

The Exact Solution to Rank-1 L1-norm TUCKER2 Decomposition

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Abstract

We study rank-1 L1-norm-based TUCKER2 (L1-TUCKER2) decomposition of 3-way tensors, treated as a collection of N $D \times M$ matrices that are to be jointly decomposed. Our contributions are as follows. i) We prove that the problem is equivalent to combinatorial optimization over N antipodal-binary variables. ii) We derive the first two algorithms in the literature for its exact solution. The first algorithm has cost exponential in N ; the second one has cost polynomial in N (under a mild assumption). Our algorithms are accompanied by formal complexity analysis. iii) We conduct numerical studies to compare the performance of exact L1-TUCKER2 (proposed) with standard HOSVD, HOOI, GLRAM, PCA, L1-PCA, and TPCA-L1. Our studies show that L1-TUCKER2 outperforms (in tensor approximation) all the above counterparts when the processed data are outlier corrupted.

Index Terms

Data analysis, L1-norm, outliers, robust, TUCKER decomposition, tensors.

I. INTRODUCTION AND PROBLEM STATEMENT

Introduced by L. R. Tucker [1] in the mid-1960s, TUCKER decomposition is a fundamental method n -way tensor analysis, with applications in a wide range of fields, including machine learning, computer vision [2], [3], wireless communications [4], biomedical signal processing [5], and social-network data analysis [6], [7] to name a few. Considering that the n -way tensor under processing is formed by the concatenation (say, across the n -th mode, with no loss of generality) of a number of coherent (same class, or distribution) $(n - 1)$ -way coherent tensor measurements, then TUCKER decomposition simplifies to TUCKER2 decomposition. TUCKER2 strives to *jointly* decompose the collected $(n - 1)$ -way tensors and unveil the low-rank multi-linear structure of their class, or distribution. Higher-Order SVD (HOSVD) and Higher-Order Orthogonal Iteration (HOOI) algorithms [8] are well-known solvers for TUCKER2 (and TUCKER) decompositions. A detailed presentation of TUCKER, TUCKER2,

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and the respective solvers is offered in [9]–[11]. Note that both types of solvers can generally only guarantee a locally optimal solution.

For $n = 2$, TUCKER/TUCKER2 take the familiar form of Principal-Component Analysis (PCA). Thus, similar to PCA, TUCKER/TUCKER2 are sensitive against outliers within the processed tensor [12]–[14]. On the other hand, L1-Principal-component Analysis (L1-PCA) [15]–[17], substituting the L2-norm in PCA by the outlier-resistant L1-norm, has illustrated remarkable outlier-resistance. Extending this formulation to tensor processing, one can similarly endow robustness to TUCKER and TUCKER2 decompositions by substituting the L2-norm in their formulations by the L1-norm. Indeed, an approximate algorithm for L1-norm-based TUCKER2 (L1-TUCKER2) was proposed in [12]. However, L1-TUCKER2 remains to date unsolved. In this work, we offer for the first time the exact solution to L1-TUCKER2 for the special case of rank-1 approximation, and provide two optimal algorithms. A formal problem statement follows.

Consider a collection of N real-valued matrices of equal size, $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_N \in \mathbb{R}^{D \times M}$. For any rank $d \leq \min\{D, M\}$, a TUCKER2 decomposition strives to jointly analyze $\{\mathbf{X}_i\}_{i=1}^N$, by maximizing $\sum_{i=1}^N \|\mathbf{U}^\top \mathbf{X}_i \mathbf{V}\|_F^2$ over $\mathbf{U} \in \mathbb{R}^{D \times d}$ and $\mathbf{V} \in \mathbb{R}^{M \times d}$, such that $\mathbf{U}^\top \mathbf{U} = \mathbf{V}^\top \mathbf{V} = \mathbf{I}_d$; then, \mathbf{X}_i is low-rank approximated as $\mathbf{U}\mathbf{U}^\top \mathbf{X}_i \mathbf{V}\mathbf{V}^\top$. The squared Frobenius norm $\|\cdot\|_F^2$ returns the summation of the squared entries of its matrix argument. Among other methods in the tensor-processing literature, TUCKER2 coincides with Multilinear PCA [18] (for zero-centered matrices) and the Generalized Low-Rank Approximation of Matrices (GLRAM) [19]. Clearly, for $N = 1$, TUCKER2 simplifies to the rank- d approximation of matrix $\mathbf{X}_1 \in \mathbb{R}^{D \times M}$, solved by means of the familiar singular-value decomposition (SVD) [20]; i.e., the optimal arguments \mathbf{U} and \mathbf{V} are built by the d left-hand and right-hand singular vectors of \mathbf{X}_1 , respectively.

