HHS Public Access
Author manuscript
IEEE Signal Process Lett. Author manuscript; available in PMC 2017 November 27.

Published in final edited form as:
IEEE Signal Process Lett. 2015 September ; 22(9): 1220–1223. doi:10.1109/LSP.2015.2393295.

Sparse Signal Recovery from a Mixture of Linear and Magnitude-Only Measurements

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Abstract

We consider the problem of exact sparse signal recovery from a combination of linear and magnitude-only (phaseless) measurements. A k-sparse signal $\mathbf{x} \in \mathbb{C}^n$ is measured as $\mathbf{r} = \mathbf{Bx}$ and $\mathbf{y} = |\mathbf{Cx}|$, where $\mathbf{B} \in \mathbb{C}^{m_1 \times n}$ and $\mathbf{C} \in \mathbb{C}^{m_2 \times n}$ are measurement matrices and $|\cdot|$ is the element-wise absolute value. We show that if $\max(2m_1, 1) + m_2 \geq 4k - 1$, then a set of generic measurements are sufficient to recover every k-sparse $\mathbf{x}$ exactly, establishing the trade-off between the number of linear and magnitude-only measurements.

Index Terms

sparse signals; sparse phase retrieval; compressed sensing; phase retrieval

I. Introduction

Let $\mathbf{x} \in \mathbb{C}^n$ be a sparse signal, where the number of non-zero coefficients of $\mathbf{x}$, denoted $\|\mathbf{x}\|_0$, is $k \ll n$.

Compressed (or compressive) sensing has emerged as a method for reconstructing sparse signals from $m < n$ linear measurements acquired as:

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \quad (1)$$

where $\mathbf{A} \in \mathbb{C}^{m \times n}$ is a sensing matrix. For exact sparse reconstruction, it is well-understood that $m = 2k$ is necessary and sufficient [1]. Moreover, this bound can be achieved with polynomial-time reconstruction algorithms for Vandermonde-based measurement matrix designs (which also include partial Fourier transforms that sample only the low-frequency components) [2]. Stability guarantees are possible when $\mathbf{A}$ is chosen randomly from the Gaussian ensemble, where $C_1 k \log(n/k)$ measurements suffice for reconstructing the sparse signal exactly (with no measurement noise), or with distortion scaling with the noise level (with measurement noise) when using reconstruction techniques based on minimizing the ℓ_1 norm [3]–[6]. Compressed sensing has been successfully used in a number of applications, including magnetic resonance imaging (MRI) [7], [8], sub-Nyquist sampling [9], radar imaging [10], and others (please see [11] and the references therein).

Recently, there has also been interest in reconstructing sparse signals from only the magnitude of the measurements. This process, referred to as sparse phase retrieval aims to estimate sparse signals from phaseless (magnitude-only) measurements

$$\mathbf{y}=|\mathbf{A}\mathbf{x}|, \quad (2)$$

where $|\cdot|$ denotes element-wise absolute value. These measurements can be used to model problems in diffractive imaging [12], astronomical imaging [13], x-ray crystallography [14] and medical imaging [15], where the measurement matrix **A** is typically the Fourier matrix. Sparsity of signals has been shown to reduce the number of measurements in practice [16], [17]. Sparse phase retrieval has also been studied theoretically in certain scenarios, e. g. $O(k \log(n/k))$ measurements were shown to be sufficient for stable sparse phase retrieval over $\mathbb{R}$ [18], matching the order of measurements in compressed sensing [4], [5]. More recently, the problem of exact reconstruction for noiseless sparse phase retrieval for complex signals has been studied [19], [20], where it was shown that $m = 4k - 2$ phaseless measurements suffice to guarantee uniqueness of the sparsest solution. It was also shown that $2k$ measurements suffice in the real case, matching the bound from compressed sensing, where linear measurements are available. We note that these latter studies [18]–[20] characterize when the measurements are injective, and are not algorithmic in nature, i.e. they do not provide tractable algorithms that can actually perform (robust) sparse phase retrieval.

