A-Optimal Sampling and Robust Reconstruction for Graph Signals via Truncated Neumann Series

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Abstract—Graph signal processing (GSP) studies signals that live on irregular data kernels described by graphs. One fundamental problem in GSP is sampling-from which subset of graph nodes to collect samples in order to reconstruct a bandlimited graph signal in high fidelity. In this paper, we seek a sampling strategy that minimizes the mean square error (MSE) of the reconstructed bandlimited graph signals assuming an independent and identically distributed (iid) noise model—leading naturally to the A-optimal design criterion. To avoid matrix inversion, we first prove that the inverse of the information matrix in the A-optimal criterion is equivalent to a Neumann matrix series. We then transform the truncated Neumann series based sampling problem into an equivalent expression that replaces eigenvectors of the Laplacian operator with a sub-matrix of an ideal low-pass graph filter. Finally, we approximate the ideal filter using a Chebyshev matrix polynomial. We design a greedy algorithm to iteratively minimize the simplified objective. For signal reconstruction, we propose an accompanied signal reconstruction strategy that reuses the approximated filter sub-matrix and is provably more robust than conventional least square recovery. Simulation results show that our sampling strategy outperforms two previous strategies in MSE performance at comparable complexity.

Index Terms—Graph signal processing (GSP), sampling, optimal design.

I. INTRODUCTION

Graph signal processing (GSP) is the study of discrete signals that live on irregular data kernels described by graphs [1], [2]. One fundamental problem in GSP is sample selection-optimally select a subset of graph nodes from which to collect samples such that an assumed bandlimited signal can be reconstructed in high fidelity¹ [3]–[6]. Under noiseless conditions, [8] proved that a qualified sampling set that leads to perfect signal reconstruction requires only full column rank of a sampling matrix, and empirically showed that it can be accomplished with high probability via random node selection for a connected graph via large experiments. Random node selection can result in a poor condition number in the reconstruction matrix, however. In response, authors in [9]-[11] proposed an efficient sampling strategy based on spectral proxies that selects stable nodes for unique reconstruction without full eigen-decomposition. If the observed samples are corrupted by noise, [12] adopts an E-optimality criterion for sampling, which minimizes the worst case reconstruction error.

¹Sampling can also be done via *aggregation* [7]: observe the same signal after different graph shifts but only at one node.

However, most graph sampling works² do not adopt and optimize a minimum mean square error (MMSE) objective directly, which leads naturally to an A-optimality design criterion assuming an independent and identically distributed (iid) additive noise model [13]. Beyond the fact that graph sampling is inherently combinatorial in nature, one main difficulty lies in the computation of the inverse of an information matrix, which in general has complexity $O(n^3)$. While [14] showed that greedy methods optimizing the A-optimality criterion can have near-optimal performance, no efficient implementation was proposed, which in general requires eigen-decomposition and/or inversion of large matrices.

In this letter, we propose a computation-efficient graph sampling strategy that addresses the A-optimality criterion directly. We first prove that the inverse information matrix in the A-optimality criterion is equivalent to a Neumann matrix series. We next transform the truncated Neumann series based sampling problem into an equivalent expression that replaces eigenvectors of the Laplacian operator with a submatrix of an ideal graph low-pass filter [1], [10]. Finally, we approximate the low-pass filter with a Chebyshev matrix polynomial [15]—arriving at a simplified proxy that approximates the A-optimality objective but requires neither full eigendecomposition nor matrix inversion. We propose a greedy algorithm to minimize the simplified objective. For signal reconstruction, we design an accompanied reconstruction strategy that is provably more "robust" ³ to large noise than the least square (LS) solution while reusing the earlier computed approximate low-pass filter sub-matrix. Experimental results show that our A-optimality based sampling strategy outperforms previous schemes in MSE at comparable complexity.

II. SIGNAL PROCESSING ON GRAPHS

Denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$ a graph containing a set of nodes indexed by $\mathcal{V} = \{1, 2, ..., n\}$. \mathcal{E} is the set of weighted edges. An edge weight $\mathbf{W}(i, j) = w_{ij}$ reflects the similarity between nodes *i* and *j*. In this letter, we focus on connected, undirected graphs with no multiple edges and adopt the symmetric normalized Laplacian matrix $\mathcal{L} = \mathbf{I} - \mathbf{D}^{-1/2} \mathbf{W} \mathbf{D}^{-1/2}$ as the variation operator, where $\mathbf{D} = \text{diag}\{d_1, ..., d_n\}$ and $d_i = \sum_j w_{ij}$. Assuming that the eigen-decomposition of \mathcal{L} is $\mathcal{L} = \mathbf{V} \Sigma \mathbf{V}^T$ where $\Sigma = \text{diag}\{\lambda_i\}$ with $\lambda_1 \leq ... \leq \lambda_n$, $\mathbf{V} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $\mathbf{V} \mathbf{V}^T = \mathbf{I}$, then the graph Fourier

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 $^{^{2}}$ [5] proposed a generic greedy procedure, termed as minimum Frobenius norm (MFN) based selection, to minimize the A-optimality criterion but no fast algorithms. We compare against [5] in our experiments.