To counteract against the impact of any outliers in $\{\mathbf{X}_i\}_{i=1}^N$, in this work, we consider the L1-norm-based TUCKER2 reformulation

$$\text{L1-TUCKER2:} \quad \begin{array}{l} \text{maximize} \\ \mathbf{U} \in \mathbb{R}^{D \times d}, \mathbf{U}^\top \mathbf{U} = \mathbf{I}_d \\ \mathbf{V} \in \mathbb{R}^{M \times d}, \mathbf{V}^\top \mathbf{V} = \mathbf{I}_d \end{array} \sum_{i=1}^N \|\mathbf{U}^\top \mathbf{X}_i \mathbf{V}\|_1, \quad (1)$$

where the L1-norm $\|\cdot\|_1$ returns the summation of the absolute values of its matrix argument. The problem in (1) was studied in [12] under the title L1-Tensor Principal-Component Analysis (TPCA-L1).¹ Authors in [12] presented an approximate algorithm for its solution which they employed for image reconstruction. To date, (1) has not been solved exactly in the literature, even for the special case of rank-1 approximation (i.e., $d = 1$). In this work, we deliver, for the first time, the exact solution to L1-TUCKER2 for $d = 1$, by means of two novel algorithms. In addition, we provide numerical studies that demonstrate the outlier-resistance of exact L1-TUCKER2, and its

¹ In this work, we refer to the problem as L1-TUCKER2, so as to highlight its connection with the TUCKER2 formulation (instead of the general TUCKER formulation).

superiority (in joint-matrix decomposition and reconstruction) over L2-norm-based (standard) TUCKER2, GLRAM, TPCA-L1, PCA, and L1-PCA.

II. EXACT SOLUTION

A. Reformulation into combinatorial optimization

For rank $d = 1$, L1-TUCKER2 in (1) takes the form

$$\max_{\mathbf{u} \in \mathbb{R}^{D \times 1}; \mathbf{v} \in \mathbb{R}^{M \times 1}; \|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1} \sum_{i=1}^N |\mathbf{u}^\top \mathbf{X}_i \mathbf{v}| \quad (2)$$

First, we focus on the absolute value in (2) and notice that, for any $\mathbf{a} \in \mathbb{R}^N$, $\sum_{i=1}^N |a_i| = \sum_{i=1}^N \text{sgn}(a_i) a_i = \text{sgn}(\mathbf{a})^\top \mathbf{a} = \max_{\mathbf{b} \in \{\pm 1\}^N} \mathbf{b}^\top \mathbf{a}$, where $\text{sgn}(\cdot)$ returns the $\{\pm 1\}$ -sign of its (vector) argument. In view of the above, Lemma 1 follows.

Lemma 1. *For any given $\mathbf{u} \in \mathbb{R}^D$ and $\mathbf{v} \in \mathbb{R}^M$, it holds that*

$$\sum_{i=1}^N |\mathbf{u}^\top \mathbf{X}_i \mathbf{v}| = \max_{\mathbf{b} \in \{\pm 1\}^N} \mathbf{u}^\top \left(\sum_{i=1}^N b_i \mathbf{X}_i \right) \mathbf{v}. \quad (3)$$

The maximum in (3) is attained for $\mathbf{b} = [\text{sgn}(\mathbf{u}^\top \mathbf{X}_1 \mathbf{v}), \text{sgn}(\mathbf{u}^\top \mathbf{X}_2 \mathbf{v}), \dots, \text{sgn}(\mathbf{u}^\top \mathbf{X}_N \mathbf{v})]^\top$. \square

In addition, the following well-known Lemma 2 derives by the matrix-approximation optimality of SVD [20].

Lemma 2. *For any given $\mathbf{b} \in \{\pm 1\}^N$, it holds that*

$$\max_{\substack{\mathbf{u} \in \mathbb{R}^{D \times 1}; \|\mathbf{u}\|_2 = 1 \\ \mathbf{v} \in \mathbb{R}^{M \times 1}; \|\mathbf{v}\|_2 = 1}} \mathbf{u}^\top \left(\sum_{i=1}^N b_i \mathbf{X}_i \right) \mathbf{v} = \sigma_{\max} \left(\sum_{i=1}^N b_i \mathbf{X}_i \right) \quad (4)$$

where $\sigma_{\max}(\cdot)$ returns the highest singular value of its matrix argument. The maximum in (4) is attained if \mathbf{u} and \mathbf{v} are the left-hand and right-hand dominant singular vectors of $\sum_{i=1}^N b_i \mathbf{X}_i$, respectively. \square

To compact our notation, we concatenate $\{\mathbf{X}_i\}_{i=1}^N$ into $\mathbf{X} \doteq [\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_N] \in \mathbb{R}^{D \times MN}$. Then, for any $\mathbf{b} \in \{\pm 1\}^N$, it holds $\sum_{i=1}^N b_i \mathbf{X}_i = \mathbf{X}(\mathbf{b} \otimes \mathbf{I}_M)$, where \otimes denotes the Kronecker matrix product [21]. Then, in view of Lemma 1 and Lemma 2, we can rewrite the L1-TUCKER2 in (2) as

$$\max_{\substack{\mathbf{u} \in \mathbb{R}^{D \times 1}; \|\mathbf{u}\|_2 = 1 \\ \mathbf{v} \in \mathbb{R}^{M \times 1}; \|\mathbf{v}\|_2 = 1}} \sum_{i=1}^N |\mathbf{u}^\top \mathbf{X}_i \mathbf{v}| \quad (5)$$

$$= \max_{\substack{\mathbf{b} \in \{\pm 1\}^N \\ \mathbf{u} \in \mathbb{R}^{D \times 1}; \|\mathbf{u}\|_2 = 1 \\ \mathbf{v} \in \mathbb{R}^{M \times 1}; \|\mathbf{v}\|_2 = 1}} \mathbf{u}^\top (\mathbf{X}(\mathbf{b} \otimes \mathbf{I}_M)) \mathbf{v} \quad (6)$$

$$= \max_{\mathbf{b} \in \{\pm 1\}^N} \sigma_{\max}(\mathbf{X}(\mathbf{b} \otimes \mathbf{I}_M)). \quad (7)$$

It is clear that (7) is a combinatorial problem over the size- 2^N feasibility set $\{\pm 1\}^N$. The following Proposition 1 derives straightforwardly from Lemma 1, Lemma 2, and (5)-(7) and concludes our transformation of (2) into a combinatorial problem.

