Maximal Gap of a Sampling Set for the Exact Iterative Reconstruction Algorithm in Shift Invariant Spaces

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***Abstract*—A conventional A/D converter prefilters a signal by an ideal lowpass filter and performs sampling for bandlimited signals by the Nyquist sampling rate. Recent research reveals that A/D conversion in a shift invariant space provides more flexible choices for designing a practical A/D conversion system of high accuracy. This paper focuses on the maximal gap of a sampling set for the iterative algorithm in shift invariant spaces, which provides an explicit formula to calculate the maximal gap of a sampling set in terms of a generator of the undertaken shift invariant spaces.**

***Index Terms*—A/D conversion, maximal gap, nonuniform sampling, prefilter, shift invariant space, sampling set.**

I. Introduction

In digital signal processing and digital communications, an analog signal is converted into a digital signal by an A/D (analog-to-digital) conversion device.

A signal f is said to be of *finite energy* if $\|f\| < \infty$, where $\|\cdot\|$ is the square norm defined by $\|f\| = (\int_{\mathbb{R}} |f(t)|^2 dt)^{1/2}$. We also denote by $L^2(\mathbb{R})$ the collection of all signals of finite energy, that is, $\{f: \|f\| < \infty\}$. f is said to be *bandlimited* if $\hat{f}(\omega) = 0$ whenever $|\omega| > \sigma$ for some $\sigma > 0$, where $\hat{f}$ is the *Fourier transform* of f defined by $\hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-it\omega}\, dt$. In this case, f is also called a *σ-band signal*.

A conventional A/D converter prefilters a signal of finite energy by an ideal lowpass filter and performs uniform sampling by the Nyquist sampling rate [33]. Since in some practical systems sampling cannot be always made uniformly, one has to consider the nonuniformly sampled signal [6], [20].

A discrete set $X = \{t_k\}_k \subset \mathbb{R}$ is called a *sampling set* for a signal f if f is completely determined by the sample set $f(X) = \{f(t_k)\}_k$; X is said to be *separated* if $\inf_{k\neq j} |t_k - t_j| = d > 0$, where d is called the *separation* of X. A sampling set $X = \{t_k\}_k$ is said to be *stable* for a signal space S if there is a constant $C \geq 1$ such that

$$C^{-1}\|f\| \leq \left(\sum_k |f(t_k)|^2\right)^{1/2} \leq C\|f\| \tag{1}$$

holds for any signal $f \in S$.

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The nonuniform sampling for the bandlimited signals is studied by Beurling, Landau and others [6], [23], [29]. It is understood that a separated X is a stable sampling set for the σ-band signal space if the *Beurling lower density*[1] $D^-(X) > \pi/\sigma$. Conversely, $D^-(X) \geq \pi/\sigma$ if X is a stable sampling set for the σ-band signal space. Since Beurling-Landau's theorem does not provide a reconstruction formula, Feichtinger and Gröchenig established an iterative reconstruction algorithm to handle the nonuniformly sampled bandlimited signals [20].

For a $\lambda \geq 1$, the *shift invariant space* $V_\lambda(\varphi)$ generated by a function φ is defined by [8], [26]

$$V_\lambda(\varphi) = \left\{\sum_k c_k\varphi(\lambda\cdot - k): \sum_k |c_k|^2 < \infty\right\}. \tag{3}$$

The φ and λ are, respectively, called a *generator* and the *dilation* of the shift invariant space $V_\lambda(\varphi)$. Let $\operatorname{sinc} t = (\sin \pi t/\pi t)$. Then $V_\lambda(\operatorname{sinc})$ contains exactly all $\pi\lambda$-band signals of finite energy. Then the conventional A/D conversion can be formulated as A/D conversion in $V_\lambda(\operatorname{sinc})$. One can therefore consider A/D conversion in a general shift invariant space $V_\lambda(\varphi)$ for a general generator φ.