³By "robustness", we mean resilience of an estimator's presumably good MSE performance for small noise as noise variance increases.

transform (GFT) of a graph signal $\mathbf{x} \in \mathbb{R}^n$ is defined as $\tilde{\mathbf{x}} = \mathbf{V}^T \mathbf{x}$ and the inverse GFT is $\mathbf{x} = \mathbf{V} \tilde{\mathbf{x}}$. A graph signal is called *bandlimited* if there exists a number $K \in \mathcal{V}$ such that $\tilde{x}_i = 0$, for all i > K [12]. The smallest such K is called the *bandwidth* of \mathbf{x} . Graph signals with bandwidth at most K are called K-bandlimited (K-BL) graph signals, which are expressed as $\mathbf{x} = \mathbf{V}_K \tilde{\mathbf{x}}_K$. \mathbf{V}_K means the first K columns of \mathbf{V} and $\tilde{\mathbf{x}}_K$ denotes the first K elements of $\tilde{\mathbf{x}}$.

Definition 1 [16]: In order to select m elements from \mathbf{x} to produce $\mathbf{x}_{\mathcal{S}} = \mathbf{C}\mathbf{x} \in \mathbb{R}^m$ with $|\mathcal{S}| = m$ and $\mathcal{S} \subseteq \mathcal{V}$, we define the sampling matrix $\mathbf{C} \in \mathbb{F}^{m \times n}$ as

$$\mathbf{C}_{ij} = \begin{cases} 1, & j = \mathcal{S}_i; \\ 0, & \text{otherwise,} \end{cases}$$
(1)

where S is the set of sampling indices, S_i means the *i*-th element of set S, and |S| is the number of elements in S.

 S^c denotes the complement set of S. $|\mathcal{E}|$ is the number of edges. For any matrix **A**, we adopt the notation $\mathbf{A}_{S_1S_2}$ to denote the sub-matrix of **A** with rows indexed by S_1 and columns indexed by S_2 . \mathbf{A}_{SS} is simplified to \mathbf{A}_S . **I** is the identity matrix whose dimension depends on the context.

III. SAMPLE SELECTION FOR NOISY K-BL GRAPH SIGNALS

A sampled *K*-BL graph signal can now be written as $\mathbf{x}_{S} = \mathbf{C}\mathbf{V}_{K}\tilde{\mathbf{x}}_{K}$. In noiseless condition, \mathbf{x} can be perfectly recovered from \mathbf{x}_{S} when rank $(\mathbf{C}\mathbf{V}_{K}) = K$, using the LS solution [12]:

$$\bar{\mathbf{x}} = \mathbf{V}_K (\mathbf{C} \mathbf{V}_K)^{\dagger} \mathbf{x}_{\mathcal{S}},\tag{2}$$

where $(\cdot)^{\dagger}$ means the pseudo-inverse operator.

When corrupted by noise, a sampled *K*-BL graph signal is $\mathbf{y}_{S} = \mathbf{x}_{S} + \mathbf{n}$. Using (2) as the recovery method, we get an estimator $\hat{\mathbf{x}} = \mathbf{V}_{K} (\mathbf{C} \mathbf{V}_{K})^{\dagger} (\mathbf{x}_{S} + \mathbf{n})$. Assuming that noise \mathbf{n} is iid with zero mean and unit variance, the covariance matrix of the reconstruction error is $C_{\hat{\mathbf{x}}} = \mathbf{V}_{K} [(\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{C} \mathbf{V}_{K}]^{-1} \mathbf{V}_{K}^{T}$ [11]. By the theory of optimal experiments design [17], minimizing the trace of the covariance matrix leads to the known *A-optimality* criterion:

$$\mathbf{C}^* = \operatorname*{arg\,min}_{\mathbf{C} \in \mathbb{F}^{m \times n}} \operatorname{tr} \left([(\mathbf{C} \mathbf{V}_K)^T \mathbf{C} \mathbf{V}_K]^{-1} \right).$$
(3)

Notice that the A-optimality criterion coincides with the MMSE criterion.

An alternative criterion is to minimize the largest eigenvalue of $C_{\hat{\mathbf{x}}}$ —as adopted in [12]—known as the *E-optimality* criterion, which minimizes the worst case signal reconstruction.

Unlike [12], we address directly the MMSE criterion in (3) to achieve smallest signal reconstruction error on average, but doing so without any matrix inversion or eigen-decomposition, where complexity is $O(n^3)$ in general.

A. Matrix Inversion Approximation

Proposition 1: The inverse matrix in (3) exists and is equivalent to its Neumann series, *i.e.*,

$$[(\mathbf{C}\mathbf{V}_K)^T\mathbf{C}\mathbf{V}_K]^{-1} = \sum_{l=0}^{\infty} [\mathbf{I} - (\mathbf{C}\mathbf{V}_K)^T\mathbf{C}\mathbf{V}_K]^l, \quad (4)$$

if the selected matrix \mathbf{CV}_K is full column rank, *i.e.*,

$$\operatorname{ank}(\mathbf{CV}_K) = K. \tag{5}$$

Proof: For simplicity, we denote $\Psi = (\mathbf{C}\mathbf{V}_K)^T \mathbf{C}\mathbf{V}_K$, $\Phi = \mathbf{I} - \Psi$ and let $\delta_1 \leq \ldots \leq \delta_K$ be the eigenvalues of Φ . The *Neumann series theorem* [18, Section 5.6] states that if $\rho(\Phi) = \max_i |\delta_i| < 1$, then the Neumann series $\mathbf{I} + \Phi + \Phi^2 + \cdots$ will converge to $(\mathbf{I} - \Phi)^{-1}$, which exactly implies (4). From the definition of \mathbf{C} , $\mathbf{C}^T \mathbf{C} = \begin{bmatrix} \mathbf{I}_S & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ under appropriate permutation. Hence, $\forall \mathbf{x} \in \mathbb{R}^K$ and $\|\mathbf{x}\|_2 = 1$,

$$\mathbf{x}^{T} \boldsymbol{\Psi} \mathbf{x} = (\mathbf{V}_{K} \mathbf{x})^{T} \begin{pmatrix} \mathbf{C}^{T} \mathbf{C} \end{pmatrix} (\mathbf{V}_{K} \mathbf{x})$$
$$= \mathbf{b}^{T} \begin{bmatrix} \mathbf{I}_{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{b} = \mathbf{b}_{S}^{T} \mathbf{b}_{S},$$
(6)

where $\mathbf{b} = \mathbf{V}_K \mathbf{x}$.