Proposition 1. *Let \mathbf{b}_{opt} be a solution to the combinatorial*

$$\underset{\mathbf{b} \in \{\pm 1\}^N}{\text{maximize}} \sigma_{\max}(\mathbf{X}(\mathbf{b} \otimes \mathbf{I}_M)) \quad (8)$$

and denote by $\mathbf{u}_{\text{opt}} \in \mathbb{R}^D$ and $\mathbf{v}_{\text{opt}} \in \mathbb{R}^M$ the left- and right-hand singular vectors of $\mathbf{X}(\mathbf{b}_{\text{opt}} \otimes \mathbf{I}_M) \in \mathbb{R}^{D \times M}$, respectively. Then, $(\mathbf{u}_{\text{opt}}, \mathbf{v}_{\text{opt}})$ is an optimal solution to (2). Also, $\mathbf{b}_{\text{opt}} = [\text{sgn}(\mathbf{u}_{\text{opt}}^\top \mathbf{X}_1 \mathbf{v}_{\text{opt}}), \dots, \text{sgn}(\mathbf{u}_{\text{opt}}^\top \mathbf{X}_N \mathbf{v}_{\text{opt}})]^\top$ and $\sum_{i=1}^N |\mathbf{u}_{\text{opt}}^\top \mathbf{X}_i \mathbf{v}_{\text{opt}}| = \mathbf{u}_{\text{opt}}^\top (\mathbf{X}(\mathbf{b}_{\text{opt}} \otimes \mathbf{I}_M)) \mathbf{v}_{\text{opt}} = \sigma_{\max}(\mathbf{X}(\mathbf{b}_{\text{opt}} \otimes \mathbf{I}_M))$. In the special case that $\mathbf{u}_{\text{opt}}^\top \mathbf{X}_i \mathbf{v}_{\text{opt}} = 0$, for some $i \in \{1, 2, \dots, N\}$, $[\mathbf{b}_{\text{opt}}]_i$ can be set to $+1$, having no effect to the metric of (8). \square

Given \mathbf{b}_{opt} , $(\mathbf{u}_{\text{opt}}, \mathbf{v}_{\text{opt}})$ are obtained by SVD of $\mathbf{X}(\mathbf{b}_{\text{opt}} \otimes \mathbf{I}_M)$. Thus, by Proposition 1, the solution to L1-TUCKER2 for low-rank $d = 1$ is obtained by the solution of the combinatorial problem (8) and a D -by- M SVD.

B. Connection to L1-PCA and hardness

In the sequel, we show that for $M = 1$ and $d = 1$, L1-TUCKER2 in (2) simplifies to L1-PCA [15]–[17]. Specifically, for $M = 1$, matrix \mathbf{X}_i is a $D \times 1$ vector, satisfying $\mathbf{X}_i = \mathbf{x}_i \doteq \text{vec}(\mathbf{X}_i)$, and (2) can be rewritten as

$$\underset{\mathbf{u} \in \mathbb{R}^D; v \in \mathbb{R}; \|\mathbf{u}\|_2 = |v| = 1}{\max} \sum_{i=1}^N |\mathbf{u}^\top \mathbf{x}_i v|. \quad (9)$$

It is clear that for every \mathbf{u} , an optimal value for v is trivially $v = 1$ (or, equivalently, $v = -1$); thus, for $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N] \in \mathbb{R}^{D \times N}$, (9) becomes

$$\underset{\mathbf{u} \in \mathbb{R}^D; \|\mathbf{u}\|_2 = 1}{\max} \sum_{i=1}^N |\mathbf{u}^\top \mathbf{x}_i| = \underset{\mathbf{u} \in \mathbb{R}^D; \|\mathbf{u}\|_2 = 1}{\max} \|\mathbf{X}^\top \mathbf{u}\|_1, \quad (10)$$

which is the exact formulation of the well-studied L1-PCA problem [15]–[17]. We notice also that for $M = 1$ the combinatorial optimization (8) in Proposition 1 becomes

$$\underset{\mathbf{b} \in \{\pm 1\}^N}{\max} \sigma_{\max}(\mathbf{X}(\mathbf{b} \otimes 1)) = \underset{\mathbf{b} \in \{\pm 1\}^N}{\max} \|\mathbf{X}\mathbf{b}\|_2, \quad (11)$$

since the maximum singular-value of a vector coincides with its Euclidean norm, which is in accordance to the L1-PCA analysis in [16], [17]. Based of the equivalence of L1-PCA to (11), [16] has proven that L1-PCA of \mathbf{X} is formally NP -hard in N , for jointly asymptotic N and $\text{rank}(\mathbf{X})$. Thus, by its equivalence to L1-PCA for $d = 1$ and $M = 1$, L1-TUCKER2 is also NP -hard in N , for jointly asymptotic N and $\text{rank}(\mathbf{X})$.

