# On the Properties of Cubic Metric for OFDM Signals

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#### Abstract

As a metric for amplitude fluctuation of orthogonal frequency division multiplexing (OFDM) signal, cubic metric (CM) has received an increasing attention because it is more closely related to the distortion induced by nonlinear devices than the well-known peak-to-average power ratio (PAPR). In this paper, the properties of CM of OFDM signal is investigated. First, asymptotic distribution of CM is derived. Second, it is verified that 1.7 times oversampling rate is good enough to capture the CM of continuous OFDM signals in terms of mean square error, which is also practically meaningful because the fast Fourier transform size is typically 1.7 times larger than the nominal bandwidth in the long-term evolution (LTE) of cellular communication systems.

#### **Index Terms**

Cubic metric (CM), fast Fourier transform (FFT), orthogonal frequency division multiplexing (OFDM), oversampling, peak-to-average power ratio (PAPR).

#### I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) is an attractive multicarrier modulation technique for broadband wireless access systems due to its strong immunity to multipath fading and high spectral efficiency. However, OFDM signals suffer from high amplitude fluctuation which causes performance degradation due to nonlinear devices. A well-known metric for

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amplitude fluctuation of OFDM signal is peak-to-average power ratio (PAPR). Many research efforts have been carried out to find efficient PAPR reduction techniques [1]–[6]. Also, the distribution of the PAPR of continuous OFDM signal was derived [7] and it is widely accepted that four times oversampling is enough to capture the PAPR of continuous OFDM signals [8].

Another metric for amplitude fluctuation of OFDM signals has been considered [9][10], which is known as cubic metric (CM) [11]. Studies on PAPR and CM suggest that, except for large power backoff, CM is more closely related to the amount of distortion induced by a nonlinear power amplifier than PAPR [10]. Moreover, after analyzing certain OFDM-type signals that are considered to meet the goal of the long-term evolution (LTE), it was shown in the 3GPP that CM predicts amplifier power de-rating more accurately than PAPR [12].

Thus, recent research to reduce the CM for the LTE systems has been carried out [13]–[16]. For example, in [13], the clipping and filtering method to reduce the CM instead of PAPR is proposed, where the descent clipper different to the conventional clipper is designed. In [14], the tone reservation method is proposed, where minimizing the CM value is set to the objective function of convex optimization.

A great deal of the literature has been devoted to research on the properties of the PAPR metric as in [7][8]. On the contrary, such analysis on the CM has not been done before. Thus, it is worth revealing more about the behavior of this metric. In this paper, an asymptotic probability distribution of CM for continuous OFDM signals is derived. Also, sufficient oversampling rate for capturing the CM of continuous OFDM signals is obtained.

## II. CUBIC METRIC

The CM of OFDM signals is defined as [12]

$$\left. \mathrm{CM} \right|_{\mathrm{dB}} \triangleq \frac{\mathrm{RCM} \Big|_{\mathrm{dB}} - \mathrm{RCM}_{\mathrm{ref}} \Big|_{\mathrm{dB}}}{K}$$

where RCM is the raw CM defined by

$$\operatorname{RCM}[s(t)]\Big|_{\mathrm{dB}} \triangleq 20 \log \left[ \operatorname{rms}\left[ \left( \frac{|s(t)|}{\operatorname{rms}[s(t)]} \right)^3 \right] \right]$$
(1)

for a continuous OFDM signal s(t) and both  $\text{RCM}_{\text{ref}}|_{dB}$  and K are determined according to the considered OFDM systems [12]. As an example, in the downlink of LTE,  $\text{RCM}_{\text{ref}}|_{dB} = 1.52 \text{ dB}$ 

and K = 1.56 are used. Thus, from (1), we are only interested in

$$\operatorname{RCM}[s(t)] = \sqrt{\mathbb{E}\left[\left(\frac{|s(t)|}{\sqrt{P_{\mathrm{av}}}}\right)^{6}\right]}$$

where  $P_{av}$  is the average power of the continuous OFDM signal s(t). To simplify analysis, we will use the square of it as

$$\xi \triangleq (\operatorname{RCM}[s(t)])^2 = \mathbb{E}\left[\left(\frac{|s(t)|}{\sqrt{P_{\operatorname{av}}}}\right)^{\operatorname{b}}\right]$$

In practice, instead of calculating the CM of continuous OFDM signals, we calculate the CM of discrete OFDM signals. Let s(t) be a continuous OFDM signal and its L times oversampled OFDM signal sequence be represented as

$$s_{n,L} \triangleq s(nT_s/LN), \qquad 0 \le n \le LN - 1$$

where  $T_s$  is the OFDM signal period, N is the number of subcarriers, and L is a real number larger than or equal to one.

