on *lax extensions*, e.g. by Marti and Venema [31].

In the setting of  $\mathcal{HO}$  specifications, where  $F = \mathcal{PB}_0(X, -)$ , a weak simulation on a  $B_0(X, -)$ -coalgebra (C, c) as per Definition III.2 is precisely a simulation on the weak transition system  $(C, \tilde{c})$ . As observed in Remark III.3(3), in order to check the weak simulation conditions for R(p, q), it suffices to show that strong transitions from p are simulated by weak transitions from q. This turns out to be the only property of weak simulations needed for our categorical congruence proof, and so we take it as our abstract definition:

**Definition VI.4.** A *weakening* of a coalgebra  $c: C \to FC$  is a coalgebra  $\tilde{c}: C \to FC$  such that for every relation (C, R)the following two statements are equivalent:

- (1) (C, R) is a simulation on  $(C, \tilde{c})$ ;
- (2) there exists a morphism  $\tilde{c}_R$  making (VI.2) commute.

$$C \xleftarrow{\operatorname{outl}_R} R \xrightarrow{\operatorname{outr}_R} C$$

$$c \downarrow \qquad \qquad \downarrow_{\widetilde{c}_R} \qquad \downarrow_{\widetilde{c}} \qquad (VI.2)$$

$$FC \xleftarrow{\operatorname{outl}_{E_R}} E_R \xrightarrow{\operatorname{outr}_{E_R}} FC$$

A weak simulation on (C, c), with respect to a given weakening  $(C, \tilde{c})$ , is a relation (C, R) satisfying the two equivalent properties above. If it exists, the greatest weak simulation is called *weak similarity*, denoted  $\leq_{(C,c)}$  or just  $\leq$ . **Remark VI.5.** (1) For the trivial weakening  $\tilde{c} = c$ , weak simulations are just (strong) simulations.

(2) The above definition is agnostic about how the weakening  $\tilde{c}$  is actually constructed from c. The construction of weak coalgebras has been studied in specific order-enriched settings [9], [10], [21]. Our present abstract approach is flexible in the choice of  $\tilde{c}$ . For example, the weak transition system  $\tilde{c}$  of Notation III.1 is an instance of the framework of [9], but the choice  $\tilde{c} = c$  as in part (1) above is not.

#### VII. HOWE'S METHOD, CATEGORICALLY

Next, we set up our version of Howe's method, which regards Howe closures abstractly as initial algebras. In a restricted setting of presheaf categories, this idea already appears in the work of Borthelle et al. [8] and Hirschowitz and Lafont [23].

**Notation VII.1.** Let  $\Sigma : \mathbb{C} \to \mathbb{C}$  be an endofunctor with its canonical relation lifting  $\overline{\Sigma} = \overline{\Sigma}_{\mathbf{Rel}}$  (Construction V.2). For every  $(X, R) \in \mathbf{Rel}_X(\mathbb{C})$  and every  $\Sigma$ -algebra  $(X, \xi)$  with monic structure  $\xi : \Sigma X \to X$ , let

$$\overline{\Sigma}_{R,\xi} \colon \mathbf{Rel}_X(\mathbb{C}) \to \mathbf{Rel}_X(\mathbb{C})$$

be the endofunctor (= monotone map) given by

$$(X,S) \mapsto (X,R) \lor_X ((\xi_\star \overline{\Sigma}(X,S)) \bullet (X,R)),$$

see Section IV-A for the notation. (The assumption that  $\xi$  is monic ensures that  $\xi_{\star}$  maps relations to relations.) Since  $\operatorname{\mathbf{Rel}}_X(\mathbb{C})$  is equivalent to a complete lattice, the initial algebra of  $\overline{\Sigma}_{R,\xi}$  exists, and we denote it by

$$(X,R) \vee_X \left( (\xi_{\star} \overline{\Sigma}(X,\widehat{R})) \bullet (X,R) \right) \xrightarrow{\alpha_{R,\xi}} (X,\widehat{R}).$$
 (VII.1)

The relation  $(X, \widehat{R})$  is called the *Howe closure* of (X, R) with respect to the algebra  $(X, \xi)$ .

**Remark VII.2.** We will instantiate the above to the initial algebra  $(X, \xi) = (\mu \Sigma, \iota)$ ; note that the structure  $\iota$  is an isomorphism. For  $\mathbb{C} = \mathbf{Set}$  and  $\Sigma$  a polynomial functor, the above definition of  $\widehat{R}$  is equivalent to the one of Notation III.9.

Lemma VII.4 below establishes some basic properties of Howe closures, generalizing Remark III.10. For that purpose let us recall the notion of *congruence* for functor algebras:

**Definition VII.3.** A congruence on a  $\Sigma$ -algebra (A, a) is a relation (A, R) such that  $a_{\star}\overline{\Sigma}_{\mathbf{Rel}}(A, R) \leq (A, R)$ .

For a polynomial functor  $\Sigma$  on Set, this matches the definition of congruence from universal algebra (cf. Section II-A).

**Lemma VII.4.** Let  $\Sigma : \mathbb{C} \to \mathbb{C}$  be an endofunctor. Then for each  $(X, R) \in \mathbf{Rel}(\mathbb{C})$  and each monic algebra  $\xi : \Sigma X \to X$ , (1) if (X, R) is reflexive, then  $(X, \widehat{R})$  is reflexive and a congruence on  $(X, \xi)$ ;

(2) if (X, R) is transitive, then  $(X, \widehat{R})$  is weakly transitive, that is,  $(X, \widehat{R}) \bullet (X, R) \le (X, \widehat{R})$ .

#### VIII. COMPOSITIONALITY

We proceed to establish our main theorem, which asserts that under natural conditions, weak similarity is a congruence on the operational model of a higher-order GSOS law.

Assumptions VIII.1. In this section we fix the following data: (1) a functor  $\Sigma = V + \Sigma' : \mathbb{C} \to \mathbb{C}$  that preserves strong epimorphisms and generates a free monad  $\Sigma^*$ ;

(2) a preordered bifunctor  $B \colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$  with a relation lifting  $\overline{B}$  that is *good for simulations* (Definition VIII.2);

(3) a V-pointed higher-order GSOS law  $\rho$  of  $\Sigma$  over B that admits a (necessarily unique) relation lifting  $\overline{\rho}$ .

It remains to explain Assumption VIII.1(2):

**Definition VIII.2.** A relation lifting  $\overline{B}$  of B is good for simulations if, for  $X, Y \in \mathbb{C}$  and  $(X, R), (Y, S), (Y, S') \in \mathbf{Rel}(\mathbb{C})$ ,

- (G1) the relation  $\overline{B}((X, R), (Y, S))$  is good for simulations;
- (G2) the relation  $\overline{B}((X,X),(Y,Y))$  is reflexive;
- (G3)  $\overline{B}((X,R),(Y,S)) \bullet \overline{B}((X,X),(Y,S')) \le \overline{B}((X,R),(Y,S) \bullet (Y,S')).$

**Remark VIII.3.** To motivate Assumption VIII.1(3), let us revisit the setting of  $\mathcal{HO}$  specifications, where  $B(X, Y) = \mathcal{P}(Y + Y^X)$  and  $\varrho$  is given by (III.5). Existence of a relation lifting of  $\varrho$  means that for  $(X, R), (Y, S) \in \mathbf{Rel}$  the map  $\varrho_{X,Y}$ is a **Rel**-morphism w.r.t. the lifting  $\overline{B}$  of Remark III.3(1). In the proof of Theorem III.8 (induction base for (III.9)) a syntactic argument shows that  $p_d \lesssim p_e$  for  $d \lesssim e$ , which amounts to the above property for  $(X, R) = (Y, S) = (\mu \Sigma, \widehat{\lesssim})$ . Hence, the purpose of Assumption VIII.1(3) is to replace the syntactic part of that proof by an abstract condition on the law  $\varrho$ .

In the following we study weak simulations on the operational model  $(\mu\Sigma, \gamma)$  of the higher-order GSOS law  $\varrho$ , understood w.r.t. to the relation lifting  $\overline{B}((\mu\Sigma, \mu\Sigma), -)$ :  $\mathbf{Rel}(\mathbb{C}) \rightarrow \mathbf{Rel}(\mathbb{C})$  of the endofunctor  $B(\mu\Sigma, -)$ :  $\mathbb{C} \rightarrow \mathbb{C}$  and a given weakening  $(\mu\Sigma, \tilde{\gamma})$  of  $(\mu\Sigma, \gamma)$ . By (G2) and (G3) the lifted endofunctor satisfies the conditions of Lemma VI.3, hence the weak similarity relation on  $(\mu\Sigma, \gamma)$  exists.

The core ingredient for our congruence theorem is a higherorder variation of *lax models* for monotone GSOS laws [6]:

**Definition VIII.4.** A lax  $\rho$ -bialgebra (X, a, c) is given by an object  $X \in \mathbb{C}$  and morphisms  $a: \Sigma X \to X$  and  $c: X \to B(X, X)$  such that the diagram below commutes laxly. Note that X is V-pointed; the point  $p_X: V \to X$  is induced by the algebra  $a: \Sigma X \to X$  (Notation II.4).

$$\begin{array}{ccc} \Sigma X & \stackrel{a}{\longrightarrow} X & \stackrel{c}{\longrightarrow} B(X,X) \\ & & & & | \Upsilon & & \\ & & & | \Upsilon & & \\ \Sigma(X \times B(X,X)) & \stackrel{\mathfrak{g}^{\mathcal{F}},\overset{\tilde{\mathcal{F}}}{\longrightarrow}}{\longrightarrow} B(X,\Sigma^{\star}(X+X)) & \stackrel{\mathfrak{g}^{(\mathrm{td}},\widehat{a})}{\longrightarrow} B(X,\Sigma^{\star}X) \end{array}$$

This generalizes the notion of  $\rho$ -bialgebra [20] which requires strict commutativity of the above diagram.

Our congruence theorem rests on the assumption that  $(\mu\Sigma, \iota, \tilde{\gamma})$  is a lax  $\varrho$ -bialgebra. As indicated in Remark III.7,

this expresses in abstract terms that the operational rules encoded by the higher-order GSOS law  $\rho$  are sound for weak transitions in the operational model. In the setting of  $\mathcal{HO}$  specifications, we proved that this entails the congruence property for weak similarity (Theorem III.8). The next proposition is key to our categorical generalization of that result.

**Proposition VIII.5.** Suppose that  $\tilde{\gamma}$  is a weakening of the operational model  $(\mu\Sigma, \gamma)$  such that  $(\mu\Sigma, \iota, \tilde{\gamma})$  is a lax  $\varrho$ -bialgebra. Then for every reflexive and transitive weak simulation  $(\mu\Sigma, R)$  on  $(\mu\Sigma, \gamma)$ , the Howe closure  $(\mu\Sigma, \hat{R})$  w.r.t.  $(\mu\Sigma, \iota)$  is a weak simulation.

In the proof below we denote the relation  $\overline{B}((X,S),(Y,T))$  by  $(B(X,Y), E_{S,T})$  and its projections by  $\operatorname{outl}_{S,T}, \operatorname{outr}_{S,T}$ .

*Proof sketch.* Form the relation  $(\mu \Sigma, P)$  via the pullback

$$\begin{array}{c} P \xrightarrow{p} E_{\widehat{R},\widehat{R}} \\ \langle \mathsf{outl}_{P},\mathsf{outr}_{P} \rangle \downarrow & \downarrow \langle \mathsf{outl}_{\widehat{R},\widehat{R}},\mathsf{outr}_{\widehat{R},\widehat{R}} \rangle \\ \mu\Sigma \times \mu\Sigma \xrightarrow{\langle \gamma, \widetilde{\gamma} \rangle} B(\mu\Sigma, \mu\Sigma) \times B(\mu\Sigma, \mu\Sigma) \end{array}$$

The crucial step is to show existence of  $\mathbf{Rel}_{\mu\Sigma}(\mathbb{C})$ -morphisms

$$\beta^{\mathsf{r}} \colon (\mu\Sigma, R) \to (\mu\Sigma, P), \beta^{\mathsf{r}} \colon \iota_{\star}\overline{\Sigma}((\mu\Sigma, \widehat{R}) \times_{\mu\Sigma} (\mu\Sigma, P)) \bullet (\mu\Sigma, R) \to (\mu\Sigma, P),$$

where  $\times_{\mu\Sigma}$  is the product in  $\operatorname{\mathbf{Rel}}_{\mu\Sigma}(\mathbb{C})$  (Section IV-A3). Their construction imitates the arguments for the induction base and induction step, respectively, in the proof of Theorem III.8.

Once this is achieved, we can conclude the proof as follows. By copairing  $\beta^{l}$  and  $\beta^{r}$  we obtain the  $\mathbf{Rel}_{\mu\Sigma}(\mathbb{C})$ -morphism

$$\beta = [\beta^{\mathsf{l}}, \beta^{\mathsf{r}}] \colon \overline{\Sigma}_{R,\iota} \big( (\mu\Sigma, \widehat{R}) \times_{\mu\Sigma} (\mu\Sigma, P) \big) \to (\mu\Sigma, P)$$

(cf. Notation VII.1) and thus primitive recursion (II.1) yields the  $\operatorname{\mathbf{Rel}}_{\mu\Sigma}(\mathbb{C})$ -morphism pr  $\beta: \mu\overline{\Sigma}_{R,\iota} = (\mu\Sigma, \widehat{R}) \to (\mu\Sigma, P)$ . Choose a morphism  $r: (\mu\Sigma, \mu\Sigma) \to (\mu\Sigma, \widehat{R})$  witnessing that  $(\mu\Sigma, \widehat{R})$  is reflexive (Lemma VII.4). Then the commutative diagram below proves  $(\mu\Sigma, \widehat{R})$  to be a weak simulation.



We are ready to state our main result. Recall that we work under the Assumptions IV.1 and VIII.1.

**Theorem VIII.6** (Compositionality). Suppose that  $\tilde{\gamma}$  is a weakening of the operational model  $(\mu\Sigma, \gamma)$  such that  $(\mu\Sigma, \iota, \tilde{\gamma})$  is a lax  $\varrho$ -bialgebra. Then the weak similarity relation on  $(\mu\Sigma, \gamma)$  is a congruence. *Proof.* Let  $(\mu\Sigma, \leq)$  be the weak similarity relation on  $(\mu\Sigma, \gamma)$ . Its Howe closure  $(\mu\Sigma, \widehat{\leq})$  satisfies

$$(\mu\Sigma, \lesssim) \le (\mu\Sigma, \widehat{\lesssim}) \le (\mu\Sigma, \lesssim)$$

The first inequality is witnessed by the morphism  $\alpha_{\leq,\iota} \cdot inl$ , for  $\alpha_{\leq,\iota}$  from (VII.1). For the second one we use that the relation  $(\mu\Sigma, \widehat{\leq})$  is a weak simulation by Proposition VIII.5 (note that it is reflexive and transitive by Lemma VI.3, so the proposition applies) and that  $\leq$  is the greatest weak simulation. Thus  $(\mu\Sigma, \leq) \cong (\mu\Sigma, \leq)$ , and since  $(\mu\Sigma, \leq)$  is a congruence by Lemma VII.4, we conclude that so is  $(\mu\Sigma, \leq)$ . 

By choosing the trivial weakening  $\tilde{\gamma} = \gamma$  and equipping B with the equality preorder, we obtain similarity as an instance of weak similarity (Remark VI.5(1)), and the laxness condition on the bialgebra  $(\mu\Sigma, \iota, \gamma)$  holds trivially by (II.3). We obtain

#### **Corollary VIII.7.** Similarity on $(\mu \Sigma, \gamma)$ is a congruence.

This is a variant of the main result of [20]. In fact, the present version is more general since its notion of similarity is parametric in a lifting of B, while the result in op. cit. is about coalgebraic behavioural equivalence, which corresponds to the *canonical* lifting of B (see Appendix, Section C).

#### IX. APPLICATIONS

We conclude with two applications of Theorem VIII.6.

#### A. HO Specifications

To recover the results of Section III, fix an  $\mathcal{HO}$  specification  $\mathcal{R}$  over the signature  $\Sigma$ , corresponding to a higher-order GSOS law  $\rho^0$  of  $\Sigma$  over  $B_0(X, Y) = Y + Y^X$ . We take  $\mathbb{C} = \mathbf{Set}$  and instantiate the data of Assumptions VIII.1 to

(1) the given polynomial functor  $\Sigma$ ;

(2) the behaviour functor  $B(X, Y) = \mathcal{P}(Y+Y^X)$ , preordered

by inclusion, with its relation lifting  $\overline{B}$  as in Remark III.3(1);

(3) the higher-order GSOS law  $\rho$  of  $\Sigma$  over B given by (III.5).

It is not difficult to verify that the above data satisfies the Assumptions VIII.1. Then by choosing the weakening  $\tilde{\gamma}$  to be the weak transition system associated to  $\gamma_0$ , see Notation III.1, we recover Theorem III.8 as a special case of Theorem VIII.6.