In practice, there are a number of applications, including astronomical and medical imaging [15], where a mixture of linear and phaseless measurements are available as follows:

$$\begin{aligned} \mathbf{r} &= \mathbf{B}\mathbf{x} \qquad \text{and} \\ \mathbf{y} &= |\mathbf{C}\mathbf{x}| \end{aligned} \quad (3)$$

For instance, in MRI, where the measurements are taken in the Fourier domain, a translational motion of the scanned object during the examination does not affect the magnitude of the measurements [15], [21]. Hence, in a scan where translational motion has occurred, we have both motion-free (linear) measurements and motion-corrupted (phaseless) measurements. Typically, the motion-corrupted measurements would be re-acquired. However, the measurement model of (3) allows (using the transform-domain sparsity of MR images) for reconstruction from these motion-corrupted data, along with the motion-free data, thus reducing scan time.

In this note, we study the exact recovery of a k-sparse $\mathbf{x} \in \mathbb{C}^n$ from a combination of m_1 linear and m_2 phaseless measurements. We show that if the rows of **B** and **C** are a generic choice of vectors in $\mathbb{C}^n$, then $\max(2m_1, 1)+m_2 = 4k-1$ measurements are sufficient to recover every k-sparse signal uniquely (up to global phase if $m_1 = 0$). The outline of the paper is given next. We state our result in Section II. We provide the proof of our main result in Section III.

II. Main Results

For sparse reconstruction with a mixture of linear and phaseless measurements (referred to as a “mixed measurement system”), the reconstructor solves

$$\min \|\mathbf{x}\|_0 \ \text{s. t.} \quad \begin{pmatrix} \mathbf{r} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{Bx} \\ |\mathbf{Cx}| \end{pmatrix}, \qquad (4)$$

where $\mathbf{r} \in \mathbb{C}^{m_1}$ and $\mathbf{y} \in (\mathbb{R}^+)^{m_2}$.

We aim to characterize the number of sufficient measurements, $m = m_1 + m_2$, as well as the tradeoff between the number of linear and phaseless measurement, m_1 and m_2 respectively, in terms of the sparsity k (and possibly the dimensionality of the sparse signal n) for which there is a unique solution to the optimization problem in (4) (up to global phase if $m_1 = 0$). Our main result for k-sparse $\mathbf{x} \in \mathbb{C}^n$ is as follows:

Theorem 1

For $\mathbf{B} \in \mathbb{C}^{m_1 \times n}$ *and* $\mathbf{C} \in \mathbb{C}^{m_2 \times n}$, *whose rows are a generic choice of vectors in* $\mathbb{C}^n$, $\max(2m_1, 1) + m_2 \geq 4k - 1$ *measurements are sufficient to guarantee unique signal recovery* (*up to global phase if* $m_1 = 0$) *for every k-sparse signal* $\mathbf{x} \in \mathbb{C}^n$.

Here, a generic choice of vectors indicate a dense open set in $\mathbb{C}^n$ [22]. Intuitively, this suggests that a complex linear measurement is twice as important as a phaseless measurement. The trade-off between the number of linear and phaseless measurements, and the achievability region are depicted in Figure 1. We also state the results for the real case for completeness:

Theorem 2

For $\mathbf{B} \in \mathbb{R}^{m_1 \times n}$ *and* $\mathbf{C} \in \mathbb{R}^{m_2 \times n}$, *whose rows are a generic choice of vectors in* $\mathbb{R}^n$, $m_1 + m_2 \geq 2k$ *measurements are sufficient to guarantee unique signal recovery* (*up to global sign if* $m_1 = 0$) *for every k-sparse signal* $\mathbf{x} \in \mathbb{R}^n$. *Furthermore,* $m_1 + m_2 = 2k$ *measurements are necessary.*

Proof—This follows from the results in [1], [4], [19].

III. Proof of the Main Result

A. Notation

We define $\mathbb{T}(\mathbb{C}) = \{x \in \mathbb{C} : |x| = 1\}$. The space of diagonal phase matrices is defined as

$$\mathscr{P}_m(\mathbb{C})=\{\mathbf{P}\in((\mathbb{C})\cup\{0\})^{m\times m}:p_{ij}=0\,\forall i\neq j\text{ and }|p_{ii}|=1\,\forall i\}$$

where p_{ij} is the $(i, j)^{\text{th}}$ element of $\mathbf{P}$.