In real world application, such an extension of A/D conversion is useful and necessary, e.g., for performing nonideal prefiltering [34], for avoiding Gibbs phenomenon in FFT [17], for using the impulse response of fast decay [32], for taking into account real acquisition and reconstruction devices [38], for considering arbitrary band signals [24], for obtaining smoother frequency cutoff or for numerical implementation [4], [5], [37], [38], [40]. This is formulated by choosing an appropriate function φ with some desirable shape corresponding to a particular "impulse response" of a device, such as a compactly supported function, a function with polynomial or exponential decay, or a function φ with smooth cutoff frequency $\hat{\varphi}$. Then one prefilters a signal by the *shift invariant space* $V_\lambda(\varphi)$ [7], [11], [37], and performs sampling in $V_\lambda(\varphi)$ [5], [10], [12], [14], [15], [37], [39]. Furthermore, A/D conversion in shift invariant spaces does provide more flexible choices to design various A/D conversion systems of high accuracy [5], [10]–[15], [36], [37], [39], [40].

The prefiltering, uniform sampling and oversampling has been well handled in the shift invariant spaces [7], [11]–[13], [39], [27], [37], [40]. However, the nonuniform sampling in shift invariant spaces is relatively tough. For instance, the

[1]Let $v^-(r)$ denote the minimum number of points of X to be found in the interval $(-r, r)$, formally, $v^-(r) = \min_{x\in R} \#(X \cap (x - r, x + r))$. The Beurling lower density is defined by

$$D^-(X) = \underline{\lim}_{r\to\infty} \frac{v^-(r)}{2r}. \tag{2}$$

Beurling-Landau type sampling theorem in shift invariant spaces is not yet established so far [3]. In 1996, Liu [31] established the iterative reconstruction algorithm [20], [22] in the spline shift invariant spaces, and then, in 1998, Aldroubi and Feichitinger [2] established the iterative reconstruction algorithm in a general shift invariant space. But the maximal gap of the a sampling set for the iterative algorithm in a general shift invariant space is unknown so far. Our objective in this paper is to find the maximal gap of sampling set for the iterative reconstruction algorithm in a general shift invariant space. We shall also calculate some typical examples to illustrate our result.

II. Exact Iterative Reconstruction Algorithm in Shift Invariant Spaces and theMaximal Gap of a Sampling Set

To prefilter a signal f by a shift invariant space $V_\lambda(\varphi)$ is equivalent to making a *quasiprojection* $P\colon L^2(\mathbb{R}) \to V_\lambda(\varphi)$ defined by

$$P(f) = \sum_k \langle f, \varphi(\cdot - k)\rangle \varphi(\cdot - k)$$

where $\langle \cdot, \cdot \rangle$ is the *inner product* in $L^2(\mathbb{R})$ defined by

$$\langle f, g\rangle = \int_{\mathbb{R}} f(t)g(t)\,dt, \quad f, g \in L^2(\mathbb{R}).$$

The aliasing error $e = \|f - P(f)\|$ can be made arbitrarily small as long as the dilation λ is sufficiently large [7], [11], [27], [28]. In this paper, we focus on the nonuniform sampling in shift invariant spaces. Hence we shall ignore the prefiltering and assume that the signal f is taken from a shift invariant space $V_\lambda(\varphi)$. In this section, we are going to find the maximal gap of sampling set for the iterative algorithm in shift invariant spaces. We shall work in the framework of *Wiener amalgam spaces* [19], [21], which is commonly used in sampling theorem for shift invariant spaces. A Wiener amalgam space W^p for some $p \geq 1$ consists of all measurable functions φ for which the norm

$$\|\varphi\|_{W^p} = \left(\sum_k \sup_{t\in[0,1]} |\varphi(t-k)|^p\right)^{1/p} < \infty.$$

We now introduce the exact iterative algorithm [20] in shift invariant spaces in a different sense to [1], [2].

