Quantization Errors of fGn and fBm Signals

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Abstract

In this Letter, we show that under the assumption of high resolution, the quantization errors of fGn and fBm signals with uniform quantizer can be treated as uncorrelated white noises.

1 Introduction

Fractional Gaussian noise (fGn) and fractional Brownian motion (fBm) provide convenient ways to describe stochastic processes with long-range dependencies. Thus, they have received continuing interests in various fields and have many applications, e.g. modeling the communication networks flow and economic times series [1]-[3].

Of particular interest is the estimation of the Hurst exponent H of a fGn or fBm process. In practice, such estimations are usually done on the sampled and quantized time series. For example, texture images are often viewed as 2D fBm signals uniformly quantized to the $0 - 255$ scale [4], [5]. As will be shown later, the quantization error might significantly affect the estimation result.

To the best of our knowledge, no reports discussed the effect of quantization errors of fGn and fBm processes. In this Letter, we will show that under the assumption of high resolution, the quantization error can be viewed as a white noise added to the sampled fGn or fBm signal.

2 The Discrete-Time fGn and fBm Signals

There exist different kinds of discrete-time approximation for the continuous-time fGn and fBm processes, e.g. [6]-[8]. In this Letter, we will use the two-step discrete-time approximation signals defined in [9]:

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1) First, the standard discrete-time fGn process $W^H(n)$ with Hurst exponent $H \in (0,1)$ is defined as a weighted sequence of a standard Gaussian white noise $W(n)$

$$W^H(n) = \begin{cases} \sum_{i=0}^{n} h_{n-i}^{H-\frac{1}{2}} W(i) & \text{for } H \neq \frac{1}{2} \\ W(n) & \text{for } H = \frac{1}{2} \end{cases} \tag{1}$$

where $W(n) \sim N(0,1)$ and $h_n^{H-\frac{1}{2}} = \frac{\Gamma(n+H-\frac{1}{2})}{\Gamma(H-\frac{1}{2})\Gamma(n+1)}$, $n \in \mathbb{N} \cup \{0\}$.

2) Second, the discrete-time fBm process $B^H(n)$ is represented as the running sum of $W^H(n)$

$$B^H(n) = \sum_{i=0}^{n} W^H(i) = \sum_{i=0}^{n} \sum_{j=0}^{i} h_{i-j}^{H-\frac{1}{2}} W(j) \tag{2}$$

As pointed out in [10], [11], Eq.(1)-(2) are indeed ARFIMA processes, which can be rewritten in terms of the lag operator L:

$$W^H(n) = \begin{cases} (1-L)^{\frac{1}{2}-H} W(n) & \text{for } H \neq \frac{1}{2} \\ W(n) & \text{for } H = \frac{1}{2} \end{cases} \tag{3}$$

and

$$(1-L)B^H(n) = B^H(n) - B^H(n-1) = W^H(n) \tag{4}$$

where $(1-L)^d = \sum_{k=0}^{\infty} \frac{\Gamma(k-d)}{\Gamma(-d)\Gamma(k+1)} L^k$. The truncated formulas in (1)-(4) are equivalent to infinite formulas with $W(i) = 0$ for $i \leq 0$; they are used here since we usually consider the case with $W^H(i) = 0$ for $i \leq 0$ [1]-[3].

For clarity, we focus on the above standard discrete-time fBm process but the conclusions can be easily extended to general cases.

3 The High-resolution Quantization Errors of the fGn and fBm Signals

The use of high resolution theory for error process analysis can date back to late 1940s [12], [13]. In [13], Bennett demonstrated that under the assumption of high resolution and smooth density of the sampled random process, the quantization error behaves like an additive white noise. In other words, the quantization error has small correlation with the signal and an approximately white spectrum; see also the good surveys in [14]-[16]. In the sequel, we will show that this conclusion also holds for fGn and fBm signals. Our proof mainly uses the results in [17].

Suppose the original discrete-time fGn or fBm sequence $S^H(n)$ is bounded within $[-b, b]$ in a finite time horizon $[0, t]$ and an M-level uniform quantizer in $[-b, b]$ is applied. We also assume the sample rate and the resolution of the quantizer are high enough.