Without loss of generality, the input symbols in frequency domain are assumed to be statistically independent, identically distributed (i.i.d.) random variables with zero mean, where the input symbol is the complex data of each subcarrier. Then the OFDM signal components in time domain are given by the sum of i.i.d. random variables. Thus, from the central limit theorem (CLT), the magnitude of  $s_{n,L}$  is Rayleigh distributed [7]. Therefore, if it is normalized as

$$\begin{split} r(t) &\triangleq |s(t)| / \sqrt{P_{\rm av}} \\ r_{n,L} &\triangleq |s_{n,L}| / \sqrt{P_{\rm av}}, \end{split}$$

the probability distribution functions (PDFs) of r(t) and  $r_{n,L}$  are given as

$$f_{r(t)}(r) = f_{r_{n,L}}(r) = 2 r e^{-r^2}$$

Finally, for the discrete OFDM signal sequence obtained by L times oversampling,  $\xi$  and RCM are expressed as

$$\xi_L = \frac{1}{LN} \sum_{n=0}^{LN-1} r_{n,L}^6$$
(2)

and

$$\operatorname{RCM}_L = \sqrt{\xi_L}.$$

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# III. PROPERTIES OF $r_{n,L}^6$

In (2),  $\xi_L$  is sample mean of  $r_{n,L}^6$ 's and thus in this section we investigate the properties of the random variable  $r_{n,L}^6$ . By the definition of Weibull distribution, the power transformation  $w_{n,L} \triangleq r_{n,L}^6$  of the Rayleigh distributed random variable  $r_{n,L}$  is known as Weibull distribution [17]. The PDF and cumulative distribution function (CDF) of  $w_{n,L}$  are given as

$$f_{w_{n,L}}(w) = \frac{1}{3}w^{-\frac{2}{3}}\exp(-w^{\frac{1}{3}})$$

and

$$F_{w_{n,L}}(w) = 1 - \exp(-w^{\frac{1}{3}}),$$

respectively. The *k*th-order moment of  $w_{n,L}$  is known as  $\mathbb{E}[w_{n,L}^k] = \Gamma(1+3k)$ , where  $\Gamma(a) = (a-1)!$  is the Gamma function for an integer x and  $\mathbb{E}[\cdot]$  denotes expectation value. Then we have

$$\mathbb{E}[w_{n,L}] = 3! = 6 \tag{3}$$

$$\mathbb{E}[w_{n,L}^2] = 6! = 720$$

$$Var(w_{n,L}) = 684 \tag{4}$$

where  $Var(\cdot)$  denotes variance.

Now, we calculate the covariance of two random variables  $w_{n,L}$  and  $w_{n',L'}$ , which will be denoted as  $Cov(w_{n,L}, w_{n',L'})$ . For this, we first obtain it with the continuous time lag  $\tau$ ,  $Cov(w(t), w(t + \tau))$ , where  $w(t) = r^6(t)$ . The joint moment of w(t) and  $w(t + \tau)$  is expressed as [17]

$$\mathbb{E}[w^{p}(t)w^{q}(t+\tau)] = (1-\rho_{\tau}^{2})^{1+3p+3q}\Gamma(1+3p)\Gamma(1+3q) {}_{2}F_{1}(1+3p,1+3q;1;\rho_{\tau}^{2})$$
(5)

where  ${}_{2}F_{1}(\cdot, \cdot; \cdot; \cdot)$  is the Gauss hypergeometric function and  $\rho_{\tau} = \rho_{s(t),s(t+\tau)}$  is the Pearson's correlation coefficient between s(t) and  $s(t+\tau)$  [17]. From (3) and (5), we have

$$Cov(w(t), w(t+\tau)) = \mathbb{E}[w(t)w(t+\tau)] - \mathbb{E}[w(t)]\mathbb{E}[w(t+\tau)] = 36(9\rho_{\tau}^2 + 9\rho_{\tau}^4 + \rho_{\tau}^6).$$