For a discrete set $X = \{t_k\}_k$, the δ-*ball* $B_\delta(t_k)$ for a $\delta > 0$ is defined by

$$B_\delta(t_k) = \{t\colon |t_k - t| \leq \delta\}. \tag{4}$$

X is called δ-*dense* if $\cup_k B_\delta(t_k) = \mathbb{R}$, where δ is called the *maximal gap*[2] of X if δ is the smallest one such that X is δ-dense. Take a sequence $\{U_k\}_k$ of nonnegative functions such that U_k is supported in $B_\delta(t_k)$ and

$$\sum_k U_k = 1. \tag{5}$$

Then the $\{U_k\}_k$ is called a *bounded uniform partition of unity (BUPU)* with respect to X. For a signal $f \in W^2$, we define the interpolation operator S_X by

$$S_X(f) = \sum_k f(t_k)U_k. \tag{6}$$

It is well-known that the operator S_X maps the space W^2 into the space $W^2 \subset L^2(\mathbb{R})$ if X is separated [18]. We also use the symbol G_φ with the norms $\|\cdot\|_0$ and $\|\cdot\|_\infty$ being defined by

$$G_\varphi = \left(\sum_k |\hat{\varphi}(\cdot + 2k\pi)|^2\right)^{1/2} \tag{7}$$

$$\|G_\varphi\|_0 = \inf_{\omega\in[0,2\pi]} G_\varphi(\omega) \tag{8}$$

$$\|G_\varphi\|_\infty = \sup_{\omega\in[0,2\pi]} G_\varphi(\omega). \tag{9}$$

For a signal $f \in V_\lambda(\varphi)$, it is easy to show $f \in W^2$ if $\varphi \in W^1$ [2], [10], [18]. Assume $\varphi \in W^1$. Then we can interpolate the samples $\{f(t_k)\}_k$ of a signal $f \in V_\lambda(\varphi)$ with $S_X(f)$. By projecting the interpolation function $S_X(f)$ into $V_\lambda(\varphi)$, we get an approximation $f_1 = PS_X(f) \in V_\lambda(\varphi)$ of the original signal f. Then we interpolate the samples $\{E_1(t_k) = f(t_k) - f_1(t_k)\}_k$ of the error $E_1 = f - f_1 \in V_\lambda(\varphi)$ with $S_X(E_1)$. By projecting the interpolation function $S_X(E_1)$ of the error E_1, we get an approximation $e_1 = PS_X(E_1) \in V_\lambda(\varphi)$ of the error E_1. Adding e_1 to f_1, we get a new approximation $f_2 = f_1 + e_1$ to the original signal $f \in V_\lambda(\varphi)$. Let $E_2 = f - f_2$, and repeat the interpolating and projecting procedures. We obtain an approximation sequence $f_{n+1} = f_1 + \sum_{k=1}^n e_k$ of the original signal f. In the following we will show that f_n converges to f in $L^2(\mathbb{R})$.

By the algorithm defined above, we know $E_n = f - f_n$. Then $E_{n+1} - E_n = f_n - f_{n+1} = -e_n = -PS_X(f - f_n) = -PS_X(E_n)$, that is $E_{n+1} = E_n - PS_X(E_n)$. Let $T = I - PS_X$, where I is the identity operator. Then $E_{n+1} = T(E_n)$, and consequently $\|E_{n+1}\| \leq \|T\|\|E_n\|$ where $\|T\|$ is the norm of the operator T defined by

$$\|T\| = \sup_{f\in L^2(\mathbb{R})} \frac{\|T(f)\|}{\|f\|}.$$

If we can show that T is a contraction, that is, $\|T\| < 1$. Then $\|E_n\| \to 0$. Consequently $f_n \to f$ in $L^2(\mathbb{R})$.

In order to show that T is a contraction, we define the oscillation function $\mathrm{osc}_\delta(f)$ of a signal f as

$$\mathrm{osc}_\delta(f) = \sup_{|s|\leq\delta} |f - f(\cdot + s)|. \tag{10}$$

For a signal $f \in V_\lambda(\varphi)$, there is a $\{c_k\}_k \in l^2$ such that $f = \sum_k c_k\varphi(\lambda \cdot - k)$. Therefore,

$$\mathrm{osc}_\delta(f)(t) \leq \int_t^{t+\delta} |f'(s)|\,ds = \int_{\mathbb{R}} |f'(s+t)|\chi_{[0,\delta]}(s)\,ds.$$

Then

$$\|\mathrm{osc}_\delta(f)\| \leq \int_{\mathbb{R}} \|f'(s + \cdot)\|\chi_{[0,\delta]}(s)\,ds = \delta\|f'\|.$$

[2]Maximal gap in this paper does not mean the "optimal maximal gap," which is the maximum one among the gaps between two consecutive samples of X, i.e., $\delta = \sup_k |t_k - t_{k-1}|$ if $X = \{t_k\}_k$.