#### B. The $\lambda$ -Calculus

We briefly sketch how our framework applies to the  $\lambda$ -calculus, building on ideas from the work of Fiore et al. [16] and our previous work [20]. The (untyped call-by-name)  $\lambda$ -calculus is given by the small-step operational rules shown below, where s, s', t range over possibly open  $\lambda$ -terms and [t/x] denotes capture-avoiding substitution.

$$app1 \frac{s \to s'}{s t \to s' t} \qquad app2 \frac{}{(\lambda x.s) t \to s[t/x]} \quad (IX.1)$$

Take the presheaf category  $\mathbb{C} = \mathbf{Set}^{\mathbb{F}}$ , where  $\mathbb{F}$  is the category of finite cardinals and functions, and the functors

$$\begin{split} \Sigma \colon \mathbb{C} \to \mathbb{C}, & \Sigma X = V + \delta X + X \times X, \\ B_0 \colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}, & B_0(X,Y) = \langle\!\langle X, Y \rangle\!\rangle \times (Y + Y^X + 1). \end{split}$$

Here  $Y^X$  denotes the exponential object in the topos  $\mathbf{Set}^{\mathbb{F}}$ , and the presheaves V,  $\delta X$  and  $\langle\!\langle X, Y \rangle\!\rangle$  are meant to represent variables,  $\lambda$ -abstraction, and substitution, respectively:

$$V(n) = n, \ \delta X(n) = X(n+1), \ \langle\!\langle X, Y \rangle\!\rangle(n) = \mathbf{Set}^{\mathbb{F}}(X^n, Y).$$

The initial algebra for  $\Sigma$  is the presheaf  $\Lambda \in \mathbf{Set}^{\mathbb{F}}$  given by

 $\Lambda(n) = \lambda$ -terms in free variables from  $n = \{0, \dots, n-1\},\$ 

where  $\lambda$ -terms are formed modulo  $\alpha$ -equivalence [16]. In [20, Def. 5.8] we devised a V-pointed higher-order GSOS law  $\rho^0$ of  $\Sigma$  over  $B_0$  whose operational model

$$\gamma^{0} = \langle \gamma_{1}^{0}, \gamma_{2}^{0} \rangle \colon \Lambda \to \langle \! \langle \Lambda, \Lambda \rangle \! \rangle \times (\Lambda + \Lambda^{\Lambda} + 1)$$
 (IX.2)

captures the operational semantics of (IX.1), in the sense that for every  $m, n \in \mathbb{F}$ ,  $t \in \Lambda(n)$  and  $\vec{u} \in \Lambda(m)^n$ :

• 
$$\gamma_1^0(t)(\vec{u}) = t[\vec{u}] = t[(u_0, \dots, u_{n-1})/(0, \dots, n-1)];$$

•  $t \to t' \implies \gamma_2^0(t) = t' \in \Lambda(n);$ •  $t = \lambda x.t' \implies \gamma_2^0(t) \in \Lambda^{\Lambda}(n) \land \forall e \in \Lambda(n).\gamma_2^0(t)(e) = t'[e];$ •  $\gamma_2^0(t) = *$  otherwise (that is, if t is stuck).

Here  $\gamma^0_2(t)(e) = \operatorname{ev}(\gamma^0_2(t), e)$  for the evaluation map  $\operatorname{ev} \colon \Lambda^\Lambda \times$  $\Lambda \rightarrow \Lambda$ . We instantiate the data of Assumptions VIII.1 to

(1) the above functor  $\Sigma X = V + \delta X + X \times X$ ;

(2) the functor  $B(X,Y) = \langle\!\langle X,Y \rangle\!\rangle \times \mathcal{P}_{\star}(Y+Y^X)$ , where  $\mathcal{P}_{\star} \colon \mathbf{Set}^{\mathbb{F}} \to \mathbf{Set}^{\mathbb{F}}$  is the pointwise powerset functor given by  $X \mapsto \mathcal{P} \cdot X$  for  $X \in \mathbf{Set}^{\mathbb{F}}$ . (Note that we dropped the "+1" summand from the behaviour, whose role is taken over by the empty set.) The relation lifting  $\overline{B}$  is constructed similarly to the one of  $B(X, Y) = \mathcal{P}(Y + Y^X)$  in Remark III.3(1);

(3) a higher-order GSOS law  $\rho$  of  $\Sigma$  over B which is derived from  $\rho^0$  in a way similar to the construction of Remark III.7. The weak operational model is the  $B(\Lambda, -)$ -coalgebra

$$\widetilde{\gamma} = \langle \widetilde{\gamma}_1, \widetilde{\gamma}_2 \rangle \colon \Lambda \to \langle\!\langle \Lambda, \Lambda \rangle\!\rangle \times \mathcal{P}_{\star}(\Lambda + \Lambda^{\Lambda})$$

given for  $t \in \Lambda(n)$  by  $\widetilde{\gamma}_1(t) = \gamma_1^0(t)$  and

$$\widetilde{\gamma}_2(t) = \{ \overline{t} \in \Lambda(n) : t \Rightarrow \overline{t} \} \cup \\ \{ f \in \Lambda^{\Lambda}(n) : \exists \overline{t}. t \Rightarrow \overline{t} \land \gamma_2^0(\overline{t}) = f \}.$$

Here  $\Rightarrow$  is the reflexive transitive hull of the reduction relation  $\rightarrow$ . Weak similarity on (IX.2) then can be shown to coincide with the following concept due to Abramsky [1]:

Definition IX.1. Applicative similarity is the greatest relation  $\lesssim_0^{\mathrm{ap}} \subseteq \Lambda(0) \times \Lambda(0)$  on the set of closed  $\lambda$ -terms such that if  $t_1 \lesssim_0^{\mathrm{ap}} t_2$  and  $t_1 \Rightarrow \lambda x.t_1'$ , then there is a term  $t_2'$  such that

$$t_2 \Rightarrow \lambda x.t'_2 \land \forall e \in \Lambda(0). t'_1[e/x] \lesssim_0^{\operatorname{ap}} t'_2[e/x].$$

Its open extension is the relation  $\leq^{ap} \subseteq \Lambda \times \Lambda$  whose component  $\leq_n^{\mathrm{ap}} \subseteq \Lambda(n) \times \Lambda(n)$  for n > 0 is given by

$$t_1 \lesssim_n^{\mathrm{ap}} t_2$$
 iff  $t_1[\vec{u}] \lesssim_0^{\mathrm{ap}} t_2[\vec{u}]$  for every  $\vec{u} \in \Lambda(0)^n$ .

One can verify that  $(\Lambda, \iota, \widetilde{\gamma})$  forms a lax  $\rho$ -bialgebra, which amounts to observing that the rules (IX.1) are sound for weak transitions. Consequently, Theorem VIII.6 instantiates to a well-known and fundamental result about the  $\lambda$ -calculus [1]: **Theorem IX.2.** The open extension of applicative similarity is a congruence: for all  $\lambda$ -terms s, t, t', one has

$$t \lesssim^{\mathrm{ap}} t' \implies st \lesssim^{\mathrm{ap}} st' \land ts \lesssim^{\mathrm{ap}} t's \land \lambda x.t \lesssim^{\mathrm{ap}} \lambda x.t'.$$

It follows that the open extension of applicative *bi*similarity, viz. the relation  $\approx^{ap} = \leq^{ap} \cap \gtrsim^{ap}$ , is also a congruence.

#### X. CONCLUSIONS AND FUTURE WORK

We have developed relation liftings of bifunctors and an abstract analogue of Howe's method to prove congruence of coalgebraic weak similarity for higher-order GSOS laws. We have thus taken the first steps towards operational reasoning in the higher-order abstract GSOS framework.

Logical relations [36], [35], [33], [15] are another important operational reasoning technique that we would like to cover in the future. Logical relations are typically type-indexed, while higher-order abstract GSOS has so far been applied to untyped languages. We aim to investigate typed languages in the context of higher-order abstract GSOS and develop abstract analogues of logical relations. It is worth noting that, even in the untyped setting, relation liftings of bifunctors already share a key characteristic with logical relations, namely that functions send related inputs to related outputs (Remark III.3(1)).

Another goal is to apply our methods to call-by-value languages. As already noted in our previous work [20, Sec. 5.4], this appears to be more subtle than the call-by-name case. We envision a multi-sorted setting as a possible approach.

Finally, we aim to explore effectful languages. For instance, by taking the behaviours  $\mathcal{P}(Y+Y^X)$  or  $\mathcal{S}(Y+Y^X)$ , where  $\mathcal{S}$  is the subdistribution functor, our results already yield a form of compositionality for nondeterministic and probabilistic combinatory logic. For the latter, exploring behavioural distances instead of (bi)similarity is also a natural direction; we expect that existing work on probabilistic  $\lambda$ -calculi [12], [18] can provide some guidance.

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#### APPENDIX

This appendix is structured as follows:

- Section A establishes a number of auxiliary results on graphs and relations.
- Section B provides complete proofs of all results from the extended abstract.
- Section C explains how to construct canonical graph and relation liftings of mixed-variance bifunctors.
- Section D addresses canonical graph and relation liftings of higher-order GSOS laws.
- Section E is a more detailed version of Section IX-B on the  $\lambda$ -calculus.

Recall that throughout our paper, including this appendix, we work under the Assumptions IV.1 on the base category C.

### Appendix A

#### More on Graphs and Relations

In Section IV-A3 we stated that the category  $\mathbf{Gra}(\mathbb{C})$  is complete and cocomplete, with limits and colimits formed at the level of  $\mathbb{C}$ . In particular, this means that the product  $(X, R) \times (Y, S)$  and coproduct (X, R) + (Y, S) are given by the graphs

$$\begin{array}{ccc} R \times S & R + S \\ \mathsf{outl}_R \times \mathsf{outl}_S \Big( \Big) \mathsf{outr}_S \times \mathsf{outr}_S & \text{and} & \mathsf{outl}_R + \mathsf{outl}_S \Big( \Big) \mathsf{outr}_R + \mathsf{outr}_S \\ X \times Y & X + Y \end{array}$$

Colimits in  $\operatorname{Gra}_X(\mathbb{C})$  are also formed at the level of  $\mathbb{C}$ . In particular, the coproduct of  $(X, R), (X, R') \in \operatorname{Gra}_X(\mathbb{C})$ , denoted by  $(X, R) +_X (X, R')$ , is the graph

$$R + R'$$

$$[\operatorname{outl}_R, \operatorname{outl}_{R'}] \left( \begin{array}{c} \\ \\ \\ \end{array} \right) [\operatorname{outr}_R, \operatorname{outr}_{R'}] \\ X$$

(Co-)limits in  $\operatorname{Rel}(\mathbb{C})$  and  $\operatorname{Rel}_X(\mathbb{C})$  are formed by taking the (co-)limit in  $\operatorname{Gra}(\mathbb{C})$  and  $\operatorname{Gra}_X(\mathbb{C})$ , respectively, and applying the reflector  $(-)^{\dagger}$ . Note that  $\operatorname{Rel}(\mathbb{C})$  is closed under products in  $\operatorname{Gra}(\mathbb{C})$ , since products of monomorphisms are monomorphisms.

The remaining results of this section are about composition of graphs and relations. We first mention a useful property of pullbacks that follows from our assumptions:

**Lemma A.1.** For every commutative diagram (A.2), if the outside and the inner square are pullbacks and  $e_0, e_1$  are strong epimorphisms, then e is a strong epimorphism.

$A \xrightarrow{f} B$	
$\left  \begin{array}{c} \searrow \\ A' \xrightarrow{f'} B' \end{array} \right $	
$g \qquad g' \downarrow \qquad \downarrow h' \qquad h$	(A.1)
$C' \xrightarrow{k'} D$	
$\downarrow e_1$ , $\land id \downarrow$	
$C \xrightarrow{k} D$	

*Proof.* Recall that pullbacks correspond to products in a slice category. Therefore, the lemma states that for any two morphisms  $e_0: h \to h'$  and  $e_1: g \to g'$  in  $\mathbb{C}/D$  with  $e_0, e_1$  strongly epic in  $\mathbb{C}$ , their product  $e = e_0 \times_D e_1$  in  $\mathbb{C}/D$  is also a strong epimorphism in  $\mathbb{C}$ . Since  $e_0 \times_D e_1 = (id \times_D e_1) \circ (e_0 \times_D id)$ , and strong epimorphisms are closed under composition, it suffices to assume that one of the  $e_i$  is an identity morphism, w.l.o.g.  $e_1 = id$ . Then we have the following commutative diagram:

The lower rectangle and the outside are pullbacks, so the upper rectangle is also a pullback [7, Prop. 2.5.9]. Since  $e_0$  is strongly epic and strong epimorphisms are pullback-stable by Assumption IV.1(2), we conclude that e is strongly epic.

**Notation A.2.** Recall that the graph (X, R); (X, R') is defined by via the pullback (IV.3). We often write RR' for R; R'.

Graph composition extends to a bifunctor

$$(-); (-): \mathbf{Gra}_X(\mathbb{C}) \times \mathbf{Gra}_X(\mathbb{C}) \to \mathbf{Gra}_X(\mathbb{C})$$

as follows. Given  $f \colon (X, R) \to (X, S)$  and  $f' \colon (X, R') \to (X, S')$  in  $\mathbf{Gra}_X(\mathbb{C})$ , the morphism

$$f; f' \colon (X, R); (X, R') \to (X, S); (X, S')$$

is defined by  $(f: f')_0 = id_X$ , and  $(f; f')_1$  is the unique  $\mathbb{C}$ -morphism making the two squares below commute, using the universal property of the pullback SS':

$$\begin{array}{cccc} RR' \xrightarrow{(f;f')_1} SS' & RR' \xrightarrow{(f;f')_1} SS' \\ \pi_{RR',R} \downarrow & \downarrow \pi_{SS',S} & \pi_{RR',R'} \downarrow & \downarrow \pi_{SS',S'} \\ R \xrightarrow{f_1} S & R' \xrightarrow{f_1'} S' \end{array}$$
(A.3)

**Lemma A.3.** If  $f_1$  and  $f'_1$  are strong epimorphisms, then  $(f; f')_1$  is a strong epimorphism. *Proof.* This follows from Lemma A.1 applied to the commutative diagram



Analogously, composition of relations extends to the bifunctor

$$(-) \bullet (-) = (\operatorname{Rel}_X(\mathbb{C}) \times \operatorname{Rel}_X(\mathbb{C}) \hookrightarrow \operatorname{Gra}_X(\mathbb{C}) \times \operatorname{Gra}_X(\mathbb{C}) \xrightarrow{(-);(-)} \operatorname{Gra}_X(\mathbb{C}) \xrightarrow{(-)^{\intercal}} \operatorname{Rel}_X(\mathbb{C})).$$

**Lemma A.4.** For every  $(X, R), (X, R') \in \mathbf{Gra}_X(\mathbb{C})$ , we have

$$(X,R)^{\dagger} \bullet (X,R')^{\dagger} \cong ((X,R);(X,R'))^{\dagger}.$$

*Proof.* Since  $(id, e_R)$ :  $(X, R) \to (X, R^{\dagger})$  and  $(id, e_{R'})$ :  $(X, R') \to (X, {R'}^{\dagger})$  are graph morphisms and by (IV.2), the following diagram commutes:



Note that  $((id, e_R); (id, e_{R'}))_1$  is a strong epimorphism by Lemma A.3. Thus the desired isomorphism follows from the uniqueness of (strong epi, mono)-factorizations.

The following lemma shows that graph and relation composition distribute over coproducts. This is where our assumption that  $\mathbb{C}$  is locally distributive is used.

**Lemma A.5.** For  $X \in \mathbb{C}$  we have the following natural isomorphisms in  $\operatorname{Gra}_X(\mathbb{C})$  and  $\operatorname{Rel}_X(\mathbb{C})$ , respectively:

$$(X,R);((X,S) + (X,T)) \cong (X,R);(X,S) + (X,R);(X,T),$$
(A.4)

$$((X,S) + (X,T)); (X,R) \cong (X,S); (X,R) + (X,T); (X,R),$$
(A.5)

$$(X,R) \bullet ((X,S) \lor (X,T)) \cong (X,R) \bullet (X,S) \lor (X,R) \bullet (X,T), \tag{A.6}$$

$$((X,S) \lor (X,T)) \bullet (X,R) \cong (X,S) \bullet (X,R) \lor (X,T) \bullet (X,R).$$
(A.7)

_	_

*Here we write* + *and*  $\vee$  *for*  $+_X$  *and*  $\vee_X$ .

Proof. We construct the isomorphisms (A.4) and (A.6); the other two are analogous.

*Proof of* (A.4). By the universal property of the pullback R(S+T), there exists a unique  $\mathbb{C}$ -morphism  $i: RS \to R(S+T)$  making the following diagrams commute:

The morphism  $j: RT \to R(S+T)$  is defined analogously. Since the slice category  $\mathbb{C}/X$  is distributive, and products in  $\mathbb{C}/X$  correspond precisely to pullbacks in  $\mathbb{C}$ , it follows that

$$[i,j] \colon RS + RT \xrightarrow{\cong} R(S+T)$$

is an isomorphism in C. Thus

 $(\mathsf{id}_X, [i, j]) \colon (X, R) \; ; \; (X, S) + (X, R) \; ; \; (X, T) \xrightarrow{\cong} (X, R) \; ; \; ((X, S) + (X, T))$ 

is an isomorphism in  $\mathbf{Gra}_X(\mathbb{C})$ ; the two diagrams below show that is indeed a  $\mathbf{Gra}_X(\mathbb{C})$ -morphism. (Below and henceforth we indicate the reason why the parts of a commutative diagram commutes, and we write  $\circlearrowleft$  if a part obviously commutes.)



Naturality of the isomorphism  $(id_X, [i, j])$  amounts to showing that the square below commutes for all  $\mathbf{Gra}_X(\mathbb{C})$ -morphisms  $f: (X, R) \to (X, R'), g: (X, S) \to (X, S')$  and  $h: (X, T) \to (X, T')$ , where [i', j'] is defined analogously to [i, j] above.

$$RS + RT \xrightarrow{[i,j]} R(S+T)$$

$$(fg)_1 + (fh)_1 \downarrow \qquad \qquad \qquad \downarrow (f(g+h))_1$$

$$R'S' + R'T' \xrightarrow{[i',j']} R'(S'+T')$$

It suffices to show that the square commutes when postcomposed with the pullback projections  $\pi_{R'(S'+T'),R'}$  and  $\pi_{R'(S'+T'),S'+T'}$ ; the latter follows from the two commutative diagrams below.

*Proof of* (A.6). For every  $(X, R), (X, S), (X, T) \in \mathbf{Rel}_X(\mathbb{C})$  we compute

$$(X,R) \bullet ((X,S) \lor (X,T)) \cong (X,R)^{\dagger} \bullet ((X,S) + (X,T))^{\dagger}$$
$$\cong ((X,R); ((X,S) + (X,T)))^{\dagger}$$
$$\cong ((X,R); (X,S) + (X,R); (X,T))^{\dagger}$$
$$\cong ((X,R); (X,S))^{\dagger} \lor ((X,R); (X,T))^{\dagger}$$
$$\cong (X,R) \bullet (X,S) \lor (X,R) \bullet (X,T).$$

In the first step we use the definition of  $\lor$  and that  $(X, R)^{\dagger} \cong (X, R)$  because (X, R) is a relation; the second step follows from Lemma A.4; the third step uses the isomorphism (A.4) established above; the fourth step uses that the reflector  $(-)^{\dagger}$  preserves coproducts, being a left adjoint; the last step follows by definition of  $\bullet$ .