It can be assumed as in [7] that the power spectrum of baseband OFDM signal has conjugate symmetry, which is valid because the power spectrum of baseband OFDM signal can be designed to have symmetry at the center of the bandwidth by giving proper frequency offset. This assumption guarantees the autocorrelation function of s(t),  $\mathbb{E}[s(t)s(t+\tau)^*]$ , to be a real function. In any case, the frequency offset is immaterial to our analysis, because it does not change the magnitude of the envelope of OFDM signal. In this case, the normalized autocorrelation function  $\rho_{\tau}$  is known as [7]

$$\rho_{\tau} = \frac{\sin(\pi N \tau / T_s)}{N \, \sin(\pi \tau / T_s)}$$

for  $\tau \neq 0$  and clearly  $\rho_{\tau} = 1$  for  $\tau = 0$ . Note that  $\rho_{\tau} = 0$  when

$$\tau = \pm \frac{T_s}{N}, \pm \frac{2T_s}{N}, \pm \frac{3T_s}{N}, \cdots$$

which implies the well known fact that the elements of the Nyquist sampled OFDM signal sequence are mutually independent.

The time lag between two discrete samples  $w_{n,L}$  and  $w_{n',L'}$  is

$$\tau = \frac{nT_s}{LN} - \frac{n'T_s}{L'N}.$$

Finally, the covariance of  $w_{n,L}$  and  $w_{n',L'}$  is given as

$$\operatorname{Cov}(w_{n,L}, w_{n',L'}) = 36(9\rho_{\tau}^2 + 9\rho_{\tau}^4 + \rho_{\tau}^6) \bigg|_{\tau = \frac{nT_s}{LN} - \frac{n'T_s}{L'N}}.$$
(6)

## IV. DISTRIBUTION OF RCM

In this section, we obtain the asymptotic distribution of  $RCM_L$  by investigating the distribution of  $\xi_L$  first.

## A. Mean and Variance of $\xi_L$

The mean of  $\xi_L$  is clearly 6 from (3) because  $\xi_L$  is sample mean of  $w_{n,L}$ . To find the variance of  $\xi_L$ , suppose that LN is an odd integer. Even though LN can be any real number larger than or equal to N, it is not difficult to show that discrepancy by the assumption is negligible. Since  $s_{n,L}$  is a complex stationary Gaussian process, both  $r_{n,L}$  and  $w_{n,L}$  are also stationary random process. Therefore, the variance of  $\xi_L$  becomes

$$\operatorname{Var}(\xi_L) = \sigma_L^2 = \frac{\operatorname{Var}(w_{n,L})}{LN} + \frac{2}{(LN)^2} \sum_{k=1}^{LN-1} (LN-k) \operatorname{Cov}(w_{0,L}, w_{k,L}).$$
(7)

We can separate the summation in (7) into two parts and change the index of variable as

$$\sigma_L^2 = \frac{\operatorname{Var}(w_{n,L})}{LN} + \frac{2}{(LN)^2} \left( \sum_{k=1}^{\frac{LN-1}{2}} (LN-k) \operatorname{Cov}(w_{0,L}, w_{k,L}) + \sum_{k=1}^{\frac{LN-1}{2}} k \operatorname{Cov}(w_{0,L}, w_{LN-k,L}) \right).$$
(8)

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Using (4) and the fact that the covariance  $Cov(w_{0,L}, w_{k,L})$  in (6) is symmetric and periodic with the period LN, (8) is rewritten as

$$\sigma_L^2 = \frac{684}{LN} + \frac{2}{LN} \sum_{k=1}^{\frac{LN-1}{2}} \operatorname{Cov}(w_{0,L}, w_{k,L}).$$

Consider the following two extreme cases using (6).