Since $\mathbf{b}^T \mathbf{b} = (\mathbf{V}_K \mathbf{x})^T (\mathbf{V}_K \mathbf{x}) = 1, \ 0 \le \mathbf{x}^T \mathbf{\Psi} \mathbf{x} \le 1$. Because rank $(\mathbf{C} \mathbf{V}_K) = K, \ \mathbf{\Psi}$ is positive definite, which results in $0 < \mathbf{x}^T \mathbf{\Psi} \mathbf{x} \le 1$ and $0 \le \mathbf{x}^T \mathbf{\Phi} \mathbf{x} < 1$. Based on the *Rayleigh quotient theorem* [18], $0 \le \delta_i < 1$ and $\rho(\mathbf{\Phi}) < 1$. \Box

As discussed in [12], (5) is the definition of *qualified* sampling operators, and it is ensured with high probability via random node selection when $m \ge K$. The necessary condition for a *qualified sampling operator* is thus $m \ge K$, which is the focus region of all sampling strategies. In the following analysis, we assume that (5) is satisfied.

We propose the following sampling method by substituting the inverse matrix in (3) with a truncated Neumann series.

$$\mathbf{C}^{*} = \operatorname*{arg\,min}_{\mathbf{C} \in \mathbb{F}^{m \times n}} \operatorname{tr} \left(\sum_{l=0}^{L} \left[\mathbf{I} - \left(\mathbf{C} \mathbf{V}_{K} \right)^{T} \mathbf{C} \mathbf{V}_{K} \right]^{l} \right), \quad (7)$$

where L is a truncation parameter.

Proposition 2: When Ψ^{-1} is approximated by its truncated Neumann series, the estimation error between (3) and (7) is

$$\left| \operatorname{tr} \left[\sum_{l=0}^{\infty} \mathbf{\Phi}^l \right] - \operatorname{tr} \left[\sum_{l=0}^{L} \mathbf{\Phi}^l \right] \right| = \sum_{i=1}^{K} \frac{\delta_i^{L+1}}{1 - \delta_i}.$$
 (8)

See Appendix A in the supporting document for the proof of this result. Equation (8) implies that the proper design of L depends on the sizes of δ_i which are influenced by the sampling strategy, sampling size m and V_K . Larger L would result in a smaller truncation error. In this letter, we set L = 10and show that its estimate error is reasonably small in Section V.

Leveraging on a property of the trace operator, we further transform (7) into an equivalent problem that involves an ideal low-pass graph filter \mathbf{T} with cutoff frequency λ_K .

Theorem 1: The sampling problem (7) is equivalent to

$$S^* = \operatorname*{arg\,min}_{S:|\mathcal{S}|=m} \operatorname{tr}\left[\sum_{l=0}^{L} \left(\mathbf{I}_{\mathcal{S}} - \mathbf{T}_{\mathcal{S}}\right)^l\right],\tag{9}$$

where $\mathbf{T} = \mathbf{V}_K \mathbf{V}_K^T \in \mathbb{R}^{n \times n}$ is an ideal low-pass graph filter (implementation to be discussed in details) and the relationship between \mathbf{C} and \mathcal{S} is defined in (1).

Proof: Denote by $\mathbf{P} = \mathbf{C}^T \mathbf{C} \mathbf{V}_K \mathbf{V}_K^T$. Because $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$ and $\Psi = \mathbf{V}_K^T \mathbf{C}^T \mathbf{C} \mathbf{V}_K$, $\operatorname{tr}(\Psi^l) = \operatorname{tr}(\mathbf{V}_K^T \mathbf{C}^T \mathbf{C} \mathbf{V}_K \dots$

 TABLE I

 Outline of The Proposed MIA Sampling Algorithm

Input:	\mathcal{L} , bandwidth K, sample size m and truncation parameter L		
Output:	Sampling set S		
Step 1.	$S = \{\emptyset\}, \text{ compute } \lambda_K \text{ of } \mathcal{L}$		
Step 2.	Calculate the truncated Chebyshev polynomial approximation of $h(\lambda)$ and then compute $\mathbf{T}^{\text{Poly}} = \sum_{j=0}^{p} \beta_j \mathcal{L}^j$		
Step 3.	While $ \mathcal{S} < m$		
	$oldsymbol{\Gamma}_i = \mathbf{I}_{\mathcal{S}+\{i\}} - \mathbf{T}^{ extsf{Poly}}_{\mathcal{S}+\{i\}} \; orall i \in \mathcal{S}^c$		
	$ \begin{split} \mathbf{\Gamma}_{i} &= \mathbf{I}_{\mathcal{S} + \{i\}} - \mathbf{T}_{\mathcal{S} + \{i\}}^{\text{Poly}} \forall i \in \mathcal{S}^{c} \\ u &= \operatorname*{argmin}_{i \in \mathcal{S}^{c}} \operatorname{tr} \left(\sum_{l=0}^{L} \mathbf{\Gamma}_{i}^{l} \right) \\ \mathcal{S} \leftarrow \mathcal{S} + \{u\} \end{split} $		
	$\mathcal{S} \leftarrow \mathcal{S} + \{u\}$		
	end		
Step 4.	Return S and $\tilde{\mathbf{\Gamma}} = \sum_{l=0}^{L} \left(\mathbf{I}_{S} - \mathbf{T}_{S}^{Poly} \right)^{l}$		