Recall from Section IV-A6 the reindexing functor  $f_*$ :  $\mathbf{Gra}_X(\mathbb{C}) \to \mathbf{Gra}_Y(\mathbb{C})$  induced by a morphism  $f: X \to Y$ . Lemma A.6. For every  $f: X \to Y$  in  $\mathbb{C}$  and  $(X, R), (X, R') \in \mathbf{Gra}_X(\mathbb{C})$  we have

$$f_{\star}((X,R);(X,R')) \le f_{\star}(X,R); f_{\star}(X,R'),$$

$$f_{\star}((X,R)^{\dagger}) \le (f_{\star}(X,R))^{\dagger}.$$
(A.9)
(A.10)

*Proof. Proof of* (A.9). By definition,  $f_{\star}(X, R)$ ;  $f_{\star}(X, R')$  is the relation (Y, S) obtained via the following pullback:



Since  $\operatorname{outr}_R$ ,  $\operatorname{outl}_{R'}$  merge  $\pi_{R;R',R}$  and  $\pi_{R;R',R'}$ , so do  $f \cdot \operatorname{outr}_R$ ,  $f \cdot \operatorname{outl}_{R'}$ , hence the universal property of the pullback S yields a unique  $h_1 \colon R : R' \to S$  such that

$$\pi_{R;R',R} = \pi_{S,R} \cdot h_1$$
 and  $\pi_{R;R',R'} = \pi_{S,R'} \cdot h_1$ .

It follows that

$$(\mathsf{id},h_1)\colon f_\star((X,R)\,;\,(X,R'))\to f_\star(X,R)\,;\,f_\star(X,R')$$

is a  $\mathbf{Gra}_Y(\mathbb{C})$ -morphism, as required.

*Proof of* (A.10). By definition,  $(f_{\star}(X, R))^{\dagger}$  is the relation (Y, S) obtained via the (strong epi, mono)-factorization e and  $\langle \text{outl}_S, \text{outr}_S \rangle$  of the morphism  $\langle f \cdot \text{outl}_R, f \cdot \text{outr}_R \rangle = (f \times f) \cdot \langle \text{outl}_R, \text{outr}_R \rangle$ . We thus obtain a diagonal fill-in  $d: R^{\dagger} \to S$  making the diagram below commute:

It follows that

$$(\mathsf{id}, d) \colon f_{\star}((X, R)^{\dagger}) \to (f_{\star}(X, R))^{\dagger}$$

is a  $\mathbf{Gra}_Y(\mathbb{C})$ -morphism, as required.

**Corollary A.7.** For every  $f: X \to Y$  in  $\mathbb{C}$  and  $(X, R), (X, R') \in \mathbf{Rel}_X(\mathbb{C})$  we have

$$f_{\star}((X,R) \bullet (X,R')) \le f_{\star}(X,R) \bullet f_{\star}(X,R').$$
 (A.11)

Proof. This follows from the computation

$$f_{\star}((X,R) \bullet (X,R')) = f_{\star}(((X,R);(X,R'))^{\dagger}) \\\leq (f_{\star}((X,R);(X,R'))^{\dagger} \\\leq (f_{\star}(X,R);f_{\star}(X,R'))^{\dagger} \\= f_{\star}(X,R) \bullet f_{\star}(X,R').$$

The second step uses (A.10) and the third one uses (A.9) and functoriality of  $(-)^{\dagger}$ .

#### APPENDIX B **OMITTED PROOFS**

## **PROOF OF PROPOSITION III.13**

The rules (III.10) for a passive operator are clearly sound, as is the first rule in (III.11) for an active operator. To see that the second rule in (III.11) is sound, suppose that  $p_1, \ldots, p_n, \overline{p_j} \in \mu\Sigma$  are programs such that  $p_j \Downarrow \overline{p_j}$ . This means that

$$p_j = p_j^0 \to p_j^1 \to \dots \to p_j^k = \overline{p_j}$$
 for some  $k \ge 0$  and  $p_j^0, \dots, p_j^k \in \mu \Sigma$ .

By applying the first of (III.11) (the first ones k times) we obtain

$$f(p_1, \dots, p_j, \dots, p_n) \to \dots \to f(p_1, \dots, \overline{p_j}, \dots, p_n) \\ \to t[p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n, (\overline{p_j})_{p_1}, \dots, (\overline{p_j})_{p_{j-1}}, (\overline{p_j})_{p_{j+1}}, \dots, (\overline{p_j})_{p_n}].$$

(The bracket  $[\cdots]$  indicates how the variables of t are substituted. Note that t does not depend on  $x_j$  and  $y_j^{x_j}$ .) It follows  $f(p_1, \ldots, p_n) \Rightarrow t[p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n, (\overline{p_j})_{p_1}, \ldots, (\overline{p_j})_{p_{j-1}}, (\overline{p_j})_{p_j+1}, \ldots, (\overline{p_j})_{p_n}]$ , as required. The soundness of the third rule in (III.11) is shown analogously.

#### **PROOF OF PROPOSITION V.4**

We first establish some auxiliary results.

**Remark B.1.** By putting D = 1 in Lemma A.1, we see that strong epimorphisms in  $\mathbb{C}$  are product-stable:  $e_0 \times e_1$  is strongly epic whenever  $e_0$  and  $e_1$  are strongly epic.

**Lemma B.2.** If  $\Sigma$  preserves strong epimorphisms, we have the following isomorphism for every  $(X, R), (Y, S) \in \mathbf{Gra}(\mathbb{C})$ :

$$\overline{\Sigma}_{\mathbf{Rel}}((X,R)^{\dagger} \times (Y,S)^{\dagger}) \cong (\overline{\Sigma}_{\mathbf{Gra}}((X,R) \times (Y,S))^{\dagger}.$$

Proof. This follows from the commutative diagram below and the uniqueness of (strong epi, mono)-factorizations. Note that  $e_R \times e_S$  is strongly epic by Remark B.1.



**Lemma B.3.** If  $\Sigma$  preserves strong epimorphisms, the following diagrams commute:

$$\begin{array}{ccc} \mathbf{Gra}(\mathbb{C}) \xrightarrow{\overline{\Sigma}_{\mathbf{Gra}}} \mathbf{Gra}(\mathbb{C}) \\ & \xrightarrow{(-)^{\dagger}} & & \downarrow^{(-)^{\dagger}} \\ \mathbf{Rel}(\mathbb{C}) \xrightarrow{\overline{\Sigma}_{\mathbf{Rel}}} \mathbf{Rel}(\mathbb{C}) \end{array}$$

*Proof.* For each  $(X, R) \in \mathbf{Gra}(\mathbb{C})$  we have

$$(\overline{\Sigma}_{\mathbf{Gra}}(X,R))^{\dagger} = (\Sigma X, \Sigma R)^{\dagger} \cong (\Sigma X, (\Sigma R^{\dagger}))^{\dagger} = \overline{\Sigma}_{\mathbf{Rel}}((X,R)^{\dagger})$$

The isomorphism in the second step follows from the commutative diagram below and the uniqueness of (strong epi, mono)factorizations.



Since  $(-)^{\dagger}$  is a left adjoint, the above lemma and [22, Thm. 2.14] yield

**Corollary B.4.** If  $\Sigma$  preserves strong epimorphisms, there is an adjunction

$$\mathbf{Alg}(\overline{\Sigma}_{\mathbf{Gra}}) \xrightarrow[U]{F} \mathbf{Alg}(\overline{\Sigma}_{\mathbf{Rel}})$$

where the right adjoint U is given by

$$(\overline{\Sigma}_{\mathbf{Rel}}(A,R) \xrightarrow{a} (A,R)) \mapsto (\overline{\Sigma}_{\mathbf{Gra}}(A,R) \xrightarrow{(\mathsf{id},e_{\Sigma R})} \overline{\Sigma}_{\mathbf{Rel}}(A,R) \xrightarrow{a} (A,R)) \quad and \quad h \mapsto h$$

the left adjoint F by

$$(\overline{\Sigma}_{\mathbf{Gra}}(A,R) \xrightarrow{a} (A,R)) \mapsto (\overline{\Sigma}_{\mathbf{Rel}}(A,R^{\dagger}) \xrightarrow{a^{\dagger}} (A,R^{\dagger})) \quad and \quad h \mapsto h^{\dagger},$$

and the unit at  $((A, R), a) \in \mathbf{Alg}(\overline{\Sigma}_{\mathbf{Gra}})$  by

$$(\mathsf{id}, e_R) \colon ((A, R), a) \to UF((A, R), a)$$

With this preparation, we are ready to prove the proposition.

*Proof of Proposition V.4.* Let  $(\Sigma^*, \eta, \mu)$  be the free monad generated by  $\Sigma$ , with associated natural transformation

$$\iota\colon \Sigma\Sigma^{\star}\to\Sigma^{\star};$$

that is,  $\iota_X \colon \Sigma\Sigma^*X \to \Sigma^*X$  is the free  $\Sigma$ -algebra on X with the universal morphism  $\eta_X \colon X \to \Sigma^*X$ . (1) We first consider the graph lifting  $\overline{\Sigma} = \overline{\Sigma}_{\mathbf{Gra}}$ . It suffices to show that a free  $\overline{\Sigma}$ -algebra on  $(X, R) \in \mathbf{Gra}(\mathbb{C})$  is given by

$$\overline{\Sigma}\,\overline{\Sigma^{\star}}(X,R) = (\Sigma\Sigma^{\star}X,\Sigma\Sigma^{\star}R) \xrightarrow{(\iota_X,\iota_R)} (\Sigma^{\star}X,\Sigma^{\star}R) = \overline{\Sigma^{\star}}(X,R)$$

with the universal morphism

$$(X,R) \xrightarrow{(\eta_X,\eta_R)} (\Sigma^* X, \Sigma^* R) = \overline{\Sigma^*}(X,R).$$

To prove this, we verify the required universal property. Thus suppose that we are given a  $\overline{\Sigma}$ -algebra  $a: \overline{\Sigma}(A, S) \to (A, S)$  and a  $\mathbf{Gra}(\mathbb{C})$ -morphism  $h: (X, R) \to (A, S)$ . The universal property of the free  $\Sigma$ -algebra  $(\Sigma^*X, \iota_X)$  yields a unique  $\Sigma$ -algebra morphism

$$\overline{h}_0 \colon (\Sigma^* X, \iota_X) \to (A, a_0) \qquad \text{such that} \qquad \overline{h}_0 \circ \eta_X = h_0,$$

and similarly the universal property of  $(\Sigma^* R, \iota_R)$  yields a unique  $\Sigma$ -algebra morphism

$$\overline{h}_1 \colon (\Sigma^* R, \iota_R) \to (S, a_1)$$
 such that  $\overline{h}_1 \circ \eta_R = h_1.$ 

This implies that the  $\overline{\Sigma}$ -algebra morphism

$$\overline{h} = (\overline{h}_0, \overline{h}_1) \colon (\Sigma^{\star}(X, R), (i_X, i_R)) \to ((A, S), a)$$

satisfies  $\overline{h} \circ (\eta_X, \eta_R) = h$ , and it is clearly unique with that property. (2) Now consider the relation lifting  $\widetilde{\Sigma} = \overline{\Sigma}_{\mathbf{Rel}}$ . It suffices to show that a free  $\widetilde{\Sigma}$ -algebra on  $(X, R) \in \mathbf{Rel}(\mathbb{C})$  is given by

$$\widetilde{\Sigma}\widetilde{\Sigma^{\star}}(X,R) \xrightarrow{(\iota_X,\iota_R)^{\dagger}} \widetilde{\Sigma^{\star}}(X,R)$$

with the universal morphism

$$(X,R) \xrightarrow{(\eta_X,\eta_R)^{\dagger}} \widetilde{\Sigma^{\star}}(X,R)$$

But this is immediate from part (1) above and Corollary B.4.

We prove a more refined result:

**Lemma B.5.** Suppose that the functor  $\overline{F}$  satisfies the following conditions for all  $X \in \mathbb{C}$  and  $(X, R), (X, S) \in \operatorname{Rel}_X(\mathbb{C})$ : (1) the relation  $\overline{F}(X, X)$  is reflexive;

(2)  $\overline{F}(X,R) \bullet \overline{F}(X,S) < \overline{F}((X,R) \bullet (X,S)).$ 

Then for every coalgebra  $c: C \to FC$ :

(a) If  $(C, R_i)$ ,  $i \in I$ , are simulations on (C, c), then the coproduct  $\bigvee_{i \in I} (C, R_i)$  in  $\operatorname{Rel}_C(\mathbb{C})$  is a simulation.

- (b) The identity relation (C, C) is a simulation on (C, c).
- (c) If (C, R) and (C, S) are simulations on (C, c), then the composite  $(C, R) \bullet (C, S)$  is a simulation.
- (d) The similarity relation on (C, c) exists, and it is reflexive and transitive.

*Proof.* (a) Let  $(C, R) = \bigvee_i (C, R_i)$  and denote by  $in_i : (C, R_i) \to (C, R)$  the coproduct injection. We first compute

$$c_{\star}(\coprod_{i}(C,R_{i})) = \coprod_{i} c_{\star}(C,R_{i}) \leq \coprod_{i} \overline{F}(C,R_{i}) \leq \overline{F}(C,R)$$

where the first  $\coprod$  refers to the coproduct in  $\operatorname{Gra}_C(\mathbb{C})$  and the other two to the coproduct in  $\operatorname{Gra}_{FC}(\mathbb{C})$ . The first step follows from the definition of  $\coprod$  and  $c_*$ , the second step uses that  $(C, R_i)$  is a simulation, and the third step is witnessed by the morphism  $[\overline{F}(\operatorname{in}_i)]_i$ . It follows that

$$c_{\star}(C,R) = c_{\star}((\coprod_{i}(C,R_{i}))^{\dagger}) \leq (c_{\star}(\coprod_{i}(C,R_{i})))^{\dagger} \leq \overline{F}(C,R)^{\dagger} = \overline{F}(C,R).$$

The first step uses that  $(C, R) = \bigvee_i (C, R_i)$  and the definition of  $\bigvee$ , the second one follows from (A.10), the third one from the computation above, and the last one since  $\overline{F}(C, R)$  is a relation.

(b) This follows from the computation

$$c_{\star}(C,C) \le (FC,FC) \le \overline{F}(C,C)$$

where the first step is witnessed by the  $\operatorname{Gra}_{FC}(\mathbb{C})$ -morphism  $(\operatorname{id}, c): c_{\star}(C, C) \to (FC, FC)$  and the second one uses that  $\overline{F}(C, C)$  is reflexive by condition (1).

(c) This follows from the computation

$$c_{\star}((C,R) \bullet (C,S)) \le c_{\star}(C,R) \bullet c_{\star}(C,S) \le \overline{F}(C,R) \bullet \overline{F}(C,S) \le \overline{F}((C,R) \bullet (C,S)).$$

The first step follows from (A.7), the second step uses that  $c_{\star}(C, R) \leq \overline{F}(C, R)$  and  $c_{\star}(C, S) \leq \overline{F}(C, S)$  because (C, R) and (C, S) are simulations and that  $\bullet$  is functorial, and the third step uses condition (2).

(d) Since the category  $\mathbb{C}$  is well-powered (Assumptions IV.1), the collection  $\{(C, R_i) : i \in I\}$  of all simulations, taken up to isomorphism in  $\operatorname{Rel}_C(\mathbb{C})$ , forms a small set. Thus the greatest simulation is given by the coproduct  $(C, R) = \bigvee_i (C, R_i)$ , see part (a). It is reflexive because (C, C) is a simulation by part (b) and thus  $(C, C) \leq (C, R)$  because (C, R) is the greatest simulation. It is transitive because  $(C, R) \bullet (C, R)$  is a simulation by part (c) and thus  $(C, R) \bullet (C, R) \leq (C, R)$ .

Remark B.6. By (A.10) the congruence property in Definition VII.3 is equivalent to

$$a_{\star}\overline{\Sigma}_{\mathbf{Gra}}(A,R) \le (A,R),$$

that is, existence of a morphism  $a_R \colon \Sigma R \to R$  such that the diagram below commutes.

$$\begin{array}{c} \Sigma A \xleftarrow{\Sigma \text{outl}_R} \Sigma R \xrightarrow{\Sigma \text{outr}_R} \Sigma A \\ a \downarrow \qquad \qquad \downarrow a_R \qquad \qquad \downarrow a \\ A \xleftarrow{\text{outl}_R} R \xrightarrow{\text{outr}_R} A \end{array}$$

#### PROOF OF LEMMA VII.4

(1) Suppose that (X, R) is reflexive. Then  $(X, \hat{R})$  is reflexive because

$$(X,X) \le (X,R) \le (X,R)$$

The first step uses that (X, R) is reflexive and the second step is witnessed by  $\alpha_{R,\xi} \cdot \text{inl}$ , see (VII.1). Similarly,  $(X, \hat{R})$  is a congruence on  $(X, \xi)$  because

$$\xi_{\star}\overline{\Sigma}(X,\widehat{R}) = (\xi_{\star}\overline{\Sigma}(X,\widehat{R})) \bullet (X,X) \le (\xi_{\star}\overline{\Sigma}(X,\widehat{R})) \bullet (X,R) \le (X,\widehat{R}).$$

The first step follows from the definition of relation composition, the second step uses that (X, R) is reflexive and that relation composition is functorial, and the third step is witnessed by  $\alpha_{R,\xi}$  · inr.