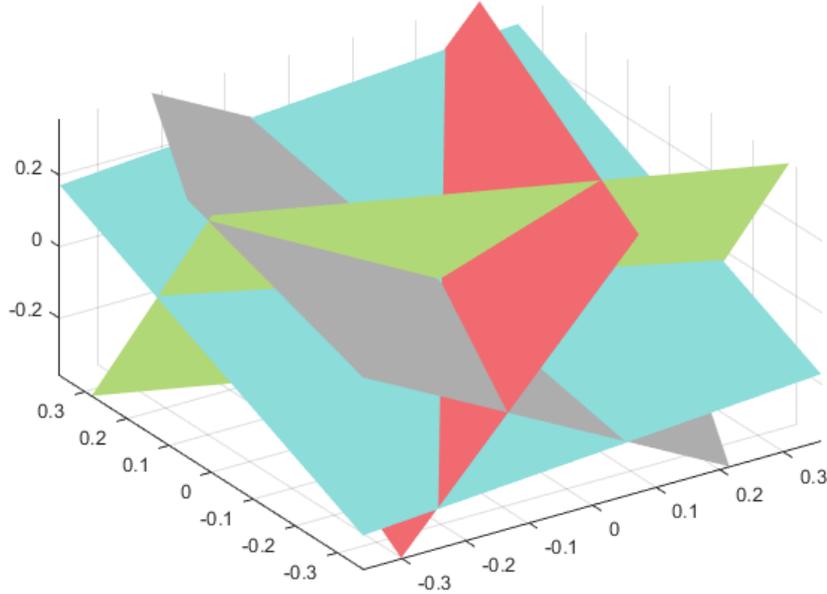


Fig. 1. For $\rho = 3$ and $N = 4$, we draw $\mathbf{W} \in \mathbb{R}^{\rho \times N}$, such that $\mathbf{W}\mathbf{W}^\top = \mathbf{I}_3$ and Assumption 1 holds true. Then, we plot the nullspaces of all 4 columns of \mathbf{W} (colored planes). We observe that the planes partition \mathbb{R}^3 into $K = 2(\binom{3}{0} + \binom{3}{1} + \binom{3}{2}) = 2(1 + 3 + 3) = 14$ coherent cells (i.e., 7 visible cells above the cyan hyperplane and 7 cells below.)

C. Exact Algorithm 1: Exhaustive search

Proposition 1 shows how the solution to (2) can be obtained through the solution to the combinatorial problem in (8). Our first exact algorithm solves (8) straightforwardly by an exhaustive search over its feasibility set. In fact, noticing that $\sigma_{\max}(\cdot)$ is invariant to negations of its matrix argument, we obtain a solution \mathbf{b}_{opt} to (8) by an exhaustive search in the size- 2^{N-1} set $\mathcal{B}_{\text{ex}} = \{\mathbf{b} \in \{\pm 1\}^N : b_1 = 1\}$. For every value that \mathbf{b} takes in \mathcal{B}_{ex} , we conduct SVD to $\mathbf{X}(\mathbf{b} \otimes \mathbf{I}_M)$ to calculate $\sigma_{\max}(\mathbf{X}(\mathbf{b} \otimes \mathbf{I}_M))$, with cost $\mathcal{O}(\min\{D, M\}DM)$ [20]. Since it entails 2^{N-1} SVD calculations, the cost of this exhaustive-search algorithm is $\mathcal{O}(2^{N-1} \min\{D, M\}DM)$; thus, it is exponential to the number of jointly processed matrices, N , and at most quadratic to the matrix sizes, D and M .

D. Exact Algorithm 2: Search with cost polynomial in N

In the sequel, we focus on the case where N is low-bounded by the constant DM and present an algorithm that solves (2) with polynomial cost in N . $DM < N$ emerges as a case of interest in signal processing applications when $\{\mathbf{X}_i\}_{i=1}^N$ are measurements of a $D \times M$ fixed-size sensing system (e.g., $D \times M$ images). By Proposition 1, for the optimal solutions \mathbf{b}_{opt} and $(\mathbf{u}_{\text{opt}}, \mathbf{v}_{\text{opt}})$ of (8) and (2), respectively, it holds

$$\mathbf{b}_{\text{opt}} = [\text{sgn}(\mathbf{v}_{\text{opt}}^\top \mathbf{X}_1^\top \mathbf{u}_{\text{opt}}), \dots, \text{sgn}(\mathbf{v}_{\text{opt}}^\top \mathbf{X}_N^\top \mathbf{u}_{\text{opt}})]^\top, \quad (12)$$

with $\text{sgn}(\mathbf{u}_{\text{opt}}^\top \mathbf{X}_i \mathbf{v}_{\text{opt}}) = +1$, if $\mathbf{u}_{\text{opt}}^\top \mathbf{X}_i \mathbf{v}_{\text{opt}} = 0$. In addition, for every $i \in \{1, 2, \dots, N\}$, we find that

$$\mathbf{v}_{\text{opt}}^\top \mathbf{X}_i^\top \mathbf{u}_{\text{opt}} = \text{Tr}(\mathbf{X}_i^\top \mathbf{u}_{\text{opt}} \mathbf{v}_{\text{opt}}^\top) = \mathbf{x}_i^\top (\mathbf{v}_{\text{opt}} \otimes \mathbf{u}_{\text{opt}}). \quad (13)$$