As shown in [17], by defining $\Delta = \frac{2b}{M-1}$, the normalized quantization noise $e(n)$ of $S^H(n)$ can be represented as the normalized quantization noise of the sigma-delta modulator for $S^H(n)$:

$$e(n) \triangleq \frac{1}{2} - \langle \sum_{i=0}^{n} \left(\frac{(M-1)S^H(n)}{\Delta} + \frac{1}{2} \right) \rangle \tag{5}$$

where $\langle z \rangle = x \bmod 1$ is the fractional part of x.

In the following subsections, we will discuss the quantization errors of the fGn and fBm signals, respectively.

3.1 fGn Signals

For the sigma-delta modulator, we have the following useful lemmas.

Lemma 1 *[17] Define an casual stable MA process $x(n)$*

$$x(n) = \psi(L)z(n) = \sum_{i=0}^{n} \psi_i z(n-i) \tag{6}$$

where $z(n)$ is an i.i.d process having a certain distribution with smooth density. If the regression coefficients ψ_i of this MA process satisfy that there exist $\eta > 0$ and infinitely many values of r such that $|\psi_0 + ... + \psi_r| > \eta$, then the distribution of the normalized quantization error $e(n)$ under modulo sigma-delta modulation converges to the uniform distribution in $[-\frac{1}{2}, \frac{1}{2}]$ under the assumption of high resolution. [1]

Lemma 2 *[18] The following series ${}_1F_0(\alpha, z)$ is a special hypergeometric series*

$$\sum_{i=0}^{\infty} \frac{\Gamma(\alpha + i)}{\Gamma(\alpha)\Gamma(i+1)} z^i = {}_1F_0(\alpha, z) \tag{7}$$

where $\alpha, z \in \mathbb{C}$.

If $1 \geq Re(\alpha) > 0$, the series converges throughout the entire unit circle $|z| = 1$ except for the point $z = 1$. If $Re(\alpha) < 0$, the series converges (absolutely) throughout the entire unit circle $|z| = 1$. Particularly, the special hypergeometric series ${}_1F_0(\alpha, 1)$ converges to 0, when $Re(\alpha) < 0$.

Now we can prove the first main result of this Letter using *Lemma 1* and *Lemma 2*.

[1] *Lemma 1* is actually a slightly modified version of *Property 3* in [17], but it is not difficult to see that the proof in [17] can be applied to *Lemma 1* with little modification.

Theorem 1 *The quantization error of fGn process with uniform quantizer is asymptotically uniformly distributed in $[-\frac{1}{2}, \frac{1}{2}]$ under the assumption of high resolution.*

Proof 1 *We will discuss the following three cases of $0 < H < \frac{1}{2}$, $H = \frac{1}{2}$, and $\frac{1}{2} < H < 1$, respectively. From Eq.(3), we know that the coefficients of an fGn signal are $\psi_{G,i} = h_i^{H-\frac{1}{2}}$.*

i) For $H \in (0, \frac{1}{2})$, $\Gamma(H - \frac{1}{2}) < 0$. Because $0 > H - \frac{1}{2} > -\frac{1}{2}$, $\sum_{i=0}^{\infty} \psi_{G,i} = \sum_{i=0}^{\infty} \frac{\Gamma(i+H-\frac{1}{2})}{\Gamma(H-\frac{1}{2})\Gamma(i+1)} = 0$ by Lemma 2. Thus we cannot directly apply Lemma 1 here. However, the convergence speed of this series satisfies $\sum_{i=0}^{n} \frac{\Gamma(i+H-\frac{1}{2})}{\Gamma(H-\frac{1}{2})\Gamma(i+1)} \geq \frac{1}{\sqrt{n}}$ for $H \in (0, \frac{1}{2})$ [19]. Based on these observations, we will derive the limit distribution of the quantization error through the limit of its characteristic function.

As proven in [17], we can rewrite Eq.(5) as

$$e(n) = 1 - \frac{1}{2}\langle\theta(n)\rangle \tag{8}$$

where $\theta(n) \triangleq \sum_{i=0}^{n} \left(\frac{W^H(i)}{\Delta} + \frac{1}{2}\right)$.