By Parserval identity, we have

$$\begin{aligned}\|f'\| &= \lambda \left\| \sum_k c_k \varphi'(\lambda \cdot -k) \right\| \\ &= \frac{\lambda}{\sqrt{2\pi}} \left\| \frac{1}{\lambda} \widehat{\varphi}'(\cdot/\lambda) \sum_k c_k e^{-ik\cdot/\lambda} \right\| \\ &= \frac{\lambda}{\sqrt{2\pi}} \left\| \sqrt{\frac{1}{\lambda}} \hat{\varphi}' \sum_k c_k e^{-ik\cdot} \right\| \\ &= \frac{\lambda}{\sqrt{2\pi}} \sqrt{\int_0^{2\pi} \left| \frac{G_{\varphi'}(\omega)}{\sqrt{\lambda} G_\varphi(\omega)} G_\varphi(\omega) \sum_k c_k e^{-ik\cdot} \right|^2 d\omega} \\ &\leq \lambda \left\| \frac{G_{\varphi'}}{G_\varphi} \right\|_\infty \|f\|.\end{aligned}$$

Therefore,

$$\|\text{osc}_\delta(f)\| \leq \delta\lambda \left\| \frac{G_{\varphi'}}{G_\varphi} \right\|_\infty \|f\|. \tag{11}$$

Now we have to find the condition on δ such that T is a contraction

$$\begin{aligned}\|T(f)\| &= \|f - PS_X(f)\| \\ &= \|P(f) - PS_X(f)\| \\ &\leq \|P\| \|f - S_X(f)\| \\ &\leq \|P\| \|\text{osc}_\delta(f)\| \\ &\leq \lambda\delta \|P\| \left\| \frac{G_{\varphi'}}{G_\varphi} \right\|_\infty \|f\|.\end{aligned} \tag{12}$$

Since P is an orthogonal projection, we have $\|P\| = 1$. Therefore T is a contraction if the maximal gap δ is small enough such that

$$\lambda\delta \left\| \frac{G_{\varphi'}}{G_\varphi} \right\|_\infty < 1. \tag{13}$$

This is formulated in the following theorem.

Theorem 1: Assume that $\varphi \in W^1$ is a differentiable generator. Then any $f \in V_\lambda(\varphi)$ can be reconstructed from any δ-dense set X by the following iterative algorithm:

$$\begin{cases} f_{n+1} = PS_X(f) + (I - PS_X)(f_n) \\ f_0 = PS_X(f), \end{cases} \tag{14}$$

provided that

$$\delta < \left\| \frac{G_\varphi}{\lambda G_{\varphi'}} \right\|_0. \tag{15}$$

In practical application, one needs to know the explicit expression for P and S_X. Define ψ by $\hat{\psi} = \hat{\varphi}/G_\varphi^2$. Then for any $f \in L^2(\mathbb{R})$, one has

$$P(f) = \sum_k \langle f, \psi(\cdot - k) \rangle \varphi(\cdot - k).$$

Define the *Voronoi domain* V_k of the sampling points t_k as

$$V_k = \{t \colon |t_k - t| < |t_j - t|, k \neq j\}. \tag{16}$$

Let χ_{V_k} be the characteristic function of V_k. Then for any sampling set $X = \{t_k\}_k$, one can choose a simple interpolation function $S_X(f)$ as

$$S_X(f) = \sum_k f(t_k) \chi_{V_k}. \tag{17}$$

In this special case, one can improve the estimate (15) by a factor π by using Wirtinger's inequality as in [31], that is, (15) becomes

$$\delta < \left\| \frac{\pi G_\varphi}{\lambda G_{\varphi'}} \right\|_0. \tag{18}$$