1) For the Nyquist Sampling Rate: We have

$$\sigma_1^2 = \frac{684}{N}.$$

2) For the Continuous OFDM Signal: We have

$$\sigma_{\infty}^{2} = \lim_{L \to \infty} \sigma_{L}^{2} = \lim_{L \to \infty} \frac{2}{LN} \sum_{k=1}^{\frac{LN-1}{2}} \operatorname{Cov}(w_{0,L}, w_{k,L})$$

$$= \frac{72}{N\pi} \lim_{L \to \infty} \sum_{k=1}^{\frac{LN-1}{2}} \left(9\left(\frac{\sin(\frac{k\pi}{L})}{N\sin(\frac{k\pi}{LN})}\right)^{2} + 9\left(\frac{\sin(\frac{k\pi}{L})}{N\sin(\frac{k\pi}{LN})}\right)^{4} + \left(\frac{\sin(\frac{k\pi}{L})}{N\sin(\frac{k\pi}{LN})}\right)^{6}\right) \frac{\pi}{L}$$

$$= \frac{72}{N\pi} \int_{0}^{\frac{N\pi}{2}} 9\left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^{2} + 9\left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^{4} + \left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^{6} dx \tag{9}$$

$$= \frac{36}{5N^{5}} + \frac{117}{N^{3}} + \frac{2799}{5N}.$$

Unless N is too small, it becomes approximately  $\sigma_{\infty}^2 \approx 2799/5N$ . The detailed derivation of the integration in (9) is explained in Appendix.

# B. Distribution of $\xi_L$

1) The Nyquist Sampling Rate Case (L = 1): In this case,  $w_{n,1}$ 's are *i.i.d.* and thus  $\xi_1$  is asymptotically Gaussian distributed due to CLT. That is,

$$\xi_1 \stackrel{a.s.}{\sim} \mathcal{N}\left(6, \sigma_1^2\right)$$

where *a.s.* means that the random variable is asymptotically distributed.

2) Natural Number Sampling Rate Case (L is a natural number): It is easy to see that in this case,  $\xi_L$  is asymptotically Gaussian distributed. We divide LN OFDM signal components  $s_{n,L}$  into L subsets  $S_0, S_1, \dots, S_{L-1}$  in the interleaved pattern as

$$S_q = \{s_{n,L} \mid n = Lp + q, \ 0 \le p \le N - 1\}, \quad 0 \le q \le L - 1.$$

6

Since the components  $s_{n,L}$  in the set  $S_q$  are mutually independent, CLT can be applied to each set. Then we have L asymptotically Gaussian distributions that are correlated. The sum of correlated Gaussian random variables is still Gaussian. Therefore, when L is a natural number, we have

$$\xi_L \stackrel{a.s.}{\sim} \mathcal{N}\left(6, \sigma_L^2\right).$$

3) Continuous OFDM Signal Case  $(L \to \infty)$ : Clearly, the distribution of  $\xi_{\infty}$  converges to some distribution as L increases and we already checked that the distributions of  $\xi_L$  for all natural number L are asymptotically Gaussian distribution. Therefore, the distribution of  $\xi_{\infty}$  is also Gaussian such that

$$\xi_{\infty} \stackrel{a.s.}{\sim} \mathcal{N}\left(6, \sigma_{\infty}^{2}\right).$$

4) L is a Real Number: Under some assumptions, the above three cases can be integrated into one general result. The CLT can be applied to m-dependent random process [18], where m-dependent means that two samples from a random process with the interval larger than m have no statistical dependency.  $w_{n,L}$  is an m-dependent random process from the fact that the correlation of the stationary random process  $w_{n,L}$  may rapidly diminish as the interval exceeds  $T_s/N$  [7]. Therefore, for any real number L larger than one,  $\xi_L$  can be considered as Gaussian distributed.