(2) Suppose that (X, R) is transitive. Putting  $\lor = \lor_X$ , we obtain

$$\begin{split} (X,R) \bullet (X,R) &\cong ((X,R) \lor (\xi_{\star} \Sigma(X,R)) \bullet (X,R)) \bullet (X,R) \\ &\cong (X,R) \bullet (X,R) \lor (\xi_{\star} \overline{\Sigma}(X,\widehat{R})) \bullet (X,R) \bullet (X,R) \\ &\leq (X,R) \lor (\xi_{\star} \overline{\Sigma}(X,\widehat{R})) \bullet (X,R) \\ &\cong (X,\widehat{R}). \end{split}$$

The first and the last step are witnessed by the isomorphism (VII.1), the second one uses Lemma A.5, and the third one uses that  $(X, R) \bullet (X, R) \le (X, R)$  by transitivity of (X, R) and that  $\lor$  and  $\bullet$  are monotone.

#### **PROOF OF PROPOSITION VIII.5**

The proof of the proposition requires some notation:

Notation B.7. (1) For each  $(X, R), (Y, S) \in \mathbf{Rel}(\mathbb{C})$  we denote the relation  $\overline{B}((X, R), (Y, S))$  by

$$E_{R,S}$$
outl $_{R,S}$   $\int$  outr $_{R,S}$   $B(X,Y)$ 

Thus the domain and codomain of the  $\mathbf{Rel}(\mathbb{C})$ -morphism

$$\overline{\varrho}_{(X,R),(Y,S)} \colon \overline{\Sigma}((X,R) \times \overline{B}((X,R),(Y,S))) \to \overline{B}((X,R),\overline{\Sigma}^{\star}((X,R)+(Y,S)))$$

are the relations

$$\overline{\Sigma}((X,R) \times \overline{B}((X,R),(Y,S))) = (\Sigma(X \times B(X,Y)), (\Sigma(R \times E_{R,S}))^{\dagger})$$

and

$$\overline{B}((X,R),\overline{\Sigma}^{\star}((X,R)+(Y,S))) = (B(X,\Sigma^{\star}(X+Y)),E_{R,(\Sigma^{\star}(R\vee S))^{\dagger}})$$

We put

$$t_{R,S} = \left(\Sigma(R \times E_{R,S}) \xrightarrow{e_{\Sigma(R \times E_{R,S})}} \left(\Sigma(R \times E_{R,S})\right)^{\dagger} \xrightarrow{(\varrho_{(X,R),(Y,S)})_{1}} E_{R,(\Sigma^{\star}(R \vee S))^{\dagger}}\right)$$

Then we have the following commutative diagram, where out  $\in \{\text{outl}, \text{outr}\}$ ; the lower right-hand part commutes because  $\overline{\varrho}_{(X,R),(Y,S)}$  is a  $\text{Rel}(\mathbb{C})$ -morphism and  $(\overline{\varrho}_{(X,R),(Y,S)})_0 = \varrho_{X,Y}$ .

$$\Sigma(R \times E_{R,S}) \xrightarrow{e_{\Sigma(R \times E_{R,S})}} (\Sigma(R \times E_{R,S}))^{\dagger} \xrightarrow{(\overline{\varrho}_{(X,R),(Y,S)})_1} E_{R,(\Sigma^*(R \vee S))^{\dagger}} \xrightarrow{(\mathbb{B}(1))} \Sigma(\operatorname{out}_{R \times \operatorname{out}_{R,S}}) \xrightarrow{\operatorname{out}_{(\Sigma(R \times E_{R,S}))^{\dagger}}} \xrightarrow{\varrho_{X,Y}} B(X, \Sigma^*(X + Y)) \xrightarrow{(\mathbb{B}(1))} E_{R,(\Sigma^*(R \vee S))^{\dagger}} \xrightarrow{(\mathbb{B}(1))} E_{R,(\Sigma^*(R \vee S))^{\dagger}} \xrightarrow{(\mathbb{B}(1))} \Sigma(X \times B(X,Y)) \xrightarrow{(\mathbb{B}(1))} E_{X,Y} \xrightarrow{(\mathbb{B}(1))} \xrightarrow{(\mathbb{B}(1))} E_{X,Y} \xrightarrow{(\mathbb{B}(1))} E_{X,Y} \xrightarrow{(\mathbb{B}(1))} E_{X,Y} \xrightarrow{(\mathbb{B}(1))} \xrightarrow{(\mathbb{B}(1))} E_{X,Y} \xrightarrow{(\mathbb{B}(1))} E_{X,Y} \xrightarrow{(\mathbb{B}(1))} \xrightarrow{(\mathbb{B}(1))} \xrightarrow{(\mathbb{B}(1))} E_{X,Y} \xrightarrow{(\mathbb{B}(1))} \xrightarrow{(\mathbb{B}$$

(2) Since  $(\mu\Sigma, \hat{R})$  is a congruence on  $(\mu\Sigma, \iota)$  by Lemma VII.4, there exists a  $\mathbb{C}$ -morphism  $\iota_{\hat{R}}$  such that diagram below commutes, cf. Remark B.6.

$$\Sigma(\mu\Sigma) \stackrel{\Sigma \text{outl}_{\widehat{R}}}{\longleftarrow} \Sigma\widehat{R} \xrightarrow{\Sigma \text{outr}_{\widehat{R}}} \Sigma(\mu\Sigma)$$

$$\downarrow \qquad \qquad \downarrow^{\iota_{\widehat{R}}} \qquad \qquad \downarrow^{\iota}$$

$$\mu\Sigma \stackrel{\text{outl}_{\widehat{R}}}{\longleftarrow} \widehat{R} \xrightarrow{\text{outr}_{\widehat{R}}} \mu\Sigma$$
(B.2)

(3) Since  $(\mu\Sigma, R)$  is a weak simulation on  $(\mu\Sigma, \gamma)$ , there exist  $\mathbb{C}$ -morphisms  $\tilde{\gamma}_R$  and  $\tilde{\tilde{\gamma}}_R$  such that the following diagrams commute, cf. Definition VI.4.

Proof of Proposition VIII.5. Suppose that  $(\mu\Sigma, R)$  is a reflexive and transitive weak simulation on  $(\mu\Sigma, \gamma)$ . Our task is to show that the Howe closure  $(\mu\Sigma, \hat{R})$  is a weak simulation on  $(\mu\Sigma, \gamma)$ . To this end, form the relation  $(\mu\Sigma, P) \in \mathbf{Rel}_{\mu\Sigma}(\mathbb{C})$  via the pullback below; it is a relation because monomorphisms are stable under pullbacks.

$$P \xrightarrow{p} E_{\widehat{R},\widehat{R}} \\ \langle \mathsf{outl}_{P},\mathsf{outr}_{P} \rangle \downarrow \downarrow \downarrow \langle \mathsf{outl}_{\widehat{R},\widehat{R}},\mathsf{outr}_{\widehat{R},\widehat{R}} \rangle \\ \mu\Sigma \times \mu\Sigma \xrightarrow{\langle \gamma,\widetilde{\gamma} \rangle} B(\mu\Sigma,\mu\Sigma) \times B(\mu\Sigma,\mu\Sigma)$$
(B.4)

The key to the proof is showing the existence of  $\mathbf{Rel}_{\mu\Sigma}(\mathbb{C})$ -morphisms

$$\beta^{\mathsf{l}} \colon (\mu\Sigma, R) \to (\mu\Sigma, P) \quad \text{and} \quad \beta^{\mathsf{r}} \colon \iota_{\star}\overline{\Sigma}((\mu\Sigma, \widehat{R}) \times_{\mu\Sigma} (\mu\Sigma, P)) \bullet (\mu\Sigma, R) \to (\mu\Sigma, P)$$
(B.5)

where  $\times_{\mu\Sigma}$  denotes the product in  $\operatorname{\mathbf{Rel}}_{\mu\Sigma}(\mathbb{C})$ , see Section IV-A3. Once this is achieved, we can conclude the proof as follows. By copairing  $\beta^{\mathsf{I}}$  and  $\beta^{\mathsf{r}}$  we obtain the  $\operatorname{\mathbf{Rel}}_{\mu\Sigma}(\mathbb{C})$ -morphism

$$\beta = [\beta^{\mathsf{l}}, \beta^{\mathsf{r}}] \colon \overline{\Sigma}_{R,\iota} ((\mu\Sigma, \widehat{R}) \times_{\mu\Sigma} (\mu\Sigma, P)) \to (\mu\Sigma, P)$$

and thus primitive recursion (II.1) yields the  $\mathbf{Rel}_{\mu\Sigma}(\mathbb{C})$ -morphism

$$\operatorname{pr}\beta\colon\mu\overline{\Sigma}_{R,\iota}=(\mu\Sigma,\widehat{R})\to(\mu\Sigma,P)$$

Choose a witness  $r: (\mu\Sigma, \mu\Sigma) \to (\mu\Sigma, \widehat{R})$  that  $(\mu\Sigma, \widehat{R})$  is reflexive, see Lemma VII.4. Then the following commutative diagram proves  $(\mu\Sigma, \widehat{R})$  to be a weak simulation on  $(\mu\Sigma, \gamma)$ :



It only remains to define the  $\operatorname{Rel}_{\mu\Sigma}(\mathbb{C})$ -morphisms  $\beta^{l}$  and  $\beta^{r}$  in (B.5). Their construction imitates the arguments for the induction base and induction step, resp., in the proof of Theorem III.8.

**Construction of**  $\beta^{l}$ :  $(\mu\Sigma, R) \rightarrow (\mu\Sigma, P)$ :

(1) We define the morphism  $f: \mu \Sigma \to E_{\widehat{R},\widehat{R}}$  via primitive recursion as follows:

Here  $t_{\widehat{R},\widehat{R}}$  is defined in Notation B.7, and  $(\iota,\iota_{\widehat{R}})^{\dagger}:\overline{\Sigma}^{\star}(\mu\Sigma,\widehat{R}) \to (\mu\Sigma,\widehat{R})$  is the  $\overline{\Sigma}^{\star}$ -algebra corresponding to the  $\overline{\Sigma}$ -algebra  $(\iota,\iota_{\widehat{R}})^{\dagger}:\overline{\Sigma}(\mu\Sigma,\widehat{R}) \to (\mu\Sigma,\widehat{R})$ , cf. Proposition V.4 and its proof.

(2) The morphism  $f: \mu \Sigma \to E_{\widehat{R},\widehat{R}}$  satisfies

$$\gamma = \operatorname{outl}_{\widehat{R},\widehat{R}} \cdot f \quad \text{and} \quad \gamma = \operatorname{outr}_{\widehat{R},\widehat{R}} \cdot f.$$
 (B.7)

By symmetry it suffices to prove the first equation, which follows from the observation that the morphism  $\operatorname{out}_{\widehat{R},\widehat{R}} \cdot f$  satisfies the commutative diagram (II.3) defining  $\gamma$ :

(3) Now consider the composite graph  $\overline{B}((\mu\Sigma, \widehat{R}), (\mu\Sigma, \widehat{R})); \overline{B}((\mu\Sigma, \mu\Sigma), (\mu\Sigma, R))$ . The universal property of the corresponding pullback  $E_{\widehat{R},\widehat{R}}; E_{\mu\Sigma,R}$  yields a unique morphism g making the diagram below commute, where  $\pi$  and  $\pi'$  are the left and right projection of the pullback:

Indeed, by definition  $E_{\widehat{R},\widehat{R}}$ ;  $E_{\mu\Sigma,R}$  is the pullback of  $\operatorname{outr}_{\widehat{R},\widehat{R}}$  and  $\operatorname{outl}_{\mu\Sigma,R}$ , and the commutative diagram below shows that these two morphisms merge  $f \cdot \operatorname{outl}_R$  and  $\widetilde{\gamma}_R$ .



(4) Next, we observe that there exists a  $\mathbf{Gra}_{B(\mu\Sigma,\mu\Sigma)}(\mathbb{C})$ -morphism

$$k \colon \overline{B}\big((\mu\Sigma, \widehat{R}), (\mu\Sigma, \widehat{R})\big) ; \overline{B}\big((\mu\Sigma, \mu\Sigma), (\mu\Sigma, R)\big) \to \overline{B}\big((\mu\Sigma, \widehat{R}), (\mu\Sigma, \widehat{R})\big).$$
(B.9)

This is shown by the following computation:

$$\overline{B}((\mu\Sigma, \widehat{R}), (\mu\Sigma, \widehat{R})); \overline{B}((\mu\Sigma, \mu\Sigma), (\mu\Sigma, R))$$

$$\leq \overline{B}((\mu\Sigma, \widehat{R}), (\mu\Sigma, \widehat{R})) \bullet \overline{B}((\mu\Sigma, \mu\Sigma), (\mu\Sigma, R))$$

$$\leq \overline{B}((\mu\Sigma, \widehat{R}), (\mu\Sigma, \widehat{R}) \bullet (\mu\Sigma, R))$$

$$\leq \overline{B}((\mu\Sigma, \widehat{R}), (\mu\Sigma, \widehat{R})).$$

The first step holds by definition of relation composition •; the second step follows from (G3); the third step uses that  $(\mu\Sigma, \hat{R})$  is weakly transitive by Lemma VII.4.

(5) Finally, we have the following commutative diagram:



The universal property of the pullback (B.4) yields a unique  $\beta_1^l : R \to P$  such that  $\operatorname{outl}_R = \operatorname{outl}_P \cdot \beta_1^l$  and  $\operatorname{outr}_R = \operatorname{outr}_P \cdot \beta_1^l$ and  $k_1 \cdot g = p \cdot \beta_1^l$ . By the first two equations, we thus obtain the  $\operatorname{\mathbf{Rel}}_{\mu\Sigma}(\mathbb{C})$ -morphism

$$\beta^{\mathsf{I}} = (\mathsf{id}, \beta_1^{\mathsf{I}}) \colon (\mu \Sigma, R) \to (\mu \Sigma, P).$$

Construction of  $\beta^{\mathsf{r}} \colon \iota_{\star}\overline{\Sigma}((\mu\Sigma,\widehat{R}) \times_{\mu\Sigma} (\mu\Sigma,P)) \bullet (\mu\Sigma,R) \to (\mu\Sigma,P)$ :

(1) Recall that the product  $(\mu\Sigma, \widehat{R}) \times_{\mu\Sigma} (\mu\Sigma, P)$  is the relation  $(\mu\Sigma, Q)$  where Q is the pullback

$$\begin{array}{cccc}
Q & \xrightarrow{q_P} & P \\
 & & \downarrow & & \downarrow & \langle \mathsf{outl}_P, \mathsf{outr}_P \rangle \\
 & & \widehat{R} & \xrightarrow{\langle \mathsf{outl}_{\widehat{R}}, \mathsf{outr}_{\widehat{R}} \rangle} & \mu\Sigma \times \mu\Sigma \\
\end{array} \tag{B.10}$$

and

$$\operatorname{\mathsf{outl}}_Q = \operatorname{\mathsf{outl}}_{\widehat{R}} \cdot q_{\widehat{R}}, \quad \text{and} \quad \operatorname{\mathsf{outr}}_Q = \operatorname{\mathsf{outr}}_{\widehat{R}} \cdot q_{\widehat{R}}.$$
 (B.11)

(2) Consider the  $\mathbb{C}$ -morphism  $h \colon \Sigma Q \to E_{\widehat{R},\widehat{R}}$  defined as the composite

$$h \ = \ \big( \Sigma Q \overset{\Sigma\langle q_{\widehat{R}}, q_{P} \rangle}{\longrightarrow} \Sigma(\widehat{R} \times P) \overset{\Sigma(\mathsf{id} \times p)}{\longrightarrow} \Sigma(\widehat{R} \times E_{\widehat{R}, \widehat{R}}) \xrightarrow{t_{\widehat{R}, \widehat{R}}} E_{\widehat{R}, (\Sigma^{\star}(\widehat{R} \vee \widehat{R}))^{\dagger}} \overset{\overline{B}(\mathsf{id}, (\overline{\lambda^{\star}}, \overline{\lambda^{\star}})^{\dagger})_{1}}{\longrightarrow} E_{\widehat{R}, (\Sigma^{\star}(\widehat{R} \vee \widehat{R}))^{\dagger}} \overset{\overline{B}(\mathsf{id}, (\overline{\lambda^{\star}}, \overline{\lambda^{\star}})^{\dagger})_{1}}{\longrightarrow} E_{\widehat{R}, \widehat{R}} \big).$$

We claim that the diagram below commutes laxly, where  $\operatorname{outl}_{\Sigma Q} = \iota \cdot \Sigma \operatorname{outl}_{Q}$  and  $\operatorname{outr}_{\Sigma Q} = \iota \cdot \Sigma \operatorname{outr}_{Q}$  are the projections of the relation  $\iota_{\star} \overline{\Sigma}(\mu \Sigma, Q)$ :

$$\begin{array}{cccc}
\mu\Sigma & \stackrel{\operatorname{outl}_{\Sigma Q}}{\longleftarrow} & \Sigma Q & \stackrel{\operatorname{outr}_{\Sigma Q}}{\longrightarrow} & \mu\Sigma \\
\gamma & & & & \downarrow_{h} & \stackrel{\searrow}{\searrow} & & \downarrow_{\widetilde{\gamma}} \\
B(\mu\Sigma, \mu\Sigma) & \stackrel{\operatorname{outl}_{\widehat{R},\widehat{R}}}{\longleftarrow} & E_{\widehat{R},\widehat{R}} & \stackrel{\operatorname{outr}_{\widehat{R},\widehat{R}}}{\longrightarrow} & B(\mu\Sigma, \mu\Sigma)
\end{array}$$
(B.12)

This is proven by the two diagrams below. In the part (\*) of the second diagram, we make use of our assumption that  $(\mu\Sigma, \iota, \tilde{\gamma})$  is a lax  $\rho$ -bialgebra.