 $\mathbf{V}_K^T \mathbf{C}^T \mathbf{C} \mathbf{V}_K) = \operatorname{tr}(\mathbf{C}^T \mathbf{C} \mathbf{V}_K ... \mathbf{V}_K^T \mathbf{C}^T \mathbf{C} \mathbf{V}_K \mathbf{V}_K^T) = \operatorname{tr}(\mathbf{P}^l).$ As a result,

$$\operatorname{tr}(\mathbf{I} - \boldsymbol{\Psi})^{l} = \operatorname{tr}[\mathbf{I} + \sum_{d=1}^{l} {l \choose d} (-\boldsymbol{\Psi})^{d}]$$
(10)

$$\stackrel{\triangle}{=} \operatorname{tr}[\mathbf{I} + \sum_{d=1}^{l} {l \choose d} (-\mathbf{P})^{d}] - n + K = \operatorname{tr}(\mathbf{I} - \mathbf{P})^{l} - n + K,$$

where $\binom{l}{d}$ means the binomial coefficients and $\stackrel{\triangle}{=}$ is derived from the property of trace operation.

Since $\mathbf{C}^T \mathbf{C} = \begin{bmatrix} \mathbf{I}_{\mathcal{S}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ under appropriate permutation, $\mathbf{P} = \begin{bmatrix} \mathbf{T}_{\mathcal{SV}} \\ \mathbf{0} \end{bmatrix}$. Hence, $\mathbf{I} - \mathbf{P} = \begin{bmatrix} \mathbf{I}_{\mathcal{S}} - \mathbf{T}_{\mathcal{S}} & -\mathbf{T}_{\mathcal{SS}^c} \\ \mathbf{0} & \mathbf{I}_{\mathcal{S}^c} \end{bmatrix}$, which will lead to

$$\left(\mathbf{I} - \mathbf{P}\right)^{l} = \begin{bmatrix} (\mathbf{I}_{\mathcal{S}} - \mathbf{T}_{\mathcal{S}})^{l} \bullet \\ \mathbf{0} & \mathbf{I}_{\mathcal{S}^{c}} \end{bmatrix},$$
(11)

where "•" denotes a nonzero matrix whose dimension is $m \times (n-m)$.

Therefore, $\operatorname{tr}(\mathbf{I} - \mathbf{P})^l = \operatorname{tr}(\mathbf{I}_{\mathcal{S}} - \mathbf{T}_{\mathcal{S}})^l + n - m$. Combined with (10), $\operatorname{tr}(\mathbf{I} - \Psi)^l = \operatorname{tr}(\mathbf{I}_{\mathcal{S}} - \mathbf{T}_{\mathcal{S}})^l - m + K$ where m and K are constant during sampling, which implies Theorem 1. \Box

B. Chebyshev Approximation of Low-pass Filter T

The low-pass graph filter **T** has a kernel function $h(\lambda) = \begin{cases} 1, & \lambda \leq \lambda_K \\ 0, & \lambda > \lambda_K \end{cases}$. We approximate this spectral kernel function via Chebyshev polynomial approximation [15], after which **T** can be expressed by a matrix polynomial in terms of \mathcal{L} , *i.e.*, $\mathbf{T}^{\text{Poly}} = \sum_{i=1}^{n} \left(\sum_{j=0}^{p} \beta_j \lambda_i^j \right) \mathbf{v}_i \mathbf{v}_i^T = \sum_{j=0}^{p} \beta_j \mathcal{L}^j$ [19]. Finally, the sampling problem is formulated as

$$\mathcal{S}^* = \operatorname*{arg\,min}_{\mathcal{S}:|\mathcal{S}|=m} \operatorname{tr} \left[\sum_{l=0}^{L} \left(\mathbf{I}_{\mathcal{S}} - \mathbf{T}_{\mathcal{S}}^{\operatorname{Poly}} \right)^l \right].$$
(12)

If we solve (7) directly, the eigenvector matrix \mathbf{V}_K of dimension $n \times K$ is required. After transforming (7) to (12), the required information is only λ_K . We compute λ_K as follows. We first compute an *M*-by-*M* tridiagonal matrix using the Lanczos algorithm, which preserves λ_K of \mathcal{L} [20]. For a large sparse graph, the complexity of the Lanczos algorithm is

 TABLE II

 COMPLEXITY COMPARISON OF DIFFERENT SAMPLING STRATEGIES

	Preparation	Selection step
Spectral Proxies	NONE	$\mathcal{O}\left(k\left \mathcal{E}\right mT_{2}\left(k\right)+nm\right)$
E-optimal	$\mathcal{O}\left(\left(\left \mathcal{E}\right m+Rm^{3}\right)T_{1}\right)$	$\mathcal{O}\left(nm^4\right)$
MFN	$\mathcal{O}\left(\left(\left \mathcal{E}\right m+Rm^{3}\right)T_{1}\right)$	$\mathcal{O}\left(nm^4\right)$
MIA	$\mathcal{O}(pn \mathcal{E})$	$\mathcal{O}\left(nLm^{3.373} ight)$

 $\mathcal{O}(Mn)$. We then compute the eigenvalues of the tridiagonal matrix using a fast multipole method, whose complexity is $\mathcal{O}(M\log M)$ [21]. Hence, the combined complexity for computing λ_K is $\mathcal{O}(Mn)$, where $K < M \ll n$. Along with λ_K , the computation of \mathbf{T}^{Poly} may be done only once during preparation, and implemented efficiently due to the sparsity of \mathcal{L} .