Let $c = \lambda\delta \|G_{\varphi'} G_\varphi^{-1}\|_\infty$. Then $c < 1$ if δ satisfies (15). From (12), we have

$$(1 - c)\|f\| \leq \|S_X(f)\| \leq (1 + c)\|f\|. \tag{19}$$

Moreover, if X is separated with separation d, then

$$\|S_X(f)\|^2 = \left\| \sum_k f(t_k) \chi_{X_k} \right\|^2 \geq d \sum_k |f(t_k)|^2. \tag{20}$$

On the other hand, since X is δ-dense, we have

$$\|S_X(f)\|^2 = \left\| \sum_k f(t_k) \chi_{X_k} \right\|^2 \leq \delta \sum_k |f(t_k)|^2. \tag{21}$$

By (19), (20) and (21), we derive

$$\frac{1 - c}{\sqrt{\delta}} \|f\| \leq \left(\sum_k |f(t_k)|^2 \right)^{1/2} \leq \frac{1 + c}{\sqrt{d}} \|f\|. \tag{22}$$

This is summarized in the following corollary.

Corollary 1: In Theorem 1, the X is a stable sampling set for $V_\lambda(\varphi)$ if X is separated.

When X is a stable sampling set for $V_\lambda(\varphi)$, an explicit reconstruction formula is found by us in [10].

If B satisfies the *Strang-Fix condition* [35], that is $\hat{B}(2k\pi) = \delta(k)$. Then $\sum_k B(\cdot - k) = 1$ by the *Poisson Summation Formula*. For a uniform sampling set $X = \{\tau k\}_k$, if B is supported on $[-D, D]$, then we can define S_X by

$$S_X(f) = \sum_k f(\tau k) B(\cdot/\tau - k).$$

Then Theorem 1 holds if $\tau < \|(G_\varphi / \lambda D G_{\varphi'})\|_0$.

III. Conclusion and Examples

Prefiltering a signal by a shift invariant space $V_\lambda(\varphi)$ provides more flexible choices to design an A/D conversion system of high accuracy. This paper focuses on the nonuniform sampling in a shift invariant space $V_\lambda(\varphi)$. We obtain a formula to calculate the maximal gap of a sampling set for the iterative reconstruction algorithm in the shift invariant space $V_\lambda(\varphi)$ in terms of the generator φ, that is

$$\delta < \left\| \frac{G_\varphi}{\lambda G_{\varphi'}} \right\|_0. \tag{23}$$

When the interpolation function is chosen to be a simple function, one can improve (23) by a factor of π and obtain the improved estimate

$$\delta < \left\| \frac{\pi G_\varphi}{\lambda G_{\varphi'}} \right\|_0. \tag{24}$$

To construct a practical A/D conversion in a shift invariant spaces, it is necessary to understand the maximal gap of sampling set for performing sampling in a shift invariant space. Let's look at some practical examples.

Example 1: Meyer scaling function φ is defined by

$$\hat{\varphi} = \begin{cases} 1, & |\omega| \le \frac{2\pi}{3}, \\ \cos\left[\frac{\pi}{2} v\left(\frac{3}{2v}|\omega| - 1\right)\right], & \frac{2\pi}{3} \le |\omega| \le \frac{4\pi}{3}, \\ 0, & \text{otherwise} \end{cases}$$

where $v \in C^\infty, v(\omega) = 1$ for $\omega \ge 1, v(\omega) = 0$ for $\omega \le 0$, and $v(\omega) + v(1-\omega) = 1$. Then $G_\varphi = 1$. Since $G_{\varphi'} \le 4\pi/3$, we obtain the maximal gap

$$\delta < \frac{3}{4\pi\lambda}.$$

Example 2: B-spline β^N of degree N is defined by the N times convolution of the characteristic function of the interval $[0,1]$, i.e., $\beta^N = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$. Then $\|G_\varphi\|_0 \ge ((2/\pi))^{N+1}$ and $\|G_{\varphi'}\|_\infty \le \sqrt{(2N+1/2N-1)}$. Therefore, the maximal gap is