From (B.12) and (G1), it follows that there exists a  $\mathbb{C}$ -morphism  $l: \Sigma Q \to E_{\widehat{R},\widehat{R}}$  such that the following diagram commutes:

(3) Again consider the composite graph  $\overline{B}((\mu\Sigma, \widehat{R}), (\mu\Sigma, \widehat{R}))$ ;  $\overline{B}((\mu\Sigma, \mu\Sigma), (\mu\Sigma, R))$ . The universal property of the corresponding pullback  $E_{\widehat{R},\widehat{R}}$ ;  $E_{\mu\Sigma,R}$  yields a unique morphism m making the diagram below commute, where the upper row refers to the composite graph  $\iota_{\star}\overline{\Sigma}(\mu\Sigma, Q)$ ;  $(\mu\Sigma, R)$ .

$$\Sigma Q \xleftarrow{\pi_{\Sigma Q;R,\Sigma Q}}{\Sigma Q; R \xrightarrow{\pi_{\Sigma Q;R,R}}} R \\
\iota \downarrow \qquad \qquad \downarrow m \qquad \qquad \downarrow \tilde{\gamma}_{R} \\
E_{\widehat{R},\widehat{R}} \xleftarrow{\pi}{E_{\widehat{R},\widehat{R}}; E_{\mu\Sigma,R} \xrightarrow{\pi'}{E_{\mu\Sigma,R}} E_{\mu\Sigma,R} } (B.14)$$

Indeed, by definition  $E_{\widehat{R},\widehat{R}}$ ;  $E_{\mu\Sigma,R}$  is the pullback of  $\operatorname{outr}_{\widehat{R},\widehat{R}}$  and  $\operatorname{outl}_{\mu\Sigma,R}$ , and the diagram below shows that these two morphisms merge  $l \cdot \pi_{\Sigma Q;R,\Sigma Q}$  and  $\tilde{\widetilde{\gamma}}_R \cdot \pi_{\Sigma Q;R,R}$ .

$$\begin{array}{c|c} \Sigma Q ; R \xrightarrow{\pi_{\Sigma Q; R, \Sigma Q}} \Sigma Q & \xrightarrow{l} E_{\widehat{R}, \widehat{R}} \\ & & \downarrow^{\operatorname{outr}_{\Sigma Q}} \\ & & \mu \Sigma & (B.13) \\ & & \mu \Sigma & (B.13) \\ & & \downarrow^{\operatorname{outl}_{R}} & (B.3) \\ & & & \tilde{\gamma} \\ & & & R \xrightarrow{\tilde{\gamma}_{R}} E_{\mu \Sigma, R} \xrightarrow{\operatorname{outl}_{\mu \Sigma, R}} B(\mu \Sigma, \mu \Sigma) \end{array}$$

(4) Finally, we have the commutative diagram below:

The universal property of the pullback (B.4) yields a unique  $\gamma_1^r \colon \Sigma Q ; R \to P$  such that  $\operatorname{outl}_{\Sigma Q} \cdot \pi_{\Sigma Q;R,\Sigma Q} = \operatorname{outl}_P \cdot \gamma_1^r$  and  $\operatorname{outr}_R \cdot \pi_{\Sigma Q;R,R} = \operatorname{outr}_P \cdot \gamma_1^r$  and  $k_1 \cdot m = p \cdot \gamma_1^r$ . Thus

$$\gamma^{\mathsf{r}} = (\mathsf{id}, \gamma_1^{\mathsf{r}}) \colon \iota_{\star} \overline{\Sigma}((\mu\Sigma, \widehat{R}) \times_{\mu\Sigma} (\mu\Sigma, P)) ; (\mu\Sigma, R) \to (\mu\Sigma, P)$$

is a  $\mathbf{Gra}_{\mu\Sigma}(\mathbb{C})\text{-morphism},$  which yields the  $\mathbf{Rel}_{\mu\Sigma}(\mathbb{C})\text{-morphism}$ 

$$\beta^{\mathsf{r}} = (\gamma^{\mathsf{r}})^{\dagger} \colon \iota_{\star} \overline{\Sigma}((\mu\Sigma, \widehat{R}) \times_{\mu\Sigma} (\mu\Sigma, P)) \bullet (\mu\Sigma, R) \to (\mu\Sigma, P).$$

This concludes the proof.

#### PROOF DETAILS FOR SECTION IX-A

We will prove that the given data satisfy the Assumptions VIII.1:

- (1) The functor  $\Sigma$  preserves strong epimorphisms.
- (2) The functor  $\overline{B}$  is good for simulations.
- (3) The higher-order GSOS law  $\overline{\varrho}$  admits a relation lifting.

*Proof.* (1) Every set functor preserves epimorphisms (= surjections) since the latter are precisely the right-invertible maps.

(2) By definition, we have  $\overline{B} = \overline{P} \cdot \overline{B}_0$  where  $\overline{P}$  and  $\overline{B}_0$  are the relation liftings of Remark III.3. Condition (G1) holds by Example IV.6. For (G2) we first observe that for each  $X \in \mathbf{Set}$  the relation  $\overline{\mathcal{P}}(X, X)$  is given by  $(\mathcal{P}X, \subseteq)$ , hence it is reflexive. The following computation then proves  $\overline{B}((X, X), (Y, Y))$  to be reflexive:

$$(B(X,Y)), B(X,Y)) = (\mathcal{P}B_0(X,Y), \mathcal{P}B_0(X,Y))$$
  

$$\leq \overline{\mathcal{P}}(B_0(X,Y), B_0(X,Y))$$
  

$$\cong \overline{\mathcal{P}} \overline{B}_0((X,X), (Y,Y))$$
  

$$= \overline{B}((X,X), (Y,Y)).$$

In the second step we use the above observation about  $\overline{P}$ . The third step follows from Proposition C.8, using the fact that  $\overline{B}_0$  is the canonical lifting of  $B_0$ . For (G3) note that

$$\overline{\mathcal{P}}(X,R) \bullet \overline{\mathcal{P}}(X,S) \le \overline{\mathcal{P}}((X,R) \bullet (X,S)) \qquad \text{for every } (X,R), (X,S) \in \mathbf{Rel}_X(\mathbb{C}), \tag{B.15}$$

which is immediate from the definition of the Egli-Milner relation. Thus

$$\overline{B}((X,R),(Y,S)) \bullet \overline{B}((X,X),(Y,S')) = \overline{\mathcal{P}}(\overline{B}_0((X,R),(Y,S))) \bullet \overline{\mathcal{P}}(\overline{B}_0((X,X),(Y,S'))) \\
\leq \overline{\mathcal{P}}(\overline{B}_0((X,R),(Y,S)) \bullet \overline{B}_0((X,X),(Y,S'))) \\
\leq \overline{\mathcal{P}}(\overline{B}_0((X,R),(Y,S) \bullet (Y,S'))) \\
= \overline{B}((X,R),(Y,S) \bullet (Y,S')).$$

The second step uses (B.15). The third step follows from Proposition C.9 and the fact that  $\overline{B}_0$  is the canonical relation lifting of  $B_0$ , see Section C.

(3) By definition of  $\rho$  as the composite (III.5), it suffices to show that for all relations (X, R) and (Y, S) the maps

$$st_{X,Y} \colon X \times \mathcal{P}Y \to \mathcal{P}(X \times Y), \\ \delta_X \colon \Sigma \mathcal{P}X \to \mathcal{P}\Sigma X, \\ \varrho^0_{X,Y} \colon \Sigma(X \times B_0(X,Y)) \to B_0(X, \Sigma^*(X+Y)).$$

are relation-preserving with respect to the relations induced by the liftings  $\overline{\Sigma}$ ,  $\overline{\mathcal{P}}$ , and  $\overline{B}_0$ . For the first two maps this is clear by definition of the Egli-Milner relation, and for the third one it follows from Construction D.5, using again that  $\overline{B}_0$  is the canonical lifting of  $B_0$ 

#### APPENDIX C

#### CANONICAL LIFTINGS OF BIFUNCTORS

In this section we demonstrate how to construct canonical graph and relation liftings of mixed-variance bifunctors on C.

**Definition C.1.** Let  $B \colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$  be a bifunctor.

- (1) A graph lifting of B is a bifunctor  $\overline{B}$  on  $\mathbf{Gra}(\mathbb{C})$  making the first diagram commute.
- (2) A relation lifting of B is a bifunctor  $\overline{B}$  on  $\operatorname{Rel}(\mathbb{C})$  making the second diagram commute.

$$\begin{array}{ccc} \mathbf{Gra}(\mathbb{C})^{\mathsf{op}} \times \mathbf{Gra}(\mathbb{C}) & \overset{B}{\longrightarrow} \mathbf{Gra}(\mathbb{C}) & \mathbf{Rel}(\mathbb{C})^{\mathsf{op}} \times \mathbf{Rel}(\mathbb{C}) & \overset{B}{\longrightarrow} \mathbf{Rel}(\mathbb{C}) \\ |-|^{\mathsf{op}} \times |-| & & & & & & & \\ \mathbb{C}^{\mathsf{op}} \times \mathbb{C} & \overset{B}{\longrightarrow} \mathbb{C} & & & & & & \\ \end{array}$$

Recall the relation lifting of  $B_0(X, Y) = Y + Y^X$  in Remark III.3(1). Its construction can be generalized to the present categorical setting:

**Construction C.2.** For every bifunctor  $B: \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$  we construct a *canonical graph lifting* 

$$\overline{B} = \overline{B}_{\mathbf{Gra}} \colon \mathbf{Gra}(\mathbb{C})^{\mathsf{op}} \times \mathbf{Gra}(\mathbb{C}) \to \mathbf{Gra}(\mathbb{C}).$$

(1) Given  $(X, R), (Y, S) \in \mathbf{Gra}(\mathbb{C})$ , we define  $\overline{B}((X, R), (Y, S)) \in \mathbf{Gra}(\mathbb{C})$  to be the graph

į

$$T_{R,S}$$

$$\mathsf{outl}_{R,S} \bigcup \mathsf{outr}_{R,S}$$

$$B(X,Y)$$
(C.1)

where  $T_{R,S}$  is obtained by the pullback below and  $outl_{R,S}$  and  $outr_{R,S}$  are given by

$$\operatorname{outl}_{R,S} = \overline{p}_{R,S} \circ p_{R,S} \quad \text{and} \quad \operatorname{outr}_{R,S} = \overline{q}_{R,S} \circ q_{R,S}.$$

$$(C.2)$$

$$T_{R,S} \xrightarrow{\qquad} B(R,S) \xrightarrow{\qquad} q_{R,S}$$

$$\overline{q}_{R,S} \xrightarrow{\qquad} B(R,S) \xrightarrow{\qquad} p_{R,S} \xrightarrow{\qquad} B(\operatorname{id},\operatorname{outr}_{S}) \xrightarrow{\qquad} B(\operatorname{id},\operatorname{outr}_{S}) \xrightarrow{\qquad} F_{R,S} \xrightarrow{\qquad} B(R,Y) \xrightarrow{\qquad} C_{R,S} \xrightarrow{\qquad} C_{R,S}$$

$$(C.3)$$

$$\overline{T}_{R,S} \xrightarrow{\qquad} B(R,Y) \xrightarrow{\qquad} C_{R,S} \xrightarrow{\qquad} B(\operatorname{outr}_{R,\operatorname{id}}) \xrightarrow{\qquad} B(\operatorname{outr}_{R,\operatorname{id}}) \xrightarrow{\qquad} B(\operatorname{outr}_{R,\operatorname{id}}) \xrightarrow{\qquad} B(X,Y) \xrightarrow{\qquad} C_{R,S}$$

In addition, we put

$$b_{R,S} := \overleftarrow{q}_{R,S} \circ p_{R,S} = \overrightarrow{p}_{R,S} \circ q_{R,S}.$$
 (C.4)

(2) Given  $\mathbf{Gra}(\mathbb{C})$ -morphisms  $h: (X', R') \to (X, R)$  and  $k: (Y, S) \to (Y', S')$ , we define

$$\overline{B}(h,k)\colon \overline{B}((X,R),(Y,S))\to \overline{B}((X',R'),(Y',S')),$$

which is a pair of morphisms  $\overline{B}(h,k)_0$  and  $\overline{B}(h,k)_1$  making the following diagram commute:

$$T_{R,S} \xrightarrow{\overline{B}(h,k)_{1}} T_{R',S'}$$

$$\mathsf{outl}_{R,S} \left( \bigcup \mathsf{outr}_{R,S} \quad \mathsf{outl}_{R',S'} \left( \bigcup \bigcup \mathsf{outr}_{R',S'} \right) \mathsf{outr}_{R',S'} \mathsf{Outr}_{R$$

We put

$$\overline{B}(h,k)_0 = \left( B(X,Y) \xrightarrow{B(h_0,k_0)} B(X',Y') \right).$$

For the definition of  $\overline{B}(h,k)_1$ , we first use the universal property of the pullback  $\overline{T}_{R',S'}$  to obtain a unique morphism  $\overline{d}_{h,k}$  making the outside and the upper rectangle of the diagram below commute. Note that the central and the lower rectangle commute because h and k are  $\mathbf{Gra}(\mathbb{C})$ -morphisms and the left and right parts commute by (C.3)

$$\begin{array}{c|c} & T_{R,S} & \xrightarrow{\overleftarrow{d}_{h,k}} & \overleftarrow{T}_{R',S'} \\ & & & & \downarrow^{\overleftarrow{q}_{R',S'}} \\ & & & & \downarrow^{\overleftarrow{q}_{R',S'}} \\ (C.3) & B(R,S) & \xrightarrow{B(h_1,k_1)} & B(R',S') & (C.3) \\ & B(id,outl_S) \downarrow & (IV.1) & \downarrow^{B(id,outl_{S'})} \\ & B(R,Y) & \xrightarrow{B(h_1,k_0)} & B(R',Y') \\ & B(outl_R,id) \uparrow & (IV.1) & \uparrow^{B(outl_{R'},id)} \\ & & \to B(X,Y) & \xrightarrow{B(h_0,k_0)} & B(X',Y') \leftarrow \end{array}$$

Analogously, we obtain a unique morphism  $\vec{d}_{h,k} \colon T_{R,S} \to \vec{T}_{R',S'}$  such that

$$\vec{q}_{R',S'} \circ \vec{d}_{h,k} = B(h_0,k_0) \circ \mathsf{outr}_{R,S}$$
 and  $\vec{p}_{R',S'} \circ \vec{d}_{h,k} = B(h_1,k_1) \circ b_{R,S}.$  (C.7)

In particular, we have

$$\overleftarrow{q}_{R',S'} \circ \overleftarrow{d}_{h,k} = B(h_1,k_1) \circ b_{R,S} = \overrightarrow{p}_{R',S'} \circ \overrightarrow{d}_{h,k}$$

so the universal property of the pullback  $T_{R',S'}$  yields a unique morphism  $\overline{B}(h,k)_1$  making both triangles in the diagram below commute:

**Lemma C.3.** The assignment  $\overline{B}$  is a functor, and it forms a graph lifting of B.

**Remark C.4.** In the proof and later on, we will repeatedly make use of the following diagram which commutes for every  $h: (X, R') \to (X, R)$  and  $k: (Y, S) \to (Y, S')$ :

*Proof.* (1) We show that  $\overline{B}(h,k)$  defined in Construction C.2(2) is a  $\mathbf{Gra}(\mathbb{C})$ -morphism. In fact, the commutative diagram below shows that  $\overline{B}(h,k)$  is compatible with left projections; the argument for right projections is symmetric.

(2) We show that  $\overline{B}$  preserves identity morphisms, that is

$$B(\mathsf{id}_{(X,R)},\mathsf{id}_{(Y,S)}) = \mathsf{id}_{\overline{B}((X,R),(X,R))}$$

for all graphs (X, R) and (Y, S). In the following we omit the subscripts of id. By the uniqueness part of the definition of  $\overline{B}(id, id)_1$ , it suffices to prove that the following diagram commutes:

$$\begin{array}{c|c} & T_{R,S} \\ & & \downarrow_{\mathsf{id}} \\ & & \downarrow_{\mathsf{id}} \\ & & \uparrow_{R,S} \\ & & T_{R,S} \\ \end{array} \\ \end{array} \\ \begin{array}{c} & & \vec{T}_{R,S} \\ & & \vec{T}_{R,S} \\ \end{array} \\ \end{array}$$

The left-hand triangle commutes because it commutes when postcomposed with the pullback projections  $\bar{q}_{R,S}$  and  $\bar{p}_{R,S}$ , as shown by the two commutative diagrams below. The argument for the right-hand triangle is analogous.

(Above and subsequently, when we indicate the reason why a part of a diagram commutes unless it is easy to see without further reference.)

(3) We show that  $\overline{B}$  preserves composition, i.e.

$$\overline{B}(h \circ h', k' \circ k) = \overline{B}(h', k') \circ \overline{B}(h, k)$$

for all graph morphisms  $h: (X', R') \to (X, R), h': (X'', R'') \to (X', R'), k: (Y, S) \to (Y', S')$  and  $k': (Y', S') \to (Y'', S'')$ . By the uniqueness part of the definition of  $\overline{B}(h \circ h', k' \circ k)_1$ , it suffices to prove that the following diagram commutes:



The left-hand part commutes because it commutes when postcomposed with the pullback projections  $\bar{q}_{R'',S''}$  and  $\bar{p}_{R'',S''}$ , as shown by the two commutative diagrams below; in the second diagram, we use that  $\bar{B}(h,k)$  is a graph morphism, see Item (1). The argument for the right-hand part is analogous.



(4) The functor  $\overline{B}$  is a lifting of B, i.e. the first diagram in Definition V.5 commutes. Indeed,

$$|B((X,R),(Y,S))| = B(X,Y) = B(|(X,R)|,|(Y,S)|)$$

and

$$|\bar{B}(h,k)| = \bar{B}(h,k)_0 = B(h_0,k_0) = B(|h|,|k|).$$

**Example C.5** ( $\mathbb{C} = \mathbf{Set}$ ). Let us spell out the construction of the lifted bifunctor

$$\overline{B} = \overline{B}_{\mathbf{Gra}} \colon \mathbf{Gra}(\mathbf{Set})^{\mathsf{op}} \times \mathbf{Gra}(\mathbf{Set}) \to \mathbf{Gra}(\mathbf{Set})$$

for  $B: \mathbf{Set}^{\mathsf{op}} \times \mathbf{Set} \to \mathbf{Set}$ . Recall that in Set, the pullback of  $f_i: A_i \to B$  (i = 1, 2) is given by

$$P = \{ (a_1, a_2) \in A_1 \times A_2 : f_1(a_1) = f_2(a_2) \}$$

with the projections

$$p_i \colon P \to A_i, \qquad (a_1, a_2) \mapsto a_i.$$

Thus, we arrive at the following concrete description of  $\overline{B}$ :

(1) The pullback  $T_{R,S}$  in (C.3) is the set of all triples (f, z, g) such that  $f, g \in B(X, Y), z \in B(R, S)$ ,

$$B(R, \mathsf{outl}_S)(z) = B(\mathsf{outl}_R, Y)(f)$$
 and  $B(R, \mathsf{outr}_S)(z) = B(\mathsf{outr}_R, Y)(g)$ 

and  $\operatorname{outl}_{R,S}$  and  $\operatorname{outr}_{R,S}$  are the projections  $(f, z, g) \mapsto f$  and  $(f, z, g) \mapsto g$ , respectively.