The corresponding characteristic function can be written as

$$\begin{aligned} &\left|\Phi_{\langle\theta(n)\rangle}(2\pi l)\right| \\ &= \left|E\left\{\exp\left[2\pi\tfrac{l}{\Delta}\left(W^H(n) + ... + W^H(0)\right)\right]\right\}\right| \\ &= \left|E\left\{\exp\left[2\pi\tfrac{l}{\Delta}\left(\textstyle\sum_{i=0}^{n} h_i^{H-\frac{1}{2}} W(n-i) + ... + W(0)\right)\right]\right\}\right| \end{aligned} \tag{9}$$

The innermost sum in Eq.(9) can be grouped as

$$\begin{aligned} &\textstyle\sum_{i=0}^{n} h_i^{H-1/2} W(n-i) + ... + W(0) \\ &= h_0^{H-\frac{1}{2}} W(n) + (h_0^{H-\frac{1}{2}} + h_1^{H-\frac{1}{2}})W(n-1) + \\ &\quad ... + (h_0^{H-\frac{1}{2}} + ... + h_n^{H-\frac{1}{2}})W(0) \end{aligned} \tag{10}$$

Hence

$$\begin{aligned} &\lim_{n\to\infty} \Phi_{\langle\theta_n\rangle}(2\pi l) \\ = \quad &\lim_{n\to\infty} E\left\{\prod_{i=0}^{n} \Phi_W\left(2\pi\frac{l}{\Delta}\sum_{j=0}^{i} h_j^{H-\frac{1}{2}}\right)\right\} \end{aligned} \tag{11}$$

Notice that the characteristic function of a standard Gaussian process is $\Phi_W(\omega) = \exp(-\frac{1}{2}\omega^2)$, $\omega \in \mathbb{R}$. We have

$$\left|\Phi_W\left(2\pi\frac{l}{\Delta}\sum_{j=0}^{i} h_j^{H-\frac{1}{2}}\right)\right|$$

$$
\begin{aligned}
&= \left|\Phi_W\left(2\pi\frac{l}{\Delta}\sum_{j=0}^{i}\frac{\Gamma(H-\frac{1}{2}+j)}{\Gamma(H-\frac{1}{2})\Gamma(j+1)}\right)\right| \\
&\leq \left|\Phi_W\left(2\pi\frac{l}{\Delta}\frac{1}{\sqrt{i}}\right)\right| \qquad (12)
\end{aligned}
$$

The harmonic series $\sum_{i=1}^{\infty}\frac{1}{i}$ diverges; in other words, for any small positive number $\epsilon > 0$, we can always find a large enough integer n^* such that

$$
\sum_{i=1}^{n^*}\frac{1}{i} > -\ln(\epsilon)\left(\frac{\Delta}{2\pi l}\right)^2 \qquad (13)
$$

or equivalently

$$
\exp\left[-\left(\frac{2\pi l}{\Delta}\right)^2\sum_{i=1}^{n^*}\frac{1}{i}\right] < \epsilon \qquad (14)
$$

From (11), (12) and (14), it follows that for any small $\epsilon > 0$, there exists a large enough integer n^* such that for all $n > n^*$,

$$
\begin{aligned}
\left|\Phi_{\langle\theta_n\rangle}(2\pi l)\right| &\leq \left|E\left\{\prod_{i=0}^{n-1}\Phi_W\left(2\pi\frac{l}{\Delta}\frac{1}{\sqrt{i}}\right)\right\}\right| \\
&= \left|E\left\{\exp\left[-\left(2\pi\frac{l}{\Delta}\right)^2\sum_{i=0}^{n}\frac{1}{i}\right]\right\}\right| \\
&< \epsilon \qquad (15)
\end{aligned}
$$

which means

$$
\lim_{n\to\infty}\Phi_{\langle\theta_n\rangle}(2\pi l) = \begin{cases} 1 & ,l = 0 \\ 0 & ,l \neq 0 \end{cases} \qquad (16)
$$