$$\delta < \sqrt{\frac{2N-1}{2N+1}} \left(\frac{2}{\pi}\right)^{N+1}.$$

Example 3: Gaussian kernel K is defined by $K(t) = 1/(\sqrt{2\pi})e^{-t^2/2}$. Then $\hat{K}(\omega) = e^{-\omega^2}$, and we have $\|G_\varphi\|_0 \ge e^{-\pi^2}$ and $\|G_{\varphi'}\|_\infty \le 1 + 2e^{-\pi^2}$. Therefore, we obtain the maximal gap

$$\delta < \frac{e^{-\pi^2}}{\lambda(1+2e^{-\pi^2})}.$$

REFERENCES

[1] A. Aldroubi, "Non-uniform weighted average sampling and exact reconstruction," *Appl. Comput. Harmon. Anal.*, vol. 13, no. 2, pp. 151–161, 2002.

[2] A. Aldroubi and H. Feichtinger, "Exact iterative reconstruction algorithm for multivariate irregularly sampled functions in spline-like spaces: the L_p theory," *Proc. Amer. Math. Soc.*, vol. 126, no. 9, pp. 2677–2686, 1998.

[3] A. Aldroubi and K. Gröchenig, "Beurling-Landau type theorems for nonuniform sampling in shift invariant spline spaces," *J. Fourier Anal. Applicat.*, vol. 6, no. 1, pp. 93–103, 2000.

[4] ——, "Non-uniform sampling and reconstruction in shift invariant spaces," in *SIAM Rev.*, vol. 43, 2001, pp. 858–620.

[5] A. Aldroubi and M. Unser, "Sampling procedure in function spaces and asymptotic equivalence with Shannon's sampling theory," *Numer. Anal. Optim.*, vol. 15, no. 1, pp. 1 –21, 1994.

[6] A. Beurling, *The Collected Works of A. Beurling*. Boston, MA: Birkhauser, 1989, vol. 2, pp. 341–365.

[7] T. Blu and M. Unser, "Approximation error for quasiinterpolators and (multi-)wavelet expansions," *Appl. Comput. Harmon. Anal.*, vol. 6, no. 2, pp. 219–251, 1999.

[8] C. de Boor, C. DeVore, and A. Ron, "The structure of finitely generated shift invariant spaces in $L^2(R^d)$," *J. Funct. Anal.*, vol. 119, no. 1, pp. 37–78, 1994.

[9] C. K. Chui, "An introduction to wavelets," in *Wavelet Analysis and its Applications*. New York: Academic, 1992, vol. 1.

[10] W. Chen, B. Han, and R.-Q. Jia. On simple oversampled A/D conversion in shift invariant spaces [Online]. Available: http://www.ualberta.ca/~wenc/files/publications.html

[11] ——, "Estimate of aliasing error for signals prefiltered by quasi-projections into shift invariant spaces," *IEEE Trans. Signal Processing*, to be published.

[12] W. Chen and S. Itoh, "A sampling theorem for shift invariant subspace," *IEEE Trans. Signal Processing*, vol. 46, pp. 2822–2824, Oct. 1998.

[13] ——, "Oversampling theorem for wavelet subspaces," *IEICE Trans. Fund.*, vol. 81, no. 1, pp. 131–138, 1998.

[14] W. Chen, S. Itoh, and J. Shiki, "Irregular sampling theorem for wavelet subspaces," *IEEE Trans. Inform. Theory*, vol. 44, no. 3, pp. 1131–1142, 1998.

[15] ——, "On sampling in shift invariant spaces," *IEEE Trans. Inform. Theory*, vol. 48, no. 10, pp. 2802–2810, 2002.

[16] I. Daubechies, *Ten Lectures on Wavelets*. Philadelphia, PA: SIAM, 1992.

[17] D. J. DeFatta, J. G. Lucas, and W. S. Hodgkiss, *Digital Signal Processing: A System Design Approach*: John Wiley & Sons, Inc., 1988.

[18] H. G. Feichtinger, "Banach convolution algebra of Wiener type," in *Proc. Conf. Functions, Series, Operators*, Budapest, Hungary, 1980, pp. 509–524.