(2) The components of the morphism  $\overline{B}(h,k)$  are given by

$$\overline{B}(h,k)_0 = B(h_0,k_0) \colon B(X,Y) \to B(X',Y')$$

and

$$\overline{B}(h,k)_1 \colon T_{R,S} \to T_{R',S'}, \qquad (f,z,g) \mapsto (B(h_0,k_0)(f), B(h_1,k_1)(z), B(h_0,k_0)(g))$$

**Construction C.6.** For every bifunctor  $B \colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$  the *canonical relation lifting*  $\overline{B}_{Rel}$  is defined via the following commutative diagram:

**Example C.7.** The canonical relation lifting of  $B_0(X, Y) = Y + Y^X$  on Set is the lifting  $\overline{B}_0$  described in Remark III.3(1).

The following two propositions show that the canonical liftings satisfy the properties (G2) and (G3) of Definition VIII.2: **Proposition C.8.** For every  $B \colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$  and  $X, Y \in \mathbb{C}$ ,

$$(B(X,Y),B(X,Y)) \cong \overline{B}_{\mathbf{Gra}}((X,X),(Y,Y)),$$
  
$$(B(X,Y),B(X,Y)) \cong \overline{B}_{\mathbf{Rel}}((X,X),(Y,Y)).$$

Proof. Immediate from the definitions.

**Proposition C.9.** If  $B: \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$  weakly preserves pullbacks in the second component, then

$$B_{\mathbf{Gra}}((X,R),(Y,S)); B_{\mathbf{Gra}}((X,R'),(Y,S')) \leq B_{\mathbf{Gra}}((X,R);(X,R'),(Y,S);(Y,S')),$$
(C.11)  
$$\overline{B}_{\mathbf{Rel}}((X,R),(Y,S)) \bullet \overline{B}_{\mathbf{Rel}}((X,X),(Y,S')) \leq \overline{B}_{\mathbf{Rel}}((X,R),(Y,S) \bullet (Y,S')),$$
(C.12)

where  $(X, R), (X, R'), (Y, S), (Y, S') \in \mathbf{Gra}(\mathbb{C})$  in (C.11) and  $(X, R), (Y, S), (Y, S') \in \mathbf{Rel}(\mathbb{C})$  in (C.12).

*Proof.* (1) Consider the first diagram in Figure 2, where the morphisms  $\pi, \pi'$  are the projections of the pullback  $T_{R,S}$ ;  $T_{R',S'}$ . Since the endofunctor  $B(RR', -): \mathbb{C} \to \mathbb{C}$  weakly preserves pullbacks by assumption, the part (\*) is a weak pullback square. Note that the outside and all remaining parts commute by definition. Thus, there exists a morphism  $f_{RR',SS'}$  making the two upper parts of the diagram commute.

(2) Next, consider the diagram in Figure 3. Note that the outside and all remaining parts commute by definition. By the universal property of the pullback  $\tilde{T}_{RR',SS'}$ , there exists a unique morphism  $\tilde{g}_{RR',SS'}$  making the top left and bottom left parts of the diagram commute. Symmetrically, we obtain a unique  $\vec{g}_{RR',SS'}: T_{R,S}T_{R',S'} \to \vec{T}_{RR',SS'}$  such that

$$\vec{p}_{RR,SS'} \circ \vec{g}_{RR',SS'} = f_{RR',SS'}$$
 and  $\vec{q}_{RR',SS'} \circ \vec{g}_{RR',SS'} = \mathsf{outr}_{R',S'} \circ \pi'$ 

(3) The universal property of the pullback  $T_{RR,SS'}$  now yields a unique  $g_{RR',SS'}$  making both triangles in the diagram below commute:

(4) To prove (C.11), we show that

$$\overline{B}_{\mathbf{Gra}}((X,R),(Y,S)); \overline{B}_{\mathbf{Gra}}((X,R'),(Y,S')) \xrightarrow{(\mathsf{id}_{B(X,Y)},g_{RR',SS'})} \overline{B}_{\mathbf{Gra}}((X,R);(X,R'),(Y,S);(Y,S'))$$



Fig. 2. First diagram for the proof of Proposition C.9



Fig. 3. Second diagram for the proof of Proposition C.9

is a  $\mathbf{Gra}(\mathbb{C})$ -morphism. To this end, consider the commutative diagram below, all whose parts commute by definition:



The morphism  $\operatorname{outl}_{R,S} \cdot \pi$  is the left projection of the graph  $\overline{B}_{\mathbf{Gra}}((X,R),(Y,S))$ ;  $\overline{B}_{\mathbf{Gra}}((X,R'),(Y,S'))$ , so the above commutative diagram shows that  $(\operatorname{id}, g_{RR',SS'})$  is compatible with left projections. The proof that  $(\operatorname{id}, g_{RR',SS'})$  is compatible with right projections is symmetric.

(5) Finally, (C.12) follows from the computation

$$\begin{aligned} B_{\mathbf{Rel}}((X,R),(Y,S)) \bullet B_{\mathbf{Rel}}((X,X),(Y,S')) &= B_{\mathbf{Gra}}((X,R),(Y,S))^{\dagger} \bullet B_{\mathbf{Gra}}((X,X),(Y,S'))^{\dagger} \\ &= (\overline{B}_{\mathbf{Gra}}((X,R),(Y,S)); \overline{B}_{\mathbf{Gra}}((X,X),(Y,S')))^{\dagger} \\ &\leq \overline{B}_{\mathbf{Gra}}((X,R);(X,X),(Y,S);(Y,S'))^{\dagger} \\ &\cong \overline{B}_{\mathbf{Gra}}((X,R),(Y,S);(Y,S'))^{\dagger} \\ &\leq \overline{B}_{\mathbf{Gra}}((X,R),(Y,S) \bullet (Y,S'))^{\dagger} \\ &= \overline{B}_{\mathbf{Rel}}((X,R),(Y,S) \bullet (Y,S')). \end{aligned}$$

The first step uses the definition of  $\overline{B}_{Rel}$ ; the second step follows from Lemma A.4; the third step uses (C.11); the fourth step uses that  $(X, R); (X, X) \cong (X, R)$  by definition of graph composition; the fifth step uses that  $(Y, S); (Y, S') \leq (Y, S) \bullet (Y, S')$  by definition of relation composition; the last step uses the definition of  $\overline{B}_{Rel}$ .

#### APPENDIX D CANONICAL LIFTINGS OF HIGHER-ORDER GSOS LAWS

Next, we show how to lift higher-order abstract GSOS laws from  $\mathbb{C}$  to  $\mathbf{Gra}(\mathbb{C})$  and  $\mathbf{Rel}(\mathbb{C})$ .

**Remark D.1.** The notion of relation lifting of a higher-order GSOS law is given in Definition V.6. *Graph liftings* are defined analogously: replace Rel by Gra and  $\lor$  by +. Note that unlike relation liftings, graphs liftings are usually not unique.

**Construction D.2.** Let  $\Sigma \colon \mathbb{C} \to \mathbb{C}$  and  $B \colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$  be functors with their canonical graph liftings

$$\overline{\Sigma} = \overline{\Sigma}_{\mathbf{Gra}} \colon \mathbf{Gra}(\mathbb{C}) \to \mathbf{Gra}(\mathbb{C}) \qquad \text{and} \qquad \overline{B} = \overline{B}_{\mathbf{Gra}} \colon \mathbf{Gra}(\mathbb{C})^{\mathsf{op}} \times \mathbf{Gra}(\mathbb{C}) \to \mathbf{Gra}(\mathbb{C})$$

given by Construction V.2 and Construction C.2, and let

$$\varrho_{X,Y} \colon \Sigma(X \times B(X,Y)) \to B(X, \Sigma^{\star}(X+Y)) \qquad ((X,p_X) \in V/\mathbb{C}, Y \in \mathbb{C})$$

be a V-pointed higher-order GSOS law of  $\Sigma$  over B. The canonical graph lifting of  $\rho$  is the (V, V)-pointed higher-order GSOS law  $\overline{\rho} = \overline{\rho}^{\mathbf{Gra}}$  of  $\overline{\Sigma}$  over  $\overline{B}$  whose components

are defined as follows. First, put

$$(\overline{\varrho}_{(X,R),(Y,S)})_0 \colon \left(\Sigma(X \times B(X,Y)) \xrightarrow{\varrho_{X,Y}} B(X,\Sigma^*(X+Y))\right)$$

Second, we define

$$(\overline{\varrho}_{(X,R),(Y,S)})_1 \colon \Sigma(R \times T_{R,S}) \to T_{R,\Sigma^*(R+S)}$$

in two steps via the universal properties of the pullbacks occurring in the construction of  $\overline{B}$ :

(1) Consider the diagram below, where we regard the objects X and R as V-pointed by the morphisms  $p_X = (p_{(X,R)})_0 \colon V \to X$  and  $p_R = (p_{(X,R)})_1 \colon V \to R$ .



Its outside commutes due to (C.3) and using  $\operatorname{out}_{R+S} = \operatorname{out}_R + \operatorname{out}_S$  (see Section IV-A3), and for the part marked (\*) we remove  $\Sigma$  and consider the product components separately: the left-hand one is the identity on R, and for the right-hand one we have the commutative diagram below:



The universal property of the pullback  $\overline{T}_{R,\Sigma^{\star}(R+S)}$  now yields a unique morphism  $(\overline{\varrho}_{(X,R),(Y,S)})_1$  making the top and bottom part of the diagram ab commute. Analogously, we obtain a unique morphism  $(\overline{\varrho}_{R,\Sigma^{\star}(R+S)})_1 \colon \Sigma(R \times T_{R,S}) \to \overline{T}_{R,\Sigma^{\star}(R+S)}$  such that

$$\vec{p}_{R,\Sigma^{\star}(R+S)} \circ (\vec{\varrho}_{R,\Sigma^{\star}(R+S)})_1 = \varrho_{R,S} \circ \Sigma(R \times b_{R,S}), \text{and} \vec{q}_{R,\Sigma^{\star}(R+S)} \circ (\vec{\varrho}_{R,\Sigma^{\star}(R+S)})_1 = \varrho_{X,Y} \circ \Sigma(\mathsf{outr}_R \times \mathsf{outr}_{R,S}).$$

(2) We take  $(\overline{\varrho}_{(X,R),(Y,S)})_1$  to be the unique morphism making both triangles in the diagram below commute, using the universal property of the pullback  $T_{R,\Sigma^*(R+S)}$ :

$$\begin{array}{c} \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\swarrow} & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\swarrow} & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\swarrow} \\ & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\swarrow} & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\checkmark} \\ & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\checkmark} & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\checkmark} \\ & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\checkmark} & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\checkmark} \\ & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\checkmark} & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{\checkmark} \\ & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{} & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{} \\ & \overset{(\tilde{\varrho}_{(X,R),(Y,S)})_{1}}{} & \overset{($$

**Proposition D.3.** The family  $\overline{\varrho}^{\mathbf{Gra}}$  is a (V, V)-pointed higher-order GSOS law of  $\overline{\Sigma}_{\mathbf{Gra}}$  over  $\overline{B}_{\mathbf{Gra}}$ . **Remark D.4.** We shall use in the proof below that the following diagram commutes:

$$\Sigma(R \times T_{R,S}) \xrightarrow{(\overline{\varrho}_{(X,R),(Y,S)})_1} T_{R,\Sigma^*(R+S)} \xrightarrow{(\mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

*Proof.* (1) We show that  $\overline{\varrho}_{(X,R),(Y,S)}$  is a  $\mathbf{Gra}(\mathbb{C})$ -morphism for every  $(X,R), p_{(X,R)}) \in (V,V)/\mathbf{Gra}(\mathbb{C})$  and  $(Y,S) \in \mathbf{Gra}(\mathbb{C})$ . In fact, that  $\overline{\varrho}_{(X,R),(Y,S)}$  is compatible with left projections is shown by the diagram below, all of whose parts commute by definition. The proof that  $\overline{\varrho}_{(X,R),(Y,S)}$  is compatible with right projections is symmetric.

$$\Sigma(R \times T_{R,S}) \xrightarrow{(\overline{\varrho}_{(X,R),(Y,S)})_{1}} T_{R,\Sigma^{*}(R+S)} \xrightarrow{T_{R,\Sigma^{*}(R+S)}} T_{R,\Sigma^{*}(R+S)} \xrightarrow{(\overline{\varrho}_{(X,R),(Y,S)})_{1}} (D.2) \xrightarrow{p_{R,\Sigma^{*}(R+S)}} outl_{R,\Sigma^{*}(R+S)} \xrightarrow{(\overline{\varrho}_{(X,R),(Y,S)})_{0} = \varrho_{X,Y}} D(L.2) \xrightarrow{p_{R,\Sigma^{*}(R+S)}} \xrightarrow{T_{R,\Sigma^{*}(R+S)}} (D.4) \xrightarrow{\overline{p}_{R,\Sigma^{*}(R+S)}} \sum(X \times B(X,Y)) \xrightarrow{(\overline{\varrho}_{(X,R),(Y,S)})_{0} = \varrho_{X,Y}} B(X,\Sigma^{*}(X+Y))$$

(2) To prove naturality, we need to show that for every  $(X, R), p_{(X,R)}) \in (V, V)/\mathbf{Gra}(\mathbb{C})$  and every  $\mathbf{Gra}(\mathbb{C})$ -morphism  $k \colon (Y, S) \to (Y', S')$  the diagram below commutes:

Commutativity in the  $(-)_0$ -component is clear because  $(\overline{\varrho}_{(X,R),(Y,S)})_0 = \varrho_{X,Y}$  and  $(\overline{\varrho}_{(X,R),(Y',S')})_0 = \varrho_{X,Y'}$  by definition and  $\varrho_{X,-}$  is natural. Commutativity in the  $(-)_1$ -component amounts to showing that the following rectangle commutes:

Indeed, the diagrams below show that (D.5) commutes when postcomposed with

$$\mathsf{outl}_{R,\Sigma^{\star}(R+S')} = \overleftarrow{p}_{R,\Sigma^{\star}(R+S')} \circ p_{R,\Sigma^{\star}(R+S')} \qquad \text{and} \qquad b_{R,\Sigma^{\star}(R+S')} = \overleftarrow{q}_{R,\Sigma^{\star}(R+S')} \circ p_{R,\Sigma^{\star}(R+S')}.$$





Since  $\bar{q}_{R,\Sigma^{\star}(R+S')}$  and  $\bar{p}_{R,\Sigma^{\star}(R+S')}$  are the projections of the pullback  $\bar{T}_{R,\Sigma^{\star}(R+S')}$  and thus jointly monic, it follows that (D.5) commutes when postcomposed with  $p_{R,\Sigma^{\star}(R+S')}$ . A symmetric argument shows that (D.5) commutes when postcomposed with  $q_{R,\Sigma^{\star}(R+S')}$ . Finally, since  $p_{R,\Sigma^{\star}(R+S')}$  and  $q_{R,\Sigma^{\star}(R+S')}$  are the projections of the pullback  $T_{R,\Sigma^{\star}(R+S')}$  and thus jointly monic, we conclude that (D.5) commutes.

(3) To prove dinaturality, we need to show that for every  $(Y, S) \in \mathbf{Gra}(\mathbb{C})$  and every  $(V, V)/\mathbf{Gra}(\mathbb{C})$ -morphism  $h: (X', R') \to (X, R)$  the diagram below commutes:

$$\begin{split} \overline{\Sigma}((X,R) \times \overline{B}((X,R),(Y,S))) & \xrightarrow{\mathcal{V}(X,R),(Y,S)} \overline{B}((X,R),\overline{\Sigma}^{\star}((X,R)+(Y,S))) \\ \overline{\Sigma}(h \times \overline{B}((X,R),(Y,S))) & & & & \\ \overline{\Sigma}((X',R') \times \overline{B}((X,R),(Y,S))) & & & & \\ \overline{\Sigma}((X',R') \times \overline{B}(h,(Y,S))) & & & & \\ \overline{\Sigma}((X',R') \times \overline{B}(h,(Y,S))) & & & & \\ \overline{\Sigma}((X',R') \times \overline{B}((X',R'),(Y,S))) & & & & \\ \overline{\Sigma}((X',R') \times \overline{B}((X',R'),(Y,S))) & & & & \\ \overline{\Sigma}((X',R') \times \overline{B}((X',R'),(Y,S))) & \xrightarrow{\mathcal{V}(X',R'),(Y,S)} \overline{B}((X',R'),\overline{\Sigma}^{\star}((X',R')+(Y,S))) \\ \end{array}$$

Commutativity in the  $(-)_0$ -component is clear because  $(\overline{\varrho}_{(X,R),(Y,S)})_0 = \varrho_{X,Y}$  and  $(\overline{\varrho}_{(X',R'),(Y,S)})_0 = \varrho_{X',Y}$  by definition and  $\varrho_{-,Y}$  is dinatural. Commutativity in the  $(-)_1$ -component amounts to showing that the following diagram commutes:

$$\begin{split} \Sigma(R \times T_{R,S}) & \xrightarrow{(\varrho_{(X,R),(Y,S)})_1} T_{R,\Sigma^*(R+S)} \\ \Sigma(h_1 \times T_{R,S}) & & \downarrow \overline{B}(h,\overline{\Sigma}^*((X,R)+(Y,S)))_1 \\ \Sigma(R' \times T_{R,S}) & & T_{R',\Sigma^*(R+S)} \\ \Sigma(R' \times \overline{B}(h,(Y,S))_1) & & \uparrow \overline{B}((X',R'),\overline{\Sigma}^*(h+(Y,S)))_1 \\ \Sigma(R' \times T_{R',S}) & \xrightarrow{(\varrho_{(X',R'),(Y,S)})_1} T_{R',\Sigma^*(R'+S)} \end{split}$$
(D.6)

The argument is similar to the one for naturality: The two diagrams below show that (D.6) commutes when postcomposed with

$$\mathsf{out}|_{R',\Sigma^*(R+S)} = \overleftarrow{p}_{R',\Sigma^*(R+S)} \circ p_{R',\Sigma^*(R+S)} \quad \text{and} \quad b_{R',\Sigma^*(R+S)} = \overleftarrow{q}_{R',\Sigma^*(R+S)} \circ p_{R',\Sigma^*(R+S)} \circ p_{R',\Sigma$$

Thus it commutes when postcomposed with  $p_{R',\Sigma^{\star}(R'+S)}$  (and analogously for  $q_{R',\Sigma^{\star}(R'+S)}$ ), and so it commutes.