For any matrix $\mathbf{A}$, let a_{ij} be the $(i, j)^{\text{th}}$ element of $\mathbf{A}$, $\mathbf{a}_j$ be the j^{th} column of $\mathbf{A}$, and $\mathbf{a}^{(j)}$ be the j^{th} row. Let $\mathbf{A}_{\mathscr{J}}$ be matrix whose columns are $\{\mathbf{a}_j : j \in \mathscr{J}\}$, and $\mathbf{A}^{(\mathscr{J})}$ be the matrix whose rows are $\{\mathbf{a}^{(j)} : j \in \mathscr{J}\}$. We let $[l] = \{1, 2, \ldots, l\}$ for any positive integer l. Let $\mathbf{A}^T$ and $\mathbf{A}^*$ denote the transpose and conjugate transpose of $\mathbf{A}$. Finally, $Gr(k,m)$ denotes the Grassmannian manifold of k-dimensional subspaces of $\mathbb{C}^m$, endowed with the projection Frobenius (chordal) distance.

B. Proof of Theorem 1

We modify and extend the proof technique in [22]. We consider different regions of m_1 and m_2:

1) $m_1 > k$—Let the rows of $\mathbf{A}$ Õ $[\mathbf{B}^T,\mathbf{C}^T]^T \in \mathbb{C}^{m\times n}$ with $m = m_1 + m_2$ be a generic choice of vectors in $\mathbb{C}^n$. We note that any $k \times k$ submatrix of $\mathbf{A}$ is invertible. Let $\mathscr{Q},\mathscr{J} \subset [n]$ be two index sets of cardinality k. Let $\mathscr{V} = \operatorname{span}(\mathbf{a}_j : j \in \mathscr{Q})$ and $\mathscr{W} = \operatorname{span}(\mathbf{a}_j : j \in \mathscr{J})$.

Suppose there are two distinct k-sparse vectors, $\mathbf{x}, \mathbf{z} \in \mathbb{C}^n$, with supports $\mathscr{Q},\mathscr{J}$ respectively, such that

$$\begin{pmatrix}\mathbf{r}\\ \mathbf{y}\end{pmatrix}=\begin{pmatrix}\mathbf{B}_{\mathscr{Q}}\mathbf{x}_{\mathscr{Q}}\\ |\mathbf{C}_{\mathscr{Q}}\mathbf{x}_{\mathscr{Q}}|\end{pmatrix}=\begin{pmatrix}\mathbf{B}_{\mathscr{J}}\mathbf{z}_{\mathscr{J}}\\ |\mathbf{C}_{\mathscr{J}}\mathbf{z}_{\mathscr{J}}|\end{pmatrix}, \quad (5)$$

In other words,

$$\begin{pmatrix}\mathbf{r}\\ \mathbf{b}\end{pmatrix}=\begin{pmatrix}\mathbf{B}_{\mathscr{Q}}\mathbf{x}_{\mathscr{Q}}\\ \mathbf{C}_{\mathscr{Q}}\mathbf{x}_{\mathscr{Q}}\end{pmatrix}=\begin{pmatrix}\mathbf{B}_{\mathscr{J}}\mathbf{z}_{\mathscr{J}}\\ \mathbf{P}\mathbf{C}_{\mathscr{J}}\mathbf{z}_{\mathscr{J}}\end{pmatrix}, \quad (6)$$

for $\mathbf{y} = |\mathbf{b}|$ and for some $\mathbf{P} \in \mathscr{P}_{m_2}(\mathbb{C})$. We first note that $\mathscr{Q} \neq \mathscr{J}$ since with $m_1 > k$, $\mathbf{x}_{\mathscr{Q}}$ is uniquely determined as $(\mathbf{B}^*_{\mathscr{J}}\mathbf{B}_{\mathscr{J}})^{-1}\mathbf{B}^*_{\mathscr{J}}\mathbf{r}$. Thus $\mathscr{Q} = \mathscr{J}$ would imply $\mathbf{x}_{\mathscr{Q}} = \mathbf{z}_{\mathscr{J}}$, or $\mathbf{x} = \mathbf{z}$. We also note that $\mathbf{r}$ has at least $(m_1 - k + 1) > 1$ non-zero elements, otherwise this would imply the existence of a rank-deficient $k \times k$ submatrix of $\mathbf{B}_{\mathscr{Q}}$. Hence, without loss of generality we assume $r_1, r_2 \neq 0$. Since the optimization in (4) is scale-invariant, we can divide both sides by $r_1 \neq 0$, thus we assume $r_1 = 1$. We now re-write this set of equations as:

$$\mathbf{r}=\mathbf{B}_{\mathscr{J}}\left(\mathbf{B}_{\mathscr{J}}^{([k])}\right)^{-1}\mathbf{B}_{\mathscr{J}}^{([k])}\mathbf{x}_{\mathscr{J}}\triangleq\begin{bmatrix}\mathbf{I}\\ \mathbf{V}^{(1)}\end{bmatrix}\mathbf{d}, \quad (7)$$