# C. Distribution of $\operatorname{RCM}_L|_{dB}$

For the Gaussian distributed  $\xi_L$  with mean 6 and variance  $\sigma_L^2$ , complementary CDF (CCDF) of  $\xi_L$  is given as

$$P(\xi_L > a) = 1 - F_{\xi_L}(a) = \frac{1}{2} \left[ 1 - \operatorname{erf}\left(\frac{a-6}{\sigma_L \sqrt{2}}\right) \right].$$

Thus, CCDF of  $\operatorname{RCM}_L|_{dB}$  is

$$P(\text{RCM}_L|_{\text{dB}} > a) = P(\xi_L > 10^{\frac{a}{10}})$$
$$= \frac{1}{2} \left[ 1 - \text{erf}\left(\frac{10^{\frac{a}{10}} - 6}{\sigma_L \sqrt{2}}\right) \right]$$

where  $\operatorname{erf}(a) = \frac{2}{\sqrt{\pi}} \int_0^a e^{-t^2} dt$  is the error function.



Fig. 1. Comparison of simulated and analytical CCDFs of  $\text{RCM}_{\infty}|_{dB}$ .

Fig. 1 compares the simulated and analytical CCDFs of  $\text{RCM}_{\infty}|_{dB}$ . Though we have obtained the distribution of  $\text{RCM}_L|_{dB}$  for general value of L, due to lack of space, we only present the comparison when  $L \to \infty$ , which is of importance practically. In the simulation, L is set to 32, which is enough to represent the continuous OFDM signal, and 16-quadrature amplitude modulation is used. It is widely known that the metrics describing the envelope behavior of OFDM signals such as CM and PAPR do not depend on the modulation order. Note that analysis is based on Gaussian approximation in Section IV-B. Unfortunately, Weibull distribution is a kind of heavy-tailed distribution and thus the sum of Weibull random variables slowly converges to Gaussian distribution as N increases but it shows good agreement when N is large as shown in Fig. 1.

## V. OVERSAMPLING RATE TO CAPTURE THE CM OF CONTINUOUS OFDM SIGNAL

Radically, the metrics such as PAPR and CM have to be calculated from continuous OFDM signals. However, it is impossible to handle with the continuous OFDM signal and instead one calculates the metrics from the oversampled discrete OFDM signal with sufficient oversampling rate. In this case, the metrics calculated from the oversampled OFDM signal can be viewed as an estimator for the metrics of the continuous OFDM signal. For instance, in the case of the CM calculation,  $\text{RCM}_L$  is the estimator for true parameter  $\text{RCM}_{\infty}$ . It is natural that a high sampling rate guarantees low estimation error but it also entails high complexity. Thus, finding a sufficient sampling rate is of great importance.

In [8], upper bounds on the estimation error of PAPR according to L is derived and it is proposed that four times oversampling rate is enough to capture the PAPR of continuous OFDM signals. However, in the case of CM, this approach is not useful. Differing from the PAPR, the estimator  $\text{RCM}_L$  rarely shows extremely large estimation error. Therefore, the total inspection approach is not suitable in the case of CM. This is due to the fact that the Weibull distribution is a kind of heavy-tailed distribution which means that its variance is quite large. For instance, when the OFDM signal sequence is an impulse signal with N = 1024, it has  $\text{RCM}_1 = 1024$ and  $\text{RCM}_{\infty} \simeq 759$ . In this case, the square error  $|\text{RCM}_1 - \text{RCM}_{\infty}|^2 \simeq 7 \times 10^4$  is much larger than the mean square error (MSE)  $\mathbb{E}[|\text{RCM}_1 - \text{RCM}_{\infty}|^2] \simeq 5 \times 10^{-3}$ . Thus, in this section we statistically approach this problem by deriving the MSE  $\mathbb{E}[|\text{RCM}_L - \text{RCM}_{\infty}|^2]$  according to L.