Therefore, the distribution of $\langle\theta(n)\rangle$ converges to the uniform distribution in $[0,1]$ and the limit distribution of $e(n)$ is $U[-\frac{1}{2},\frac{1}{2}]$.

ii) For $H = \frac{1}{2}$, we directly have $\sum_{i=0}^{\infty}\psi_{G,i} = 1 \neq 0$, so Lemma 1 applies.

iii) For $H \in (\frac{1}{2}, 1)$, $\sum_{i=0}^{\infty}\psi_{G,i} = \sum_{i=0}^{\infty}\frac{\Gamma(i+H-\frac{1}{2})}{\Gamma(H-\frac{1}{2})\Gamma(i+1)}$ diverges by Lemma 2. It then follows from the definition of divergence that the premise of Lemma 1 are satisfied.

3.2 fBm Signals

To analyze the quantization error of fBm processes, we need another lemma from [17].

Lemma 3 *[17] Define an AR(1) process $x(n)$ as*

$$x(n) - x(n-1) = z(n) \tag{17}$$

If the input $z(n)$ is a stationary independent increments, the normalized quantization noise $e(n)$ of the modulo sigma-delta modulation converges to the uniform distribution in $[-\frac{1}{2}, \frac{1}{2}]$ and has a white spectrum under assumption of high resolution.

Lemma 3 directly applies to the quantization error of fBm processes with $H = \frac{1}{2}$ (indeed, the Brownian motion). For $H \neq 1/2$ the techniques used for proving this lemma in [17] can also be adopted to characterize the quantization error.

Theorem 2 *The quantization noise with uniform quantizer of fBm process is asymptotically uniformly distributed and white under the assumption of high resolution.*

Proof 2 *i) For $H = \frac{1}{2}$, we have*

$$B^H(n) - B^H(n-1) = W^H(n) \tag{18}$$

which follows directly from Lemma 3.

ii) For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, we will derive the limit distribution using the characteristic function. We can define

$$e(n) = 1 - \frac{1}{2}\langle \delta(n) \rangle \tag{19}$$

where $\delta(n) \triangleq \sum_{i=0}^{n} \left(\frac{B^H(i)}{\Delta} + \frac{1}{2} \right)$.

From Eq.(1) and Eq.(2), we know the regression coefficients of an fBm signal are $\psi_{B,i} = \sum_{j=0}^{i} (i-j) h_j^{H-\frac{1}{2}}$. Hence in the case of $H \in (\frac{1}{2}, 1)$, there exists $\eta > 0$ and infinitely many values of r such that

$$\begin{aligned} |\psi_{B,0} + \cdots + \psi_{B,r}| &= \left| r h_0^{H-\frac{1}{2}} + \cdots h_r^{H-\frac{1}{2}} \right| \\ &> \left| h_0^{H-\frac{1}{2}} + \cdots h_r^{H-\frac{1}{2}} \right| > \eta \end{aligned} \tag{20}$$

where the last inequality follows from part iii) of the proof of Theorem 1. Now Lemma 1 applies and the distribution of $e(n)$ converges to $U[-\frac{1}{2}, \frac{1}{2}]$.

Similarly, we can prove the case of $H \in (0, \frac{1}{2})$. Because $h_0^{H-\frac{1}{2}} > 0$, $h_i^{H-\frac{1}{2}} < 0$ for $i > 0$, we have

$$r h_0^{H-\frac{1}{2}} + \cdots + h_r^{H-\frac{1}{2}} > 0 \tag{21}$$

for $r \in \mathbb{N}$ *and thus*

$$\begin{aligned} &\lim_{r->\infty} |\psi_{B,0} + ... + \psi_{B,r}| \\ &= \lim_{r\to\infty} \left| rh_0^{H-\frac{1}{2}} + (r-1)h_1^{H-\frac{1}{2}} + \cdots + h_r^{H-\frac{1}{2}} \right| \\ &> \lim_{r\to\infty} \left| h_0^{H-\frac{1}{2}} + ... + h_r^{H-\frac{1}{2}} \right| = 0 \end{aligned} \tag{22}$$

Again Lemma 1 applies.