$$\mathbf{b}=\mathbf{C}_{\mathscr{I}}\left(\mathbf{B}_{\mathscr{I}}^{([k])}\right)^{-1}\mathbf{B}_{\mathscr{I}}^{([k])}\mathbf{x}_{\mathscr{I}}\triangleq\mathbf{V}^{(\mathbf{2})}\ \mathbf{d}, \quad (8)$$

and

$$\mathbf{r}=\mathbf{B}_{\mathscr{I}}\left(\mathbf{B}_{\mathscr{I}}^{([k])}\right)^{-1}\mathbf{B}_{\mathscr{I}}^{([k])}\mathbf{z}_{\mathscr{I}}\triangleq\begin{bmatrix}\mathbf{I}\\ \mathbf{W}^{(1)}\end{bmatrix}\mathbf{e}, \quad (9)$$

$$\mathbf{b}=\mathbf{P}\mathbf{C}_{\mathscr{I}}\left(\mathbf{B}_{\mathscr{I}}^{([k])}\right)^{-1}\mathbf{B}_{\mathscr{I}}^{([k])}\mathbf{z}_{\mathscr{I}}\triangleq\mathbf{P}\mathbf{W}^{(\mathbf{2})}\,\mathbf{e}, \quad (10)$$

where $\mathbf{I}$ is the identity matrix. We note $\mathbf{d} = \mathbf{r}_{[k]}$ and $\mathbf{e} = \mathbf{r}_{[k]}$.

We say two distinct k-planes $(\mathscr{V},\mathscr{W})$, both in $\mathbb{C}^{m_1+m_2}$ satisfy the distinct-(m_1,m_2)-mapping property if there are distinct non-parallel vectors $[\mathbf{r}^T, \mathbf{b}^T]^T \in \mathscr{V}$ and $[\mathbf{r}^T, \mathbf{c}^T]^T \in \mathscr{W}$ with $\mathbf{r} \in \mathbb{C}^{m_1}$ and $\mathbf{b}, \mathbf{c} \in \mathbb{C}^{m_2}$, such that $|b_j| = |c_j|$ for $1 \le j \le m_2$.

From Equations (7) and (8), for $[\mathbf{r}^T, \mathbf{b}^T]^T \in \mathscr{V}$

$$\begin{aligned} r_i &= \sum_{j=1}^{k} r_j v_{ij}^{(1)} \quad \text{for } k<i\le m_1,\\ b_i &= \sum_{j=1}^{k} r_j v_{ij}^{(2)} \quad \text{for } 1\le i\le m_2, \end{aligned}$$

where $v_{ij}^{(1)}$ and $v_{ij}^{(2)}$ are the $(i, j)^{\text{th}}$ elements of $\mathbf{V}^{(\mathbf{1})}$ and $\mathbf{V}^{(\mathbf{2})}$ respectively. Similarly from Equations (9) and (10):

$$\begin{aligned} r_i &= \sum_{j=1}^{k} r_j w_{ij}^{(1)} \quad \text{for } k<i\le m_1,\\ c_i &= p_{ii}\sum_{j=1}^{k} r_j w_{ij}^{(2)} \quad \text{for } 1\le i\le m_2, \end{aligned}$$

where $w_{ij}^{(1)}$ and $w_{ij}^{(2)}$ are the $(i, j)^{\text{th}}$ elements of $\mathbf{W}^{(\mathbf{1})}$ and $\mathbf{W}^{(\mathbf{2})}$ respectively.

Hence if $(\mathscr{V},\mathscr{W})$ satisfies the distinct-(m_1,m_2)-mapping property, there exists $r_2, \ldots, r_k \in \mathbb{C}$ (since $r_1 = 1$) such that

$$\sum_{j=1}^{k} r_j (v_{ij}^{(1)} - w_{ij}^{(1)}) = 0 \qquad (11)$$

for $k < i \leq m_1$, and

$$\left| \sum_{j=1}^{k} r_j v_{ij}^{(2)} \right| = \left| \sum_{j=1}^{k} r_j w_{ij}^{(2)} \right| \qquad (12)$$

for $1 \leq i \leq m_2$.