# A. Joint PDF of $\xi_L$ and $\xi_{\infty}$

To obtain the MSE  $\mathbb{E}[|\text{RCM}_L - \text{RCM}_{\infty}|^2] = \mathbb{E}[|\sqrt{\xi_L} - \sqrt{\xi_{\infty}}|^2]$ , first we find the joint PDF of  $\xi_L$  and  $\xi_{\infty}$  which is clearly bivariate Gaussian distribution from our investigation in Section IV-B. We already checked their mean values  $\mathbb{E}[\xi_L] = \mathbb{E}[\xi_{\infty}] = 6$ . Next, the correlation coefficient

between  $\xi_L$  and  $\xi_\infty$  is given as

$$\begin{split} \rho_{\xi_L,\xi_{\infty}} &= \frac{\operatorname{Cov}(\xi_L,\xi_{\infty})}{\sigma_L \sigma_{\infty}} \\ &= \lim_{L' \to \infty} \frac{\sum_{n=0}^{LN-1} \sum_{n'=0}^{L'N-1} \operatorname{Cov}(w_{n,L}, w_{n',L'})}{\sigma_L \sigma_{\infty} L L' N^2} \\ &= \lim_{L' \to \infty} \frac{\sum_{n=0}^{LN-1} \sum_{n'=0}^{L'N-1} 36(9\rho_{\tau}^2 + 9\rho_{\tau}^4 + \rho_{\tau}^6) \big|_{\tau = \frac{nT_s}{LN} - \frac{n'T_s}{L'N}}}{\sigma_L \sigma_{\infty} L L' N^2} \\ &= \frac{36}{\sigma_L \sigma_{\infty} \pi L N^2} \cdot \sum_{n=0}^{LN-1} \int_{-\frac{n\pi}{L}}^{-\frac{n\pi}{L} + N\pi} 9\left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^2 + 9\left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^4 + \left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^6 dx \end{split}$$

where the equation in the integration is periodic with the period  $N\pi$ . Thus, we have

$$\rho_{\xi_L,\xi_\infty} = \frac{36}{\sigma_L \sigma_\infty \pi N} \cdot \int_0^{N\pi} 9\left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^2 + 9\left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^4 + \left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^6 dx$$
$$= \frac{\sigma_\infty}{\sigma_L}.$$
(10)

In terms of estimation theory,  $\rho_{\xi_L,\xi_\infty} = \sigma_\infty/\sigma_L$  implies that  $\xi_L$  can be considered as an unbiased minimum MSE estimator of  $\xi_\infty$ . That is, orthogonality principle  $\mathbb{E}[(\xi_L - \xi_\infty) \ \xi_\infty] = 0$  and unbiased property  $\mathbb{E}[\xi_L] = \mathbb{E}[\xi_\infty]$  are satisfied.

# B. MSE between $\xi_L$ and $\xi_{\infty}$

Using the orthogonality principle, the MSE between  $\xi_L$  and  $\xi_\infty$  is obtained as

$$\mathbb{E}[|\xi_L - \xi_\infty|^2] = \sigma_L^2 - \sigma_\infty^2. \tag{11}$$



Fig. 2. Comparison of simulated and analytical  $\mathbb{E}[|\xi_L - \xi_{\infty}|^2]$  for various L.

Fig. 2 compares  $\mathbb{E}[|\xi_L - \xi_{\infty}|^2]$  values obtained by simulation and analysis, which shows a good agreement. Analytical results are given by using (11) and simulation results are obtained by testing randomly generated 10<sup>5</sup> OFDM signal sequences.

# C. MSE between $\operatorname{RCM}_L$ and $\operatorname{RCM}_{\infty}$

The MSE between  $\operatorname{RCM}_L$  and  $\operatorname{RCM}_\infty$  can be obtained as

$$\mathbb{E}[|\mathrm{RCM}_L - \mathrm{RCM}_{\infty}|^2] = \mathbb{E}[|\sqrt{\xi_L} - \sqrt{\xi_{\infty}}|^2]$$
$$= 12 - 2 \cdot \mathbb{E}[\sqrt{\xi_L \xi_{\infty}}]$$
$$\stackrel{a.s.}{\sim} 12 - 2 \int_0^\infty \int_0^\infty \sqrt{\xi_L \xi_{\infty}} f_{\xi_L,\xi_{\infty}}(\xi_1,\xi_2) \ d\xi_1 \ d\xi_2$$

where  $f_{\xi_L,\xi_\infty}(\xi_1,\xi_2)$  is the joint Gaussian PDF of  $\xi_1$  and  $\xi_2$  with the correlation coefficient  $\rho_{\xi_L,\xi_\infty}$  in (10).