C. Complexity Analysis

Optimizing the proposed criterion (12) is still combinatorial, so we adopt a greedy algorithm to obtain its solution, which we call the *matrix inversion approximation* (MIA) sampling algorithm. Details of the algorithm are presented in Table I. The complexity for computing λ_K is $\mathcal{O}(Mn)$. Calculating the coefficients β and $\mathbf{T}^{\text{Poly}} = \sum_{j=0}^{p} \beta_j \mathcal{L}^j$ has complexity $\mathcal{O}(pn)$ and $\mathcal{O}(pn|\mathcal{E}|)$ [15]. Hence, in the preparation step, the complexity of MIA is $\mathcal{O}(pn|\mathcal{E}|)$. In each sampling step, the algorithm involves matrix multiplication, where $\mathbf{I}_S - \mathbf{T}_S^{\text{Poly}} \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{S}|}$ and $|\mathcal{S}|$ gradually increases until m, having an asymptotic complexity of $\mathcal{O}(m^{2.373})$ [22]. Considering the impact from L and n, the complexity of each search step is $\mathcal{O}(nLm^{2.373})$. Since finally $|\mathcal{S}| = m$, the whole complexity of the sampling step is $\mathcal{O}(nLm^{3.373})$.

Table II compares the computational complexity among different sampling strategies, in which we assume m = K and adopt some results in [11]. In the preparation step, the spectral proxies algorithm utilizes \mathcal{L} directly, while \mathbf{V}_K is necessary for the E-optimal and the MFN algorithms. In the selection step, the E-optimal and the MFN algorithms need singular value decomposition and the first eigen-pair of $((\mathcal{L}^T)^k \mathcal{L}^k)_{S^c}$ is required for the spectral proxies algorithm.

IV. ACCOMPANIED RECONSTRUCTION STRATEGY

Assuming rank $(\mathbf{CV}_K) = K$, then according to Proposition 1, the LS solution of a graph signal is equivalent to

$$\hat{\mathbf{x}} = \mathbf{V}_{K} \left[(\mathbf{C}\mathbf{V}_{K})^{T} \mathbf{C}\mathbf{V}_{K} \right]^{-1} (\mathbf{C}\mathbf{V}_{K})^{T} \mathbf{y}_{S}$$
$$= \mathbf{V}_{K} \sum_{l=0}^{\infty} [\mathbf{I} - \boldsymbol{\Psi}]^{l} \mathbf{V}_{K}^{T} \mathbf{C}^{T} \mathbf{y}_{S}.$$
(13)

It is easy to derive that

$$\mathbf{V}_{K}(\mathbf{I} - \boldsymbol{\Psi})^{l} \mathbf{V}_{K}^{T} = \mathbf{V}_{K} \left[\sum_{d=0}^{l} {l \choose d} \mathbf{I}^{l-d} (-1)^{d} \boldsymbol{\Psi}^{d} \right] \mathbf{V}_{K}^{T}$$
$$= \sum_{d=0}^{l} {l \choose d} \mathbf{I}^{l-d} (-1)^{d} \left(\mathbf{V}_{K} \boldsymbol{\Psi}^{d} \mathbf{V}_{K}^{T} \right) \qquad (14)$$
$$\stackrel{\triangle}{=} \sum_{d=0}^{l} {l \choose d} \mathbf{I}^{l-d} (-1)^{d} \left(\mathbf{V}_{K} \mathbf{V}_{K}^{T} \mathbf{P}^{d} \right) = \mathbf{T} (\mathbf{I} - \mathbf{P})^{l},$$

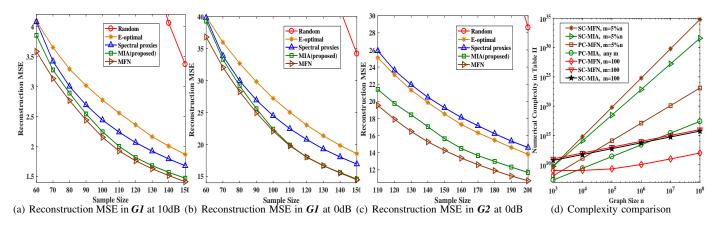


Fig. 1. (a) (b) (c) Simulation results for different sampling algorithms where graph signals are all recovered by the LS reconstruction, (d) Numerical comparison of complexity between the MFN and the MIA algorithms where "PC" and "SC" are the abbreviation of the complexity of the preparation step and that of the sampling step respectively.

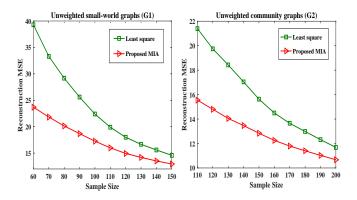


Fig. 2. Reconstruction MSE for different reconstruction algorithms in GI and G2 at 0dB where the sampling algorithm are all the MIA sampling.

where
$$\stackrel{\Delta}{=}$$
 holds since $\mathbf{V}_K(\mathbf{V}_K^T\mathbf{C}^T\mathbf{C}\mathbf{V}_K)...(\mathbf{V}_K^T\mathbf{C}^T\mathbf{C}\mathbf{V}_K)\mathbf{V}_K^T$
= $\mathbf{V}_K\mathbf{V}_K^T\mathbf{P}^d$.