Relation liftings of higher-order GSOS laws can be derived from their canonical graph liftings. **Construction D.5.** Let  $\Sigma \colon \mathbb{C} \to \mathbb{C}$  and  $B \colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$  be functors with their canonical relation liftings

 $\overline{\Sigma}_{\mathbf{Rel}} \colon \mathbf{Rel}(\mathbb{C}) \to \mathbf{Rel}(\mathbb{C}) \qquad \text{and} \qquad \overline{B}_{\mathbf{Rel}} \colon \mathbf{Rel}(\mathbb{C})^{\mathsf{op}} \times \mathbf{Rel}(\mathbb{C}) \to \mathbf{Rel}(\mathbb{C}),$ 

and suppose that  $\Sigma$  preserves strong epimorphisms. Every V-pointed higher-order GSOS law

$$\varrho_{X,Y} \colon \Sigma(X \times B(X,Y)) \to B(X, \Sigma^*(X+Y)) \qquad ((X,p_X) \in V/\mathbb{C}, Y \in \mathbb{C})$$

of  $\Sigma$  over B has a (necessarily unique) lifting to a (V, V)-pointed higher-order GSOS law  $\overline{\varrho}^{\mathbf{Rel}}$  of  $\overline{\Sigma}_{\mathbf{Rel}}$  over  $\overline{B}_{\mathbf{Rel}}$ . Its

component at  $((X, R), (p_{(X,R)}) \in (V, V)/\mathbf{Rel}(\mathbb{C})$  and  $(Y, S) \in \mathbf{Rel}(\mathbb{C}))$  is given by the composite

$$\begin{split} \Sigma_{\mathbf{Rel}}((X,R) \times B_{\mathbf{Rel}}((X,R),(Y,S))) \\ & \parallel \\ \overline{\Sigma}_{\mathbf{Rel}}((X,R) \times \overline{B}_{\mathbf{Gra}}((X,R),(Y,S))^{\dagger}) \\ & \downarrow \cong \\ (\overline{\Sigma}_{\mathbf{Gra}}((X,R) \times \overline{B}_{\mathbf{Gra}}((X,R),(Y,S))))^{\dagger} \\ & \downarrow (\overline{e}_{(X,R),(Y,S)}^{\mathbf{Gra}})^{\dagger} \\ \overline{B}_{\mathbf{Gra}}((X,R),\overline{\Sigma}_{\mathbf{Gra}}^{\star}((X,R)+(Y,S)))^{\dagger} \\ & \downarrow \overline{B}_{\mathbf{Gra}}(\mathrm{id},\overline{\Sigma}_{\mathbf{Gra}}^{\star}e)^{\dagger} \\ \overline{B}_{\mathbf{Gra}}((X,R),\overline{\Sigma}_{\mathbf{Gra}}^{\star}((X,R) \vee (Y,S)))^{\dagger} \\ & \downarrow \overline{B}_{\mathbf{Gra}}(\mathrm{id},h)^{\dagger} \\ \overline{B}_{\mathbf{Gra}}((X,R),\overline{\Sigma}_{\mathbf{Rel}}^{\star}((X,R) \vee (Y,S)))^{\dagger} \\ & \parallel \\ \overline{B}_{\mathbf{Rel}}((X,R),\overline{\Sigma}_{\mathbf{Rel}}^{\star}((X,R) \vee (Y,S))) \end{split}$$

Here the isomorphism in the second step follows from Lemma B.2, and

$$e: (X,R) + (Y,S) \twoheadrightarrow ((X,R) + (X,S))^{\dagger} = (X,R) \lor (Y,S)$$

and

$$h \colon \overline{\Sigma}^{\star}_{\mathbf{Gra}}((X,R) \vee (X,S)) \twoheadrightarrow \overline{\Sigma}^{\star}_{\mathbf{Rel}}((X,R) \vee (X,S))$$

are the reflections. For the latter recall that the free  $\overline{\Sigma}_{Rel}$ -algebra on a relation (X,T) is given by applying  $(-)^{\dagger}$  to the free  $\overline{\Sigma}_{Gra}$ -algebra on (X,T), see Corollary B.4.

#### APPENDIX E

#### The $\lambda$ -Calculus

We give a more detailed account of the  $\lambda$ -calculus in the higher-order abstract GSOS framework. Recall from Section IX-B that we work with the functors

$$\Sigma X = V + \delta X + X \times X$$
 and  $B_0(X, Y) = \langle\!\langle X, Y \rangle\!\rangle \times (Y + Y^X + 1)$ 

on the presheaf category  $\mathbf{Set}^{\mathbb{F}}$ , where  $\mathbb{F}$  is the category of finite cardinals and functions and the presheaves V,  $\delta X$  and  $\langle\!\langle X, Y \rangle\!\rangle$  are given by

 $V(n) = n, \qquad \delta X(n) = X(n+1), \qquad \langle\!\langle X, Y \rangle\!\rangle(n) = \mathbf{Set}^{\mathbb{F}}(X^n, Y).$ 

The initial algebra for  $\Sigma$  is the presheaf  $\Lambda$  of  $\lambda$ -terms modulo  $\alpha$ -equivalence [16]. To introduce the higher-order GSOS law for the  $\lambda$ -calculus, we need some notation.

Notation E.1. (1) Given  $X \in \mathbf{Set}^{\mathbb{F}}$  we sometimes write  $X_n$  for X(n). For a presheaf morphism (i.e. a natural transformation)  $f: X \to Y$ , we drop subscripts of components and write f for  $f_n: X_n \to Y_n$ .

(2) We let  $ev: Y^X \times X \to X$  denote the evaluation morphism of the exponential object  $Y^X$ . Given  $n \in \mathbb{F}$ ,  $f \in Y^X(n)$  and  $e \in X(n)$  we write f(e) for ev(f, e).

(3) We define the maps  $n \xrightarrow{\text{old}_n} n + 1 \xleftarrow{\text{new}_n} 1$  by  $\text{old}_n(i) = i$  and  $\text{new}_n(0) = n$ .

(4) For a presheaf  $X \in \mathbf{Set}^{\mathbb{F}}$  we define  $up_{X,n} = (X(n) \xrightarrow{X(old_n)} X(n+1)).$ 

(5) Given a pointed presheaf  $(X, var) \in \mathcal{V}/\mathbf{Set}^{\mathbb{F}}$  and a presheaf  $Y \in \mathbf{Set}^{\mathbb{F}}$  we define  $\varrho_1 \colon \delta\langle\langle X, Y \rangle\rangle \to \langle\langle X, \delta Y \rangle\rangle$  to be the map sending a natural transformation  $f \colon X^{n+1} \to Y$  to the natural transformation  $\varrho_1(f) \colon X^n \to \delta Y$  given by

$$\vec{u} \in X(m)^n \quad \mapsto \quad f_{m+1}(\mathsf{up}_{X,m}(\vec{u}),\mathsf{var}_{m+1}(\mathsf{new}_m)) \in Y(m+1).$$

(6) Similarly, for a pointed presheaf  $(X, \mathsf{var}) \in \mathcal{V}/\mathbf{Set}^{\mathbb{F}}$  and a presheaf  $Y \in \mathbf{Set}^{\mathbb{F}}$  we define the map  $\varrho_2 \colon \delta\langle\langle X, Y \rangle\rangle \to Y^X$  by

$$\varrho_2(f)(e) = f_n(\operatorname{var}_n(0), \dots, \operatorname{var}_n(n-1), e) \quad \text{for } f \colon X^{n+1} \to Y \text{ and } e \in X(n).$$

(7) We write  $\lambda$ .(-):  $\delta \Sigma^* \to \Sigma^*$  and  $\circ: \Sigma^* \times \Sigma^* \to \Sigma^*$  for the natural transformations whose components come from the  $\Sigma$ -algebra structure on free  $\Sigma$ -algebras; here  $\circ$  denotes application. In the following we will consider free algebras of the form  $\Sigma^*(X + Y)$ . For simplicity, we usually keep inclusion maps implicit: Given  $t_1, t_2 \in X(n)$  and  $t'_1 \in Y(n)$  we write  $t_1 t_2$  for  $[\eta \cdot \operatorname{inl}(t_1)] \circ [\eta \cdot \operatorname{inl}(t_2)]$ , and similarly  $t_1 t'_1$  for  $[\eta \cdot \operatorname{inl}(t_1)] \circ [\eta \cdot \operatorname{inr}(t'_1)]$  etc., where inl and inr are the coproduct injections and  $\eta$ : Id  $\to \Sigma^*$  is the unit of the free monad  $\Sigma^*$ .

(8) Finally, define  $\pi: V \to \langle\!\langle X, \Sigma^*(X+Y) \rangle\!\rangle$  to be the map sending  $v \in V(n) = n$  to the natural transformation  $\pi(v)(n): X^n \to \Sigma^*(X+Y)$  given by the v-th projection  $X^n \to X$  followed by  $\eta \cdot \text{inl.}$ 

With these preparations at hand, we are now ready to phrase the small-step operational semantics of the call-by-name  $\lambda$ -calculus in terms of a V-pointed higher-order GSOS law of the syntax endofunctor  $\Sigma X = V + \delta X + X \times X$  over the behaviour bifunctor  $B_0(X, Y) = \langle \! \langle X, Y \rangle \! \rangle \times (Y + Y^X + 1)$ . A law of this type is given by a family of presheaf maps

dinatural in  $(X, \operatorname{var}_X) \in V/\operatorname{\mathbf{Set}}^{\mathbb{F}}$  and natural in  $Y \in \operatorname{\mathbf{Set}}^{\mathbb{F}}$ . We let  $\varrho_{X,Y,n}$  denote the component of  $\varrho_{X,Y}$  at  $n \in \mathbb{F}$ .

**Definition E.2** (V-pointed higher-order GSOS law for the call-by-name  $\lambda$ -calculus).

where  $t \in \delta X(n)$ ,  $f \in \delta \langle \!\langle X, Y \rangle \!\rangle(n)$ ,  $g, h \in \langle \!\langle X, Y \rangle \!\rangle(n)$ ,  $\vec{u} \in X(m)^n$  for  $m \in \mathbb{N}$ ,  $k \in Y^X(n)$ ,  $t_1, t_2 \in X(n)$  and  $t'_1 \in Y(n)$  (we have omitted the brackets around the pairs on the right).

Remark E.3. By the above definition the first component

$$\mathsf{fst} \cdot \varrho^0_{X,Y} \colon V + \delta(X \times \langle\!\langle X, Y \rangle\!\rangle \times (Y + Y^X + 1)) + (X \times \langle\!\langle X, Y \rangle\!\rangle \times (Y + Y^X + 1))^2 \to \langle\!\langle X, \Sigma^\star(X + Y) \rangle\!\rangle$$

of  $\rho_{X,Y}^0$  only depends on f in the clause for abstraction, and only on g and h in the clauses for application. Therefore fst  $\rho_{X,Y}^0$  can be expressed as a composite

for suitable  $\tau_{X,Y}^0$ , where p is the middle product projection.

The higher-order GSOS law  $\rho^0$  correctly captures the operational semantics of the call-by-name  $\lambda$ -calculus:

Proposition E.4 ([20]). The operational model

$$\gamma^{0} = \langle \gamma_{1}^{0}, \gamma_{2}^{0} \rangle \colon \Lambda \to \langle\!\langle \Lambda, \Lambda \rangle\!\rangle \times (\Lambda + \Lambda^{\Lambda} + 1)$$
(E.1)

of the higher-order GSOS law  $\varrho^0$  satisfies the following for  $n \in \mathbb{F}$  and  $t \in \Lambda(n)$ :

- (1)  $\gamma_1^0(t)(\vec{u}) = t[u_0, \dots, u_{n-1}/0, \dots, n-1]$  for all  $m \in \mathbb{F}$  and  $\vec{u} \in \Lambda(m)^n$ .
- (2) If  $t \to t'$ , then  $\gamma_2^0(t) = t' \in \Lambda(n)$ .
- (3) If  $t = \lambda x.t'$ , then  $\gamma_2^0(t) \in \Lambda^{\Lambda}(n)$  and  $\gamma_2(t)(e) = t'[e]$  for all  $e \in \Lambda(n)$ .
- (4) Otherwise (that is, if t is stuck), one has  $\gamma_2^0(t) = *$ .

**Remark E.5.** In order to deal with weak similarity, we need to work with a nondeterministic version B of the bifunctor  $B_0(X,Y) = \langle\!\langle X,Y \rangle\!\rangle \times (Y+Y^X+1)$ . The nondeterminism is introduced via the pointwise powerset functor  $\mathcal{P}_{\star} \colon \mathbf{Set}^{\mathbb{F}} \to \mathbf{Set}^{\mathbb{F}}$  given by  $X \mapsto \mathcal{P} \cdot X$ , and we put

$$B(X,Y) = \langle\!\langle X,Y \rangle\!\rangle \times \mathcal{P}_{\star}(Y+Y^X).$$

Note that we dropped the "+1"; the reason is our intended notion of weak similarity, viz. the open extension of applicative similarity, which does not detect whether a term is stuck. We extend the law  $\rho^0$  of Definition E.2 to a higher-order GSOS law  $\rho$  of  $\Sigma$  over B as follows. Given  $(X, \operatorname{var}_X) \in V/\operatorname{Set}^{\mathbb{F}}$  and  $Y \in \operatorname{Set}^{\mathbb{F}}$ , the first component fst  $\rho_{X,Y}$  is the following composite, where p is the middle product projection and  $\tau_{X,Y}^0$  has been introduced in Remark E.3:

$$\begin{split} \Sigma(X\times \langle\!\!\langle X,Y\rangle\!\!\rangle \times \mathcal{P}_{\star}(Y+Y^X)) \\ & \parallel \\ V+\delta(X\times \langle\!\!\langle X,Y\rangle\!\!\rangle \times \mathcal{P}_{\star}(Y+Y^X)) + (X\times \langle\!\!\langle X,Y\rangle\!\!\rangle \times \mathcal{P}(Y+Y^X))^2 \\ & \downarrow^{\mathsf{id}+\delta(p)+p^2} \\ V+\delta \langle\!\!\langle X,Y\rangle\!\!\rangle + \langle\!\!\langle X,Y\rangle\!\!\rangle^2 \\ & \downarrow^{\tau^0_{X,Y}} \\ & \langle\!\!\langle X,\Sigma^{\star}(X+Y)\rangle\!\!\rangle \end{split}$$

For the second component snd  $\cdot \rho_{X,Y}$  we need a number of auxiliary natural transformations involving the powerset functor:

$$\begin{array}{ll} \operatorname{st}_{A,B} \colon A \times \mathcal{P}(B) \to \mathcal{P}(A \times B), & (a,S) \mapsto \{(a,b) \colon b \in S\}; \\ \delta_A \colon \mathcal{P}A \times \mathcal{P}A \to \mathcal{P}(A \times A), & (S,T) \mapsto \{(s,t) \colon s \in S, t \in T\}; \\ \phi_A \colon \mathcal{P}(A+1) \to \mathcal{P}(A), & S \mapsto S \smallsetminus \{*\}; \\ \varepsilon_A \colon \mathcal{P}(A) \to \mathcal{P}(A+1), & \emptyset \mapsto 1, S \mapsto S \ (S \neq \emptyset); \\ \eta_A \colon A \to \mathcal{P}A, & a \mapsto \{a\}; \\ \operatorname{can}_{A,B,C} \colon \mathcal{P}A + \mathcal{P}B + \mathcal{P}Z \to \mathcal{P}(A+B+C) & S \mapsto S. \end{array}$$

The map snd  $\cdot \rho_{X,Y}$  is the defined to be composite given as follows for  $n \in \mathbb{F}$ . We drop subscripts of st,  $\delta, \phi, \varepsilon, \eta$ .

**Lemma E.6.** The operational model of the higher-order GSOS law  $\rho$  is given by

$$\gamma = \langle \gamma_1, \gamma_2 \rangle \colon \Lambda \to \langle \! \langle \Lambda, \Lambda \rangle \! \rangle \times \mathcal{P}_{\star}(\Lambda + \Lambda^{\Lambda})$$

where  $\gamma_1 = \gamma_1^0$  and  $\gamma_2$  is the following composite at  $n \in \mathbb{F}$ :

$$\gamma_2 = (\Lambda(n) \xrightarrow{\gamma_2^0} \Lambda(n) + \Lambda^{\Lambda}(n) + 1 \xrightarrow{\eta} \mathcal{P}(\Lambda(n) + \Lambda^{\Lambda}(n) + 1) \xrightarrow{\phi} \mathcal{P}(\Lambda(n) + \Lambda^{\Lambda}(n))).$$

*Proof.* One only needs to show that the map  $\langle \gamma_1^0, \phi \cdot \eta \cdot \gamma_2^0 \rangle$  satisfies the diagram (II.3) defining  $\gamma$ . This follows via a lengthy routine verification from the definition of  $\rho$ , using elementary properties of the involved maps st,  $\delta$ , can,  $\varepsilon$ ,  $\phi$ .