Fig. 3 compares  $\mathbb{E}[|\text{RCM}_L - \text{RCM}_{\infty}|^2]$  values obtained by simulation and analysis using Gaussian approximation. From a practical viewpoint, MSE is usually normalized as  $\mathbb{E}[|\text{RCM}_L -$ 

 $\operatorname{RCM}_{\infty}|^{2}]/\mathbb{E}[|\operatorname{RCM}_{\infty}|^{2}] = \mathbb{E}[|\operatorname{RCM}_{L} - \operatorname{RCM}_{\infty}|^{2}]/6$ . Thus, one can conclude that 1.7 times oversampling gives the normalized MSE smaller than about  $10^{-4}$  for practical value of  $N \geq 256$ , which is remarkable because the FFT size is typically 1.7 times larger than the nominal bandwidth for the LTE cellular communication systems [19].



Fig. 3. Comparison of simulated and analytical  $\mathbb{E}[|\text{RCM}_L - \text{RCM}_{\infty}|^2]$  for various L using Gaussian approximation.

## VI. CONCLUSION

In this paper, the properties of CM are investigated. First, asymptotic distribution of the CM is obtained, which shows a good agreement to simulation results when the number of subcarriers is large. Second, the oversampling rate good enough to capture the CM of the continuous OFDM signal is investigated. We confirmed that 1.7 times oversampling rate is good enough for that purpose from a viewpoint of MSE, which is of great importance because the FFT size is typically 1.7 times larger than the nominal bandwidth for the LTE cellular communication systems.

### APPENDIX

Derivation of  $\int_0^{\frac{N\pi}{2}} (\frac{\sin(x)}{N\sin(\frac{x}{N})})^r dx$  When r = 2, 4, and 6

It is known that there are several methods to derive  $\int_0^{\frac{N\pi}{2}} (\frac{\sin(x)}{N\sin(\frac{\pi}{N})})^r dx$  when r = 2. By expanding one of them, both cases of r = 4 and 6 are derived analogously as follows. Without loss of generality, by using N = 2M and change of variables, we have more simplified expression as

$$\int_0^{\frac{N\pi}{2}} \left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^r dx = \frac{1}{2^r M^{r-1}} \int_0^\pi \left(\frac{\sin(2Mx)}{\sin(x)}\right)^r dx$$

where

$$\left(\frac{\sin(2Mx)}{\sin(x)}\right)^r = 2^r (\cos(x) + \cos(3x) + \dots + \cos((2M-1)x))^r.$$
 (12)

Since the integral is over  $[0, \pi]$ , only part of coefficients contribute to the integration. For example, when r = 2, all the cross terms from the polynomial expansion in (12) becomes zero after integration, i.e.,  $\int_0^{\pi} \cos(x) \cos(3x) dx = 0$ . After some manipulations, we have

$$\left(\frac{\sin(2Mx)}{\sin(x)}\right)^2 = 2^2 \left(\frac{M}{2} + z_2(x)\right)$$
$$\left(\frac{\sin(2Mx)}{\sin(x)}\right)^4 = 2^4 \left(\frac{1}{24}\left(M + 8M^3\right) + z_4(x)\right)$$
$$\left(\frac{\sin(2Mx)}{\sin(x)}\right)^6 = 2^6 \left(\frac{1}{160}\left(M + 5M^3 + 44M^5\right) + z_6(x)\right)$$

where  $z_r(x)$  denotes all terms which become zero after integration over  $[0, \pi]$ . Finally, we have

$$\int_{0}^{\frac{N\pi}{2}} \left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^{2} dx = \frac{\pi}{2}$$
$$\int_{0}^{\frac{N\pi}{2}} \left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^{4} dx = \pi \left(\frac{1}{3} + \frac{1}{6N^{2}}\right)$$
$$\int_{0}^{\frac{N\pi}{2}} \left(\frac{\sin(x)}{N\sin(\frac{x}{N})}\right)^{6} dx = \pi \left(\frac{11}{40} + \frac{1}{8N^{2}} + \frac{1}{10N^{4}}\right)$$

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