Therefore, defining $\mathbf{Y} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N] \in \mathbb{R}^{DM \times N}$, (12) can be rewritten as

$$\mathbf{b}_{\text{opt}} = \text{sgn} \left(\mathbf{Y}^\top (\mathbf{v}_{\text{opt}} \otimes \mathbf{u}_{\text{opt}}) \right). \quad (14)$$

Consider now that \mathbf{Y} is of some rank $\rho \leq \min\{DM, N\}$ and admits SVD $\mathbf{Y} \stackrel{\text{svd}}{=} \mathbf{Q}\mathbf{S}\mathbf{W}$, where $\mathbf{Q}^\top \mathbf{Q} = \mathbf{W}\mathbf{W}^\top = \mathbf{I}_\rho$ and \mathbf{S} is the $\rho \times \rho$ diagonal matrix that carries the ρ non-zero singular-values of \mathbf{Y} . Defining $\mathbf{p}_{\text{opt}} \doteq \mathbf{S}^\top \mathbf{Q}^\top (\mathbf{v}_{\text{opt}} \otimes \mathbf{u}_{\text{opt}})$, (15) can be rewritten as

$$\mathbf{b}_{\text{opt}} = \text{sgn} \left(\mathbf{W}^\top \mathbf{p}_{\text{opt}} \right). \quad (15)$$

In view of (15) and since $\text{sgn}(\cdot)$ is invariant to positive scalings of its vector argument, an optimal solution to (8), \mathbf{b}_{opt} , can be found in the binary set

$$\mathcal{B} = \{\mathbf{b} \in \{\pm 1\}^N : \mathbf{b} = \text{sgn} \left(\mathbf{W}^\top \mathbf{c} \right), \mathbf{c} \in \mathbb{R}^\rho\}. \quad (16)$$

Certainly, by definition, (16) is a subset of $\{\pm 1\}^N$ and, thus, has finite size upper bounded by 2^N . This, in turn, implies that there exist instances of $\mathbf{c} \in \mathbb{R}^\rho$ that yield the same value in $\text{sgn}(\mathbf{W}^\top \mathbf{c})$. Below, we delve into this observation to build a tight superset of \mathcal{B} that has polynomial size in N , under the following mild “general position” assumption [22].

Assumption 1. For every $\mathcal{I} \subset \{1, 2, \dots, N\}$ with $|\mathcal{I}| = \rho - 1$, it holds that $\text{rank}([\mathbf{W}]_{:, \mathcal{I}}) = \rho - 1$; i.e., any collection of $\rho - 1$ columns of \mathbf{W} are linearly independent.

For any $i \in \{1, 2, \dots, N\}$, define $\mathbf{w}_i \doteq [\mathbf{W}]_{:, i}$ and denote by \mathcal{N}_i the nullspace of \mathbf{w}_i . Then, for every $\mathbf{c} \in \mathcal{N}_i$, the (non-negative) angle between \mathbf{c} and \mathbf{w}_i , $\phi(\mathbf{c}, \mathbf{w}_i)$, is equal to $\frac{\pi}{2}$ and, accordingly, $\mathbf{w}_i^\top \mathbf{c} = \|\mathbf{c}\|_2 \|\mathbf{w}_i\|_2 \cos(\phi(\mathbf{c}, \mathbf{w}_i)) = 0$. Clearly, the hyperplane \mathcal{N}_i partitions \mathbb{R}^ρ in two non-overlapping halfspaces, \mathcal{H}_i^+ and \mathcal{H}_i^- [23], such that $\text{sgn}(\mathbf{c}^\top \mathbf{w}_i) = +1$ for every $\mathbf{c} \in \mathcal{H}_i^+$ and $\text{sgn}(\mathbf{c}^\top \mathbf{w}_i) = -1$ for every $\mathbf{c} \in \mathcal{H}_i^-$. In accordance with Proposition 1, we consider that \mathcal{H}_i^+ is a closed set that includes its boundary \mathcal{N}_i , whereas \mathcal{H}_i^- is open and does not overlap with \mathcal{N}_i . In view of these definitions, we proceed with the following illustrative example. Consider some $\rho > 2$ and two column indices $m < i \in \{1, 2, \dots, N\}$. Then, hyperplanes \mathcal{N}_m and \mathcal{N}_i divide \mathbb{R}^ρ in the halfspace pairs $\{\mathcal{H}_m^+, \mathcal{H}_m^-\}$ and $\{\mathcal{H}_i^+, \mathcal{H}_i^-\}$, respectively. By Assumption 1,² each one of the two halfspaces defined by \mathcal{N}_m will intersect with both halfspaces defined by \mathcal{N}_i , forming the four halfspace-intersection “cells” $\mathcal{C}_1 = \mathcal{H}_m^+ \cap \mathcal{H}_i^+$, $\mathcal{C}_2 = \mathcal{H}_m^+ \cap \mathcal{H}_i^-$, $\mathcal{C}_3 = \mathcal{H}_m^- \cap \mathcal{H}_i^-$, $\mathcal{C}_4 = \mathcal{H}_m^- \cap \mathcal{H}_i^+$. It is now clear that, for any $k \in \{1, 2, 3, 4\}$, $[\text{sgn}([\mathbf{W}]^\top \mathbf{c})]_{m, i}$ is the same for every $\mathbf{c} \in \mathcal{C}_k$. For example, for every $\mathbf{c} \in \mathcal{C}_2$, it holds that $[\text{sgn}([\mathbf{W}]^\top \mathbf{c})]_m = +1$ and $[\text{sgn}([\mathbf{W}]^\top \mathbf{c})]_i = -1$.