We now instantiate the data of Assumptions VIII.1 to

(1) the functor  $\Sigma X = V + \delta X + X \times X$ ;

(2) the functor  $B(X,Y) = \langle\!\langle X,Y \rangle\!\rangle \times \mathcal{P}_{\star}(Y+Y^X)$  of Remark E.5, preordered by equality in the first component and inclusion in the second one. Its relation lifting is  $\overline{B} = F \times (\overline{\mathcal{P}}_{\star} \cdot G)$ , where F and G are the canonical relation liftings of the bifunctors  $(X,Y) \mapsto \langle\!\langle X,Y \rangle\!\rangle$  and  $(X,Y) \mapsto (Y+Y^X)$  and  $\overline{\mathcal{P}}_{\star}$  is the lifting of  $\mathcal{P}_{\star}$  given by

$$\overline{\mathcal{P}_{\star}}(X,R) = (\mathcal{P}_{\star}(X), S_R),$$

where  $S_R(n) \subseteq \mathcal{P}(X(n)) \times \mathcal{P}(X(n))$  is the (one-sided) Egli-Milner relation induced by  $R(n) \subseteq X(n) \times X(n)$ , cf. Remark III.3; (3) the higher-order GSOS law  $\rho$  of  $\overline{\Sigma}$  over  $\overline{B}$  as described in Remark E.5. Let us verify that this data satisfies the required properties:

**Lemma E.7.** (1) The functor  $\Sigma$  preserves strong epimorphisms.

- (2) The functor  $\overline{B}$  is good for simulations.
- (3) The higher-order GSOS law  $\varrho$  admits a relation lifting.

*Proof.* (1) Since strong epimorphisms (i.e. componentwise surjective natural transformations) are stable under coproducts, it suffices to show that the functors  $\delta$  and Id × Id preserve strong epimorphisms. For the functor  $\delta$  this follows from the fact that it is a left adjoint [16]. For Id × Id use that strong epimorphisms are stable under products, see Remark B.1.

(2) This is shown as in the proof for Section IX-A verifying the Assumptions VIII.1 Item (2), using that all the structure involved (including relation composition  $\bullet$  in  $\mathbf{Set}^{\mathbb{F}}$ ) is just formed componentwise in  $\mathbf{Set}$ .

(3) Given a (V, V)-pointed relation (X, R), p<sub>(X,R)</sub>) ∈ (V, V)/Rel(Set<sup>F</sup>) and a relation (Y, S) ∈ Rel(Set<sup>F</sup>) we need to show that the map ρ<sub>X,Y</sub> is relation-preserving with respect to the relations on the domain Σ(X × ⟨⟨X,Y⟩⟩ × P<sub>\*</sub>(Y + Y<sup>X</sup>))(n) and codomain ⟨⟨X, Σ<sup>\*</sup>(X + Y)⟩⟩ × P(Σ<sup>\*</sup>(X + Y)(n) + Σ<sup>\*</sup>(X + Y)<sup>X</sup>(n)) obtained by applying the relation liftings Σ, P<sub>\*</sub>, F, G of the functors Σ, P<sub>\*</sub>, (X, Y) ↦ ⟨⟨X, Y⟩⟩, (X, Y) ↦ Y + Y<sup>X</sup>. There are several cases; we follow the notation of Definition E.2.
(a) Suppose that λ.(t, f, A) and λ.(t', f', A') ∈ X(n + 1) × ⟨⟨X, Y⟩⟩(n + 1) × P(Y(n + 1) + Y<sup>X</sup>(n + 1)) are related. Then ρ<sub>X,Y</sub> sends λ.(t, f, A) to the pair (⟨⟨X, λ.(−) · η · inr⟩⟩(ρ<sub>1</sub>(f)), (η · inr)<sup>X</sup>(ρ<sub>2</sub>(f'))). These pairs are related because f and f' are related in Y<sup>X</sup>(n + 1) ⊆ Y(n + 1) + Y<sup>X</sup>(n + 1) and ρ<sub>1</sub>, ρ<sub>2</sub> are relation-preserving, which is easy to see by their definition.

(b) Suppose that  $(t_1, g, A_1) (t_2, h, A_2)$  and  $(t'_1, g', A'_1) (t'_2, h', A'_2)$  are related in  $(X(n) \times \langle\!\langle X, Y \rangle\!\rangle (n) \times \mathcal{P}(Y(n) + Y^X(n)))^2$ . Then fst  $\varrho_{X,Y}$  sends the two pairs to  $\lambda \vec{u}.(g_m(\vec{u}) h_m(\vec{u})$  and  $\lambda \vec{u}.(g'_m(\vec{u}) h'_m(\vec{u})$ , respectively, and these are related in  $\langle\!\langle X, \Sigma^*(X+Y)\rangle\!\rangle (n)$  because the  $\Sigma$ -algebra structure on  $\Sigma^*(X+Y)$  is relation-preserving by Proposition V.4. For snd  $\cdot \varrho_{X,Y}$  we consider two subcases:

- i) If  $A_1 = \emptyset$ , then  $\varrho_{X,Y}$  sends  $(t_1, g, A_1) (t_2, h, A_2)$  to  $\emptyset \in \mathcal{P}(\Sigma^*(X + Y)(n) + (\Sigma^*(X + Y))^X(n))$ , which is related to every element of  $\mathcal{P}(\Sigma^*(X + Y)(n) + (\Sigma^*(X + Y))^X(n))$  by definition of the Egli-Milner relation.
- ii) If  $A_1 \neq \emptyset$ , then  $\varrho_{X,Y}$  sends  $(t_1, g, A_1)(t_2, h, A_2)$  to  $\{st_2 : s \in A_1 \cap Y(n)\} \cup \{\eta \cdot \text{inr} \cdot k(t_2) : k \in A_1 \cap Y^X(n)\}$  and  $(t'_1, g', A'_1)(t'_2, h', A'_2)$  to  $\{s't'_2 : s' \in A'_1 \cap Y(n)\} \cup \{\eta \cdot \text{inr} \cdot k'(t'_2) : k' \in A'_1 \cap Y^X(n)\}$ , and these two sets are clearly related by the Egli-Milner relation because  $A_1, A'_1$  are related, the  $\Sigma$ -algebra structure on  $\Sigma^*(X+Y)$  is relation-preserving, and by definition of the relation lifting of  $(X, Y) \mapsto Y + Y^X$ .

Next we describe the weakening  $\tilde{\gamma}$  of the operational model (E.1).

**Definition E.8.** The weak operational model is the  $B(\Lambda, -)$ -coalgebra

$$\widetilde{\gamma} = \langle \widetilde{\gamma}_1, \widetilde{\gamma}_2 \rangle \colon \Lambda \to \langle\!\langle \Lambda, \Lambda \rangle\!\rangle \times \mathcal{P}_{\star}(\Lambda + \Lambda^{\Lambda} + 1)$$

given for  $t \in \Lambda(n)$  by

$$\widetilde{\gamma}_1(t) = \gamma_1(t) \qquad \text{and} \qquad \widetilde{\gamma}_2(t) = \ \{\overline{t} \in \Lambda(n) : t \Rightarrow \overline{t}\} \ \cup \ \{f \in \Lambda^{\Lambda}(n) : \exists \overline{t}. t \Rightarrow \overline{t} \land \gamma_2^0(\overline{t}) = f\}.$$

Here  $\Rightarrow$  is the reflexive transitive hull of the reduction relation  $\rightarrow$ .

**Lemma E.9.** The coalgebra  $\tilde{\gamma}$  is a weakening of  $\gamma$ , cf. Definition VI.4.

*Proof.* For each relation  $(\Lambda, R)$  we need to prove that existence of a morphism  $\delta$  making (E.3) commute is equivalent to existence of a morphism  $\varepsilon$  making (E.4) commute. Here we denote the relation  $\overline{B}((\Lambda, \Lambda), (\Lambda, R))$  by  $(B(\Lambda, \Lambda), E_{\Lambda, R})$ .

By Proposition E.4 the existence of  $\delta$  in (E.3) is equivalent to the following properties for every  $n \in \mathbb{F}$  and  $R_n(t_1, t_2)$ : (1)  $R_m(t_1[\vec{u}], t_2[\vec{u}])$  for all  $m \in \mathbb{F}$  and  $\vec{u} \in \Lambda(m)^n$ ; (2) t<sub>1</sub> → t'<sub>1</sub> ⇒ ∃t'<sub>2</sub>. t<sub>2</sub> ⇒ t'<sub>2</sub> ∧ R<sub>n</sub>(t'<sub>1</sub>, t'<sub>2</sub>);
(3) t<sub>1</sub> = λx.t'<sub>1</sub> ⇒ ∃t'<sub>2</sub>.t<sub>2</sub> ⇒ λx.t'<sub>2</sub> ∧ ∀e ∈ Λ(n).R<sub>n</sub>(t'<sub>1</sub>[e/x], t'<sub>2</sub>[e/x]).
Similarly, the existence of ε in (E.4) is equivalent to the following properties for every n ∈ F and R<sub>n</sub>(t<sub>1</sub>, t<sub>2</sub>):
(1') R<sub>m</sub>(t<sub>1</sub>[ū], t<sub>2</sub>[ū]) for all m ∈ F and ū ∈ Λ(m)<sup>n</sup>;
(2') t<sub>1</sub> ⇒ t'<sub>1</sub> ⇒ ∃t'<sub>2</sub>. t<sub>2</sub> ⇒ t'<sub>2</sub> ∧ R<sub>n</sub>(t'<sub>1</sub>, t'<sub>2</sub>);

 $(3') t_1 \Rightarrow \lambda x.t_1' \implies \exists t_2'.t_2 \Rightarrow \lambda x.t_2' \land \forall e \in \Lambda(n).R_n(t_1'[e/x], t_2'[e/x]).$ 

The conditions (1)–(3) are clearly equivalent to (1')–(3').

Recall from Definition IX.1 the notion of applicative similarity and its open extension.

**Proposition E.10.** Weak similarity on the operational model (E.1) coincides with the open extension of applicative similarity:

$$\lesssim = \lesssim^{ap}$$

*Proof.* By Lemma E.9 and its proof, weak similarity is the greatest relation  $\leq \subseteq \Lambda \times \Lambda$  such that for every  $n \in \mathbb{F}$  and  $t_1 \leq_n t_2$ : (1)  $t_1[\vec{u}] \leq_m t_2[\vec{u}]$  for all  $m \in \mathbb{F}$  and  $\vec{u} \in \Lambda(m)^n$ ;

- (2)  $t_1 \to t'_1 \implies \exists t'_2. t_2 \Rightarrow t'_2 \land t'_1 \lesssim_n t'_2;$
- $(3) \ t_1 = \lambda x.t_1' \implies \exists t_2'.t_2 \Rightarrow \lambda x.t_2' \land \forall e \in \Lambda(n).t_1'[e/x] \lesssim_n t_2'[e/x].$

*Proof of*  $\lesssim \subseteq \lesssim^{ap}$ . Note first that  $\lesssim_0 \subseteq \Lambda(0) \times \Lambda(0)$  is an applicative simulation by the above conditions (2) and (3) for n = 0. Therefore  $\lesssim_0 \subseteq \lesssim_0^{ap}$  because  $\lesssim_0^{ap}$  is the greatest applicative simulation. Moreover, for n > 0 and  $t_1 \lesssim_n t_2$ , we have

$$t_1[\vec{u}] \lesssim_0 t_2[\vec{u}]$$
 for every  $\vec{u} \in \Lambda(0)^n$ 

by condition (1), whence

$$t_1[\vec{u}] \lesssim_0^{\mathrm{ap}} t_2[\vec{u}]$$
 for every  $\vec{u} \in \Lambda(0)^n$ 

because  $\leq_0 \leq \leq_0^{ap}$ , and so  $t_1 \leq_n^{ap} t_2$ . This proves  $\leq_n \leq \leq_n^{ap}$  for n > 0 and thus  $\leq \leq \leq^{ap}$  overall.

*Proof of*  $\lesssim^{\mathrm{ap}} \subseteq \lesssim$ . Since  $\lesssim$  is the greatest weak simulation, it suffices to show that  $\lesssim^{\mathrm{ap}}$  is a weak simulation. Thus suppose that  $n \in \mathbb{F}$  and  $t_1 \lesssim^{\mathrm{ap}}_n t_2$ ; we need to verify the above conditions (1)–(3) with  $\lesssim_n$  replaced by  $\lesssim^{\mathrm{ap}}_n$ .

Let us first consider the case n = 0, i.e.  $t_1 \leq_0^{\text{ap}} t_2$ .

(1) Since  $t_1$  and  $t_2$  are closed terms, this condition simply states that  $t_1 \leq_m^{\text{ap}} t_2$  for every m > 0. This holds by definition of  $\leq_m^{\text{ap}}$  because  $t_1[\vec{u}] = t_1 \leq_0^{\text{ap}} t_2 = t_2[\vec{u}]$  for every  $\vec{u} \in \Lambda(0)^m$ .

(2) holds because  $\leq_0^{\text{ap}}$  is closed under reduction:  $t_1 \to t'_1$  and  $t_1 \leq_0^{\text{ap}} t_2$  implies  $t'_1 \leq_0^{\text{ap}} t_2$ . Thus we can take  $t'_2 = t_2$ .

(3) holds by definition of  $\leq_0^{ap}$ .

Now suppose that  $t_1 \leq_n^{\text{ap}} t_2$  for some n > 0:

(1) Let  $\vec{u} = (u_0, \ldots, u_{n-1}) \in \Lambda(m)^n$ . If m = 0 we have  $t_1[\vec{u}] \lesssim_0^{\operatorname{ap}} t_2[\vec{u}]$  by definition of  $\lesssim_n^{\operatorname{ap}}$ . If m > 0 and  $\vec{v} \in \Lambda(0)^m$  we have

$$t_1[\vec{u}][\vec{v}] = t_1[u_0[\vec{v}], \dots, u_{n-1}[\vec{v}]] \lesssim_0^{ap} t_2[u_0[\vec{v}], \dots, u_{n-1}[\vec{v}]] = t_2[\vec{u}][\vec{v}],$$

whence  $t_1[\vec{u}] \lesssim_m^{\text{ap}} t_2[\vec{u}]$ .

(2) Suppose that  $t_1 \to t'_1$ . Then  $t_1[\vec{u}] \to t'_1[\vec{u}]$  for every  $\vec{u} \in \Lambda(0)^n$  because reductions respect substitution, and moreover  $t_1[\vec{u}] \lesssim_0^{\operatorname{ap}} t_2[\vec{u}]$  because  $t_1 \lesssim_n^{\operatorname{ap}} t_2$ . It follows that  $t'_1[\vec{u}] \lesssim_0^{\operatorname{ap}} t_2[\vec{u}]$  because  $\lesssim_0^{\operatorname{ap}}$  is closed under reduction, whence  $t'_1 \lesssim_n^{\operatorname{ap}} t_2$ . (3) Suppose that  $t_1 = \lambda x \cdot t'_1$ . To show that  $t_2 \Rightarrow \lambda x \cdot t'_2$  for some  $t'_2$ , suppose the contrary. There are two cases:

<u>Case 1:</u> The term  $t_2$  diverges, that is, its reduction sequence  $t_2 \rightarrow t'_2 \rightarrow t''_2 \rightarrow \cdots$  is infinite.

Choose an arbitrary  $\vec{u} \in \Lambda(0)^n$ . Then  $t_1[\vec{u}]$  is a  $\lambda$ -abstraction, while  $t_2[\vec{u}]$  diverges. It follows that  $t_1[\vec{u}] \not\lesssim_0^{\text{ap}} t_2[\vec{u}]$ , in contradiction to  $t_1 \lesssim_n^{\text{ap}} t_2$ .

<u>Case 2:</u> The term  $t_2$  reduces in finitely many steps to  $y s_1 \cdots s_m$  for some variable  $y \in n$  and terms  $s_1, \ldots, s_m \in \Lambda(n)$ .

Choose an arbitrary  $\vec{u} \in \Lambda(0)^n$  such that the term  $u_y$  diverges, e.g.  $u_y = (\lambda x.x x) (\lambda x.x x)$ . Then  $t_1[\vec{u}]$  is a  $\lambda$ -abstraction while  $t_2[\vec{u}]$  diverges, again contradicting  $t_1 \lesssim_n^{\text{ap}} t_2$ .

Thus  $t_2 \Rightarrow \lambda x.t'_2$  for x = n and  $t'_2 \in \Lambda(n+1)$ . Moreover, for every  $e \in \Lambda(n)$  and  $\vec{u} \in \Lambda(0)^n$  we have

$$t_1'[e/x][\vec{u}] = t_1'[\vec{u}, e[\vec{u}]] = t_1'[\vec{u}, x][e[\vec{u}]/x] \lesssim_0^{\operatorname{ap}} t_2'[\vec{u}, x][e[\vec{u}]/x] = t_2'[\vec{u}, e[\vec{u}]] = t_2'[e/x][\vec{u}] = t_2'[\vec{u}]$$

using that  $t_1[\vec{u}] \lesssim_0^{\text{ap}} t_2[\vec{u}]$  by definition of  $\lesssim_n^{\text{ap}}$ . This proves  $t'_1[e/x] \lesssim_n^{\text{ap}} t'_2[e/x]$ .

Finally, let us verify that the condition of Theorem VIII.6 is satisfied:

**Proposition E.11.** The triple  $(\Lambda, \iota, \widetilde{\gamma})$  forms a lax  $\varrho$ -bialgebra.

*Proof.* We need to prove lax commutativity of the following diagram:



In the first component, the diagram strictly commutes by definition of fst  $\rho_{X,Y}$ . In the second component, by definition of snd  $\rho_{X,Y}$ , lax commutativity amounts to the assertion that the weak versions

w-app1 
$$\frac{s \Rightarrow s'}{st \Rightarrow s't}$$
 w-app2  $\frac{s \Rightarrow \lambda x.s'}{st \Rightarrow s'[t/x]}$ 

of the rules app1 and app2 in (IX.1) are sound. This is clearly the case, since w-app1 and w-app2 amount to repeated application of app1 and app2.

We thus obtain Theorem IX.2 as an instance of Theorem VIII.6.