Combining (11), (13) and (14), a closed-form reconstruction strategy (named as MIA reconstruction) is given by

$$\hat{\mathbf{x}} = \mathbf{T} \sum_{l=0}^{\infty} \left(\mathbf{I} - \mathbf{P} \right)^{l} \mathbf{C}^{T} \mathbf{y}_{S}$$

$$= \mathbf{T} \sum_{l=0}^{\infty} \left[\begin{array}{c} \left(\mathbf{I}_{S} - \mathbf{T}_{S} \right)^{l} & \bullet \\ \mathbf{0} & \mathbf{I}_{S^{c}} \end{array} \right] \left[\begin{array}{c} \mathbf{y}_{S} \\ \mathbf{0} \end{array} \right]$$

$$= \mathbf{T} \sum_{l=0}^{\infty} \left[\begin{array}{c} \left(\mathbf{I}_{S} - \mathbf{T}_{S} \right)^{l} \mathbf{y}_{S} \\ \mathbf{0} \end{array} \right] = \mathbf{T}_{\mathcal{VS}} \mathbf{\Gamma} \mathbf{y}_{S} \approx \mathbf{T}_{\mathcal{VS}}^{\text{Poly}} \tilde{\mathbf{\Gamma}} \mathbf{y}_{S},$$
(15)

where $\mathbf{\Gamma} = \sum_{l=0}^{\infty} (\mathbf{I}_{\mathcal{S}} - \mathbf{T}_{\mathcal{S}})^{l}$ and $\tilde{\mathbf{\Gamma}} = \sum_{l=0}^{L} (\mathbf{I}_{\mathcal{S}} - \mathbf{T}_{\mathcal{S}}^{\text{Poly}})^{l}$.

 \mathbf{T}^{Poly} and $\tilde{\Gamma}$ have been computed in Table I, so the MIA recovery strategy only needs matrix-vector product, thus has low complexity. Moreover, assuming the Chebyshev polynomial approximates the ideal low-pass filter well enough, this proposed MIA reconstruction method is more robust to large noise than the LS reconstruction in theory. See Appendix B in the supporting document for the proof of the robustness.

V. EXPERIMENTAL RESULTS

We evaluated our proposed strategy via simulations. All experiments were performed in MATLAB R2017b, running on a PC with Intel Core I3 3.7 GHz CPU and 16GB RAM.

Artificial graphs: (G1) Small-world graphs [23] (unweighted) with 1000 nodes, degree 8 and connection probability 0.1; (G2) Community graphs [24] (unweighted) with 1000 nodes.

Artificial signals: The true signal is exactly bandlimited with K = 50 and the non-zero GFT coefficients are generated from $\mathcal{N}(1, 0.5^2)$. Samples are corrupted by AWGN.

Other Parameters: We set L = 10 for the MIA algorithm and k = 10 for the spectral proxies algorithm. In the complexity comparison experiments, we set R = 10, $T_1 = 100$ and $|\mathcal{E}| = \mathcal{O}(n)$. The SGWT toolbox [24] is adopted to approximate the ideal low pass filter, where p = 25 and $\alpha = 30$ [10].

Fig. 1(a), (b) and (c) show that our proposed MIA sampling algorithm achieves better MSE performance than the Eoptimal and spectral proxies algorithms and closely approximates the performance of the MFN algorithm in both smallworld graphs and community graphs at different SNRs. Fig.1 (d) shows that although the complexity of the MIA algorithm for the preparation step may be larger for a constant m, when m is a fixed percentage of n, the proposed MIA algorithm has smaller complexity for both the preparation step and the sampling step compared to the MFN algorithm, especially for large graphs. To evaluate the Neumann truncation error at L = 10, we computed the ratio between the estimate error in (8) and the MSE value in (3) in small-world graphs. Numerical results reveal that when m = 120, this ratio was 0.19.

We also performed simulations using our proposed MIA reconstruction method, where the sample sets were all collected by the MIA sampling algorithm. As depicted in Fig. 2, the MIA reconstruction outperformed the LS reconstruction in both small-world graphs and community graphs at 0dB. These results empirically validate the robustness of the proposed MIA reconstruction algorithm for large noise variance.

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APPENDIX

A. Proof of Proposition 1

Assuming that the eigen-decomposition of Φ is $\Phi = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$ where $\mathbf{\Lambda} = \text{diag}\{\delta_i\}$ and $\mathbf{U}\mathbf{U}^T = \mathbf{I}$. Then,

$$\sum_{l} \mathbf{\Phi}^{l} = \sum_{l} \mathbf{U} \mathbf{\Lambda} \mathbf{U}^{T} ... \mathbf{U} \mathbf{\Lambda} \mathbf{U}^{T} = \sum_{l} \mathbf{U} \mathbf{\Lambda}^{l} \mathbf{U}^{T} = \mathbf{U} \mathbf{\Omega} \mathbf{U}^{T},$$

where $\mathbf{\Omega} = \text{diag}\{\sum_{l} \delta_{i}^{l}\}, \delta_{i}$ is the *i*-th eigenvalue of $\mathbf{\Phi}$ and $0 \leq \delta_{i} < 1$ has been proved in Proposition 1.