Next, we go one step further and consider the arrangement of all N hyperplanes $\{\mathcal{N}_i\}_{i=1}^N$. Similar to our discussion

²If \mathbf{w}_m and \mathbf{w}_i are linearly independent, then \mathcal{N}_m and \mathcal{N}_i intersect but do not coincide.

above, these hyperplanes partition \mathbb{R}^ρ in K cells $\{\mathcal{C}_k\}_{k=1}^K$, where K depends on ρ and N . Formally, for every k , the k -th halfspace-intersection set is defined as

$$\mathcal{C}_k \doteq \bigcap_{i \in \mathcal{I}_k^+} \mathcal{H}_i^+ \bigcap_{m \in \mathcal{I}_k^-} \mathcal{H}_m^-, \quad (17)$$

for complementary index sets \mathcal{I}_k^+ and \mathcal{I}_k^- that satisfy $\mathcal{I}_k^+ \cap \mathcal{I}_k^- = \emptyset$ and $\mathcal{I}_k^+ \cup \mathcal{I}_k^- = \{1, 2, \dots, N\}$ [24], [25]. By the definition in (17), and in accordance with our example above, every $\mathbf{c} \in \mathcal{C}_k$ lies in the same intersection of halfspaces and, thus, yields the exact same value in $\text{sgn}(\mathbf{W}^\top \mathbf{c})$. Specifically, for every $\mathbf{c} \in \mathcal{C}_k$, it holds that

$$\left[\text{sgn}(\mathbf{W}^\top \mathbf{c}) \right]_i = \text{sgn}(\mathbf{w}_i^\top \mathbf{c}) = \begin{cases} +1, & i \in \mathcal{I}_k^+ \\ -1, & i \in \mathcal{I}_k^- \end{cases}. \quad (18)$$

In view of (18), for every $k \in \{1, 2, \dots, K\}$ and any $\mathbf{c} \in \mathcal{C}_k$, we define the ‘‘signature’’ of the k -th cell $\mathbf{b}_k \doteq \text{sgn}(\mathbf{W}^\top \mathbf{c})$. Moreover, we observe that $\mathcal{C}_k \cap \mathcal{C}_l = \emptyset$ for every $k \neq l$ and that $\bigcup_{k=1}^K \mathcal{C}_k = \mathbb{R}^\rho$. By the above observations and definitions, (16) can be rewritten as

$$\mathcal{B} = \bigcup_{k=1}^K \{ \text{sgn}(\mathbf{W}^\top \mathbf{c}) : \mathbf{c} \in \mathcal{C}_k \} = \{ \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_K \}. \quad (19)$$

Importantly, in [24], [26], it was shown that the exact number of coherent cells formed by the nullspaces of N points in \mathbb{R}^ρ that are in general position (under Assumption 1) is exactly

$$K = 2 \sum_{j=0}^{\rho-1} \binom{N-1}{j} \leq 2^N, \quad (20)$$

with equality in (20) if and only if $\rho = N$. Accordingly, per (20), the cardinality of \mathcal{B} in (16) is equal to $|\mathcal{B}| = 2 \sum_{j=0}^{\rho-1} \binom{N-1}{j}$. For clarity, in Fig. 1, we plot the nullspaces (colored planes) of the columns of arbitrary $\mathbf{W} \in \mathbb{R}^{3 \times 4}$ that satisfies both $\mathbf{W}\mathbf{W}^\top = \mathbf{I}_3$ and Assumption 1. It is interesting that exactly $K = 14 < 2^4 = 16$ coherent cells emerge by the intersection of the formed halfspaces. In the sequel, we rely on (19) to develop a conceptually simple method for computing a tight superset of the cell signatures in \mathcal{B} .