We consider the following variety of all tuples

$$\left((\mathscr{V}, \mathscr{W}), r_2, \ldots, r_k \right). \qquad (13)$$

Let $|\mathscr{I} \cap \mathscr{J}| = l$, and note $l < k$ since $\mathscr{I} \neq \mathscr{J}$. The variety in (13) is locally isomorphic to $\mathbb{C}^{l(m-l)+2(k-l)(m-k+l)} \times (\mathbb{C}\backslash 0) \times \mathbb{C}^{k-2}$, corresponding to a real dimension $2l(m-l) + 4(k-l)(m-k+l) + 2k - 2$. Next, we note that the set of 2-tuples in $Gr(k,m) \times Gr(k,m)$ that satisfy the distinct- (m_1,m_2)-mapping property is the image of the projection onto the first factor of the variety in (13) subject to the $m_1 - k$ equations in (11) and the m_2 equations in (12) [22].

The measurements are generic, r_1, $r_2 \neq 0$ and $\mathscr{I} \neq \mathscr{J}$, thus each of the equations in (11) and (12) are non-degenerate. Since the variables $\{ v_{i1}^{(1)}, \ldots, v_{ik}^{(1)}, v_{i1}^{(2)}, \ldots, v_{ik}^{(2)} \}$ and $\{ w_{i1}^{(1)}, \ldots, w_{ik}^{(1)}, w_{i1}^{(2)}, \ldots, w_{ik}^{(2)} \}$ appear in exactly one equation, the equations in (11) define a subspace of real codimension $\geq 2(m_1 - k)$, whereas the equations in (12) define a subspace of real codimension $\geq m_2$. This is true for all choices, implying the equations are independent [22]. Therefore, the set of 2-tuples with an l-dimensional intersection in $Gr(k,m) \times Gr(k,m)$ that satisfy the distinct-(m_1,m_2)-mapping property have real dimension $\leq 2l(m-l)+4(k-l)(m-k+l)+2k-2-2(m_1 - k) - m_2 = 2l(m-l)+4(k-l)(m-k+l)+4k-2-(2m_1+m_2)$.

Thus, if $m_1 > k$ and $2m_1 + m_2 > 4k - 2$, then this set of 2-tuples cannot be the whole set of 2-tuples in $Gr(k,m) \times Gr(k,m)$ with an l-dimensional intersection, since this space has dimension $2l(m-l)+4(k-l)(m-k+l)$. In fact, if $2m_1 + m_2 > 4k - 2$, and the measurements are generic, then the set of 2-tuples with an l-dimensional intersection in $Gr(k,m) \times Gr(k,m)$ that satisfy the distinct- (m_1,m_2)-mapping property has measure 0. There are finitely many choices for $\mathscr{I}$ and $\mathscr{J}$ (and thus l), hence the results extend to all possible choices of k columns of **A**, using a union bound argument. Thus, if the rows of $\mathbf{A} \in \mathbb{C}^{(m_1+m_2)\times n}$ (equivalently $\mathbf{B} \in \mathbb{C}^{m_1 \times n}$ and $\mathbf{C} \in \mathbb{C}^{m_2 \times n}$) are a generic choice of vectors, $m_1 > k$ and $2m_1 + m_2 > 4k - 2$, no two sparse vectors with ℓ_0 norm $\leq k$ map to the same mixed measurements acquired using **B** and **C**.

We note that for $m_1 > k$, $m_2 = 4k - 1 - 2m_1 < 2k - 1$ phaseless equations suffice. Thus, we next consider the region $m_2 \geq 2k - 1$.

2) $m_2 \geq 2k - 1$**—**Suppose there are two distinct (or non-parallel if $m_1 = 0$) vectors $\mathbf{x}, \mathbf{z} \in \mathbb{C}^n$, with supports $\mathscr{I}, \mathscr{J}$ respectively, mapping to the same mixed measurements, as in Equation (5). In other words,

$$\begin{pmatrix} \mathbf{b} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{\mathscr{I}}\mathbf{x}_{\mathscr{I}} \\ \mathbf{B}_{\mathscr{I}}\mathbf{x}_{\mathscr{I}} \end{pmatrix} = \begin{pmatrix} \mathbf{P}'\mathbf{C}_{\mathscr{J}}\mathbf{z}_{\mathscr{J}} \\ \mathbf{B}_{\mathscr{J}}\mathbf{z}_{\mathscr{J}} \end{pmatrix}, \qquad (14)$$