From Eq.(19), the limit correlation $\bar{R}\left((e(n), e(n+k)\right)$ *can be written as*

$$\begin{aligned} &\bar{R}\left((e(n), e(n+k)\right) \\ &= \tfrac{1}{4} - \bar{E}(\langle\delta(n)\rangle) + \bar{E}\{\langle\delta(n)\rangle\langle\delta(n+k)\rangle\} \end{aligned} \tag{23}$$

where the limit mean $\bar{E}(x(n)) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} E(x(n))$.

For $k \neq 0$, $(\langle\delta(n)\rangle, \langle\delta(n+k)\rangle)$ *converges in distribution to a random variable which is uniformly distributed in* $[0,1)\times[0,1)$*; thus we have* $\bar{E}(\langle\delta(n)\rangle, \langle\delta(n+k)) = \int_0^1\int_0^1 uvdudv = \frac{1}{4}$.

For $k = 0$*, we have* $\bar{E}\left(\langle\delta(n)\rangle^2\right) = \frac{1}{3}$ *and* $\bar{R}\left(\langle\delta(n)\langle\delta(n)\right) = \frac{1}{3}$.

By (23), we have

$$\bar{R}\left((e(n), e(n+k)\right) = \begin{cases} \frac{1}{12} & , k = 0 \\ 0 & , k \neq 0 \end{cases} \tag{24}$$

Thus, the normalized quantization error is white and is asymptotically uncorrelated with the output of the quantized signal. Therefore, we prove the whole conclusion.

4 Some Simulation Results

Fig.1 shows a typical Power Spectral Density (PSD) for the quantization error of a 1D quantized fBm signal, which indicates that the normalized quantization error is indeed white.

Fig.2 shows the PCA eigen-spectrum of the original fBm signal, the quantized fBm signal and the quantization error. According to the results given in [20]-[21], when the sampling data length K is a sufficiently large constant, the PCA eigenvalue spectrum of the auto-correlation of a 1D fBm process with Hurst exponent H decays as a power-law

$$\tilde{\lambda}_k \sim k^{-(2H+1)}, \qquad k = 1, ..., K \tag{25}$$

It is also proven in [20]-[21] that the numerical eigen-spectrum of a white noise should be a straight line with slope $\alpha_0 \approx 0$ in the log-log scale (the slope is not strictly 0 because of the finite sampling length effect). Moreover, when the 1D fBm signal is corrupted with additive white noise and the SNR is large enough, the eigenvalue spectrum of the corrupted signal crossovers from

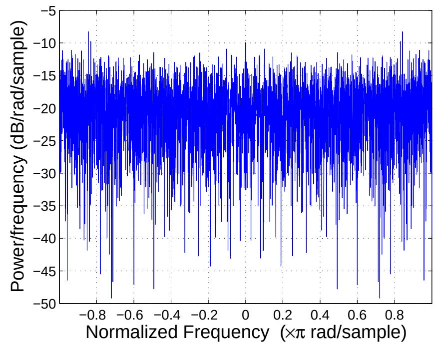

Figure 1: The PSD estimate for the quantization error of a 1D quantized fBm signal, where $H = 0.2$, the quantization scale is $\Delta = 1$. The PSD is estimated via periodogram method.

$\alpha_1 = -(2H+1)$ to α_0. Comparing Fig.2 to the simulation results provided in [20]-[21], we can see that under high resolution, the quantization error behaves exactly like a certain additive white noise.

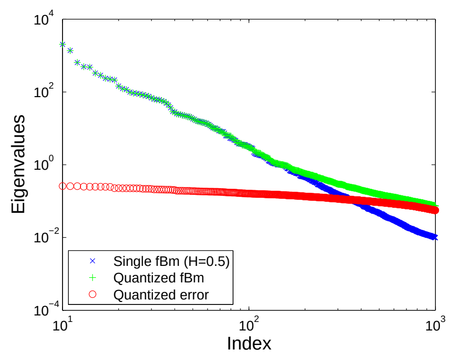

Figure 2: PCA eigen-spectrum of the auto-correlation of a standard fBm signal, its corresponding quantized signal, and the quantization error, where $H = 0.8$, and the quantization scale is $\Delta = 1$.

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