Under Assumption 1, for any $\mathcal{I} \subseteq \{1, 2, \dots, N\}$ with $|\mathcal{I}| = \rho - 1$, the hyperplane intersection $\mathcal{V}_{\mathcal{I}} \doteq \bigcap_{i \in \mathcal{I}} \mathcal{N}_i$ is a line (1-dimensional subspace) in \mathbb{R}^ρ . By its definition, this line is the verge between all cells that are jointly bounded by the $\rho - 1$ hyperplanes in $\{\mathcal{N}_i\}_{i \in \mathcal{I}}$. Consider now a vector $\mathbf{c} \in \mathbb{R}^\rho$ that crosses over the verge $\mathcal{V}_{\mathcal{I}}$ (at any point other than $\mathbf{0}_\rho$). By this crossing, the value of $[\text{sgn}(\mathbf{W}^\top \mathbf{c})]_{\mathcal{I}}$ will change so that $\text{sgn}(\mathbf{W}^\top \mathbf{c})$ adjusts to the signature of the new cell to which \mathbf{c} just entered. At the same time, a crossing over $\mathcal{V}_{\mathcal{I}}$ cannot be simultaneously over any of the hyperplanes in $\{\mathcal{N}_i\}_{i \in \mathcal{I}^c}$, for $\mathcal{I}^c \doteq \{1, 2, \dots, N\} \setminus \mathcal{I}$; this is because, under Assumption 1, it is only at $\mathbf{0}_\rho$ that more than $\rho - 1$ hyperplanes can intersect. Therefore, it is clear that $[\text{sgn}(\mathbf{W}^\top \mathbf{c})]_{\mathcal{I}^c}$ will remain

Algorithm 2: Polynomial in N

Input: $\{\mathbf{X}_i\}_{i=1}^N$
0: $\mathbf{Y} \leftarrow [\text{vec}(\mathbf{X}_1), \text{vec}(\mathbf{X}_2), \dots, \text{vec}(\mathbf{X}_N)]$
1: $(\mathbf{Q}, \mathbf{S}_{d \times d}, \mathbf{W}) \leftarrow \text{svd}(\mathbf{Y}), m_t \leftarrow 0$
2: For every $\mathcal{I} \subseteq \{1, 2, \dots, N\}, |\mathcal{I}| = d - 1$
3: Build $\mathcal{B}_{\mathcal{I}}$ in (21)
4: For every $\mathbf{b} \in \mathcal{B}_{\mathcal{I}}$
5: $(\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}) \leftarrow \text{svd}(\mathbf{X}(\mathbf{b} \otimes \mathbf{I}_M))$
6: $m \leftarrow \max\{\text{diag}(\mathbf{\Sigma})\}$
7: if $m > m_t$,
8: $m_t \leftarrow m, \mathbf{b}_t \leftarrow \mathbf{b}, \mathbf{u} \leftarrow [\mathbf{U}]_{:,1}, \mathbf{v} \leftarrow [\mathbf{V}]_{:,1}$
Output: $\mathbf{b}_{\text{opt}} \leftarrow \mathbf{b}_t, \mathbf{u}_{\text{opt}} \leftarrow \mathbf{u},$ and $\mathbf{v}_{\text{opt}} \leftarrow \mathbf{v}$

Fig. 2. Algorithm for the exact solution of rank-1 L1-TUCKER2 in (2), with cost $\mathcal{O}(N^{\rho+1})$.

invariant during this crossing and, in fact, equal to $[\text{sgn}(\mathbf{W}^\top \mathbf{v})]_{\mathcal{I}^c}$, for any $\mathbf{v} \in \mathcal{V}_{\mathcal{I}}$ with $\mathbf{v}^\top \mathbf{c} > 0$. In view of the above, for any $\mathbf{v} \in \mathcal{V}_{\mathcal{I}} \setminus \mathbf{0}_\rho$, the set

$$\mathcal{B}_{\mathcal{I}} \doteq \{\mathbf{b} \in \{\pm 1\}^N : [\mathbf{b}]_{\mathcal{I}^c} = [\text{sgn}(\mathbf{W}^\top \mathbf{v})]_{\mathcal{I}^c}\} \quad (21)$$

contains the signatures of all sets that are bounded by the verge $\mathcal{V}_{\mathcal{I}}$. Moreover, it has been shown (see, e.g., [26]) that, for every cell, there exists at least one such verge that bounds it. Therefore, it derives that the set

$$\mathcal{B}_{\text{pol}} = \bigcup_{\mathcal{I} \subset \{1, 2, \dots, N\}; |\mathcal{I}| = \rho - 1} \mathcal{B}_{\mathcal{I}} \quad (22)$$

includes all cell signatures and, thus, is a superset of \mathcal{B} . We notice that, for every \mathcal{I} , $\mathcal{B}_{\mathcal{I}}$ has size $2^{\rho-1}$. Since \mathcal{I} can take $\binom{N}{\rho-1}$ distinct values, we find that \mathcal{B}_{pol} is upper bounded by $2^{\rho-1} \binom{N}{\rho-1}$. Thus, both $|\mathcal{B}_{\text{pol}}|$ and $|\mathcal{B}|$ are polynomial, in the order of $\mathcal{O}(N^{\rho-1})$.

Practically, for every \mathcal{I} , \mathbf{v} can be calculated by Gram-Schmidt orthogonalization of $[\mathbf{W}]_{:, \mathcal{I}}$ with cost $\mathcal{O}(\rho^3)$. Keeping the dominant terms, the construction of \mathcal{B}_{pol} costs $\mathcal{O}(N^{\rho-1})$ and can be parallelized in $\binom{N}{\rho-1}$ processes. Then, testing every entry of \mathcal{B}_{pol} for optimality in (8) costs an additional $\mathcal{O}(N)$. Thus, the overall cost of our second algorithm, taking also into account the $\mathcal{O}(N)$ (for constant DM) SVD cost for the formation of \mathbf{W} , is $\mathcal{O}(N^\rho)$. The presented algorithm is summarized in Fig. 2.