for $\mathbf{y} = |\mathbf{b}|$ and for some $\mathbf{P}' \in \mathscr{P}_{m_2}(\mathbb{C})$.[1] Similar to Section III-B1, $\mathbf{b}$ has at least $(m_2 - k + 1) \geq k$ non-zero elements, thus we assume $b_1, \ldots, b_k$ are non-zero and that $b_1 = 1$. We now re-write Equation (14) as:

$$\mathbf{b}{=}\mathbf{C}_{\mathscr{I}}\left(\mathbf{C}_{\mathscr{I}}^{([k])}\right)^{-1}\mathbf{C}_{\mathscr{I}}^{([k])}\mathbf{x}_{\mathscr{I}} \triangleq \begin{bmatrix} \mathbf{I} \\ \mathbf{V}^{(3)} \end{bmatrix} \mathbf{d}', \qquad (15)$$

$$\mathbf{r}{=}\mathbf{B}_{\mathscr{I}}\left(\mathbf{C}_{\mathscr{I}}^{([k])}\right)^{-1}\mathbf{C}_{\mathscr{I}}^{([k])}\mathbf{x}_{\mathscr{I}} \triangleq \mathbf{V}^{(4)}\ \mathbf{d}', \qquad (16)$$

and

$$\mathbf{b}{=}\mathbf{P}'\mathbf{C}_{\mathscr{J}}\left(\mathbf{C}_{\mathscr{J}}^{([k])}\right)^{-1}\mathbf{C}_{\mathscr{J}}^{([k])}\mathbf{z}_{\mathscr{J}} \triangleq \mathbf{P}'\begin{bmatrix} \mathbf{I} \\ \mathbf{W}^{(3)} \end{bmatrix} \mathbf{e}', \qquad (17)$$

$$\mathbf{r}{=}\mathbf{B}_{\mathscr{J}}\left(\mathbf{C}_{\mathscr{J}}^{([k])}\right)^{-1}\mathbf{C}_{\mathscr{J}}^{([k])}\mathbf{z}_{\mathscr{J}} \triangleq \mathbf{W}^{(4)}\,\mathbf{e}'. \qquad (18)$$

We note $\mathbf{d}' = \mathbf{b}_{[k]}$ and $\mathbf{e}'{=}\left(\mathbf{P}'^{[k]}_{[k]}\right)^{-1}\mathbf{b}_{[k]}$.

Similar to Section III-B1, $(\mathscr{V}, \mathscr{W})$ satisfy the distinct-(m_1,m_2)-mapping property, if there exists $p'_{jj} \in (\mathbb{C})$ for $j \in [k]$ and $b_2, \ldots, b_k \in \mathbb{C}\backslash 0$ (since $b_1 = 1$) such that

[1]For the case $m_1 = 0$ and $\mathscr{I} = \mathscr{J}$, we have $\mathbf{P}' \in \mathscr{P}_{m_2}(\mathbb{C})\backslash\{\alpha \mathbf{I}_{m_2} : \alpha \in \mathbb{C}, |\alpha| = 1\}$, where $\mathbf{I}_{m_2}$ is the $m_2 \times m_2$ identity matrix, since one can only guarantee uniquness up to global phase.

$$\left|\sum_{j=1}^{k} b_j v_{ij}^{(3)}\right| = \left|\sum_{j=1}^{k} (b_j / p'_{jj}) w_{ij}^{(3)}\right| \quad (19)$$

for $k < i \leq m_2$, and

$$\sum_{j=1}^{k} b_j (v_{ij}^{(4)} - w_{ij}^{(4)}) = 0 \quad (20)$$

for $1 \leq i \leq m_1$, where $v_{ij}^{(3)}, v_{ij}^{(4)}, w_{ij}^{(3)}$ and $w_{ij}^{(4)}$ are the $(i, j)^{\text{th}}$ elements of $\mathbf{V}^{(3)}$, $\mathbf{V}^{(4)}$, $\mathbf{W}^{(3)}$ and $\mathbf{W}^{(4)}$ respectively.