Therefore,

$$\operatorname{tr}\left(\sum_{l=L+1}^{\infty} \boldsymbol{\Phi}^{l}\right) = \sum_{i=1}^{K} \lambda_{i} \left(\sum_{l=L+1}^{\infty} \boldsymbol{\Phi}^{l}\right)$$
$$= \sum_{i=1}^{K} \sum_{l=L+1}^{\infty} \delta_{i}^{l} \qquad (16)$$
$$= \sum_{i=1}^{K} \frac{\delta_{i}^{L+1} (1 - \delta_{i}^{\infty})}{1 - \delta_{i}}$$
$$= \sum_{i=1}^{K} \frac{\delta_{i}^{L+1}}{1 - \delta_{i}} > 0,$$

where $\lambda_i(\bullet)$ denotes the *i*-th eigenvalue of a matrix.

As a result,

$$\left| \operatorname{tr} \left[\sum_{l=0}^{\infty} \Phi^{l} \right] - \operatorname{tr} \left[\sum_{l=0}^{L} \Phi^{l} \right] \right| = \left| \operatorname{tr} \left[\sum_{l=L+1}^{\infty} \Phi^{l} \right] \right| = \sum_{i=1}^{K} \frac{\delta_{i}^{L+1}}{1 - \delta_{i}}.$$

B. Proof of the robustness of the MIA reconstruction

Assume that the graph signal has the same energy for different SNR, i.e., $\mathbf{E}[\mathbf{x}\mathbf{x}^T]$ is a constant matrix, and the distribution of noise \mathbf{n} is iid with zero mean and variance σ^2 which varies with SNR. A corrupted *K*-BL graph signal is $\mathbf{y}_S = \mathbf{x}_S + \mathbf{n}$.

1) Least square (LS) reconstruction

If original signal is recovered by the LS reconstruction method, i.e.,

$$\begin{split} \mathbf{\hat{x}}_{\text{LS}} &= \mathbf{V}_{K}(\mathbf{C}\mathbf{V}_{K})^{\dagger}\mathbf{y}_{S} \\ &= \mathbf{V}_{K}(\mathbf{C}\mathbf{V}_{K})^{\dagger}(\mathbf{x}_{S} + \mathbf{n}) \\ &= \mathbf{x} + \mathbf{V}_{K}(\mathbf{C}\mathbf{V}_{K})^{\dagger}\mathbf{n}, \end{split}$$

the expected mean square error (MSE) is

$$\mathbf{E} \| \hat{\mathbf{x}}_{LS} - \mathbf{x} \|_{2}^{2} = \mathbf{E} \| \mathbf{V}_{K} (\mathbf{C} \mathbf{V}_{K})^{\dagger} \mathbf{n} \|_{2}^{2}
= \mathbf{E} \left[\operatorname{tr} \left[\left(\mathbf{V}_{K} (\mathbf{C} \mathbf{V}_{K})^{\dagger} \mathbf{n} \right) \left(\mathbf{V}_{K} (\mathbf{C} \mathbf{V}_{K})^{\dagger} \mathbf{n} \right)^{T} \right] \right]
= \operatorname{tr} \left[\mathbf{E} \left(\mathbf{V}_{K} (\mathbf{C} \mathbf{V}_{K})^{\dagger} \mathbf{n} \mathbf{n}^{T} (\mathbf{C} \mathbf{V}_{K})^{\dagger} \mathbf{V}_{K}^{T} \right) \right]
= \operatorname{tr} \left[\mathbf{V}_{K} (\mathbf{C} \mathbf{V}_{K})^{\dagger} \mathbf{E} (\mathbf{n} \mathbf{n}^{T}) (\mathbf{C} \mathbf{V}_{K})^{\dagger} \mathbf{V}_{K}^{T} \right]$$
(17)
= $\operatorname{tr} \left[\mathbf{V}_{K} (\mathbf{C} \mathbf{V}_{K})^{\dagger} (\sigma^{2} \mathbf{I}) (\mathbf{C} \mathbf{V}_{K})^{\dagger} \mathbf{V}_{K}^{T} \right]$
= $\sigma^{2} \operatorname{tr} \left[\left((\mathbf{C} \mathbf{V}_{K})^{T} (\mathbf{C} \mathbf{V}_{K}) \right)^{-1} \right].$

Moreover,

$$\operatorname{tr}\left[\left(\left(\mathbf{C}\mathbf{V}_{K}\right)^{T}\left(\mathbf{C}\mathbf{V}_{K}\right)\right)^{-1}\right] = \operatorname{tr}\left[\sum_{l=0}^{\infty} \mathbf{\Phi}^{l}\right]$$
$$\stackrel{\triangle}{=} \sum_{i=1}^{K} \sum_{l=0}^{\infty} \delta_{i}^{l} = \sum_{i=1}^{K} \frac{1}{1-\delta_{i}},$$

where the first equality holds because of Proposition 1 and $\stackrel{\triangle}{=}$ holds by reusing the same derivation used in (16).