III. NUMERICAL STUDIES

Consider $\{\mathbf{X}_i\}_{i=1}^{14}$ such that $\mathbf{X}_i = \mathbf{A}_i + \mathbf{N}_i \in \mathbb{R}^{20 \times 20}$ where $\mathbf{A}_i = b_i \mathbf{u} \mathbf{v}^\top$ and $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1, b_i \sim \mathcal{N}(0, 49)$, and each entry of \mathbf{N}_i is additive white Gaussian noise (AWGN) from $\mathcal{N}(0, 1)$. We consider that \mathbf{A}_i is the rank-1 useful data in \mathbf{X}_i that we want to reconstruct, by joint analysis (TUCKER2-type) of $\{\mathbf{X}_i\}_{i=1}^{14}$. By irregular corruption, 30 entries in 2 out of the 14 matrices (i.e., 60 entries out of the total 5600 entries in $\{\mathbf{X}_i\}_{i=1}^{14}$) have

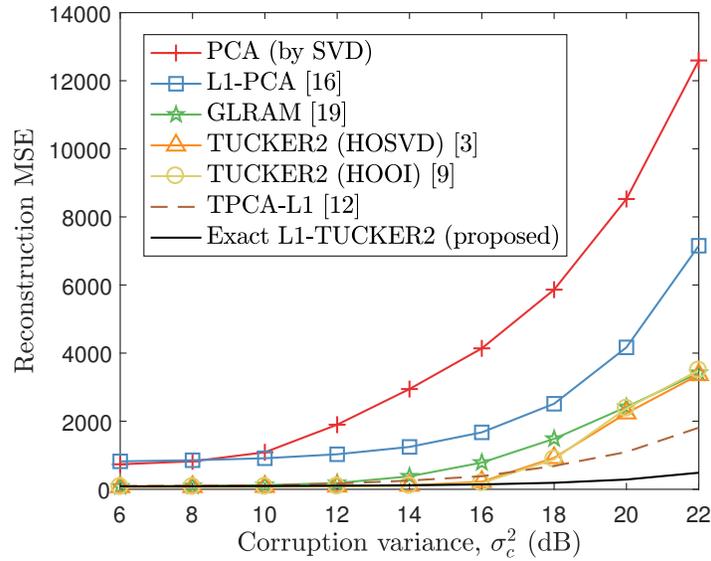


Fig. 3. Reconstruction MSE versus corruption variance σ_c^2 (dB).

been further corrupted additively by noise from $\mathcal{N}(0, \sigma_c^2)$. To reconstruct $\{\mathbf{A}_i\}_{i=1}^{14}$ from $\{\mathbf{X}_i\}_{i=1}^{14}$, we follow one of the two approaches below.

In the first approach, we vectorize the matrix samples and perform standard matrix analysis. That is, we obtain the first ($d = 1$) principal component (PC) of $[\text{vec}(\mathbf{X}_1), \text{vec}(\mathbf{X}_2), \dots, \text{vec}(\mathbf{X}_N)]$, \mathbf{q} . Then, for every i , we approximate \mathbf{A}_i by $\hat{\mathbf{A}}_i = \text{mat}(\mathbf{q}\mathbf{q}^\top \mathbf{a}_i)$, where $\text{mat}(\cdot)$ reshapes its vector argument into a 20×20 matrix, in accordance with $\text{vec}(\cdot)$. In the second approach, we process the samples in their natural form, as matrices, analyzing them by TUCKER2. If (\mathbf{u}, \mathbf{v}) is the TUCKER2 solution pair, then we approximate \mathbf{A}_i by $\hat{\mathbf{A}}_i = \mathbf{u}\mathbf{u}^\top \mathbf{X}_i \mathbf{v}\mathbf{v}^\top$. For the first approach, we obtain \mathbf{q} by PCA (i.e., SVD) and L1-PCA [16]. For the second approach, we conduct TUCKER2 by HOSVD [3], HOOI [9], GLRAM [19], TPCA-L1 [12], and the proposed exact L1-TUCKER2. Then, for each reconstruction method, we measure the mean of the squared error $\sum_{i=1}^{14} \|\mathbf{A}_i - \hat{\mathbf{A}}_i\|_F^2$ over 1000 independent realizations for corruption variance $\sigma_c^2 = 6, 8, \dots, 22$ dB. In Fig. 3, we plot the reconstruction mean squared error (MSE) for every method, versus σ_c^2 . We observe that PCA and L1-PCA exhibit the highest MSE due to the vectorization operation (L1-PCA outperforms PCA clearly, across all values of σ_c^2). Then, all TUCKER2-type methods perform similarly well when σ_c^2 is low. As the outlier variance σ_c^2 increases, the performance of L2-norm-based TUCKER2 (HOSVD, HOOI) and GLRAM deteriorates severely. On the other hand, the L1-norm-based TPCA-L1 exhibits some robustness. The proposed exact L1-TUCKER2 maintains the sturdiest resistance against the corruption, outperforming its counterparts across the board.

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