We consider the following variety of all tuples

$$\left((\mathscr{V}, \mathscr{W}), b_2, \ldots, b_k, p'_{11}, p'_{22}, \ldots, p'_{kk}\right). \quad (21)$$

For $|\mathscr{I} \cap \mathscr{J}| = l$, with $0 \leq l \leq k$, the variety in (21) is locally isomorphic to $\mathbb{C}^{l(m-l)+2(k-l)(m-k+l)} \times (\mathbb{C}\backslash 0)^{k-1} \times \mathbb{T}(\mathbb{C})^k$, corresponding to a real dimension of $2l(m-l)+4(k-l)(m-k+l)+3k-2$. The set of 2-tuples in $Gr(k,m) \times Gr(k,m)$ that satisfy the distinct-(m_1,m_2)-mapping property is the image of the projection onto the first factor of the variety in (21) subject to the $2m_1$ equations in (20) and the $m_2 - k$ equations in (19). Since $b_2, \ldots, b_k \neq 0$, and since $\mathbf{P}'$ is not a multiple of identity when $\mathscr{I} = \mathscr{J}$ and $m_1 = 0$, these equations are non-degenerate. They are also independent, similar to Section III-B1. Thus, Equations (19) and (20) define subspaces of real codimensions $\geq m_2 - k$ and $\geq 2m_1$ respectively.

Therefore, the set of 2-tuples with an l-dimensional intersection in $Gr(k,m) \times Gr(k,m)$ that satisfy the distinct-(m_1,m_2)-mapping property have real dimension $\leq 2l(m-l) + 4(k-l)(m-k+l) + 3k - 2 - (m_2 - k) - 2m_2 = 2l(m-l) + 4(k-l)(m-k+l) + 4k - 2 - (2m_1 + m_2)$. Finally, we also note that if $m_1 = 0$, then p'_{11} can be set to 1 without loss of generality, since uniqueness is guaranteed only up to global phase in this case. Hence, for $m_1 = 0$, the 2-tuples will have real dimension $\leq 2l(m-l) + 4(k-l)(m-k+l) + 4k - 3 - m_2$.

Thus if $m_2 \geq 2k-1$ and $\max(2m_1, 1)+m_2 > 4k-2$, then this set of 2-tuples cannot be the whole set of 2-tuples in $Gr(k,m) \times Gr(k,m)$ with an l-dimensional intersection. Furthermore, if the measurements are generic, then this set has measure 0, similar to Section III-B1. A union bound argument over finitely many choices of $\mathscr{I}$ and $\mathscr{J}$ (and thus l) shows that if the rows of $\mathbf{A} \in \mathbb{C}^{(m_1+m_2)\times n}$ (equivalently $\mathbf{B} \in \mathbb{C}^{m_1\times n}$ and $\mathbf{C} \in \mathbb{C}^{m_2\times n}$) are a generic choice of vectors, $m_2 \geq 2k-1$ and $\max(2m_1, 1) + m_2 > 4k-2$, no two sparse vectors with ℓ_0 norm $\leq k$ map to the same mixed measurements acquired using $\mathbf{B}$ and $\mathbf{C}$.

Acknowledgments

M. Akçakaya would like to acknowledge grant support from NIH K99HL111410-01.

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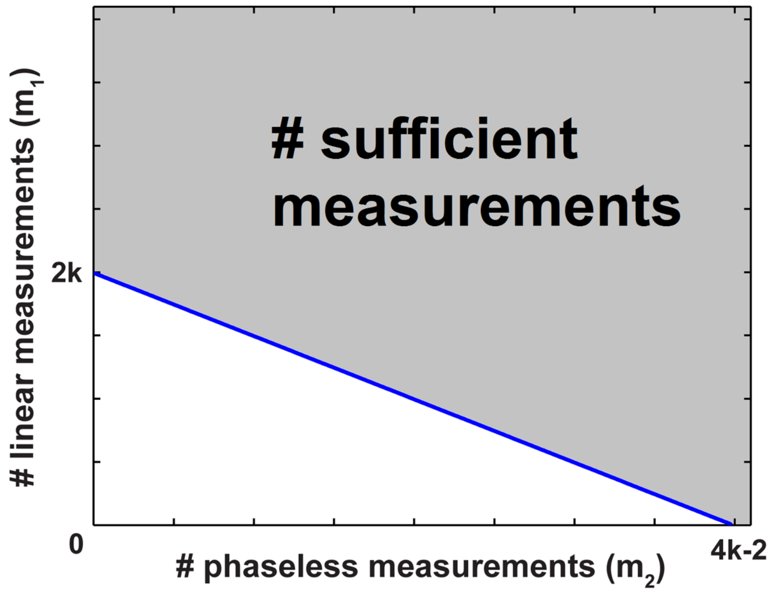

sufficient measurements
linear measurements (m1)
2k
0
phaseless measurements (m2)
4k-2

Figure 1.