Therefore,

$$\mathbf{E} \|\hat{\mathbf{x}}_{\text{LS}} - \mathbf{x}\|_{2}^{2} = \sigma^{2} \sum_{i=1}^{K} \frac{1}{1 - \delta_{i}}.$$
 (18)

2) Proposed matrix inversion approximation (MIA) reconstruction

According to Proposition 1 in our paper, the above LS reconstruction is equivalent to

$$\hat{\mathbf{x}}_{\text{LS}} = \mathbf{V}_{K} \left[(\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{C} \mathbf{V}_{K} \right]^{-1} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{y}_{S}$$
$$= \mathbf{V}_{K} \sum_{l=0}^{\infty} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{y}_{S}.$$

If an original signal is recovered by the proposed MIA method, i.e., truncating the first L items of the infinite matrix polynomial,

$$\begin{split} \hat{\mathbf{x}}_{\text{MIA}} &= \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} (\mathbf{x}_{S} + \mathbf{n}) \\ &= \mathbf{V}_{K} \left[\sum_{l=0}^{\infty} \mathbf{\Phi}^{l} - \sum_{l=L+1}^{\infty} \mathbf{\Phi}^{l} \right] (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{x}_{S} \\ &+ \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{n} \\ &= \mathbf{x} - \mathbf{V}_{K} \sum_{l=L+1}^{\infty} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{x}_{S} + \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{n} \\ &:= \mathbf{x} - \mathbf{t} + \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{n}, \end{split}$$

where t is a vector representing the Von Neumann series truncation error on the bandlimited signal x_S itself, that remains constant for different noise variance.

Then, the corresponding MSE of the MIA reconstruction is

$$\mathbf{E} \| \hat{\mathbf{x}}_{\text{MIA}} - \mathbf{x} \|_{2}^{2} = \mathbf{E} \left\| -\mathbf{t} + \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{n} \right\|_{2}^{2}$$

$$= \| \mathbf{t} \|_{2}^{2} - 2\mathbf{E} \left[\mathbf{t}^{T} \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{n} \right]$$

$$+ \mathbf{E} \left\| \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{n} \right\|_{2}^{2} \quad (19)$$

$$= \| \mathbf{t} \|_{2}^{2} + \mathbf{E} \left\| \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{n} \right\|_{2}^{2}.$$

Using the same derivation in (17) and the property of trace operation, we have

$$\begin{split} \mathbf{E} \left\| \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{n} \right\|_{2}^{2} \\ &= \operatorname{tr} \left[\mathbf{E} \left(\mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{n} \mathbf{n}^{T} \mathbf{C} \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} \mathbf{V}_{K}^{T} \right) \right] \\ &= \operatorname{tr} \left[\mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{E} (\mathbf{n} \mathbf{n}^{T}) \mathbf{C} \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} \mathbf{V}_{K}^{T} \right] \\ &= \sigma^{2} \operatorname{tr} \left[\sum_{l=0}^{L} \mathbf{\Phi}^{l} (\mathbf{C} \mathbf{V}_{K})^{T} \mathbf{C} \mathbf{V}_{K} \sum_{l=0}^{L} \mathbf{\Phi}^{l} \right] \\ &= \sigma^{2} \operatorname{tr} \left[\left(\mathbf{I} - \mathbf{\Phi} \right) \left(\sum_{l=0}^{L} \mathbf{\Phi}^{l} \right)^{2} \right]. \end{split}$$

Therefore,

$$\mathbf{E} \|\mathbf{\hat{x}}_{\text{MIA}} - \mathbf{x}\|_{2}^{2} = \|\mathbf{t}\|_{2}^{2} + \sigma^{2} \text{tr} \left[(\mathbf{I} - \mathbf{\Phi}) \left(\sum_{l=0}^{L} \mathbf{\Phi}^{l} \right)^{2} \right].$$

The Von Neumann's trace inequality states that

$$\operatorname{tr}\left[\left(\mathbf{I}-\boldsymbol{\Phi}\right)\left(\sum_{l=0}^{L}\boldsymbol{\Phi}^{l}\right)^{2}\right] \leq \sum_{i=1}^{K}\lambda_{i}\left(\mathbf{I}-\boldsymbol{\Phi}\right)\lambda_{i}\left[\left(\sum_{l=0}^{L}\boldsymbol{\Phi}^{l}\right)^{2}\right],$$

where $\lambda_i(\bullet)$ denotes the *i*-th eigenvalue of a matrix. By reusing the derivation in (16), we know

$$\lambda_i \left[\left(\sum_{l=0}^{L} \mathbf{\Phi}^l \right)^2 \right] = \lambda_i^2 \left(\sum_{l=0}^{L} \mathbf{\Phi}^l \right) = \left(\sum_{l=0}^{L} \delta_i^l \right)^2 = \frac{(1 - \delta_i^{L+1})^2}{(1 - \delta_i)^2}$$

Combined with $\lambda_i (\mathbf{I} - \mathbf{\Phi}) = 1 - \delta_i$,

$$\operatorname{tr}\left[\left(\mathbf{I}-\boldsymbol{\Phi}\right)\left(\sum_{l=0}^{L}\boldsymbol{\Phi}^{l}\right)^{2}\right] \leq \sum_{i=1}^{K}\frac{(1-\delta_{i}^{L+1})^{2}}{1-\delta_{i}},\qquad(20)$$

which results in

$$\mathbf{E} \|\mathbf{\hat{x}}_{\text{MIA}} - \mathbf{x}\|_{2}^{2} \le \|\mathbf{t}\|_{2}^{2} + \sigma^{2} \sum_{i=1}^{K} \frac{(1 - \delta_{i}^{L+1})^{2}}{1 - \delta_{i}}.$$
 (21)

We have proved in Proposition 1 that $0 \le \delta_i < 1$, so

$$\sum_{i=1}^{K} \frac{(1-\delta_i^{L+1})^2}{1-\delta_i} < \sum_{i=1}^{K} \frac{1}{1-\delta_i}.$$
(22)

Combining (18), (21) and (22), we can see that when noise

variance, i.e., σ^2 , is very large, the proposed MIA reconstruction method will achieve better MSE performance. Thus, we can safely claim that our proposed reconstruction method is more robust to large noise than the LS reconstruction.