[19] H. G. Feichtinger, "Generalized amalgams, with application to Fourier transform," *Can. J. Math.*, vol. 42, no. 3, pp. 395–409, 1990.

[20] H. G. Feichtinger and K. Gröchenig, "Iterative reconstruction of multivariate bandlimited functions from irregular sampling values," in *SIAM J. Math. Anal.*, vol. 231, 1992, pp. 244–261.

[21] J. J. Fournier and J. Stewart, "Amalgams of l^p and l^q," *Bull. Amer. Math. Soc.*, vol. 13, no. 1, pp. 1–21, 1985.

[22] K. Gröchenig, "Reconstruction algorithms in irregular sampling," *Math. Comput.*, vol. 59, no. 199, pp. 181–194, 1992.

[23] K. Gröchenig and H. Razafinjatovo, "On Landau's necessary density conditions for sampling and interpolation of bandlimited function," *J. Lond. Math. Soc.*, vol. 54, no. 3, pp. 557–567, 1996.

[24] C. Herley and P. Wong, "Minimum rate sampling and reconstruction of signals with arbitrary frequency support," *IEEE Trans. Inform. Theory*, vol. 45, no. 5, pp. 1555–1564, 1999.

[25] A. J. E. M. Janssen, "The Zak-transform and sampling theorem for wavelet subspaces," *IEEE Trans. Signal Processing*, vol. 41, pp. 3360–3364, Dec. 1993.

[26] R.-Q. Jia, "Shift-invariant spaces and linear operator equations," *Israel J. Math.*, vol. 103, pp. 259–288, 1998.

[27] R.-Q. Jia and J. Lei, "Approximation by multiinteger of translates of functions having global support," *J. Approx. Theory*, vol. 72, no. 1, pp. 2–23, 1993.

[28] R.-Q. Jia, J. Lei, and E. W. Cheney, "Approximation from shift invariant spaces by integral operators," in *SIAM J. Math. Anal.*, vol. 28, 1997, pp. 481–498.

[29] H. Landau, "Necessary density conditions for sampling and interpolation of certain entire functions," *Acta Math.*, vol. 117, pp. 37–52, 1967.

[30] J. J. Lei, "L_p-approximation by certain projection operations," *J. Math. Anal. Applicat.*, vol. 185, no. 1, pp. 1–1, 1994.

[31] Y. Liu, "Irregular sampling for spline wavelet subspaces," *IEEE Trans. Inform. Theory*, vol. 42, no. 2, pp. 623–627, 1996.

[32] J. A. Parker, R. V. Kenyon, and D. E. Troxel, "Comprarison of interpolating methods for image resampling," *IEEE Trans. Med. Imag.*, vol. 2, no. 2, pp. 31–39, 1983.

[33] C. E. Shannon, "Communications in the presence of noise," in *Proc. IRE*, vol. 37, 1949, pp. 10–21.

[34] D. Slepian, "On bandwidth," in *Proc. IRE*, vol. 64, 1976, pp. 31–39.

[35] G. Strang and G. Fix, "A Fourier analysis of finite element variational method," in *Constructive Aspects of Functional Analysis*, G. Geymonat, Ed: C. I. M. E., 1973, pp. 793–840.

[36] W. Sun and X. Zhou, "Reconstruction of functions in spline subspaces from local averages," in *Proc. AMS*, vol. 131, 2003, pp. 1561–2571.

[37] M. Unser, "Sampling—50 year after Shannon," *Proc. IEEE*, vol. 42, pp. 2915–2925, Nov. 1995.

[38] M. Unser and A. Aldroubi, "A general sampling theory for nonideal acquisition devices," *IEEE Trans. Signal Processing*, vol. 42, pp. 2915–2925, Nov. 1994.

[39] G. G. Walter, "A sampling theorem for wavelet subspaces," *IEEE Trans. Inform. Theory*, vol. 38, no. 2, pp. 881–884, 1992.

[40] X. G. Xia and Z. Z. Zhang, "On sampling theorem, wavelets, and wavelet transforms," *IEEE Trans. Signal Processing*, vol. 41, pp. 3524–2525, Dec. 1993.